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DIFFERENTIAL INVARIANCE OF THE MULTIPLICITY OF REAL AND COMPLEX ANALYTIC SETS

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Abstract: This paper is devoted to proving the differential invariance of the multiplicity of real and complex analytic sets. In particular, we prove the real version of the Gau–Lipman theorem, i.e., it is proved that the multiplicity mod 2 of real analytic sets is a differential invariant. We also prove a generalization of the Gau–Lipman theorem.

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1. Introduction

In 1983, Y.-N. Gau and J. Lipman proved ([**6**]) the following result about the differential invariance of the multiplicity of complex analytic sets (see [**1**] for a definition of multiplicity of complex analytic sets):

Theorem 1.1 (Gau–Lipman theorem)**.** *Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If there exists a homeomorphism $\varphi\colon (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ such that φ and φ^{-1} have a derivative at the origin (as mappings from $(\mathbb{R}^{2n}, 0)$ to $(\mathbb{R}^{2n}, 0)$), then $m(X, 0) = m(Y, 0)$.*

This result was a generalization of the result proved separately by R. Ephraim in [**3**] and D. Trotman in [**13**] (see also [**14**]). They showed that the following question has a positive answer when the homeomorphism φ is a C^1 diffeomorphism.

> **Question A.** Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two complex analytic functions. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then is it true that $m(V(f), 0) = m(V(g), 0)$?

This question was asked by O. Zariski in 1971 (see [**17**]) and in its stated version is known as Zariski's multiplicity conjecture. It is still an open problem.

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Here, we are interested in the case of real analytic sets. However, the problem has a negative answer in this case, as we can see in the following example.

Example 1.2. Let $X=\{(x,y)\in\mathbb{R}^2;\ y=0\}$, $Y=\{(x,y)\in\mathbb{R}^2;\ y^3=x^2\}$, and $\varphi\colon \mathbb{R}^2\to\mathbb{R}^2$ given by $\varphi(x,y)=(x,x^{\frac{2}{3}}-y)$. Then φ is a homeomorphism such that $\varphi(X)=Y$, but $m(X)\equiv 1 \bmod 2$ and $m(Y)\equiv 0 \bmod 2$.

However, some authors have approached Question A in the real case. For example, J.-J. Risler in [**10**] proved that the multiplicity mod 2 of a real analytic curve is invariant by bi-Lipschitz homeomorphisms. T. Fukui, K. Kurdyka, and L. Paunescu proposed in [**5**] the following conjecture:

Conjecture F-K-P. Let $h\colon(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be the germ of a subanalytic, arc-analytic, bi-Lipschitz homeomorphism, and let $X,Y\subset\mathbb{R}^n$ be two irreducible analytic germs. Suppose that $Y=h(X)$, then $m(X)=m(Y)$.

They proved that multiplicity of a real analytic curve is invariant by arc-analytic bi-Lipschitz homeomorphisms. G. Valette proved in [**15**] that the multiplicity mod 2 of a real analytic hypersurface is invariant by arc-analytic bi-Lipschitz homeomorphisms and the multiplicity mod 2 of a real analytic surface is invariant by subanalytic bi-Lipschitz homeomorphisms, and the present author proved in [**12**] that the multiplicity mod 2 of a real analytic surface is invariant by bi-Lipschitz homeomorphisms.

The main aim of this paper is to prove the real version of the Gau–Lipman theorem, i.e., to prove that the multiplicity mod 2 of real analytic sets is a differential invariant (see Corollary 3.2). Let us remark that Y.-N. Gau and J. Lipman's proof does not work in the real setting, since their proof uses, for instance, that the tangent cone at a point of a complex analytic set is a complex algebraic set, which may not happen for tangent cones of real analytic sets.

Let us describe how this paper is organized. In Section 2 we present some preliminaries. In Section 3 we present a result on the differential invariance of the multiplicity of real analytic sets (see Theorem 3.1) and as a corollary we obtain the real version of the Gau–Lipman theorem (see Corollary 3.2). We also present some examples in order to show that the hypotheses of Theorem 3.1 cannot be removed. In Section 4 we present a generalization of the Gau–Lipman theorem (see Theorem 4.1), which is the complex version of Theorem 3.1. An example showing that the hypotheses in Theorem 4.1 are weaker than the hypotheses in the Gau–Lipman theorem is also presented (see Example 4.2).

2. Preliminaries

Here all real analytic sets are assumed to be pure dimensional.

Definition 2.1. Let $X \subset \mathbb{R}^n$ be a subset such that $x_0 \in \overline{X}$. We say that $v \in \mathbb{R}^n$ is a tangent vector of X at $x_0 \in \mathbb{R}^n$ if there is a sequence of points $\{x_i\} \subset X$ tending to $x_0 \in \mathbb{R}^n$ and there is a sequence of positive numbers $\{t_i\} \subset \mathbb{R}^+$ such that

$$\lim_{i\to\infty} \frac{1}{t_i}(x_i - x_0) = v.$$

Let $C(X, x_0)$ denote the set of all tangent vectors of X at $x_0 \in \mathbb{R}^n$. We call $C(X, x_0)$ the *tangent cone* of X at x_0.

