

ON EULER SYSTEMS FOR THE MULTIPLICATIVE GROUP OVER GENERAL NUMBER FIELDS

DAVID BURNS, ALEXANDRE DAOUD, TAKAMICHI SANO,
AND SOOGIL SEO

Abstract: We formulate, and provide strong evidence for, a natural generalization of a conjecture of Robert Coleman concerning Euler systems for the multiplicative group over arbitrary number fields.

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1. Introduction

1.1. Ever since its introduction by Kolyvagin in the late 1980s, the theory of Euler systems has played a vital role in the proof of many celebrated results concerning the structure of Selmer groups attached to p -adic representations that are defined over number fields and satisfy a variety of technical conditions.

In order to consider a wider class of representations, the theory has also been developed in recent years to incorporate a natural notion of ‘higher rank’ Euler systems.

Given their importance, it is clearly of interest to understand the full collection of Euler systems (of the appropriate rank) that arise in any given setting.

In the concrete setting of (rank one) Euler systems that are attached to the multiplicative group \mathbb{G}_m over abelian extensions of \mathbb{Q} , a conjecture of Coleman concerning circular distributions can be seen to imply that all such systems should arise in an elementary way from the classical theory of cyclotomic units.

This conjecture of Coleman was itself motivated by an ‘Archimedean characterization’ of norm-coherent sequences of cyclotomic units that he

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had obtained in [8] and hence by attempts to understand a globalized version of the fact that all norm-compatible families of units in towers of local cyclotomic fields arise by evaluating a ‘Coleman power series’ at roots of unity.

To consider the analogous problem for an arbitrary number field K we write r_K for the number of Archimedean places of K and K^s for the maximal abelian extension of K in which all Archimedean places split completely.

Then, by adapting a general construction for p -adic representations that is described by the first and third authors in [5], we shall first define (unconditionally) a module ES_K^b of ‘basic’ Euler systems of rank r_K for \mathbb{G}_m relative to the extension K^s/K .

In the case $K = \mathbb{Q}$ we shall prove in Theorem 5.1 that ES_K^b is generated by the restriction of the cyclotomic Euler system to real abelian fields.

In the general case, we shall show in Theorem 3.8 that ES_K^b contains systems that are directly related to the leading terms of Artin L -series at $s = 0$.

We shall then predict that essentially all Euler systems of rank r_K for \mathbb{G}_m over K^s/K should belong to ES_K^b (for a precise statement see Conjecture 2.5).

In the case $K = \mathbb{Q}$ we can deduce from recent results of the first and fourth authors in [6] that Conjecture 2.5 is equivalent to Coleman’s original conjecture on circular distributions, and hence that the results of loc. cit. give strong evidence for our conjecture in this case.

In addition, to obtain evidence for the general case of Conjecture 2.5 we can incorporate the construction of basic Euler systems into the equivariant theory of higher rank Euler, Kolyvagin, and Stark systems for \mathbb{G}_m that is developed by Sakamoto and the first and third authors in [4].

In particular, in this way we shall prove in Theorem 2.7 (and Theorem 4.1) that the main result of [4] leads to some strong, and unconditional, evidence in support of Conjecture 2.5.

In fact, we find that even in the case $K = \mathbb{Q}$ the latter result constitutes a strong improvement on results in the literature.

For example, in Theorem 2.9, we shall use it to prove a natural algebraic analogue of Coleman’s ‘Archimedean characterization’ of norm-coherent sequences in towers of cyclotomic fields (which is the main result of [8]).

1.2. Notation. In the remainder of the introduction we shall, for the convenience of the reader, collect together various notations that we employ throughout this article.

1.2.1. Arithmetic. We fix an algebraic closure \mathbb{Q}^c of \mathbb{Q} and an algebraic closure \mathbb{Q}_p^c of \mathbb{Q}_p for each prime number p .

Throughout the article, K will denote a number field (that is, a finite extension of \mathbb{Q} in \mathbb{Q}^c) and K^{ab} the maximal abelian extension of K inside \mathbb{Q}^c . We write $S_\infty(K)$ for the set of Archimedean places of K and denote its cardinality by r_K .

For each extension \mathcal{K} of K in K^{ab} we consider the following collection of intermediate fields:

$$\Omega(\mathcal{K}/K) := \{F \mid K \subset F \subseteq \mathcal{K}, F/K \text{ is ramified and of finite degree}\}.$$

For each field E in $\Omega(\mathcal{K}/K)$ we set

$$\mathcal{G}_E := \text{Gal}(E/K)$$

and we write $S_{\text{ram}}(E/K)$ for the (finite, non-empty) set of places of K that ramify in E .

For any set S of places of K , we then set

$$S(E) := S \cup S_{\text{ram}}(E/K).$$

We write S_E for the set of places of E lying above those in S . The ring of S_E -integers of E is denoted by

$$\mathcal{O}_{E,S} := \{a \in E \mid \text{ord}_w(a) \geq 0 \text{ for all non-Archimedean } w \notin S_E\},$$

where ord_w is the normalized additive valuation relative to w .

Given a finite set T of places of K that is disjoint from $S \cup S_\infty(K)$, we define the (S, T) -unit group of E to be the finite index subgroup of $\mathcal{O}_{E,S}^\times$ given by

$$\mathcal{O}_{E,S,T}^\times := \{a \in \mathcal{O}_{E,S}^\times \mid a \equiv 1 \pmod{w} \text{ for all } w \in T_E\}.$$

We denote by $Y_{E,S}$ the free \mathbb{Z} -module on S_E and $X_{E,S}$ its augmentation kernel so that there is a tautological short exact sequence

$$0 \longrightarrow X_{E,S} \longrightarrow Y_{E,S} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Given a place w of E , we denote by $\mathcal{G}_{E,w}$ the decomposition subgroup of \mathcal{G}_E relative to w . If w is non-Archimedean, we denote by $\kappa(w)$ the residue field relative to w and write Nw for the cardinality of $\kappa(w)$. Let v be the place of K lying under w . If v does not ramify in E , then we write $\text{Fr}_v \in \mathcal{G}_{E,w}$ for the Frobenius automorphism relative to w .

We denote by $\mathcal{R}_{\mathcal{K}/K}$ the completed group ring $\mathbb{Z}[[\text{Gal}(\mathcal{K}/K)]]$. For each E in $\Omega(\mathcal{K}/K)$ we write μ_E for the torsion subgroup of E^\times and \mathcal{A}_E for the annihilator in $\mathbb{Z}[\mathcal{G}_E]$ of μ_E . Then for each E' with $E \subseteq E'$ one has $\pi_{E'/E}(\mathcal{A}_{E'}) = \mathcal{A}_E$ (see Lemma 3.9(ii)) and so we obtain an ideal of $\mathcal{R}_{\mathcal{K}/K}$ by setting

$$\mathcal{A}_{\mathcal{K}/K} := \varprojlim_{E \in \Omega(\mathcal{K}/K)} \mathcal{A}_E.$$

We write K^s for the maximal subfield of K^{ab} in which all Archimedean places of K split. We then abbreviate the modules $\mathcal{R}_{K^{\text{ab}}/K}$, $\mathcal{R}_{K^s/K}$, $\mathcal{A}_{K^{\text{ab}}/K}$, and $\mathcal{A}_{K^s/K}$ to \mathcal{R}_K , \mathcal{R}_K^s , \mathcal{A}_K , and \mathcal{A}_K^s respectively.

For each natural number m we write μ_m for the group of m -th roots of unity in \mathbb{Q}^c . For each n we fix a generator ζ_n of μ_n such that $\zeta_{mn}^m = \zeta_n$ for all m and n .

1.2.2. Algebra. Let R be a commutative Noetherian ring and X an R -module. We set $X^* := \text{Hom}_R(X, R)$. We write X_{tor} for the torsion subgroup of X and X_{tf} for the associated torsion-free quotient X/X_{tor} . We also write \widehat{X} for the profinite completion of X and \widehat{X}^p for its pro- p completion at a rational prime p .

For a non-negative integer r the r -th exterior power bidual of X is defined by setting

$$\bigcap_R^r X := \left(\bigwedge_R^r (X^*) \right)^*.$$

We recall that there exists a natural homomorphism of R -modules

$$\xi_X^r: \bigwedge_R^r X \longrightarrow \bigcap_R^r X; x \longmapsto (\Phi \longmapsto \Phi(x))$$

that is, in general, neither injective nor surjective (see [5, (1)]).

Exterior power biduals play an essential role in the theory of higher rank Euler, Kolyvagin, and Stark systems (see, for example, [4] and [5]).

In particular, if $R = \mathbb{Z}[G]$ for a finite abelian group G , then ξ_X^r induces an identification

$$(1) \quad \left\{ x \in \bigwedge_{\mathbb{Q}[G]}^r (\mathbb{Q} \otimes_{\mathbb{Z}} X) \mid \Phi(x) \in R \text{ for all } \Phi \in \bigwedge_R^r (X^*) \right\} \xrightarrow{\sim} \bigcap_R^r X$$

(cf. [5, Proposition A.8]). Lattices of this form were first used by Rubin in [11] to formulate an integral refinement of Stark's conjecture.

In order to ease notations for objects associated to group rings, we shall sometimes abbreviate a subscript ' $\mathbb{Z}[G]$ ' to ' G ' (for example, writing ' \bigcap_G^r ' rather than ' $\bigcap_{\mathbb{Z}[G]}^r$ ').

We denote the cardinality of a finite set X by $|X|$.

2. Statement of the conjecture and main results

2.1. Euler systems. We first introduce the various notions of Euler systems with which we shall be concerned.

In this section, we set $S := S_\infty(K)$ so that $S(E) = S_\infty(K) \cup S_{\text{ram}}(E/K)$ for any finite extension E/K .

Fix a field \mathcal{K} in K^{ab} , an integer $r \geq 0$, and a pair of fields E and E' in $\Omega(\mathcal{K}/K)$ with $E \subseteq E'$. Then the field-theoretic norm map $N_{E'/E} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{E',S(E')}^\times \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E)}^\times$ induces a commutative diagram

$$\begin{array}{ccc} \bigcap_{\mathcal{G}_{E'}}^r \mathcal{O}_{E',S(E')}^\times & \hookrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_{E'}}^r \mathcal{O}_{E',S(E')}^\times \\ \downarrow N_{E'/E}^r & & \downarrow N_{E'/E}^r \\ \bigcap_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times & \hookrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times \end{array}$$

where the horizontal inclusions are induced by the identification (1).

Definition 2.1. A ‘rational Euler system of rank r ’ for \mathbb{G}_m and \mathcal{K}/K is a collection

$$c = (c_E)_E \in \prod_{E \in \Omega(\mathcal{K}/K)} \left(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times \right)$$

that satisfies the following distribution relations: for every pair of fields E and E' in $\Omega(\mathcal{K}/K)$ with $E \subseteq E'$ one has

$$(2) \quad N_{E'/E}^r(c_{E'}) = \left(\prod_{v \in S(E') \setminus S(E)} (1 - \text{Fr}_v^{-1}) \right) c_E$$

in $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times$.

The set $\text{RES}_r(\mathcal{K}/K)$ of all rational Euler systems of rank r has a natural structure as an $\mathcal{R}_{\mathcal{K}/K}$ -module. If $r = r_K$ and $\mathcal{K} = K^{\text{s}}$, then we abbreviate $\text{RES}_r(\mathcal{K}/K)$ to RES_K .

Example 2.2. A classical example of a rational Euler system is given by the system of cyclotomic units. In fact, if for each E in $\Omega(\mathbb{Q}^{\text{s}}/\mathbb{Q})$ of (non-trivial) conductor $\mathfrak{f}(E)$, so that $E \subseteq \mathbb{Q}(\zeta_{\mathfrak{f}(E)})$, one sets $c_E := N_{\mathbb{Q}(\zeta_{\mathfrak{f}(E)})/E}(1 - \zeta_{\mathfrak{f}(E)})$, then the collection $(c_E)_E$ belongs to $\text{RES}_{\mathbb{Q}}$.

More generally, the Euler system of (conjectural) Rubin–Stark elements constitutes an element of RES_K . In particular, for each field E in $\Omega(K^{\text{s}}/K)$, the (‘ T -less’) Rubin–Stark conjecture for the data set $(E/K, S(E), \emptyset, S_\infty(K))$ is formulated in [11, Conjecture A] and predicts the existence of an element c_E in $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times$ that is related by the

Dirichlet regulator isomorphism to the r -th order $S(E)$ -truncated Stick-
elberger element of E/K . Then the argument of [11, Proposition 6.1]
shows that $(c_E)_E$ belongs to RES_K .

