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CORRIGENDUM TO: "AN INTERPOLATION PROPERTY OF LOCALLY STEIN SETS"

Viorel Vâjâitu

To the memory of my teacher and friend Ilie Bârză

Abstract: The purpose of this note is to supply a correct proof of a proposition in the article quoted above.

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Here we revise Proposition 3 from [**5**], which is restated below for the convenience of the reader, and give an accurate proof by expanding an idea proposed by Siu ([**4**, pp. 486–487, Section 4.4]), avoiding the patching technique used in [**5**].

We need to recall some notations. For a reduced complex space X, X_{reg} denotes the set of smooth points of X and $X_{\mathrm{sg}} := X \setminus X_{\mathrm{reg}}$ is the singular locus of X. An open subset Ω of X is locally Stein on a set $T \subset \partial\Omega$, if every point in T has a neighborhood U in X such that $U \cap \Omega$ is Stein.

Proposition 1. *Let X be a normal Stein space of pure dimension n and let $\Omega \subset X$ be an open set that is locally Stein on $\partial\Omega \setminus X_{\mathrm{sg}}$.*

Then, there exists a smooth function $\phi\colon \Omega \longrightarrow [0,\infty)$ that is strictly plurisubharmonic, and for every closed subset Λ of Ω whose closure in X is disjoint with X_{sg}, the restriction of ϕ to Λ is proper.

First, for the proof we need an auxiliary result.

Lemma 1. *Let X be a normal complex space of pure dimension, A a hypersurface in X, and $S \subset X$ a thin set of order 2.*

If $\varphi\colon X \setminus S \longrightarrow [0,\infty)$ is a plurisubharmonic function that vanishes on $A \setminus S$, then the trivial extension $\widetilde{\varphi}$ of φ to X with value 0 on S is plurisubharmonic on X and continuous on S. In particular, $\widetilde{\varphi}$ turns out to be continuous if φ is continuous.

Recall that a closed subset S of a complex space X is said to be *thin of order k* ($k \in \mathbb{N}$) if S is locally contained in a (not necessarily closed) analytic subset of X of codimension k.

Proof of Lemma 1: First note that, if Z is a complex space of pure dimension, granting the maximum principle for plurisubharmonic functions one has that, for every plurisubharmonic function u on Z and any point $z_0 \in Z$,

$$\limsup_{z \mapsto z_0} u(z) = u(z_0).$$

On the other hand, since S is thin of order 2, by [**1**, Satz 4, p. 181] there is a unique plurisubharmonic extension $\psi \colon X \longrightarrow [0,\infty)$ of φ. Let $a \in S$ and Γ a one-dimensional irreducible analytic set in a neighborhood of a in A such that $\Gamma \cap S = \{a\}$.

Since the restriction $\psi|_\Gamma$ is plurisubharmonic, by the above remark and hypothesis, $\psi(a) = 0$. Thus ψ vanishes on S and ψ is continuous on S, because for every point $a \in S$,

$$0 \le \liminf_{x \mapsto a} \psi(x) \le \limsup_{x \mapsto a} \psi(x) = 0 = \psi(a),$$

so that $\psi = \widetilde{\varphi}$, whence the lemma. □

Proof of Proposition 1: We divide the proof into three steps.

Step 1) Let X be a normal Stein space of pure dimension n and consider a discrete holomorphic map $\Phi \colon X \longrightarrow \mathbb{C}^n$, $\Phi = (\Phi_1, \dots, \Phi_n)$.

The *branching locus* $\mathrm{B}(\Phi)$ of Φ is the complement in X of the set of points $a \in X$ such that there are open neighborhoods U of a in X and W of $\Phi(a)$ in $\mathbb{C}^n$ such that Φ induces a biholomorphism between U and W.

Obviously, $\mathrm{B}(\Phi)$ contains the singular part X_{sg} of X and $\mathrm{B}(\Phi)$ is a closed analytic subset of X. Furthermore, $\mathrm{B}(\Phi)$ is either the empty set or a hypersurface.

Also, there is a thin subset of order 2 in X (see [**2**]) such that

$$X_{\mathrm{sg}} \subseteq S \subseteq \mathrm{B}(\Phi),$$

and for an arbitrary point $a \in \mathrm{B}(\Phi) \setminus S$ there are coordinates $(x_1, \dots, x_n)$ centered at a with respect to which the function Φ takes the form

$$\Phi(x', x_n) = (x', x_n^\mu)$$

for some $\mu \in \mathbb{N}$, $\mu \ge 2$, where $x' = (x_1, \dots, x_{n-1})$.

Besides, if $\vartheta_{k_1}, \dots, \vartheta_{k_n}$ are holomorphic vector fields on X generating the tangent vector space at every point of a discrete set $\Lambda \subset X$ disjoint with X_{sg} and containing a point of each connected component of X, then

$$\det(\vartheta_{k_i}(\Phi_j))_{i,j}$$

defines a holomophic function on X_{reg} that extends, due to the normality of X, to a holomorphic function f on X vanishing on $\mathrm{B}(\Phi)$, but f does not vanish identically on any connected component of X.

A straightforward argument gives finitely many such holomorphic functions $f_1, \dots, f_m$ on X (incidentally, the form of these functions are of crucial importance, as can be seen from the next step) such that

$$\mathrm{B}(\Phi) = \bigcap_{j=1}^{m} \{f_j = 0\}.$$

Besides, since the sheaf of holomorphic vector fields on X is an analytic coherent sheaf and X is Stein, we may apply Cartan's Theorem A and [**3**] to get global holomorphic vector fields $\vartheta_1, \dots, \vartheta_N$ on X generating the tangent vector space at every regular point of X; hence above we can take $m = \binom{N}{n}$.