Remark 2.2. It follows from the curve selection lemma for subanalytic sets that, if $X \subset \mathbb{R}^n$ is a subanalytic set and $x_0 \in \overline{X}$ is a non-isolated point, then the following holds true:

$$C(X, x_0) = \{v;\ \exists \text{ subanalytic } \alpha\colon [0, \varepsilon) \to \mathbb{R}^n \text{ s.t. } \alpha(0) = x_0,$$
$$\alpha((0, \varepsilon)) \subset X, \text{ and } \alpha(t) - x_0 = tv + o(t)\}.$$

Definition 2.3. The mapping $\beta_n\colon \mathbb{S}^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n$ given by $\beta_n(x, r) = rx$ is called *spherical blowing-up* (at the origin) of $\mathbb{R}^n$.

Note that $\beta_n\colon \mathbb{S}^{n-1} \times (0, +\infty) \to \mathbb{R}^n \setminus \{0\}$ is a homeomorphism with inverse $\beta_n^{-1}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times (0, +\infty)$ given by $\beta_n^{-1}(x) = \left(\frac{x}{\|x\|}, \|x\|\right)$.

Definition 2.4. The *strict transform* of the subset X under the spherical blowing-up β_n is $X' := \overline{\beta_n^{-1}(X \setminus \{0\})}$ and the *boundary* $\partial X'$ of the *strict transform* is $\partial X' := X' \cap (\mathbb{S}^{n-1} \times \{0\})$.

Note that $\partial X' = C_X \times \{0\}$, where $C_X = C(X, 0) \cap \mathbb{S}^{n-1}$.

2.1. Multiplicity and relative multiplicities. Let $X \subset \mathbb{R}^n$ be a d-dimensional real analytic set with $0 \in X$ and

$$X_{\mathbb{C}} = V(\mathcal{I}_{\mathbb{R}}(X, 0)),$$

where $\mathcal{I}_{\mathbb{R}}(X, 0)$ is the ideal in $\mathbb{C}\{z_1, \ldots, z_n\}$ generated by the complexifications of all germs of real analytic functions that vanish on the germ $(X, 0)$. We have that $X_{\mathbb{C}}$ is a germ of a complex analytic set and $\dim_{\mathbb{C}} X_{\mathbb{C}} = \dim_{\mathbb{R}} X$ (see [**8**, Propositions 1 and 3, pp. 91–93]). Then, for a linear projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^d$ such that $\pi^{-1}(0) \cap C(X_{\mathbb{C}}, 0) = \{0\}$, there exists an open neighborhood $U \subset \mathbb{C}^n$ of 0 such that $\#(\pi^{-1}(x) \cap (X_{\mathbb{C}} \cap U))$ is constant for a generic point $x \in \pi(U) \subset \mathbb{C}^d$. This number is the multiplicity of $X_{\mathbb{C}}$ at the origin and it is denoted by $m(X_{\mathbb{C}}, 0)$.

Definition 2.5. With the above notation, we define the multiplicity of X at the origin by $m(X) := m(X_{\mathbb{C}}, 0)$.

Definition 2.6. We shall not distinguish between a $2(n-d)$-dimensional real linear subspace in $\mathbb{C}^n$ and its canonical image in $G^{2n}_{2(n-d)}(\mathbb{R})$. Thus we regard $G^n_{n-d}(\mathbb{C})$ as a subset of $G^{2n}_{2(n-d)}(\mathbb{R})$. Let $\mathcal{E}(X_{\mathbb{C}})$ denote the subset of $G^{2n}_{2(n-d)}(\mathbb{R})$ consisting of all $L \in G^{2n}_{2(n-d)}(\mathbb{R})$ such that $L \cap C(X_{\mathbb{C}}, 0) = \{0\}$.

Remark 2.7. We have the following comments on the set $\mathcal{E}(X_{\mathbb{C}})$.

(i) $\mathcal{E}(X_{\mathbb{C}})$ is an open dense set in $G^{2n}_{2(n-d)}(\mathbb{R}) \cong G^{2n}_{2d}(\mathbb{R})$ (see [**2**, Lemme 1.4]).

(ii) For each $L \in \mathcal{E}(X_{\mathbb{C}}) \cap G^n_{n-d}(\mathbb{C})$, let $\pi_L\colon \mathbb{C}^n \to L^{\perp}$ be the orthogonal projection over L. Then there exist a polydisc $U \subset \mathbb{C}^n$ and a complex analytic set $\sigma \subset U' := \pi_L(U)$ such that $\dim \sigma < \dim X_{\mathbb{C}}$ and $\pi_L\colon (U \cap X_{\mathbb{C}}) \setminus \pi_L^{-1}(\sigma) \to U' \setminus \sigma$ is a k-sheeted cover with $k = m(X_{\mathbb{C}}, 0)$ (see [**16**, Theorem 7P, p. 234]).