Definition 2.3. ‘An Euler system of rank r ’ for the pair $(\mathbb{G}_m, \mathcal{K}/K)$
is an element c of $\text{RES}_r(\mathcal{K}/K)$ with the property that for every field E
in $\Omega(\mathcal{K}/K)$ the element c_E belongs to the lattice $\bigcap_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times$.

The collection $\text{ES}_r(\mathcal{K}/K)$ of such systems is an \mathcal{R}_K -submodule of
 $\text{RES}_r(\mathcal{K}/K)$. If $r = r_K$ and $\mathcal{K} = K^s$, then we abbreviate $\text{ES}_r(\mathcal{K}/K)$
to ES_K .

Remark 2.4. The module $\text{ES}^{\text{cl}} := \text{ES}^{\text{cl}}(\mathcal{K}/K)$ of ‘classical’ (rank one)
Euler systems for \mathbb{G}_m and \mathcal{K}/K comprises all elements $c = (c_E)_E$ of
 $\prod_E \mathcal{O}_{E,S(E)}^\times$, where E runs over $\Omega(\mathcal{K}/K)$, with the property that for all
pairs E and E' with $E \subseteq E'$ the distribution relation (2) is valid in the
group $\mathcal{O}_{E,S(E')}^\times$.

In particular, since for every E the module $\bigcap_{\mathcal{G}_E}^1 \mathcal{O}_{E,S(E)}^\times := (\mathcal{O}_{E,S(E)}^\times)^{**}$
identifies with $\mathcal{O}_{E,S(E),\text{tf}}^\times$, the natural reduction map from ES^{cl} to the
module $\text{ES}_1 := \text{ES}_1(\mathcal{K}/K)$ need be neither injective nor surjective.

However, if c belongs to ES_1 , then for any element a of $\mathcal{A}_{\mathcal{K}/K}$ there
exists a canonical system $c' = c'_a$ in ES^{cl} that projects to give c^a in ES_1 :
for each E in $\Omega(\mathcal{K}/K)$ one need only define c'_E to be $(\tilde{c}_E)^{a_E}$, where \tilde{c}_E
is any choice of element of $\mathcal{O}_{E,S(E)}^\times$ that projects to c_E in $\bigcap_{\mathcal{G}_E}^1 \mathcal{O}_{E,S(E)}^\times$.

2.2. The conjecture and main results. In §3 below we will use cer-
tain natural families of étale cohomology complexes to construct a canon-
ical ‘invertible’ \mathcal{R}_K^s -module $\text{VS}(K^s/K)$ and to prove the existence of a
canonical non-zero homomorphism

$$\Theta_K^s: \text{VS}(K^s/K) \longrightarrow \text{RES}_K$$

of \mathcal{R}_K -modules with the property that

$$(3) \quad \mathcal{A}_K^s \cdot \text{im}(\Theta_K^s) \subseteq \text{ES}_K.$$

The constructions of the module $\text{VS}(K^s/K)$ and map Θ_K^s arise by
adapting certain generic p -adic constructions from [5] that are uncondi-
tional and both essentially algebraic and quite elementary in nature.

As a result, the inclusion (3) implies that for every field K there
exists a ‘large’ module of Euler systems of rank r_K (for details see The-
orem 3.8).

We define the module of ‘basic Euler systems of rank r_K ’ for \mathbb{G}_m
over K by setting

$$\text{ES}_K^{\text{b}} := \text{im}(\Theta_K^s).$$

In this article we shall then study the following conjecture concerning a partial converse to the inclusion (3).

Conjecture 2.5. $\mathcal{A}_K^s \cdot \text{ES}_K \subseteq \mathcal{A}_K^s \cdot \text{ES}_K^b$.

This conjecture asserts that, modulo minor technical issues concerning torsion, *all* Euler systems in ES_K should arise via the elementary construction given in §3.1.

Remark 2.6. It can be shown that $\text{ES}_{\mathbb{Q}}^b$ is not contained in $\text{ES}_{\mathbb{Q}}$ (see Lemma 5.3). However, if we set $\Lambda := \mathbb{Z}[\frac{1}{2}]$, then for any totally real field K one has $\Lambda \otimes_{\mathbb{Z}} \mathcal{A}_K^s = \Lambda \otimes_{\mathbb{Z}} \mathcal{R}_K^s$ and so Conjecture 2.5 combines with (3) to predict $\Lambda \otimes_{\mathbb{Z}} \text{ES}_K = \Lambda \otimes_{\mathbb{Z}} \text{ES}_K^b$. In fact, at this stage, we do not know an example showing that the inclusion of Conjecture 2.5 should not itself always be an equality.

To obtain evidence for Conjecture 2.5 one can use the equivariant theory of higher rank Kolyvagin systems for \mathbb{G}_m , as developed by Sakamoto and the first and third authors in [4]. In particular, in this way we shall obtain the following (unconditional) result, a precise statement, and proof, of which is given in §4.

Theorem 2.7 (Theorem 4.1). *Fix an odd prime p and for each field E in $\Omega(K^s/K)$ write Δ_E for the maximal subgroup of \mathcal{G}_E of order prime to p . Then for every system c in ES_K , every field E in $\Omega(K^s/K)$, and every homomorphism $\chi: \Delta_E \rightarrow \mathbb{Q}_p^{c,\times}$ that satisfies certain mild technical hypotheses, the ‘ χ -component’ of c_E belongs to the $\mathbb{Z}_p[\mathcal{G}_E]$ -module generated by $\{c'_E \mid c' \in \text{ES}_K^b\}$.*

If K has at least one real embedding, then $\mu_L = \{\pm 1\}$ for all L in $\Omega(K^s/K)$ and so \mathcal{A}_K^s is the kernel $\mathcal{I}_{K,(2)}$ of the natural ‘mod 2 augmentation’ map $\mathcal{R}_K^s \rightarrow \mathbb{Z}/(2)$.

In particular, in this case Conjecture 2.5 predicts that every Euler system in ES_K has the form $2^{-1} \cdot \Theta_K^s(x)$ for some x in $\mathcal{I}_{K,(2)} \cdot \text{VS}(K^s/K)$.

However, as the following result shows, in special cases it can predict much more.

Theorem 2.8. *If $K = \mathbb{Q}$, then Conjecture 2.5 is equivalent to Coleman’s conjecture on circular distributions. In particular, if Conjecture 2.5 is valid, then for every $c = (c_F)_F$ in $\text{ES}^{\text{cl}}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ there exists an element r_c of $\mathcal{R}_{\mathbb{Q}}$ such that*

$$c_{\mathbb{Q}(n)} = \pm(1 - \zeta_n)^{r_c}$$

for every $n \not\equiv 2 \pmod{4}$.

This result shows that Conjecture 2.5 constitutes a natural ‘generalized Coleman conjecture’ and will be proved in §5.2.

In addition, by combining Theorem 2.7 with the analysis used to prove Theorem 2.8 we obtain the following algebraic analogue of the ‘Archimedean characterization’ of norm-coherent sequences in towers of the form $\bigcup_n \mathbb{Q}(\mu_{p^n})$ that was given by Coleman in [8]. (We recall that this result was the original motivation for Coleman’s study of circular distributions.)

Theorem 2.9. *Let p be an odd prime and $(a_n)_n$ a norm-coherent sequence in the tower of fields $\bigcup_n \mathbb{Q}(\mu_{p^n})$. Then each element a_n belongs to the $\mathcal{R}_{\mathbb{Q}}$ -module generated by $1 - \zeta_{p^n}$ if and only if there exists a circular distribution f and a non-negative integer t such that $f(\zeta_{p^n})^{2^t} = a_n$ for all n .*

This result improves upon [15, Theorem B] (by showing that the positive integer c in loc. cit. can be taken as a power of 2) and also gives an affirmative answer, modulo powers of 2, to the question raised by the fourth author in [17, end of §1]. It will be proved in §5.3.

3. Basic Euler systems

In this section we shall define, and establish the key properties of, the homomorphism Θ_K^s that occurs in Conjecture 2.5.

To do this we fix an extension \mathcal{K} in K^{ab} and a finite set of places S of K containing $S_{\infty}(K)$. (Later in the article we will specialize to $S = S_{\infty}(K)$.)

Throughout this section we shall also use the following general notation. For any finite abelian group \mathcal{G} and any commutative Noetherian ring R we write $D(R[\mathcal{G}])$ for the derived category of $R[\mathcal{G}]$ -modules and $D^{\text{perf}}(R[\mathcal{G}])$ for the full triangulated subcategory of $D(R[\mathcal{G}])$ comprising complexes that are perfect.

For a bounded above complex of \mathcal{G} -modules C we write C^* for the object $\mathbf{R}\text{Hom}_{\mathbb{Z}}(C, \mathbb{Z}[0])$ of $D(\mathbb{Z}[\mathcal{G}])$, where the linear dual is endowed with the natural contragredient action of \mathcal{G} .

We write $\mathbf{d}_{R[\mathcal{G}]}(-)$ for the determinant functor on $D^{\text{perf}}(R[\mathcal{G}])$, as constructed by Knudsen and Mumford in [10].

We write $\hat{\mathcal{G}}$ for the group of \mathbb{Q}^c -valued characters of \mathcal{G} and for any such χ we write e_{χ} for idempotent $|\mathcal{G}|^{-1} \sum_{\sigma \in \mathcal{G}} \chi(\sigma) \sigma^{-1}$ of $\mathbb{Q}^c[\mathcal{G}]$.

For any \mathcal{G} -module M we also set $M^{\chi} := e_{\chi} \cdot (\mathbb{Q}^c \cdot M)$, where $\mathbb{Q}^c \cdot M$ denotes the $\mathbb{Q}^c[\mathcal{G}]$ -module that is generated by M .

3.1. Modified étale cohomology complexes. For each E in $\Omega(\mathcal{K}/K)$ and each finite set of places T of K that is disjoint from $S(E)$, the

methods of Kurihara and the first and third authors in [3, §2] define a canonical T -modified, compactly supported ‘Weil-étale’ cohomology complex $\mathrm{R}\Gamma_{c,T}((\mathcal{O}_{E,S(E)})_{\mathcal{W}}, \mathbb{Z})$ of the constant sheaf \mathbb{Z} on $\mathrm{Spec}(\mathcal{O}_{E,S(E)})$.

In the sequel we set

$$C_{E,S(E),T} := \mathrm{R}\Gamma_{c,T}((\mathcal{O}_{E,S(E)})_{\mathcal{W}}, \mathbb{Z})^*[-2]$$

and shall use the properties of this complex that are recalled in the following result.

For any finite set T' of places of K that is disjoint from $S(E)$ we set

$$\mathbb{F}_{T'_E}^\times := \bigoplus_{w \in T'_E} \kappa(w)^\times.$$

For each non-Archimedean place v of K we write $\mathrm{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])$ for the direct sum over places w of E above v of the complexes $\mathrm{R}\Gamma(\kappa(w)_{\mathcal{W}}, \mathbb{Z})$ defined in [3, Proposition 2.4(ii)].

For any finite set of places Σ of K that contains $S_\infty(K)$ and is disjoint from T , we write $\mathrm{Cl}_\Sigma^T(E)$ for the ray class group of $\mathcal{O}_{E,\Sigma}$ modulo the product of all places of E above T .

Proposition 3.1. *For each E in $\Omega(\mathcal{K}/K)$ the complex $C_{E,S(E),T}$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}[\mathcal{G}_E])$ and has all of the following properties.*

- (i) *For any finite set of places Σ of K that contains $S(E)$ and is disjoint from T the complex $C_{E,\Sigma,T}$ is acyclic outside degrees zero and one and there are canonical identifications of \mathcal{G}_E -modules $H^0(C_{E,\Sigma,T}) = \mathcal{O}_{E,\Sigma,T}^\times$, $H^1(C_{E,\Sigma,T})_{\mathrm{tor}} = \mathrm{Cl}_\Sigma^T(E)$, and $H^1(C_{E,\Sigma,T})_{\mathrm{tf}} = X_{E,\Sigma}$.*
- (ii) *If T' is any finite set of places that contains T and is disjoint from $S(E)$, then there is a canonical exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}[\mathcal{G}_E])$*

$$C_{E,S(E),T'} \longrightarrow C_{E,S(E),T} \longrightarrow \mathbb{F}_{(T' \setminus T)_E}^\times[0] \longrightarrow$$

- (iii) *Given a finite set S' of places of K that contains S and is disjoint from T , there exists a canonical exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}[\mathcal{G}_E])$ of the form*

$$C_{E,S(E),T} \longrightarrow C_{E,S'(E),T} \longrightarrow \bigoplus_{v \in S'(E) \setminus S(E)} \mathrm{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1] \longrightarrow$$

- (iv) *For any fields E and E' in $\Omega(\mathcal{K}/K)$ with $E \subseteq E'$ there exists a natural isomorphism*

$$\mathbb{Z}[\mathcal{G}_E] \otimes_{\mathbb{Z}[\mathcal{G}_{E'}]}^L C_{E',S(E'),T} \cong C_{E,S(E'),T}$$

in $D^{\mathrm{perf}}(\mathbb{Z}[\mathcal{G}_E])$.