Step 2) Here we reconsider the setting from Step 1. Let $Z_f = \{f = 0\}$, where f is one of the holomorphic functions $f_1, \dots, f_m$.

For $x \in \Omega \setminus Z_f$ define $\delta(x)$ to be the largest positive number so that Φ maps an open neighborhood of x in $\Omega \setminus Z_f$ biholomorphic onto the ball of radius $\delta(x)$ centered at $\Phi(x)$.

Pictorially, we have the following diagram, where $\iota\colon \Omega \longrightarrow X$ is the inclusion:

$$\begin{array}{ccccc} \Omega & \xrightarrow{\ \iota\ } & X & \xrightarrow{\ f\ } & \mathbb{C} \\ & & \Big\downarrow{\scriptstyle \Phi} & & \\ & & \mathbb{C}^n & & \end{array}$$

Notice that $\delta < \infty$ everywhere on Ω, unless Ω is biholomorphic to $\mathbb{C}^n$; so there is nothing to be proved in this case.

Let $\Sigma \subset X$ be the thin set of order 2 defined as the union of S and the singular part of Z_f.

Now we claim that, for every point $a \in Z_f \setminus \Sigma$, there is an open neighborhood U of a in $\Omega \setminus \Sigma$ and a positive constant M such that

$$(\star) \qquad \forall\, x \in U \setminus Z_f, \quad |f(x)|^2 \le M\delta(x).$$

Here we check $(\star)$. Clearly, $a \in X_{\mathrm{reg}}$. Let W be an open neighborhood of a in X_{reg} with coordinates $(x_1, \dots, x_n)$ centered at a such that

$$W \cap Z_f = \{x_n = 0\}.$$

We have the following alternative: either the point a does not belong to $B(\Phi)$, or $a \in B(\Phi) \setminus S$.

In the first case, it is easily seen that, for a suitable neighborhood U of a relatively compact in W, one has $\|\Phi(x) - \Phi(a)\| \ge C\|x - a\| \ge |f(x)|$. Therefore, we obtain $(\star)$ with exponent 1 instead of 2.

In the second case, the definition of f gives the form $f(x) = \mu x_n^{\mu-1} g(x)$, for some holomorphic function g on W. Hence, for a relatively compact neighborhood U of a in W, whenever $x \in U$ in which $|x_n|$ is small, it holds that:

$$\begin{aligned}|f(x_1,\dots,x_n)|^2 &\le \|g\|_U \cdot (\mu|x_n|^{\mu-1})^2 \le M|x_n|^\mu \\ &\le M\|(x_1,x_2,\dots,x_n^\mu) - (x_1',\dots,x_{n-1}',0)\| = M\|\Phi(x)-\Phi(a)|,\end{aligned}$$

which readily implies inequality $(\star)$.

Altogether, given $c \in \mathbb{R}$, $c > 2$ (for instance, $c = \sqrt{5}$), there is an open neighborhood V of $Z_f \cap (\Omega \setminus \Sigma)$ in $\Omega \setminus \Sigma$ such that on V it holds true that

$$(\star\star) \qquad -\log\delta + c\log|f| < 0.$$

Notice also that, if $(x_k)_k$ is a sequence of points in $\Omega \setminus Z_f$ that converges to a point of $\partial\Omega \setminus X_{\mathrm{sg}}$, then $\delta(x_k) \mapsto 0$.

Step 3) Here we give the patching procedure. First note that there are finitely many holomorphic maps $\Phi_j : X \longrightarrow \mathbb{C}^n$, $1 \le j \le p$, with discrete fibers such that

$$X_{\mathrm{sg}} = \bigcap \mathrm{B}(\Phi_j).$$

Then as in Step 2, we get, for each j, finitely many holomorphic functions $f_{j1},\dots,f_{jq}$ (their form via the above vector fields is important) such that

$$\mathrm{B}(\Phi_j) = \bigcap_k Z(f_{jk}).$$

Since $X \setminus Z(f_{jk})$ is a Stein manifold, $\Omega \setminus Z(f_{jk})$ is Stein too, hence the boundary distance δ_{jk} of the domain $\Omega \setminus Z(f_{jk})$ over $\mathbb{C}^n$ via Φ_j has the property that $-\log\delta_{jk}$ is plurisubharmonic and continuous. (As noted above, we dispense with the case when Ω is biholomorphic to $\mathbb{C}^n$, when there is nothing to be done!)

Therefore, granting Lemma 1 and $(\star\star)$, the function $\varphi_{jk} : \Omega \longrightarrow [0,\infty)$ defined by

$$\varphi_{jk} := \begin{cases} \max\{-\log\delta_{jk} + \sqrt{5}\log|f_{jk}|, 0\} & \text{on } \Omega \setminus Z_{jk}, \\ 0 & \text{elsewhere}, \end{cases}$$

is plurisubharmonic and continuous.

Finally, the desired function ϕ is obtained via Richberg's regularization of the following strict plurisubharmonic function, which has the required properties except regularity, namely

$$\psi + \max\{\varphi_{jk}\},$$

where $\psi\colon X \longrightarrow [0,\infty)$ is strictly plurisubharmonic, continuous, and surjective. The proof of the proposition is concluded. □

Remark. The reasons, in the proof of [**5**, Proposition 3], on p. 720, lines 33–35, why "the function $\psi_V^{(k)} - \psi_W^{(l)}$ is bounded on $M \cap \Omega$" were not well explained.

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Université des Sciences et Technologies de Lille 1, Laboratoire Paul Painlevé, Bât. M2, F-59655 Villeneuve d'Ascq Cedex, France

E-mail address: viorel.vajaitu@univ-lille.fr

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