(iii) Since $\pi := \pi_L$ is an $\mathbb{R}$-linear mapping, we identify the d-dimensional real linear subspace $\pi(\mathbb{R}^n)$ with $\mathbb{R}^d$ and, with this identification, we obtain that $\mathbb{R}^d \cap \sigma$ is a closed nowhere dense subset of $\mathbb{R}^d \cap U'$. Indeed, it is clear that $\mathbb{R}^d \cap \sigma$ is a closed subset of $\mathbb{R}^d \cap U'$ and thus, if σ is somewhere dense in $\mathbb{R}^d \cap U'$, then σ contains an open ball $B_r(p) \subset \mathbb{R}^d \cap U'$, which implies that σ must contain a non-empty open subset of U' (see [**8**, Proposition 1, p. 91]) and thus we obtain a contradiction. Therefore, σ is nowhere dense in $\mathbb{R}^d \cap U'$ and then $\mathbb{R}^d \cap U' \setminus \sigma$ is an open dense subset of $\mathbb{R}^d \cap U'$.

(iv) For a generic point $x \in \mathbb{R}^d$ near to the origin (i.e., for $x \in (\mathbb{R}^d \cap U') \setminus \sigma$), we have

$$\begin{aligned} m(X_{\mathbb{C}}, 0) &= \#(\pi^{-1}(x) \cap (X_{\mathbb{C}} \cap U)) \\ &= \#(\mathbb{R}^n \cap \pi^{-1}(x) \cap (X_{\mathbb{C}} \cap U)) \\ &\quad + \#((\mathbb{C}^n \setminus \mathbb{R}^n) \cap \pi^{-1}(x) \cap (X_{\mathbb{C}} \cap U)) \\ &= \#(\pi^{-1}(x) \cap (X \cap U)) + \#(\pi^{-1}(x) \cap ((X_{\mathbb{C}} \setminus \mathbb{R}^n) \cap U)). \end{aligned}$$

Since for each $f \in \mathcal{I}_{\mathbb{R}}(X, 0)$ we may write $f(z) = \sum\limits_{|I|=k}^{\infty} a_I z^I$ such that $a_I \in \mathbb{R}$ for all I, then $f(z_1, \ldots, z_n) = 0$ if and only if $f(\bar{z}_1, \ldots, \bar{z}_n) = 0$, where each $\bar{z}_i$ denotes the complex conjugate of z_i. In particular, $\#(\pi^{-1}(x) \cap ((X_{\mathbb{C}} \setminus \mathbb{R}^n) \cap U))$ is an even number. Therefore, we obtain that $m(X) \equiv \#(\pi^{-1}(x) \cap (X \cap U)) \mod 2$ for a generic point $x \in \mathbb{R}^d$ near to the origin.

Definition 2.8. Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{X}$ is a non-isolated point. We say that $x \in \partial X'$ is *a simple point of* $\partial X'$, if there is an open set $U \subset \mathbb{R}^{n+1}$ with $x \in U$ such that:

(i) the connected components of $(X' \cap U) \setminus \partial X'$, say $M_1, \ldots, M_r$, are topological manifolds with $\dim M_i = \dim X$, $i = 1, \ldots, r$;
(ii) $(M_i \cup \partial X') \cap U$ are topological manifolds with boundary.

Let $\mathrm{Smp}(\partial X')$ be the set of simple points of $\partial X'$.

Remark 2.9. By Theorems 2.1 and 2.2 in [**9**], we obtain that $\mathrm{Smp}(\partial X')$ is an open dense subset of the $(d-1)$-dimensional part of $\partial X'$ whenever $\partial X'$ is a $(d-1)$-dimensional subset, where $d = \dim X$.

Definition 2.10. Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in X$. We define $k_X \colon \mathrm{Smp}(\partial X') \to \mathbb{N}$ such that $k_X(x)$ is the number of connected components of the germ $(\beta_n^{-1}(X \setminus \{0\}), x)$.

Remark 2.11. It is clear that the function k_X is locally constant. In fact, k_X is constant in each connected component C_j of $\mathrm{Smp}(\partial X')$. Then we define $k_X(C_j) := k_X(x)$ with $x \in C_j$.

Remark 2.12. The numbers $k_X(C_j)$ are equal to the numbers n_j defined by Kurdyka and Raby [**7**, p. 762].

Remark 2.13. When X is a complex analytic set, there is a complex analytic set Σ with $\dim \Sigma < \dim X$, such that $X_j \setminus \Sigma$ intersects only one connected component C_i of $\mathrm{Smp}(\partial X')$ (see [**1**, pp. 132–133]), for each irreducible component X_j of the tangent cone $C(X, 0)$. Then we define $k_X(X_j) := k_X(C_i)$.

Remark 2.14 ([**1**, p. 133, Proposition]). Let X be a complex analytic set of $\mathbb{C}^n$ with $0 \in X$ and let $X_1, \ldots, X_r$ be the irreducible components of $C(X, 0)$. Then

$$m(X, 0) = \sum_{j=1}^{r} k_X(X_j) \cdot m(X_j, 0).$$

Definition 2.15. Let $X \subset \mathbb{R}^n$ be a real analytic set with $0 \in X$. We denote by C'_X the union of all connected components C_j of $\mathrm{Smp}(\partial X')$ having odd $k_X(C_j)$. We call C'_X the *odd part of* $C_X \subset \mathbb{S}^{n-1}$.