Proof: Write $D^-(\mathbb{Z}[\mathcal{G}_E])$ for the full subcategory of $D(\mathbb{Z}[\mathcal{G}_E])$ comprising complexes that are cohomologically bounded above and ψ^* for the functor from $D^-(\mathbb{Z}[\mathcal{G}_E])$ to $D(\mathbb{Z}[\mathcal{G}_E])$ that sends each complex C to $C^*[-2]$.

Concerning claim (i), since Σ contains $S(E)$, the fact that $C_{E,\Sigma,T}$ belongs to $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$ is a consequence of [3, Proposition 2.4(iv)] and the fact that ψ^* preserves $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$. The descriptions of cohomology given in claim (i) are stated in [3, Remark 2.7].

The exact triangles in claims (ii) and (iii) result from applying ψ^* to the triangles given by the right-hand column of the diagram in [3, Proposition 2.4(i)] and to the exact triangle in [3, Proposition 2.4(ii)] respectively.

The existence of isomorphisms as in claim (iv) can be deduced by combining the commutative diagram of exact triangles in [3, Proposition 2.4(i)] together with the well known isomorphisms

$$\begin{aligned} \mathbb{Z}[\mathcal{G}_E] \otimes_{\mathbb{Z}[\mathcal{G}_{E'}]}^{\mathbb{L}} \text{R}\Gamma_c((\mathcal{O}_{E',S(E')})_{\text{ét}}, \mathbb{Z})^* \\ \cong \text{RHom}_{\mathbb{Z}[\mathcal{G}_{E'}]}(\mathbb{Z}[\mathcal{G}_E], \text{R}\Gamma_c((\mathcal{O}_{E',S(E')})_{\text{ét}}, \mathbb{Z}))^* \end{aligned}$$

and

$$\text{RHom}_{\mathbb{Z}[\mathcal{G}_{E'}]}(\mathbb{Z}[\mathcal{G}_E], \text{R}\Gamma_c((\mathcal{O}_{E',S(E')})_{\text{ét}}, \mathbb{Z})) \cong \text{R}\Gamma_c((\mathcal{O}_{E,S(E')})_{\text{ét}}, \mathbb{Z}). \quad \square$$

Remark 3.2. For each place v of K outside $S(E)$, claim (ii) of Proposition 3.1 with $T = \emptyset$ and $T' = \{v\}$ implies that $\mathbb{F}_{T'_E}^\times[0]$ belongs to $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$. It is in fact straightforward to show (and well known) that this complex is isomorphic in $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$ to the complex

$$\mathbb{Z}[\mathcal{G}_E] \xrightarrow{1 - N \cdot \text{Fr}_v^{-1}} \mathbb{Z}[\mathcal{G}_E],$$

where the first term is placed in degree minus one.

Remark 3.3. For each place v of K outside $S(E)$ the result of [1, Proposition 3.2] implies that the complex $\text{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1]$ that occurs in Proposition 3.1(iii) is canonically isomorphic to the complex

$$\mathbb{Z}[\mathcal{G}_E] \xrightarrow{1 - \text{Fr}_v^{-1}} \mathbb{Z}[\mathcal{G}_E],$$

where the first term is placed in degree zero.

Remark 3.4. If $S(E)$ contains all places above a given rational prime p , then [1, Proposition 3.3] implies that there is a canonical isomorphism in $D^{\text{perf}}(\mathbb{Z}_p[\mathcal{G}_E])$ of the form

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E,S(E)} \cong \text{RHom}_{\mathbb{Z}_p}(\text{R}\Gamma_c(\mathcal{O}_{E,S(E)}, \mathbb{Z}_p), \mathbb{Z}_p)[-2].$$

This isomorphism relates the constructions that we make below to those from [5].

3.2. Vertical determinantal systems. In this section we use the complexes discussed in §3.1 to construct a canonical module over the algebra $\mathcal{R}_{\mathcal{K}/K}$.

We start by proving a useful technical result about the set $\Omega(\mathcal{K}/K)$.

Lemma 3.5. *There exists a cofinal directed subset of $\Omega(\mathcal{K}/K)$ order-isomorphic to \mathbb{N} .*

Proof: Fix an enumeration $\mathfrak{p}_1, \mathfrak{p}_2, \dots$ of the places of K . Given a modulus \mathfrak{m} of K , let $K(\mathfrak{m})$ denote the intersection of \mathcal{K} with the ray class field modulo \mathfrak{m} for K . For each $n \geq 1$, we now set $K_n = K(\prod_{i=1}^n \mathfrak{p}_i^n)$. By class field theory, each K_n is abelian of finite degree over K and there is a chain of inclusions $K_1 \subseteq K_2 \subseteq \dots$. Given an extension $L \in \Omega(\mathcal{K}/K)$, let \mathfrak{f}_L be the conductor of L . Then L is contained in any $K(\mathfrak{m})$ such that $\mathfrak{f}_L \mid \mathfrak{m}$. Hence we can always choose an n such that $L \subseteq K_n$. We thus see that the tower of fields $\{K_n\}_{n \geq 1}$ forms a countable cofinal directed subset of $\Omega(\mathcal{K}/K)$. \square

For E in $\Omega(\mathcal{K}/K)$ we abbreviate the functor $\mathbf{d}_{\mathbb{Z}[\mathcal{G}_E]}(-)$ to $\mathbf{d}_{\mathcal{G}_E}$. For each pair of fields E and E' in $\Omega(\mathcal{K}/K)$ with $E \subseteq E'$, we then define

$$\nu_{E'/E}: \mathbf{d}_{\mathcal{G}_{E'}}(C_{E',S(E')}) \longrightarrow \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)})$$

to be the following composite homomorphism of $\mathbb{Z}[\mathcal{G}_{E'}]$ -modules:

$$\begin{aligned} \mathbf{d}_{\mathcal{G}_{E'}}(C_{E',S(E')}) &\longrightarrow \mathbb{Z}[\mathcal{G}_E] \otimes_{\mathbb{Z}[\mathcal{G}_{E'}]} \mathbf{d}_{\mathcal{G}_{E'}}(C_{E',S(E')}) \\ &\xrightarrow{\sim} \mathbf{d}_{\mathcal{G}_E}(\mathbb{Z}[\mathcal{G}_E] \otimes_{\mathbb{Z}[\mathcal{G}_{E'}]}^{\mathbb{L}} C_{E',S(E')}) \\ &\xrightarrow{\sim} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E')}) \\ &\xrightarrow{\sim} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) \otimes \bigotimes_{v \in S(E') \setminus S(E)} \mathbf{d}_{\mathcal{G}_E}(\mathrm{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1]) \\ &\xrightarrow{\sim} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}). \end{aligned}$$

Here the first map is the canonical projection, the second is induced by the standard base-change property of determinant functors, the third by the isomorphism in Proposition 3.1(iv) (with $T = \emptyset$), the fourth by the exact triangle in Proposition 3.1(iii) (with $T = \emptyset$), and the last homomorphism is induced by the identification

$$(4) \quad \mathbf{d}_{\mathcal{G}_E}(\mathrm{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1]) \cong \mathbf{d}_{\mathcal{G}_E}(\mathbb{Z}[\mathcal{G}_E]) \otimes_{\mathcal{G}_E} \mathbf{d}_{\mathcal{G}_E}^{-1}(\mathbb{Z}[\mathcal{G}_E]) \\ \cong \mathbb{Z}[\mathcal{G}_E],$$

where the first isomorphism is induced by the description, given in Remark 3.3, of $\mathrm{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1]$, and the second is the standard ‘evaluation map’ isomorphism.

Definition 3.6. The module of ‘vertical determinantal systems’ for \mathbb{G}_m and \mathcal{K}/K is the $\mathcal{R}_{\mathcal{K}/K}$ -module given by the inverse limit

$$\mathrm{VS}(\mathcal{K}/K) := \varprojlim_{E \in \Omega(\mathcal{K}/K)} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}),$$

where in the inverse limit the transition morphisms are the maps $\nu_{E'/E}$ defined above.

The following result shows that this $\mathcal{R}_{\mathcal{K}/K}$ -module is in a natural sense ‘invertible’.

Proposition 3.7. *For each prime p the pro- p -completion of $\mathrm{VS}(\mathcal{K}/K)$ is free of rank one over $\widehat{\mathcal{R}}_{\mathcal{K}/K}^p$.*

Proof: We fix a rational prime p and for E in $\Omega(\mathcal{K}/K)$ set

$$\Xi_E := \mathbf{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E,S(E)}) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}).$$

Then the pro- p -completion of $\mathrm{VS}(\mathcal{K}/K)$ is equal to

$$\begin{aligned} \varprojlim_{n \geq 1} \left(\varprojlim_E \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) \right) / p^n &\xrightarrow{\sim} \varprojlim_{n \geq 1} \varprojlim_E (\mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) / p^n) \\ &\xrightarrow{\sim} \varprojlim_E \varprojlim_{n \geq 1} (\mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) / p^n) \\ &\xrightarrow{\sim} \varprojlim_E \Xi_E, \end{aligned}$$

where in all cases E runs over $\Omega(\mathcal{K}/K)$. Here the first isomorphism follows from the fact that Lemma 3.5 combines with the surjectivity of each map $\nu_{E'/E}$ to imply that the inverse system underlying $\mathrm{VS}(\mathcal{K}/K)$ satisfies the Mittag-Leffler property, and the last from the fact that $\mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)})$ is finitely generated so that its pro- p completion is Ξ_E .

Lemma 3.5 implies that we can compute the last limit by using the tower of fields $(K_n)_{n \geq 1}$. To do this we set $\mathcal{G}_n := \mathcal{G}_{K_n}$, $\nu_n := \nu_{K_n/K_{n-1}}$ and $\Xi_n := \Xi_{K_n}$.

Note that each $\mathbb{Z}_p[\mathcal{G}_n]$ -module Ξ_n is free of rank one. We fix n and assume that for each $m < n$ there exists a $\mathbb{Z}_p[\mathcal{G}_m]$ -basis of Ξ_m with $\nu_{K_m/K_{m-1}}(z_m) = z_{m-1}$. Then it is enough to show that z_{n-1} lifts to a $\mathbb{Z}_p[\mathcal{G}_n]$ -basis of Ξ_n .

To do this we write L for the maximal p -extension inside K_n/K_{n-1} and set $\mathcal{P} := \mathrm{Gal}(L/K_{n-1})$ and $\mathcal{H} := \mathrm{Gal}(K_n/L)$.

Then if z_L is any pre-image of z_{n-1} under $\nu_{L/K_{n-1}}$, one has $\Xi_L = I(\mathcal{P}) \cdot \Xi_L + \mathbb{Z}_p[\mathcal{G}_L] \cdot z_L$, where $I(\mathcal{P})$ is the ideal of $\mathbb{Z}_p[\mathcal{G}_L]$ generated by all elements of the form $g - 1$ with g in \mathcal{P} .

In particular, since $I(\mathcal{P})$ is contained in the Jacobson radical of $\mathbb{Z}_p[\mathcal{G}_L]$, Nakayama’s lemma implies that z_L is a $\mathbb{Z}_p[\mathcal{G}_L]$ -basis of Ξ_L .

Observe moreover that, since $|\mathcal{H}|$ is invertible in \mathbb{Z}_p , there is a canonical decomposition of $\mathbb{Z}_p[\mathcal{G}_n]$ -modules

$$\begin{aligned}\Xi_n &= e_{\mathcal{H}}\Xi_n \oplus (1 - e_{\mathcal{H}})\Xi_n \\ &= (\mathbb{Z}_p[\mathcal{P}] \otimes_{\mathbb{Z}_p[\mathcal{G}_n]} \Xi_n) \oplus (1 - e_{\mathcal{H}})\Xi_n \\ &\cong \Xi_L \oplus (1 - e_{\mathcal{H}})\Xi_n,\end{aligned}$$

where $e_{\mathcal{H}}$ denotes the idempotent $|\mathcal{H}|^{-1} \sum_{\sigma \in \mathcal{H}} \sigma$ of $\mathbb{Z}_p[\mathcal{G}_n]$.

It is thus enough to choose z_n to be any basis of Ξ_n that projects under this decomposition to give z_L in the first component. \square

3.3. The construction of basic Euler systems. In this section we specialize to the case $\mathcal{K} = K^s$ and set $r := r_K := |S_{\infty}(K)|$.

In the following, we also set $S := S_{\infty}(K)$ (so that $S(E) = S_{\infty}(K) \cup S_{\text{ram}}(E/K)$ for any $E \in \Omega(K^s/K)$).

Then for every $E \in \Omega(K^s/K)$ the $\mathbb{Z}[\mathcal{G}_E]$ -module $Y_E := Y_{E, S_{\infty}(K)}$ is free of rank r . In particular, by fixing a set of representatives of the G_K -orbits of embeddings $K^s \rightarrow \mathbb{Q}^c$ we obtain (by restriction of the embeddings) a compatible family of $\mathbb{Z}[\mathcal{G}_E]$ -bases of the modules Y_E and hence a compatible family of isomorphisms

$$(5) \quad Y_E \cong \mathbb{Z}[\mathcal{G}_E]^r.$$

We also define an idempotent of $\mathbb{Q}[\mathcal{G}_E]$ by setting

$$e_{(E)} := \sum_{\chi} e_{\chi},$$

where χ runs over all characters in $\widehat{\mathcal{G}_E}$ for which $X_{E, S_{\text{ram}}(E/K)}^{\chi}$ vanishes. (Here we note that, whilst each individual idempotent e_{χ} belongs to $\mathbb{Q}^c[\mathcal{G}_E]$, the sum $e_{(E)}$ belongs to $\mathbb{Q}[\mathcal{G}_E]$ since $X_{E, S_{\text{ram}}(E/K)}$ spans a finitely generated $\mathbb{Q}[\mathcal{G}_E]$ -module.)

Then, with this definition, the natural exact sequence

$$(6) \quad 0 \longrightarrow X_{E, S_{\text{ram}}(E/K)} \longrightarrow X_{E, S(E)} \longrightarrow Y_E \longrightarrow 0$$

(the third arrow of which is surjective since, by assumption, $S_{\text{ram}}(E/K)$ is non-empty) restricts to give an identification

$$(7) \quad e_{(E)}(\mathbb{Q} \otimes_{\mathbb{Z}} X_{E, S(E)}) = e_{(E)}(\mathbb{Q} \otimes_{\mathbb{Z}} Y_E) \cong (\mathbb{Q}[\mathcal{G}_E]e_{(E)})^r,$$

where the isomorphism is induced by (5).

3.3.1. Statement of the main result. We define Θ_E to be the composite (surjective) homomorphism of $\mathbb{Z}[\mathcal{G}_E]$ -modules

$$\begin{aligned}
 & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) \\
 & \longrightarrow \mathbf{d}_{\mathbb{Q}[\mathcal{G}_E]}(\mathbb{Q} \otimes_{\mathbb{Z}} H^0(C_{E,S(E)})) \otimes_{\mathbb{Q}[\mathcal{G}_E]} \mathbf{d}_{\mathbb{Q}[\mathcal{G}_E]}^{-1}(\mathbb{Q} \otimes_{\mathbb{Z}} H^1(C_{E,S(E)})) \\
 (8) \quad & \xrightarrow{e_{(E)}^\times} e_{(E)} \left(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times \right) \otimes_{\mathbb{Q}[\mathcal{G}_E]} e_{(E)} \left(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathbb{Z}[\mathcal{G}_E]^r \right) \\
 & = e_{(E)} \left(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times \right).
 \end{aligned}$$

Here the first arrow is the composite of the natural identification $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) = \mathbf{d}_{\mathbb{Q}[\mathcal{G}_E]}(\mathbb{Q} \otimes_{\mathbb{Z}} C_{E,S(E)})$ and the standard ‘passage-to-cohomology’ map. In addition, the second arrow is induced by multiplication by $e_{(E)}$, the isomorphism

$$e_{(E)}(\mathbb{Q} \otimes_{\mathbb{Z}} H^1(C_{E,S(E)})) \cong e_{(E)}(\mathbb{Q} \otimes_{\mathbb{Z}} X_{E,S(E)}) \cong e_{(E)}\mathbb{Q}[\mathcal{G}_E]^r$$

induced by Proposition 3.1(i) and (7) and the fact that

$$e_{(E)}(\mathbb{Q} \otimes_{\mathbb{Z}} H^0(C_{E,S(E)})) = e_{(E)}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E)}^\times)$$

is a free $\mathbb{Q}[\mathcal{G}_E]e_{(E)}$ -module of rank r .

The collection of morphisms $(\Theta_E)_E$ then induces a homomorphism of \mathcal{R}_K^s -modules

$$\Theta_K^s : \text{VS}(K^s/K) \longrightarrow \prod_{E \in \Omega(K^s/K)} \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times$$

and we set

$$\text{ES}_K^b := \text{im}(\Theta_K^s).$$

Finally, we note that for every E in $\Omega(K^s/K)$ and every character χ in $\widehat{\mathcal{G}_E}$ the $S(E)$ -truncated Artin L -series $L_{S(E)}(\chi, s)$ vanishes to order at least r at $s = 0$ (see, for example, the discussion in §3.3.4 below) and so we can write $L_{S(E)}^{(r)}(\chi, 0)$ for the value at $s = 0$ of its r -th derivative.

We can now state the main result of this section.

Theorem 3.8.

- (i) ES_K^b is contained in RES_K .
- (ii) $\mathcal{A}_K^s \cdot \text{ES}_K^b$ is contained in ES_K .
- (iii) Fix a system c in RES_K and a field E in $\Omega(K^s/K)$. Then for every ramified character χ in $\widehat{\mathcal{G}}_E$ one has

$$e_\chi(c_E) \neq 0 \implies L_{S(E)}^{(r)}(\chi, 0) \neq 0.$$

- (iv) There exists a system c in ES_K with the property that for every field E in $\Omega(K^s/K)$ and every character χ in $\widehat{\mathcal{G}}_E$ one has

$$e_\chi(c_E) \neq 0 \iff L_{S(E)}^{(r)}(\chi, 0) \neq 0.$$

3.3.2. The proof of Theorem 3.8(i). Fix $(z_E)_E$ in $\text{VS}(K^s/K)$ and set $c_E := \Theta_E(z_E)$ for each E in $\Omega(K^s/K)$.

Then to show that the family $(c_E)_E$ belongs to RES_K , it suffices to prove that for every pair of fields E and E' in $\Omega(K^s/K)$ with $E \subseteq E'$ and every χ in $\widehat{\mathcal{G}}_E$ one has

$$(9) \quad e_\chi(N_{E'/E}^r(c_{E'})) = e_\chi(P_{E'/E}) \cdot e_\chi(c_E)$$

with $P_{E'/E} := \prod_{v \in S(E') \setminus S(E)} (1 - \text{Fr}_v^{-1})$.

In addition, it is enough to verify this equality in $\mathbb{Q}_p \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E, S(E')}^\times$ and with the system $(c_E)_E$ replaced by the image $(\tilde{c}_E)_E$ under the collection of maps $(\mathbb{Q}_p \otimes_{\mathbb{Q}} \Theta_E)_E$ of any $\widehat{\mathcal{R}}_K^{s-p}$ -generator of the pro- p -completion of $\text{VS}(K^s/K)$ (cf. Proposition 3.7).

Then, for every E and every χ in $\widehat{\mathcal{G}}_E$ the surjectivity of Θ_E implies that

$$e_\chi(\tilde{c}_E) \neq 0 \iff e_\chi e_{(E)} \neq 0 \iff X_{E, S_{\text{ram}}(E/K)}^\chi = 0.$$

Thus, the direct sum decomposition $X_{E, S_{\text{ram}}(E'/K)} = X_{E, S_{\text{ram}}(E/K)} \oplus Y_{E, S(E') \setminus S(E)}$ implies

$$\begin{aligned} e_\chi(N_{E'/E}^r(\tilde{c}_{E'})) \neq 0 &\iff X_{E, S_{\text{ram}}(E'/K)}^\chi = 0 \\ &\iff X_{E, S_{\text{ram}}(E/K)}^\chi = 0 \quad \text{and} \quad Y_{E, S(E') \setminus S(E)}^\chi = 0 \\ &\iff e_\chi(\tilde{c}_E) \neq 0 \quad \text{and} \quad e_\chi(P_{E'/E}) \neq 0. \end{aligned}$$

It therefore suffices to verify (9) for characters χ with $e_\chi(P_{E'/E}) \neq 0$ and in this case the required equality follows directly from the commutative diagram

$$\begin{array}{ccc} (\mathbf{d}_{\mathcal{G}_{E'}}(C_{E',S(E')}))^\chi & \xrightarrow{\Theta_{E'}} & \left(\bigwedge_{\mathcal{G}_{E'}}^r \mathcal{O}_{E',S(E')}^\times \right)^\chi \\ \downarrow \nu_{E'/E} & & \downarrow N_{E'/E}^r \\ (\mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}))^\chi & \xrightarrow{P_{E'/E} \cdot \Theta_E} & \left(\bigwedge_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times \right)^\chi \end{array}$$

The existence of this diagram follows from the fact (itself a consequence of the general observation in [1, Lemma 1]) that for each $v \in S(E') \setminus S(E)$ there is in this case a commutative diagram

$$\begin{array}{ccc} \mathbf{d}_{\mathcal{G}_E}(\mathbf{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1])^\chi & \longrightarrow & \mathbb{Z}[\mathcal{G}_E]^\chi \\ \parallel & & \downarrow 1 - \chi(\mathrm{Fr}_v) \\ \mathbf{d}_{\mathcal{G}_E}(\mathbf{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1])^\chi & \longrightarrow & \mathbb{Z}[\mathcal{G}_E]^\chi \end{array}$$

where the upper row is induced by the isomorphism (4) and the lower row by the acyclicity of the complex $(\mathbf{R}\Gamma(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1])^\chi$.

3.3.3. The proof of Theorem 3.8(ii). For $a = (a_E)_E$ in \mathcal{A}_K^s and $z = (z_E)_E$ in $\mathrm{VS}(K^s/K)$ we need to show that for every fixed E in $\Omega(K^s/K)$ one has

$$a_E \cdot \Theta_E(z_E) \in \bigcap_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times.$$

In view of Lemma 3.9 below, we can also assume $a_E = \delta_T := 1 - Nv \cdot \mathrm{Fr}_v^{-1}$, with $T = \{v\}$, where v is a place of K that is not contained in $S(E)$ and $\mathcal{O}_{E,S(E),T}^\times$ is torsion-free.

To prove this we note that

$$\mathbf{d}_{\mathcal{G}_E}(C_{E,S(E),T}) = \mathbf{d}_{\mathcal{G}_E}^{-1}(\mathbb{F}_{T_E}^\times[0]) \cdot \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) = \delta_T \cdot \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}),$$

where the first equality follows from the exact triangle in Proposition 3.1(ii) (with T and T' replaced by \emptyset and T respectively) and the second from the fact the explicit resolution of $\mathbb{F}_{T_E}^\times[0]$ described in Remark 3.2 implies that $\mathbf{d}_{\mathcal{G}_E}(\mathbb{F}_{T_E}^\times[0]) = \mathbb{Z}[\mathcal{G}_E] \cdot \delta_T^{-1}$.

Hence, since $\bigcap_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E),T}^\times$ is a subset of $\bigcap_{\mathcal{G}_E}^r \mathcal{O}_{E,S(E)}^\times$ it is enough to show that for every prime p one has

$$(\mathbb{Q}_p \otimes_{\mathbb{Q}} \Theta_E)(\mathbf{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E,S(E),T})) \subseteq \bigcap_{\mathbb{Z}_p[\mathcal{G}_E]}^r (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E),T}^\times).$$

To do this we note that, since $\mathcal{O}_{E,S(E),T}^\times$ is torsion-free, Proposition 3.1(i) implies that $\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E,S(E),T}$ is an admissible complex of $\mathbb{Z}_p[\mathcal{G}_E]$ -modules in the sense of [5, Definition 2.20].