Definition 2.16. Let $X \subset \mathbb{R}^n$ be a d-dimensional real analytic set with $0 \in X$, $L \in \mathcal{E}(X_{\mathbb{C}}) \cap G^n_{n-d}(\mathbb{C})$, and let $\pi := \pi_L \colon \mathbb{C}^n \to L^{\perp}$ be the orthogonal projection over L. Let $\pi' \colon \mathbb{S}^{n-1} \setminus L \to \mathbb{S}^{d-1}$ be the mapping given by $\pi'(u) = \frac{\pi(u)}{\|\pi(u)\|}$, where we are identifying $\pi(\mathbb{R}^n)$ with $\mathbb{R}^d$ and $\pi(\mathbb{R}^n) \cap \mathbb{S}^{2n-1}$ with $\mathbb{S}^{d-1}$ (see Remark 2.7(iii)). We define

$$\varphi_{\pi,C'_X}(x) := \#(\pi'^{-1}(x) \cap C'_X).$$

In this case, if $\varphi_{\pi,C'_X}(x)$ mod 2 is constant for a generic $x \in \mathbb{S}^{d-1}$, we write $m_\pi(C'_X) := \varphi_{\pi,C'_X}(x)$ mod 2, for a generic $x \in \mathbb{S}^{d-1}$.

3. Proof of the real version of the Gau–Lipman theorem

In this section we show that the multiplicity mod 2 of a real analytic set is a differential invariant, which is the real version of the Gau–Lipman theorem. In fact, we prove a little bit more, as we can see in the next result.

Theorem 3.1. *Let $X, Y \subset \mathbb{R}^N$ be two real analytic sets with $0 \in X \cap Y$. Assume that there exists a mapping $\varphi\colon (\mathbb{R}^N, 0) \to (\mathbb{R}^N, 0)$ such that $\varphi\colon (X, 0) \to (Y, 0)$ is a homeomorphism. If φ has a derivative at the origin and $D\varphi_0\colon \mathbb{R}^N \to \mathbb{R}^N$ is an isomorphism, then $m(X) \equiv m(Y)$ mod 2.*

Proof: Since $\phi := D\varphi_0\colon \mathbb{R}^N \to \mathbb{R}^N$ is an $\mathbb{R}$-linear isomorphism, we have that $A = \phi(X)$ is a real analytic set.

We have that the complexification of ϕ, denoted by $\phi_{\mathbb{C}}$, is a complex diffeomorphism between $X_{\mathbb{C}}$ and $A_{\mathbb{C}}$. Thus, by the proposition in ([**1**, Section 11, p. 120]), $m(X_{\mathbb{C}}, 0) = m(A_{\mathbb{C}}, 0)$. Therefore, $m(X) = m(A)$.

Thus it is enough to show that $m(Y) \equiv m(A)$ mod 2. In order to do this, we consider the mapping $\psi\colon (Y, 0) \to (A, 0)$ given by $\psi = \phi \circ \varphi^{-1}$.

Claim 3.1.1. The mapping $\psi'\colon Y' \to A'$ given by

$$\psi'(x,t) = \begin{cases} \left(\dfrac{\psi(tx)}{\|\psi(tx)\|}, \|\psi(tx)\|\right), & t \neq 0, \\ (x, 0), & t = 0 \end{cases}$$

is a homeomorphism.

Proof of Claim 3.1.1: Observe that $\nu\colon \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ given by

$$\nu(x) = \frac{\phi(x)}{\|\phi(x)\|}$$

is a homeomorphism, and using that $\varphi(tx) = t\phi(x) + o(t)$ we obtain

$$\lim_{t \to 0^+} \frac{\varphi(tx)}{\|\varphi(tx)\|} = \frac{\phi(x)}{\|\phi(x)\|} = \nu(x).$$

Therefore, the mappings $\phi'\colon \mathbb{S}^{N-1}\times[0,\infty)\to\mathbb{S}^{N-1}\times[0,\infty)$ and $\varphi'\colon X'\to Y'$ given by

$$\phi'(x,t)=\begin{cases}\left(\dfrac{\phi(tx)}{\|\phi(tx)\|},\|\phi(tx)\|\right), & t\neq 0,\\ (\nu(x),0), & t=0,\end{cases}$$

and

$$\varphi'(x,t)=\begin{cases}\left(\dfrac{\varphi(tx)}{\|\varphi(tx)\|},\|\varphi(tx)\|\right), & t\neq 0,\\ (\nu(x),0), & t=0\end{cases}$$

are homeomorphisms, which implies that the mapping $(\varphi^{-1})'\colon Y'\to X'$ given by

$$(\varphi^{-1})'(x,t)=\begin{cases}\left(\dfrac{\varphi^{-1}(tx)}{\|\varphi^{-1}(tx)\|},\|\varphi^{-1}(tx)\|\right), & t\neq 0,\\ (\nu^{-1}(x),0), & t=0\end{cases}$$

is also a homeomorphism. Since $\psi'=\phi'\circ(\varphi^{-1})'$, we finish the proof of Claim 3.1.1. □

As a direct consequence, we obtain that $\mathrm{Smp}(\partial Y')=\psi'(\mathrm{Smp}(\partial Y'))=\mathrm{Smp}(\partial A')$.

Claim 3.1.2. $k_Y(p)=k_A(p)$ for all $p\in\mathrm{Smp}(\partial Y')$.