In particular, the above inclusion follows directly upon applying [5, Proposition A.11(ii)] with the data (\mathcal{R}, C, X) taken to be $(\mathbb{Z}_p[\mathcal{G}_E], \mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E,S(E),T}, \mathbb{Z}_p \otimes_{\mathbb{Z}} Y_E)$ and the map f equal to the natural composite homomorphism

$$\begin{aligned} H^1(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E,S(E),T}) &\longrightarrow \mathbb{Z}_p \otimes H^1(C_{E,S(E),T})_{\text{tf}} \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} X_{E,S(E)} \\ &\longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} Y_E, \end{aligned}$$

where the isomorphism is by Proposition 3.1(i) and the last map is the surjective map induced by (6).

This completes the proof of Theorem 3.8(ii).

Lemma 3.9. *For each field E in $\Omega(K^s/K)$ the following claims are valid.*

- (i) *Let U be any finite set of places of K containing $S(E)$ and all places dividing $|\mu_E|$. Then \mathcal{A}_E is generated as a \mathbb{Z} -module by $\{1 - Nv \cdot \text{Fr}_v^{-1} \mid v \notin U\}$. Furthermore, for any place v of K that does not belong to U , the group $\mathcal{O}_{E,S(E),\{v\}}^\times$ is torsion-free.*
- (ii) *The natural projection map $\mathcal{A}_K^s \rightarrow \mathcal{A}_E$ is surjective.*

Proof: The first assertion of claim (i) is proved in [19, Chapitre IV, Lemme 1.1] and the second assertion is a straightforward exercise since the residue characteristic of v does not divide $|\mu_E|$.

To prove claim (ii) it suffices to show that for any fields E and E' in $\Omega(K^s/K)$ with $E \subseteq E'$ the natural projection $\mathbb{Z}[\mathcal{G}_{E'}] \rightarrow \mathbb{Z}[\mathcal{G}_E]$ sends $\mathcal{A}_{E'}$ onto \mathcal{A}_E .

This follows easily by applying the first assertion of claim (i) for both E' and E with respect to the same set U in both cases. \square

3.3.4. The proof of Theorem 3.8(iii) and (iv). We fix E in $\Omega(K^s/K)$ and a character χ in $\widehat{\mathcal{G}_E}$ that is ramified.

In this case the fixed field E_χ of E by the kernel of χ belongs to $\Omega(K^s/K)$ and so for any system $c = (c_E)_E$ in RES_K one has

$$\begin{aligned} [E : E_\chi] \cdot e_\chi(c_E) &= e_\chi(N_{E/E_\chi}^r(c_E)) \\ &= e_\chi(P_{E/E_\chi}) \cdot e_\chi(c_{E_\chi}) \\ &= \left(\prod_{v \in S(E) \setminus S(E_\chi)} (1 - \chi(\text{Fr}_v^{-1})) \right) \cdot e_\chi(c_{E_\chi}). \end{aligned}$$

In particular, if $e_\chi(c_E) \neq 0$, then one has $\chi(\text{Fr}_v^{-1}) \neq 1$ for all v in $S(E) \setminus S(E_\chi)$.

On the other hand, since $L_{S(E_\chi)}(\chi, s) = L(\chi, s)$ one has

$$(10) \quad L_{S(E)}(\chi, s) = \left(\prod_{v \in S(E) \setminus S(E_\chi)} (1 - \chi(\text{Fr}_v) N v^{-s}) \right) \cdot L(\chi, s),$$

whilst [19, Chapitre I, Proposition 3.4] implies that if Σ denotes either $S(E)$ or $S_\infty(K)$, then for all ψ in $\widehat{\mathcal{G}}_E$ one has

$$(11) \quad \begin{aligned} \text{ord}_{s=0} L_\Sigma(\psi, s) &= \dim_{\mathbb{Q}^c}(X_{E, \Sigma}^\psi) \\ &= r + \begin{cases} \dim_{\mathbb{Q}^c}(X_{E, S_{\text{ram}}(E/K)}^\psi), & \text{if } \Sigma = S(E), \\ 0, & \text{if } \Sigma = S_\infty(K) \\ & \text{and } \psi \text{ is non-trivial.} \end{cases} \end{aligned}$$

(Note that if $\Sigma = S(E)$, then the second equality here is valid for the trivial character ψ since, by assumption, $S_{\text{ram}}(E/K)$ is not empty.)

In particular, since the ramified character χ cannot be trivial, this implies $L^{(r)}(\chi, 0) \neq 0$ and hence that the value

$$L_{S(E)}^{(r)}(\chi, 0) = \left(\prod_{v \in S(E) \setminus S(E_\chi)} (1 - \chi(\text{Fr}_v^{-1})) \right) \cdot L^{(r)}(\chi, 0)$$

is not zero if and only if $\chi(\text{Fr}_v^{-1}) \neq 1$ for all v in $S(E) \setminus S(E_\chi)$. Since we have observed that the latter condition is satisfied whenever $e_\chi(c_E) \neq 0$, this proves Theorem 3.8(iii).

We claim next that to prove Theorem 3.8(iv) it is enough to show that there exist elements $a = (a_E)_E$ of \mathcal{A}_K^s and $z = (z_E)_E$ of $\text{VS}(K^s/K)$ with the property that for every E in $\Omega(K^s/K)$ one has $a_E \in \mathbb{Q}[\mathcal{G}_E]^\times$ and $\mathbb{Q}[\mathcal{G}_E] \cdot z_E = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{d}_{\mathcal{G}_E}(C_{E, S(E)})$.

In fact, if this is true, then Theorem 3.8(ii) combines with the argument in the proof of Theorem 3.8(i) and (11) to show that $c := a \cdot \Theta_K^s(z)$ is a system in ES_K with the property that for all E in $\Omega(K^s/K)$ and all χ in $\widehat{\mathcal{G}}_E$ one has $e_\chi(c_E) \neq 0$ if and only if $L_{S(E)}^{(r)}(\chi, 0) \neq 0$.

To complete the proof of Theorem 3.8 it is thus enough to construct elements a in \mathcal{A}_K^s and z in $\text{VS}(K^s/K)$ with the properties described above.

To do this we again use Lemma 3.5 to reduce to the consideration of the tower of fields $(K_n)_{n \geq 1}$. Then, to ease notation, we set $\mathcal{G}_n := \mathcal{G}_{K_n}$, $\mathcal{H}_n := \text{Gal}(K_n/K_{n-1})$, $\Xi_n := \mathfrak{d}_{\mathcal{G}_n}(C_{K_n, S(K_n)})$, and $\nu_n := \nu_{K_n/K_{n-1}}$ for each n in \mathbb{N} .

To construct z we fix n in \mathbb{N} and assume that for each $m < n$ we have fixed an element z_m of Ξ_m such that $\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_m = \mathbb{Q}[\mathcal{G}_m] \cdot z_m$ and $\nu_m(z_m) = z_{m-1}$. Then we must construct an element z_n of Ξ_n such that $\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_n = \mathbb{Q}[\mathcal{G}_n] \cdot z_n$ and $\nu_n(z_n) = z_{n-1}$.

To do this we first choose any pre-image z'_n of z_{n-1} under the (surjective) map $\nu_{K_n/K_{n-1}}$. Then, since $\nu_{K_n/K_{n-1}}$ induces an isomorphism

$$\begin{aligned} \mathbb{Q} \otimes_{\mathbb{Z}} \Xi_n &= e_{\mathcal{H}_n} \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_n) \oplus (1 - e_{\mathcal{H}_n})(\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_n) \\ &\cong (\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_{n-1}) \oplus (1 - e_{\mathcal{H}_n})(\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_n) \end{aligned}$$

there exists an element z''_n of Ξ_n such that $e_{\mathcal{H}_n}(z''_n) = 0$ and $z'_n + z''_n$ is a $\mathbb{Q}[\mathcal{G}_n]$ -generator of $\mathbb{Q} \otimes_{\mathbb{Z}} \Xi_n$. The element $z_n := z'_n + z''_n$ is then an element of the required type.

By an entirely similar argument (which we leave to the reader), one finds that Lemma 3.9(ii) implies the existence of an element a in \mathcal{A}_K^s with the required property.

This completes the proof of Theorem 3.8.

4. Higher Kolyvagin derivatives

In this section we use the theory of equivariant higher Kolyvagin derivatives to obtain some strong, and unconditional, evidence in support of Conjecture 2.5.

4.1. Statement of the main result.

4.1.1. To state a precise version of Theorem 2.7 we fix an *odd* prime p and a finite abelian p -extension F of K . For simplicity, we shall assume that F contains the maximal p -extension $K(1)$ of K inside its Hilbert class field H_K .

As before, we continue to set $S := S_{\infty}(K)$ and we recall this means that for each E' in $\Omega(K^s/K)$ the notation $S(E')$ denotes $S_{\infty}(K) \cup S_{\text{ram}}(E'/K)$.

We write G_K for the absolute Galois group of K and ω for the p -adic Teichmüller character.

We also fix a non-trivial character

$$\chi: G_K \longrightarrow \mathbb{Q}_p^{c,\times}$$

of finite prime-to- p order, write L for the abelian extension of K that corresponds to the kernel of χ , and assume that all of the following conditions are satisfied:

- (H₁) L is not contained in $K(\mu_p)$.
- (H₂) $\chi^2 \neq \omega$ if $p = 3$.
- (H₃) Any place of K that ramifies in F is not completely split in L .
- (H₄) All Archimedean places of K split in L .
- (H₅) L/K is ramified.

In the sequel we shall consider the compositum

$$E := LF$$

of L and F . We note in particular that, since p is odd, hypothesis (H₄) implies that E is contained in K^s .

We decompose the group \mathcal{G}_E as a product $\Pi \times \Delta$ with Π the Sylow p -subgroup of \mathcal{G}_E and we identify \mathcal{G}_F and \mathcal{G}_L with Π and Δ in the obvious way.

We set $\mathcal{O} := \mathbb{Z}_p[\text{im } \chi]$ and define the (p, χ) -component of a Δ -module X by setting

$$X_\chi := \mathcal{O} \otimes_{\mathbb{Z}[\Delta]} X,$$

where \mathcal{O} is regarded as a $\mathbb{Z}[\Delta]$ -algebra via χ . For an element $a \in X$, we also set

$$a^\chi := 1 \otimes a \in X_\chi.$$

The following result is a precise version of Theorem 2.7.

Theorem 4.1. *For any system c in ES_K , any field F as above, and any character χ that satisfies all of the hypotheses (H₁), (H₂), (H₃), (H₄), and (H₅) one has $c_E^\chi \in \Theta_E(\mathfrak{d}_{\mathcal{G}_E}(C_{E,S(E)}))_\chi$.*

In the remainder of §4.1 we shall reduce the proof of this result to the proof of a statement about higher rank Stark systems for F/K and χ .

4.1.2. We fix an abelian pro- p extension \mathcal{K} of K that contains both F and the maximal p -extension $K(\mathfrak{q})$ of K inside its ray class field modulo \mathfrak{q} for all but finitely many primes \mathfrak{q} of K . We note that, since p is odd, \mathcal{K} is a subfield of K^s .

We write $\Omega'(\mathcal{K}/K)$ for the set of all finite extensions of K in \mathcal{K} and for each F' in $\Omega'(\mathcal{K}/K)$ we set $U_{F'} := (\mathcal{O}_{LF'}^\times)_\chi$.

We continue to write r in place of $r_K := |S_\infty(K)|$. We recall from [4, Definition 2.4] that a ‘strict p -adic Euler system’ of rank r for the extension \mathcal{K}/K and character χ is a collection

$$c = (c_{F'})_{F'} \in \prod_{F' \in \Omega'(\mathcal{K}/K)} \bigcap_{\mathcal{O}[\mathcal{G}_{F'}]}^r U_{F'}$$

that for all F' and F'' in $\Omega'(\mathcal{K}/K)$ with $F' \subseteq F''$ satisfies the distribution relation (2) with E'/E replaced by F''/F' . The set of strict p -adic Euler systems of rank r for \mathcal{K}/K and χ is denoted by $\text{ES}_r(\mathcal{K}/K, \chi)$ and is naturally an $\mathcal{O}[[\text{Gal}(\mathcal{K}/K)]]$ -module.

We further recall from [4, §2.3] that a Kolyvagin system of rank r for F/K and χ is a collection of elements that is parametrized by certain square-free products of prime ideals of K and that the set $\text{KS}_r(F/K, \chi)$ of all such collections is naturally a module over the quotient $\mathcal{O}[\Pi]$ of $\mathcal{O}[[\text{Gal}(\mathcal{K}/K)]]$.