Proof of Claim 3.1.2: In fact, let $p\in\mathrm{Smp}(\partial Y')$ be a point and let $U\subset Y'$ be a small neighborhood of p. Since $\psi'\colon Y'\to A'$ is a homeomorphism, we have that $V=\psi'(U)$ is a small neighborhood of $p=\psi'(p)\in\partial A'$. Moreover, $\psi'(U\setminus\partial Y')=V\setminus\partial A'$, since $\psi'|_{\partial Y'}\colon\partial Y'\to\partial A'$ is a homeomorphism as well. Using once more that ψ' is a homeomorphism, we obtain that the number of connected components of $U\setminus\partial Y'$ is equal to the number of connected components of $V\setminus\partial A'$, showing that $k_Y(p)=k_A(p)$ for all $p\in\mathrm{Smp}(\partial Y')$. □

As a direct consequence, we obtain that $C'_Y=\psi'(C'_Y)=C'_A$.

Let $L\in\mathcal{E}(Y_{\mathbb{C}})\cap G^N_{N-d}(\mathbb{C})$ and let $\pi:=\pi_L\colon\mathbb{C}^N\to L^\perp$ be the orthogonal projection over L, where $d=\dim Y$ (see Remark 2.7). Let $\pi'\colon\mathbb{S}^{N-1}\setminus L\to\mathbb{S}^{d-1}$ be given by $\pi'(u)=\frac{\pi(u)}{\|\pi(u)\|}$, where we are identifying $\pi(\mathbb{R}^N)$ with $\mathbb{R}^d$ and $\pi(\mathbb{R}^N)\cap\mathbb{S}^{2N-1}$ with $\mathbb{S}^{d-1}$ as in Definition 2.16.

Claim 3.1.3. $\varphi_{\pi,C'_Y}(y)=\#(\pi'^{-1}(y)\cap C'_Y) \bmod 2$ is constant for a generic point $y\in\mathbb{S}^{d-1}$. Moreover, $m_\pi(C'_Y)\equiv m(Y) \bmod 2$.

Proof of Claim 3.1.3: If $\dim C_Y<d-1$, then $C'_Y=\emptyset$ and $\dim C(\pi(Y),0)<d$, which implies that there exist $w\in\mathbb{S}^{d-1}$ and small enough numbers $\eta,\varepsilon\in(0,1)$ such that $C_{\eta,\varepsilon}(y)\cap\pi(Y)=\emptyset$, where $C_{\eta,\varepsilon}(w)=\{v\in\mathbb{R}^d;\ \|v-tw\|\leq\eta t,\ t\in(0,\varepsilon]\}$. Therefore $\varphi_{\pi,C'_Y}(y)=0$ for any point $y\in\mathbb{S}^{d-1}$ and $m(Y)\equiv 0 \bmod 2$, since $C'_Y=\emptyset$ and $\pi^{-1}(v)\cap Y=\emptyset$, for all $v\in C_{\eta,\varepsilon}(w)$ (see Remark 2.7(iv)). In particular, $m_\pi(C'_Y)$ is defined and satisfies $m_\pi(C'_Y)\equiv m(Y) \bmod 2$.

Thus we may assume that $\dim C_Y=d-1$. By Remark 2.9, $\mathrm{Smp}(\partial Y')$ is an open dense subset of the $(d-1)$-dimensional part of $\partial Y'=C_Y\times\{0\}\cong C_Y$. Let $y\in\mathbb{S}^{d-1}$ be a generic point such that $\pi'^{-1}(y)\cap C_Y=\pi'^{-1}(y)\cap\mathrm{Smp}(\partial Y')=\{y_1,\ldots,y_p\}$ and $u=\#(\pi^{-1}(ty)\cap Y)\equiv m(Y) \bmod 2$, for all small enough $t>0$ (see Remark 2.7(iv)). Then we have the following:

$$u=\sum_{j=1}^{p}k_Y(y_j).$$

In fact, let $\eta,\varepsilon>0$ be small enough numbers such that $C_{\eta,\varepsilon}(y)\cap\pi(br(\pi|_Y))=\emptyset$, where $C_{\eta,\varepsilon}(y)=\{v\in\mathbb{R}^d;\ \|v-ty\|\leq\eta t,\ t\in(0,\varepsilon]\}$ and $br(\pi|_Y)$ denotes the set of all critical points of $\pi|_Y$. Thus denote the connected components of $(\pi|_Y)^{-1}(C_{\eta,\varepsilon}(y))$ by $Y_1,\ldots,Y_u$. Hence, $\pi|_{Y_i}\colon Y_i\to C_{\eta,\varepsilon}(y)$ is a homeomorphism, for $i=1,\ldots,u$. Thus, for each $i=1,\ldots,u$, there is a unique $\gamma_i\colon(0,\varepsilon)\to Y_i$ such that $\pi(\gamma_i(t))=ty$ for all $t\in(0,\varepsilon)$. We define for each $i=1,\ldots,u$, $\widetilde{\gamma}_i\colon[0,\varepsilon)\to\overline{\beta_N^{-1}(Y_i)}$ given by $\widetilde{\gamma}_i(s)=\lim\limits_{t\to s^+}\beta_N^{-1}\circ\gamma_i(t)$, for all $s\in[0,\varepsilon)$.