In addition, the main result of [4] implies that, under the present hypotheses on \mathcal{K}/K and χ , the $\mathcal{O}[\Pi]$ -module $\text{KS}_r(F/K, \chi)$ is free of rank one and there exists a canonical ‘ F -relative r -th order Kolyvagin derivative’ homomorphism of $\mathcal{O}[[\text{Gal}(\mathcal{K}/K)]]$ -modules

$$\mathcal{D}_{F,r}: \text{ES}_r(\mathcal{K}/K, \chi) \longrightarrow \text{KS}_r(F/K, \chi).$$

We write $\text{VS}_{LK}^{(p)}$ for the pro- p -completion of $\text{VS}(LK/K)$.

Lemma 4.2.

- (i) For c in ES_K the assignment $F' \mapsto c_{LF'}^\chi$ defines an element c_{str}^χ of $\text{ES}_r(\mathcal{K}/K, \chi)$.
- (ii) There exists a natural homomorphism of $\widehat{\mathcal{R}}_{K^s/K}^p$ -modules

$$\Theta_{LK}^\chi: \text{VS}_{LK}^{(p)} \longrightarrow \text{ES}_r(\mathcal{K}/K, \chi).$$

- (iii) Let b be a generator of $\text{VS}_{LK}^{(p)}$ over $\widehat{\mathcal{R}}_{K^s/K}^p$. Then Theorem 4.1 is valid if the $\mathcal{O}[\Pi]$ -module $\text{KS}_r(F/K, \chi)$ is generated by the image of $\Theta_{LK}^\chi(b)$ under $\mathcal{D}_{F,r}$.

Proof: Hypothesis (H₅) implies that for each F' in $\Omega'(\mathcal{K}/K)$ the field LF' belongs to $\Omega(LK/K)$. Thus, if c belongs to ES_K , then for F' in $\Omega'(\mathcal{K}/K)$ the element $c_{\text{str},F'}^\chi := c_{LF'}^\chi$ belongs to $\bigcap_{\mathcal{O}[\mathcal{G}_{F'}]}^r (\mathcal{O}_{LF',S(LF')})_\chi^\times$ and the collection

$$c_{\text{str}}^\chi := (c_{\text{str},F'}^\chi)_{F'}$$

satisfies the necessary distribution relations as the fields F' vary.

To deduce c_{str}^χ belongs to $\text{ES}_r(\mathcal{K}/K, \chi)$ it is thus enough to show that for each F' the element $c_{LF'}^\chi$ belongs to $\bigcap_{\mathcal{O}[\mathcal{G}_{F'}]}^r U_{F'}$.

In view of the equality (11), it is therefore sufficient to show that $e_{\psi\chi}(c_{LF'}) = 0$ for any character ψ in $\widehat{\mathcal{G}}_{F'}$ for which $L_{S(LF')}^{(r)}(\psi\chi, 0) = 0$, where we regard each product $\psi\chi$ as a character in $\widehat{\mathcal{G}}_{LF'}$ in the obvious way.

In addition, since (H₅) implies $\psi\chi$ is ramified, the required vanishing is a direct consequence of Theorem 3.8(iii). This proves claim (i).

To prove claim (ii) we show first that for each F' in $\Omega'(\mathcal{K}/K)$ the map $\Theta_{LF'}^\dagger := \mathbb{Q}_p^c \otimes_{\mathbb{Q}} \Theta_{LF'}$ sends $\Xi_{F',\chi} := d_{\mathcal{G}_{LF'}}(C_{LF',S(LF')})_\chi$ to $\bigcap_{\mathcal{O}[\mathcal{G}_{F'}]}^r U_{F'}$.

To show this we note that (H_1) implies $\mu_{LF',\chi}$ vanishes so that $\mathcal{A}_{LF',\chi}$ is equal to $\mathbb{Z}[\mathcal{G}_{LF'}]_\chi$, and hence contains the element e_χ .

From the argument of Theorem 3.8(iv) we can therefore deduce $\Theta_{LF'}^\dagger(\Xi_{F',\chi})$ is contained in $\bigcap_{\mathcal{O}[\mathcal{G}_{F'}]}^r U_{F'}$.

We can therefore define Θ_{LK}^χ to be the product map $\prod_{F'} \Theta_{F'}^\chi$, where F' runs over $\Omega'(\mathcal{K}/K)$ and each $\Theta_{F'}^\chi$ is the composite homomorphism

$$\mathrm{VS}_{LK}^{(p)} \longrightarrow \mathfrak{d}_{\mathcal{G}_{LF'}}(C_{LF',S(LF')})_\chi \xrightarrow{\Theta_{LF'}^\dagger} \bigcap_{\mathcal{O}[\mathcal{G}_{F'}]}^r U_{F'},$$

where the first map is the natural projection.

To prove claim (iii) we assume $\mathcal{D}_{F,r}(\Theta_{LK}^\chi(b))$ is a generator over $\mathcal{O}[\Pi]$ of $\mathrm{KS}_r(F/K, \chi)$. Then for any c in ES_K claim (i) implies there exists an element $x = x_{b,c}$ of $\mathcal{O}[\Pi]$ such that $\mathcal{D}_{F,r}(c_{\mathrm{str}}^\chi) = x \cdot \mathcal{D}_{F,r}(\Theta_{LK}^\chi(b))$.

Upon evaluating these (equivariant) Kolyvagin systems at 1 (that is, at the empty product of prime ideals of K) one deduces that

$$c_E^\chi = c_{\mathrm{str},F}^\chi = \mathcal{D}_{F,r}(c_{\mathrm{str}}^\chi)_1 = x \cdot \mathcal{D}_{F,r}(\Theta_{LK}^\chi(b))_1 = x \cdot \Theta_{LK}^\chi(b)_F = x \cdot \Theta_E(b_E)^\chi,$$

where b_E denotes the image of b in $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathfrak{d}_{\mathcal{G}_E}(C_{E,S(E)})$. This shows that c_E^χ belongs to $\Theta_E(\mathfrak{d}_{\mathcal{G}_E}(C_{E,S(E)}))_\chi$, as required. \square

In view of Lemma 4.2(ii), to prove Theorem 4.1 it is enough to show that $\mathcal{D}_{F,r}(\Theta_{LK}^\chi(b))$ generates $\mathrm{KS}_r(F/K, \chi)$ over $\mathcal{O}[\Pi]$.

To do this we use the $\mathcal{O}[\Pi]$ -module $\mathrm{SS}_r(F/K, \chi)$ of Stark systems of rank r for F/K and χ that is defined in [4, §2.3] and recall from [4, Theorem 3.3(ii)] that there exists a canonical ‘algebraic regulator’ isomorphism of $\mathcal{O}[\Pi]$ -modules of the form

$$\mathcal{R}_r^\chi: \mathrm{SS}_r(F/K, \chi) \xrightarrow{\sim} \mathrm{KS}_r(F/K, \chi).$$

The key now is to prove the following result (which, we note, also justifies [4, Remark 3.5]). We abbreviate $\mathrm{Gal}(LK/K)$ to \mathcal{G}_{LK} .

Theorem 4.3. *There exists a canonical surjective homomorphism*

$$\Delta_{F,r}^\chi: \mathrm{VS}_{LK}^{(p)} \longrightarrow \mathrm{SS}_r(F/K, \chi)$$

of $\mathcal{O}[[\mathcal{G}_{LK}]]$ -modules that makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{VS}_{LK}^{(p)} & \xrightarrow{\Theta_{LK}^\chi} & \mathrm{ES}_r(\mathcal{K}/K, \chi) \\ \Delta_{F,r}^\chi \downarrow & & \downarrow \mathcal{D}_{F,r} \\ \mathrm{SS}_r(F/K, \chi) & \xrightarrow{\mathcal{R}_r^\chi} & \mathrm{KS}_r(F/K, \chi) \end{array}$$

Before proving this result, we note that it does indeed finish the proof of Theorem 4.1 since the surjectivity of the composite map $\mathcal{R}_r^\times \circ \Delta_{F,r}^\times = \mathcal{D}_{F,r} \circ \Theta_{LK}^\times$ implies that the image $\mathcal{D}_{F,r}(\Theta_{LK}^\times(b))$ of b under this map must be a generator of the $\mathcal{O}[\Pi]$ -module $\text{KS}_r(F/K, \chi)$, as required.

4.2. Stark systems and the proof of Theorem 4.3. We first quickly review the definition of Stark systems given in [4].

4.2.1. Let m be a non-negative integer and write \mathcal{P}_m for the set of prime ideals of K that do not divide p and split completely in $H_K E(\mu_{p^m}, (\mathcal{O}_K^\times)^{p^{-m}})$, where $(\mathcal{O}_K^\times)^{p^{-m}}$ denotes the set of elements x in \mathbb{Q}^c with $x^{p^m} \in \mathcal{O}_K^\times$. (Recall here that H_K denotes the Hilbert class field of K and E the field LF .)

We write \mathcal{N}_m for the set of square-free products of primes in \mathcal{P}_m . For each product \mathfrak{n} in \mathcal{N}_m , we write $\Sigma_{\mathfrak{n}}$ for the union of $S_\infty(K)$ and the set of prime divisors of \mathfrak{n} , we set

$$U_{E,\mathfrak{n}} := \mathcal{O}_{E,\Sigma_{\mathfrak{n}}}^\times$$

and we write $\nu(\mathfrak{n})$ for the number of prime divisors of \mathfrak{n} .

For any prime \mathfrak{q} of K , we define

$$(12) \quad v_{\mathfrak{q}}: E^\times \longrightarrow \mathbb{Z}[\mathcal{G}_E]; a \longmapsto \sum_{\sigma \in \mathcal{G}_E} \text{ord}_{\mathfrak{Q}}(\sigma a) \sigma^{-1},$$

where \mathfrak{Q} is a fixed place of E lying above \mathfrak{q} and $\text{ord}_{\mathfrak{Q}}$ denotes the normalized additive valuation at \mathfrak{Q} .

We set $\mathcal{O}_m := \mathcal{O}/(p^m)$. Upon reduction modulo p^m , the above map induces a map of $\mathcal{O}_m[\Pi]$ -modules $(E^\times/p^m)_\chi \rightarrow \mathcal{O}_m[\Pi]$ that we also denote by $v_{\mathfrak{q}}$ and we then set

$$\mathcal{S}_m^n := \{a \in (E^\times/p^m)_\chi \mid v_{\mathfrak{q}}(a) = 0 \text{ for every } \mathfrak{q} \nmid \mathfrak{n}\}.$$

For each pair of products \mathfrak{m} and \mathfrak{n} in \mathcal{N}_m with $\mathfrak{n} \mid \mathfrak{m}$, there is then an exact sequence of $\mathcal{O}_m[\Pi]$ -modules

$$0 \longrightarrow \mathcal{S}_m^n \longrightarrow \mathcal{S}_m^{\mathfrak{m}} \xrightarrow{\bigoplus_{\mathfrak{q} \mid \mathfrak{m}/\mathfrak{n}} v_{\mathfrak{q}}} \bigoplus_{\mathfrak{q} \mid \mathfrak{m}/\mathfrak{n}} \mathcal{O}_m[\Pi]$$

and hence (via the general result of [5, Proposition A.3]), for any non-negative integer r a map of $\mathcal{O}_m[\Pi]$ -modules

$$v_{\mathfrak{m},\mathfrak{n}} := \pm \bigwedge_{\mathfrak{q} \mid \mathfrak{m}/\mathfrak{n}} v_{\mathfrak{q}}: \prod_{\mathcal{O}_m[\Pi]}^{r+\nu(\mathfrak{m})} \mathcal{S}_m^{\mathfrak{m}} \longrightarrow \prod_{\mathcal{O}_m[\Pi]}^{r+\nu(\mathfrak{n})} \mathcal{S}_m^n.$$

Here the sign is chosen so that $v_{\mathfrak{m}',\mathfrak{n}} = v_{\mathfrak{m},\mathfrak{n}} \circ v_{\mathfrak{m}',\mathfrak{m}}$ if $\mathfrak{m} \mid \mathfrak{m}'$ (cf. [5, §3.1]).

Then the $\mathcal{O}_m[\Pi]$ -module of Stark systems of rank r and level m for F/K and χ is defined to be the inverse limit

$$\mathrm{SS}_r(F/K, \chi)_m := \varprojlim_{\mathfrak{n} \in \mathcal{N}_m} \bigcap_{\mathcal{O}_m[\Pi]}^{r+\nu(\mathfrak{n})} \mathcal{S}_m^{\mathfrak{n}}$$

with transition maps $v_{\mathfrak{m}, \mathfrak{n}}$.

We also recall, from [4, §2.3], that the present hypotheses on χ imply that for each non-negative integer m there exists a natural surjective homomorphism of $\mathcal{O}_{m+1}[\Pi]$ -modules $\pi_m: \mathrm{SS}_r(F/K, \chi)_{m+1} \rightarrow \mathrm{SS}_r(F/K, \chi)_m$ so that one can set

$$\mathrm{SS}_r(F/K, \chi) := \varprojlim_m \mathrm{SS}_r(F/K, \chi)_m,$$

where the limit is taken with respect to the transition morphisms π_m .

We next note the argument of [13, Proposition 3.6] shows that, just as above, the maps $v_{\mathfrak{q}}$ in (12) give rise to a homomorphism of $\mathcal{O}[\Pi]$ -modules

$$v_{\mathfrak{m}, \mathfrak{n}}: \bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{m})} U_{E, \mathfrak{m}, \chi} \longrightarrow \bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \mathfrak{n}, \chi}$$

and we define the $\mathcal{O}[\Pi]$ -module of global Stark systems of rank r for F/K and χ to be

$$\mathrm{SS}_r^{\mathrm{glob}}(F/K, \chi) := \varprojlim_{\mathfrak{n} \in \mathcal{N}_0} \bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \mathfrak{n}, \chi},$$

where the limit is taken with respect to the maps $v_{\mathfrak{m}, \mathfrak{n}}$.

Lemma 4.4. *There exists a natural surjective homomorphism of $\mathcal{O}[\Pi]$ -modules*

$$\varrho_{F, \chi}: \mathrm{SS}_r^{\mathrm{glob}}(F/K, \chi) \longrightarrow \mathrm{SS}_r(F/K, \chi).$$

Proof: For each natural number m we use the cofinal subset \mathcal{N}'_m of \mathcal{N}_m described in Lemma 4.5 below.

Then the latter result implies that for each \mathfrak{m} in \mathcal{N}'_m there exists a natural surjective homomorphism of $\mathcal{O}[\Pi]$ -modules

$$\begin{aligned} \bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{m})} U_{E, \mathfrak{m}, \chi} &= \bigwedge_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{m})} U_{E, \mathfrak{m}, \chi} \twoheadrightarrow \bigwedge_{\mathcal{O}_m[\Pi]}^{r+\nu(\mathfrak{m})} (U_{E, \mathfrak{m}, \chi}/(p^m)) \\ &= \bigwedge_{\mathcal{O}_m[\Pi]}^{r+\nu(\mathfrak{m})} \mathcal{S}_m^{\mathfrak{m}} = \bigcap_{\mathcal{O}_m[\Pi]}^{r+\nu(\mathfrak{m})} \mathcal{S}_m^{\mathfrak{m}} \end{aligned}$$

that are compatible with the respective transition morphisms $v_{\mathfrak{m}, \mathfrak{n}}$ as \mathfrak{n} and \mathfrak{m} vary over \mathcal{N}'_m . This therefore gives a surjective homomorphism $\varrho_{F, \chi, m}$ of $\mathcal{O}[\Pi]$ -modules from $\mathrm{SS}_r^{\mathrm{glob}}(F/K, \chi)$ to $\mathrm{SS}_r(F/K, \chi)_m$.

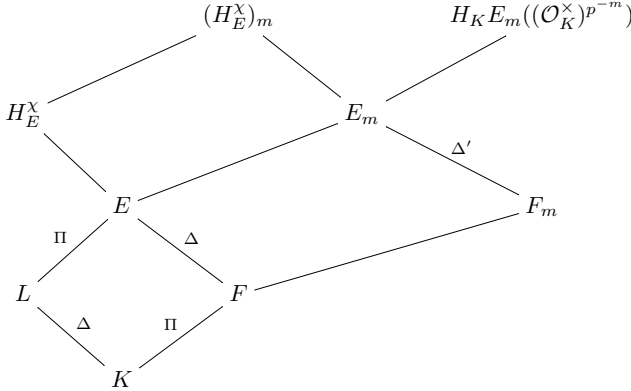
These homomorphisms $\varrho_{F,\chi,m}$ are compatible with the transition morphisms π_m as m varies and hence lead to a surjective homomorphism $\varrho_{F,\chi}$ of the required sort. \square

Lemma 4.5. *For each natural number m there exists a cofinal subset \mathcal{N}'_m of \mathcal{N}_m such that for every product \mathfrak{m} in \mathcal{N}'_m the following conditions are satisfied.*

- (i) *The $\mathcal{O}[\Pi]$ -module $U_{E,\mathfrak{m},\chi}$ is free of rank $r + \nu(\mathfrak{m})$.*
- (ii) *The $\mathcal{O}_m[\Pi]$ -module $\mathcal{S}_m^{\mathfrak{m}}$ is equal to $(U_{E,\mathfrak{m}}/p^m)_\chi$ and is free of rank $r + \nu(\mathfrak{m})$.*

Proof: For each number field N we set $N_m := N(\mu_{p^m})$. We also write H_E^χ for the subfield of the Hilbert class field of E that corresponds to the image of $\text{Cl}(\mathcal{O}_E)$ in $\text{Cl}(\mathcal{O}_E)_\chi$.

We claim that $H_E^\chi \cap H_K E_m((\mathcal{O}_K^\times)^{p^{-m}}) = E$ and to prove this we shall use the following diagram of fields.



We set $D := (H_E^\chi)_m \cap H_K E_m((\mathcal{O}_K^\times)^{p^{-m}})$ and at the outset consider the conjugation action of Δ' on the abelian group $\text{Gal}(D/E_m)$.

In fact, since D is contained in $H_K E_m((\mathcal{O}_K^\times)^{p^{-m}})$ it is abelian over F_m and so this conjugation action must be trivial.

However, since H_E^χ/E is of p -power degree, $\text{Gal}(D/E_m)$ is a p -group and so identifies with a subset of the module $\mathcal{O} \otimes_{\mathbb{Z}} \text{Gal}(D/E_m)$ upon which the conjugation action of Δ' is via the restriction χ' of χ (as $D \subseteq (H_E^\chi)_m$).

In particular, since Δ' , and hence also χ' , is non-trivial (as $L \not\subseteq K(\mu_p)$ by hypothesis (H_1)), the group $\text{Gal}(D/E_m)$ must be trivial and so $(H_E^\chi)_m \cap H_K E_m((\mathcal{O}_K^\times)^{p^{-m}}) = E_m$.

This fact implies that $H_E^\chi \cap H_K E_m((\mathcal{O}_K^\times)^{p^{-m}})$ is equal to the field $D' := H_E^\chi \cap E_m$.

Then the same approach as above shows that $D' = E$ since the conjugation action of Δ on $\text{Gal}(D'/E)$ is trivial (as $D' \subseteq E_m$) whilst $\text{Gal}(D'/E)$ also embeds into an \mathcal{O} -module upon which Δ acts via the non-trivial character χ .

We have now shown that $H_E^\chi \cap H_K E_m((\mathcal{O}_K^\times)^{p^{-m}}) = E$ and so, by Chebotarev’s density theorem, we can choose a set of primes \mathfrak{n} in \mathcal{N}'_m with the property that $\text{Cl}(\mathcal{O}_{E, \Sigma_{\mathfrak{n}}})_\chi$ is trivial.

We claim that the cofinal subset \mathcal{N}'_m of \mathcal{N}_m comprising multiples of \mathfrak{n} has both of the stated properties (i) and (ii).

To prove this we fix \mathfrak{m} in \mathcal{N}'_m . Then $\text{Cl}(\mathcal{O}_{E, \Sigma_{\mathfrak{m}}})_\chi$ is trivial and so Proposition 3.1(i), with $\Sigma = S_{\text{ram}}(E/K) \cup \Sigma_{\mathfrak{m}}$ and T empty, implies that $H^0(C_{E, \Sigma})_\chi = (\mathcal{O}_{E, \Sigma}^\times)_\chi = (U_{E, \mathfrak{m}})_\chi$ and that $H^1(C_{E, \Sigma})_\chi = X_{E, \Sigma, \chi} = Y_{E, \Sigma_{\mathfrak{m}}, \chi}$ is a free $\mathcal{O}[\Pi]$ -module of rank $r + \nu(\mathfrak{m})$. (Here we used hypothesis (H₃).)

Since $(C_{E, \Sigma})_\chi$ belongs to $D^{\text{perf}}(\mathcal{O}[\Pi])$ these facts combine to imply that $(U_{F, \mathfrak{m}})_\chi$ is free of rank $r + \nu(\mathfrak{m})$, as required to prove claim (i).

Claim (ii) then follows since the vanishing of $\text{Cl}(\mathcal{O}_{E, \Sigma_{\mathfrak{m}}})_\chi$ implies that the natural inclusion $(U_{E, \mathfrak{m}})_\chi / (p^m) = (U_{E, \mathfrak{m}} / p^m)_\chi \rightarrow \mathcal{S}_m^{\mathfrak{m}}$ is bijective. \square

4.2.2. Lemma 4.4 implies that to construct a surjective homomorphism of $\mathcal{O}[[\mathcal{G}_{LK}]]$ -modules $\Delta_{F, r}^\chi: \text{VS}_{LK}^{(p)} \rightarrow \text{SS}_r(F/K, \chi)$ it is enough to construct a surjective map of $\mathcal{O}[[\mathcal{G}_{LK}]]$ -modules $\Delta_{F, r, 1}^\chi: \text{VS}_{LK}^{(p)} \rightarrow \text{SS}_r^{\text{glob}}(F/K, \chi)$, and then to define $\Delta_{F, r}^\chi$ to be $\varrho_{F, \chi} \circ \Delta_{F, r, 1}^\chi$. To do this we shall mimic the construction of ‘horizontal determinant systems’ from [5, §3.3].

For each \mathfrak{q} in \mathcal{P}_0 we write $K(\mathfrak{q})/K$ for the maximal p -extension of K inside its ray class field modulo \mathfrak{q} . For each \mathfrak{n} in \mathcal{N}_0 we write $E(\mathfrak{n})$ for the compositum $E \prod_{\mathfrak{q}|\mathfrak{n}} K(\mathfrak{q})$ and set $S_{\mathfrak{n}} := S(E(\mathfrak{n})) = S_{\text{ram}}(E/K) \cup \Sigma_{\mathfrak{n}}$.

For each \mathfrak{n} in \mathcal{N}_0 we then define $\Delta_{\mathfrak{n}}$ to be the composite homomorphism

$$\mathfrak{d}_{\mathcal{G}_{E(\mathfrak{n})}}(C_{E(\mathfrak{n}), S_{\mathfrak{n}}})_\chi \longrightarrow \mathfrak{d}_{\mathcal{O}[\Pi]}((C_{E, S_{\mathfrak{n}}})_\chi) \xrightarrow{\Theta_{E, \mathfrak{n}}} \bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \mathfrak{n}, \chi},$$

where the first map is the natural projection and $\Theta_{E, \mathfrak{n}}$ is defined just as in (8) but with $S(E)$ replaced by $S_{\mathfrak{n}}$ and with respect to a fixed choice of $\mathcal{O}[\Pi]$ -basis of the free module $Y_{E, \Sigma_{\mathfrak{n}}, \chi}$. In particular, the fact that $\Theta_{E, \mathfrak{n}}$ maps $\mathfrak{d}_{\mathcal{O}[\Pi]}((C_{E, S_{\mathfrak{n}}})_\chi)$ to $\bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \mathfrak{n}, \chi}$ follows by an application of [5, Proposition A.11(ii)], just as in §3.3.3.

We write $\Delta'_{\mathfrak{n}}$ for the composite of $\Delta_{\mathfrak{n}}$ and the natural (surjective) projection map from $\text{VS}_{LK}^{(p)}$ to $\mathfrak{d}_{\mathcal{G}_{E(\mathfrak{n})}}(C_{E(\mathfrak{n}), S_{\mathfrak{n}}})_\chi$.

Then, provided that one makes a compatible choice of bases of the modules $Y_{E, \Sigma_n, \chi}$, these maps Δ'_n are compatible with the transition morphisms $v_{\mathfrak{m}, \mathfrak{n}}$ as \mathfrak{n} varies over \mathcal{N}'_0 and so lead to the construction of a map $\Delta_{F, r, 1}^X$.