We remark that $\widetilde{\gamma}_i(0)=\lim\limits_{t\to 0^+}\widetilde{\gamma}_i(t)\in\{y_1,\ldots,y_p\}$ for all $i=1,\ldots,u$ and thus $u\leq\sum\limits_{j=1}^{p}k_Y(y_j)$. Shrinking η, if necessary, we can suppose that each C_{Y_i} contains at most one y_j. Thus for fixed y_j and if $\gamma\colon[0,\delta)\to Y$ is a subanalytic curve such that $\lim\limits_{t\to 0^+}\beta_N^{-1}\circ\gamma(t)=y_j$, then there exists $\delta_0>0$ such that $\pi(\gamma(t))\in C_{\eta,\varepsilon}(y)$, for all $0<t<\delta_0$. So, there is $i\in\{1,\ldots,u\}$ such that $\gamma(t)\in Y_i$, with $0<t<\delta_0$. Then $\widetilde{\gamma}_i(0)=y_j$ and we obtain the equality $u=\sum\limits_{j=1}^{p}k_Y(y_j)$.

Let $C_1,\ldots,C_r$ be the connected components of $\mathrm{Smp}(\partial Y')$. By Remark 2.11, we know that k_Y is constant in each C_i and thus if $y_j,y_{j'}\in C_i$, then $k_Y(y_j) = k_Y(y_{j'})$. Since $\pi'^{-1}(y) \cap C_Y = \pi'^{-1}(y) \cap \operatorname{Smp}(\partial Y') = \{y_1, \ldots, y_p\}$, we have

$$u = \sum_{j=1}^{p} k_Y(y_j) = \sum_{i\in\Lambda} k_Y(C_i) \cdot \#(\pi'^{-1}(y) \cap C_i),$$

where $\Lambda = \{i \in \{1, \ldots, r\};\ \pi'^{-1}(y) \cap C_i \neq \emptyset\}$. Therefore, we obtain

$$u = \sum_{i=1}^{r} k_Y(C_i) \cdot \#(\pi'^{-1}(y) \cap C_i).$$

However, $\sum\limits_{i=1}^{r} k_Y(C_i) \cdot \#(\pi'^{-1}(y) \cap C_i) \equiv \#(\pi'^{-1}(y) \cap C'_Y) \bmod 2$ and $u \equiv m(Y) \bmod 2$, then

$$m(Y) \equiv \#(\pi'^{-1}(y) \cap C'_Y) \bmod 2,$$

for a generic $y \in \mathbb{S}^{d-1}$, which shows that $\varphi_{\pi,C'_Y}(y) = \#(\pi'^{-1}(y) \cap C'_Y) \bmod 2$ is constant for a generic point $y \in \mathbb{S}^{d-1}$ and thus $m_\pi(C'_Y)$ is defined and satisfies $m_\pi(C'_Y) \equiv m(Y) \bmod 2$. □

Then we obtain that $m_\pi(C'_Y)$ does not depend on a generic π, since $m(Y)$ does not depend on a generic π. Similarly, we obtain that $m_{\bar\pi}(C'_A)$ does not depend on a generic projection $\bar\pi$ and $m_{\bar\pi}(C'_A) \equiv m(A) \bmod 2$. Thus we write $m(C'_Y)$ (resp. $m(C'_A)$) instead of $m_\pi(C'_Y)$ (resp. $m_{\bar\pi}(C'_A)$).

Let $\widetilde{L} \in \mathcal{E}(Y_\mathbb{C} \cup A_\mathbb{C}) \cap G^N_{N-d}(\mathbb{C})$ and let $\widetilde{\pi} := \pi_{\widetilde{L}} \colon \mathbb{C}^N \to \widetilde{L}^\perp$ be the orthogonal projection over $\widetilde{L}$. Let $\widetilde{\pi}' \colon \mathbb{S}^{N-1} \setminus \widetilde{L} \to \mathbb{S}^{d-1}$ given by $\widetilde{\pi}'(u) = \frac{\widetilde{\pi}(u)}{\|\widetilde{\pi}(u)\|}$ as in Definition 2.16. Then, for a generic $y \in \mathbb{S}^{d-1}$, we obtain the following:

$$\begin{aligned}
m(Y) &\equiv m(C'_Y) \bmod 2 && \text{(by Claim 3.1.3)}\\
&\equiv \#(\widetilde{\pi}'^{-1}(y) \cap C'_Y) \bmod 2 && \text{(by the definition of } m(C'_Y))\\
&\equiv \#(\widetilde{\pi}'^{-1}(y) \cap C'_A) \bmod 2 && \text{(since } C'_Y = C'_A)\\
&\equiv m(C'_A) \bmod 2 && \text{(by the definition of } m(C'_A))\\
&\equiv m(A) \bmod 2 && \text{(by Claim 3.1.3),}
\end{aligned}$$

which finishes the proof. □

As consequences, we obtain the following.

Corollary 3.2. *Let $X, Y \subset \mathbb{R}^N$ be two real analytic sets containing 0. If there exists a homeomorphism $\varphi\colon (\mathbb{R}^N, X, 0) \to (\mathbb{R}^N, Y, 0)$ such that φ and φ^{-1} have a derivative at the origin, then $m(X) \equiv m(Y) \bmod 2$.*

Proof: Since φ and φ^{-1} have a derivative at 0, we have that $D\varphi_0\colon \mathbb{R}^N \to \mathbb{R}^N$ is an isomorphism and by Theorem 3.1, $m(X) \equiv m(Y) \mod 2$. □

Definition 3.3. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed subsets. We say that a continuous mapping $f\colon X \to Y$ is differentiable at $x \in X$, if there exist an open $U \subset \mathbb{R}^n$ and a continuous mapping $F\colon U \to \mathbb{R}^m$ such that $x \in U$, $F|_{X\cap U} = f|_{X\cap U}$, and F has a derivative at x.

Corollary 3.4. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be two real analytic sets containing* 0*. If there exists a homeomorphism $\phi\colon (X,0) \to (Y,0)$ such that ϕ and ϕ^{-1} are differentiable at* 0*, then $m(X) \equiv m(Y) \mod 2$.*

Proof: By hypothesis there are closed representatives A and B respectively of $(X,0)$ and $(Y,0)$ and a homeomorphism $\phi\colon A \to B$ such that $\phi(0) = 0$, and ϕ and ϕ^{-1} have a derivative at 0. Let $\widetilde{\phi}\colon \mathbb{R}^m \to \mathbb{R}^n$ (resp. $\widetilde{\psi}\colon \mathbb{R}^n \to \mathbb{R}^m$) be a continuous extension of ϕ (resp. ϕ^{-1}), which has a derivative at $0 \in \mathbb{R}^m$ (resp. $0 \in \mathbb{R}^n$). Then the mapping $\varphi\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ given by

$$\varphi(x,y) = (x - \widetilde{\psi}(y + \widetilde{\phi}(x)), y + \widetilde{\phi}(x))$$

is a homeomorphism such that $\varphi(A \times \{0\}) = \{0\} \times B$, its inverse is given by

$$\varphi^{-1}(z,w) = (z + \widetilde{\psi}(w), w - \widetilde{\phi}(z + \widetilde{\psi}(w))),$$

and both have a derivative at $0 \in \mathbb{R}^{m+n}$.

Since $m(A \times \{0\}) = m(A) = m(X)$ and $m(\{0\} \times B) = m(B) = m(Y)$, by Corollary 3.2, we obtain $m(X) \equiv m(Y) \mod 2$. □

Let us make some remarks on Theorem 3.1. Firstly, the assumption that $D\varphi_0$ is an isomorphism cannot be removed, as is shown in the next example.

Example 3.5. Let $X = \{(x,y) \in \mathbb{R}^2;\ y^3 = x^2\}$ and $Y = \{(x,y) \in \mathbb{R}^2;\ y = 0\}$. Then $\varphi\colon (\mathbb{R}^2, X, 0) \to (\mathbb{R}^2, Y, 0)$ given by $\varphi(x,y) = (x, y^3 - x^2)$ is a homeomorphism, which has a derivative at the origin, but $D\varphi_0$ is not an isomorphism. In this case, $m(X) = 2$ and $m(Y) = 1$.

Secondly, we cannot expect equality (without modulus 2), as is shown in the next example.

Example 3.6. Let $V = \{(x,y,z) \in \mathbb{R}^3;\ z^3 = x^5y + xy^5\}$. Then the mapping $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $\varphi(x,y,z) = (x, y, z - (x^5y + xy^5)^{\frac{1}{3}})$ is a homeomorphism which has a derivative at the origin and its inverse also has a derivative at the origin. Moreover, $\varphi(V) = \mathbb{R}^2 \times \{0\}$, but $m(V) = 3$ and $m(\mathbb{R}^2 \times \{0\}) = 1$.

We finish this section by presenting an example of a mapping which has a derivative at the origin and is a homeomorphism between two analytic sets, but its inverse does not have a derivative at the origin.

Example 3.7. The mapping $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi(x,y) = \begin{cases} \left(x, y + 2y^2 \sin \dfrac{1}{y}\right), & y \neq 0, \\ (x,0), & y = 0 \end{cases}$$

has a derivative at the origin, $D\varphi_0 = \mathrm{id}\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\varphi|_{\mathbb{R}\times\{0\}}\colon \mathbb{R}\times\{0\} \to \mathbb{R}\times\{0\}$ is a homeomorphism, but it does not have an inverse which has a derivative at the origin.

4. A generalization of the Gau–Lipman theorem

In this section we present a complex version of Theorem 3.1, which is a generalization of the Gau–Lipman theorem.

Theorem 4.1. *Let $X, Y \subset \mathbb{C}^N$ be two complex analytic sets with $0 \in X \cap Y$. Assume that there exists a mapping $\varphi\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ such that $\varphi|_X\colon (X,0) \to (Y,0)$ is a homeomorphism. If φ has a derivative at the origin (as a mapping from $(\mathbb{R}^{2N},0)$ to $(\mathbb{R}^{2N},0)$) and $D\varphi_0\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is an isomorphism, then $m(X,0) = m(Y,0)$.*

Proof: By using that $\phi := D\varphi_0\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is an $\mathbb{R}$-linear isomorphism, we obtain that ϕ maps bijectively the irreducible components of $C(X,0)$ over the irreducible components of $C(Y,0)$ (see Lemma A.8 in [**6**] or Proposition 2 in [**11**]) and the mapping $\varphi'\colon X' \to Y'$ given by

$$\varphi'(x,t) = \begin{cases} \left(\dfrac{\varphi(tx)}{\|\varphi(tx)\|}, \|\varphi(tx)\|\right), & t \neq 0, \\ \left(\frac{\phi(x)}{\|\phi(x)\|}, 0\right), & t = 0 \end{cases}$$

is a homeomorphism. Let $X_1,\dots,X_r$ and $Y_1,\dots,Y_r$ be the irreducible components of $C(X,0)$ and $C(Y,0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1,\dots,r$. Thus, by proceeding as in the proof of Claim 3.1.2, we obtain $k_X(X_j) = k_Y(Y_j)$ for all $j = 1,\dots,r$.

Fix $j \in \{1,\dots,r\}$ and regard X_j and Y_j as real algebraic sets in $\mathbb{R}^{2N} \cong \mathbb{C}^N$. Since ϕ is an $\mathbb{R}$-linear isomorphism, then its complexification $\phi_{\mathbb{C}}\colon \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ is a $\mathbb{C}$-linear isomorphism such that $\phi_{\mathbb{C}}(X_{j\mathbb{C}}) = Y_{j\mathbb{C}}$. By Proposition 2.9 in [**4**], $X_{j\mathbb{C}}$ (resp. $Y_{j\mathbb{C}}$) is complex analytic diffeomorphic to $X_j \times c_N(X_j)$ (resp. $Y_j \times c_N(Y_j)$), where $c_N\colon \mathbb{C}^N \to \mathbb{C}^N$ is the conjugation mapping given by $c_N(z_1,\ldots,z_N) = (\overline{z}_1,\ldots,\overline{z}_N)$. Then,

$$m(X_{j\mathbb{C}},0) = m(X_j \times c_N(X_j),0) = m(Y_j \times c_N(Y_j),0) = m(Y_{j\mathbb{C}},0),$$

since the multiplicity is invariant by complex analytic diffeomorphisms (see [**1**, Section 11, p. 120, Proposition]). However, $c_N(X_j)$ and $c_N(Y_j)$ are complex analytic sets satisfying $m(c_N(X_j),0) = m(X_j,0)$ and $m(c_N(Y_j),0) = m(Y_j,0)$, then we obtain $m(X_j \times c_N(X_j),0) = m(X_j,0)^2$ and $m(Y_j \times c_N(Y_j),0) = m(Y_j,0)^2$, so we obtain $m(X_j,0) = m(Y_j,0)$, for all $j \in \{1,\ldots,r\}$.

By Remark 2.14,

$$m(X,0) = \sum_{j=1}^{r} k_X(X_j)\cdot m(X_j,0)$$

and

$$m(Y,0) = \sum_{j=1}^{r} k_Y(Y_j)\cdot m(Y_j,0).$$

Therefore, $m(X,0) = m(Y,0)$. □

It is clear that as a consequence of Theorem 4.1, we obtain the Gau–Lipman theorem. The next example shows that Theorem 4.1 is really a generalization of the Gau–Lipman theorem.

Example 4.2. Let $X = \{(x,y) \in \mathbb{C}^2;\ y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 = 0\}$ and $\widetilde{X} = \{(x,y) \in \mathbb{C}^2;\ y^4 - 2x^3y^2 - 4x^6y + x^6 - x^9 = 0\}$. The mapping $\Phi\colon (\mathbb{C},0) \to (X,0)$ given by $\Phi(t) = (t^4, t^6 + t^7)$ is a Puiseux parametrization of X and there exists a complex analytic function $\phi\colon (\mathbb{C},0) \to (\mathbb{C},0)$ such that $\mathrm{ord}_0(\phi) > 9$ and the mapping $\widetilde{\Phi}\colon (\mathbb{C},0) \to (\widetilde{X},0)$ given by $\widetilde{\Phi}(t) = (t^4, t^6 + t^9 + \phi(t))$ is a Puiseux parametrization of $\widetilde{X}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(s) = \begin{cases} s + 2s^2 \sin\dfrac{1}{s}, & s \neq 0,\\[2ex] 0, & s = 0,\end{cases}$$

and $\varphi\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be the mapping given by

$$\varphi(x,y) = \begin{cases} \widetilde{\Phi}(t), & \text{if } (x,y) = \Phi(t) \text{ for some } t \in \mathbb{C},\\[2ex] \left(x, f\left(\dfrac{y+\overline{y}}{2}\right) + if\left(\dfrac{y-\overline{y}}{2}\right)\right), & \text{if } (x,y) \neq \Phi(t) \text{ for any } t \in \mathbb{C}.\end{cases}$$

Thus φ has a derivative at the origin, $D\varphi_0 = \mathrm{id}\colon \mathbb{R}^4 \to \mathbb{R}^4$ and $\varphi|_X\colon (X,0) \to (\widetilde{X},0)$ is a homeomorphism. Moreover, since X and $\widetilde{X}$ have different Puiseux pairs, there is no homeomorphism $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $h(X) = \widetilde{X}$.

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