In addition, since the modules $\ker(\Delta'_n)$ are compact the derived limit $\varprojlim_{\mathfrak{n} \in \mathcal{N}'_0}^1 \ker(\Delta'_n)$ vanishes.

To prove that $\Delta_{F, r, 1}^X$, and hence also $\Delta_{E, r}^X$, is surjective it is therefore enough to prove that the maps Δ_n are surjective for all \mathfrak{n} in the cofinal subset \mathcal{N}'_0 of \mathcal{N}_0 comprising products \mathfrak{m} for which $\text{Cl}(\mathcal{O}_{E, \Sigma_n})_\chi$ vanishes.

This is true since for all such \mathfrak{n} the argument of Lemma 4.5(i) shows the $\mathcal{O}[\Pi]$ -modules $H^0((C_{E, S_n})_\chi) = U_{E, \mathfrak{n}, \chi}$ and $H^1((C_{E, S_n})_\chi) = Y_{E, \Sigma_n, \chi}$ to be free of rank $r + \nu(\mathfrak{n})$ and hence that $\Theta_{E, \mathfrak{n}}$ coincides with the natural ‘passage-to-cohomology’ isomorphism

$$d_{\mathcal{O}[\Pi]}((C_{E, S_n})_\chi) \cong \left(\bigwedge_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \mathfrak{n}, \chi} \right) \otimes \left(\bigwedge_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} Y_{E, \Sigma_n, \chi}^* \right) \cong \bigwedge_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \mathfrak{n}, \chi}$$

that exists in this case.

4.2.3. To complete the proof of Theorem 4.3 it suffices to check the diagram commutes.

To do this, for each \mathfrak{n} in \mathcal{N}_0 we set $\mathcal{G}_n := \text{Gal}(E(\mathfrak{n})/K)$ and $\mathcal{H}_n := \text{Gal}(F(\mathfrak{n})/F)$ and write I_n for the kernel of the augmentation map $\mathbb{Z}[\mathcal{H}_n] \rightarrow \mathbb{Z}$.

For each product \mathfrak{n} in \mathcal{N}_0 and each prime \mathfrak{q} in \mathcal{P}_0 we define a map $\varphi_{\mathfrak{q}}^n$ as follows:

$$\varphi_{\mathfrak{q}}^n: E^\times \longrightarrow \mathbb{Z}[\mathcal{G}_E] \otimes_{\mathbb{Z}} I_n/I_n^2; a \longmapsto \sum_{\sigma \in \mathcal{G}_E} \sigma^{-1} \otimes (\text{rec}_{\Omega}(\sigma a) - 1),$$

where Ω is a fixed place of E lying above \mathfrak{q} and $\text{rec}_{\Omega}: E^\times \rightarrow \text{Gal}(E(\mathfrak{n})/E) \cong \mathcal{H}_n$ is the local reciprocity map at Ω . This map induces a map $U_{E, \Sigma_n, \chi} \rightarrow \mathcal{O}[\Pi] \otimes_{\mathbb{Z}} I_n/I_n^2$, that we continue to denote by $\varphi_{\mathfrak{q}}^n$.

Upon combining these maps for each prime divisor \mathfrak{q} of \mathfrak{n} we obtain a map of $\mathcal{O}[\Pi]$ -modules

$$\bigwedge_{\mathfrak{q}|\mathfrak{n}} \varphi_{\mathfrak{q}}^n: \bigcap_{\mathcal{O}[\Pi]}^{r+\nu(\mathfrak{n})} U_{E, \Sigma_n, \chi} \longrightarrow \bigcap_{\mathcal{O}[\Pi]}^r U_{E, \Sigma_n, \chi} \otimes I_n^{\nu(\mathfrak{n})}/I_n^{\nu(\mathfrak{n})+1}.$$

Finally we recall (from, for example, [4, (4.1.1)]) that there exists a natural injective homomorphism

$$\begin{aligned} \iota_n: \bigcap_{\mathcal{O}[\Pi]}^r U_{E, \Sigma_n, \chi} \otimes_{\mathbb{Z}} I_n^{\nu(\mathfrak{n})}/I_n^{\nu(\mathfrak{n})+1} \\ \longhookrightarrow \bigcap_{\mathcal{O}[\mathcal{G}_n]}^r U_{E(\mathfrak{n}), \Sigma_n, \chi} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{H}_n]/I_n^{\nu(\mathfrak{n})+1}. \end{aligned}$$

Then, after unwinding the definitions of the maps Θ_{LK}^X and $\Delta_{F,r}^X$, the argument that is used to prove [4, Theorem 4.13] shows that the square in Theorem 4.3 is commutative if and only if for all products \mathfrak{n} in \mathcal{N}_0 and all elements $b_{\mathfrak{n}}$ of $\mathfrak{d}_{\mathcal{G}_{E(\mathfrak{n})}}(C_{E(\mathfrak{n}),S_{\mathfrak{n}}})$ one has

$$\sum_{\sigma \in \mathcal{H}_{\mathfrak{n}}} \sigma(\Theta_{E(\mathfrak{n})}(b_{\mathfrak{n}})^X) \otimes \sigma^{-1} = \iota_{\mathfrak{n}} \left(\left(\bigwedge_{\mathfrak{q}|\mathfrak{n}} \varphi_{\mathfrak{q}}^{\mathfrak{n}} \right) (\Delta_{\mathfrak{n}}(b_{\mathfrak{n}}^X)) \right)$$

in the module $\bigcap_{\mathcal{O}[\mathcal{G}_{\mathfrak{n}}]}^r U_{E(\mathfrak{n}),\Sigma_{\mathfrak{n}},\mathcal{X}} \otimes \mathbb{Z}[\mathcal{H}_{\mathfrak{n}}]/\mathcal{I}_{\mathfrak{n}}^{\nu(\mathfrak{n})+1}$.

To complete the proof of Theorem 4.3, it is thus enough to note that the latter equality follows directly from the argument used in [3, §5.7] to prove [3, Theorem 5.16].

5. The case $K = \mathbb{Q}$

In this section we consider the special case $K = \mathbb{Q}$ and will, in particular, prove Theorems 2.8 and 2.9.

Throughout we abbreviate the rings $\mathcal{R}_{\mathbb{Q}}$ and $\mathcal{R}_{\mathbb{Q}}^s$ and ideals $\mathcal{A}_{\mathbb{Q}}$ and $\mathcal{A}_{\mathbb{Q}}^s$ to \mathcal{R} , \mathcal{R}^s , \mathcal{A} , and \mathcal{A}^s respectively.

We also set $S := S_{\infty}(\mathbb{Q})$ so that for any finite abelian extension E of \mathbb{Q} the set $S(E)$ comprises the Archimedean place of \mathbb{Q} together with all rational primes that ramify in E .

For each natural number m we set $\mathbb{Q}(m) := \mathbb{Q}(\mu_m)$ and denote its maximal real subfield by $\mathbb{Q}(m)^+$.

We write \mathbb{N}^{\dagger} for the subset of $\mathbb{N} \setminus \{1\}$ comprising all numbers that are not congruent to 2 modulo 4 and \mathbb{N}^{\ddagger} for the subset $\mathbb{N}^{\dagger} \setminus \{3, 4\}$ of \mathbb{N}^{\dagger} comprising numbers m for which $\mathbb{Q}(m)^+ \neq \mathbb{Q}$.

5.1. In this subsection we establish some essential preliminary results.

5.1.1. The first such result gives an explicit description of the image of the homomorphism $\Theta_{\mathbb{Q}}^s$ defined in §3.3.1.

Before stating this result we note the Kronecker–Weber theorem implies that the set of fields $\{\mathbb{Q}(m)^+\}_{m \in \mathbb{N}^{\ddagger}}$ is cofinal in the set $\Omega^+ := \Omega(\mathbb{Q}^s/\mathbb{Q})$ of non-trivial finite real abelian extensions of \mathbb{Q} in \mathbb{Q}^c .

The distribution relations satisfied by cyclotomic units therefore imply that there exists a unique system ε in $\text{RES}_{\mathbb{Q}}$ with the property that

$$\varepsilon_{\mathbb{Q}(m)^+} = (1 - \zeta_m)^{(1+\tau)/2}$$

for every m in \mathbb{N}^{\ddagger} , where τ denotes complex conjugation.

Theorem 5.1. *The \mathcal{R}^s -module $\text{ES}_{\mathbb{Q}}^b$ is free of rank one, with basis ε .*

Proof: For each E in Ω^+ we write ϑ_E for the composite isomorphism of $\mathbb{R}[\mathcal{G}_E]$ -modules

$$\begin{aligned} \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{d}_{\mathcal{G}_E}(C_{E,S(E)}) &\xrightarrow{\sim} \mathbf{d}_{\mathbb{R}[\mathcal{G}_E]}(\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E)}^\times) \otimes_{\mathbb{R}[\mathcal{G}_E]} \mathbf{d}_{\mathbb{R}[\mathcal{G}_E]}^{-1}(\mathbb{R} \otimes_{\mathbb{Z}} X_{E,S(E)}) \\ &\xrightarrow{\sim} \mathbf{d}_{\mathbb{R}[\mathcal{G}_E]}(\mathbb{R} \otimes_{\mathbb{Z}} X_{E,S(E)}) \otimes_{\mathbb{R}[\mathcal{G}_E]} \mathbf{d}_{\mathbb{R}[\mathcal{G}_E]}^{-1}(\mathbb{R} \otimes_{\mathbb{Z}} X_{E,S(E)}) \\ &\xrightarrow{\sim} \mathbb{R}[\mathcal{G}_E]. \end{aligned}$$

Here the first arrow is induced by the descriptions in Proposition 3.1(i) and the natural passage-to-cohomology map, the final arrow is the canonical evaluation map and the second arrow is induced by the canonical Dirichlet regulator isomorphism

$$(13) \quad \lambda_E : \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E)}^\times \cong \mathbb{R} \otimes_{\mathbb{Z}} X_{E,S(E)}$$

that sends each u in $\mathcal{O}_{E,S(E)}^\times$ to $-\sum_w \log(|u|_w) \cdot w$, where in the sum w runs over all places of E above those in $S(E)$ and $|\cdot|_w$ denotes the absolute value with respect to w (normalized as in [19, Chapitre 0, 0.2]).

We then write \mathfrak{z}_E for the pre-image under ϑ_E of the element

$$\theta_{E,S(E)}^*(0) := \sum_{\chi \in \widehat{\mathcal{G}}_E} L_{S(E)}^*(\chi^{-1}, 0) e_\chi$$

of $\mathbb{R}[\mathcal{G}_E]^\times$, where $L_{S(E)}^*(\chi^{-1}, 0)$ denotes the leading term in the Taylor expansion at $s = 0$ of the series $L_{S(E)}(\chi^{-1}, s)$.

Then, by Lemma 5.2 below, the collection $\mathfrak{z} := (\mathfrak{z}_E)_{E \in \Omega^+}$ is an \mathcal{R}^s -basis of $\text{VS}(\mathbb{Q}^s/\mathbb{Q})$. In addition, by the argument of [6, Lemma 5.4], one knows that the map $\Theta_{\mathbb{Q}}^s$ is injective.

Hence, the claimed result will follow if we can show for each m in \mathbb{N}^\dagger that the element $\Theta_{\mathbb{Q}}^s(\mathfrak{z})_{\mathbb{Q}(m)^+} = \Theta_{\mathbb{Q}(m)^+}(\mathfrak{z}_{\mathbb{Q}(m)^+})$ is equal to $(1 - \zeta_m)^{(1+\tau)/2}$.

To do this we fix such an m and set $S(m) := S(\mathbb{Q}(m)) = S(\mathbb{Q}(m)^+)$ and $\mathcal{G}_m := \mathcal{G}_{\mathbb{Q}(m)^+}$. We then fix an embedding $j: \mathbb{Q}^c \rightarrow \mathbb{C}$ and recall (from, for example [19, Chapitre 3, §5]) that for each χ in $\widehat{\mathcal{G}}_m$ the first derivative $L'_{S(m)}(\chi, s)$ is holomorphic at $s = 0$ and such that

$$L'_{S(m)}(\chi, 0) = -\frac{1}{2} \sum_{\sigma \in \mathcal{G}_m} \chi(\sigma) \log |(1 - \zeta_m^\sigma)^{1+\tau}|,$$

where both $\chi(\sigma)$ and $1 - \zeta_m^\sigma$ are regarded as complex numbers via j .

This equality implies, firstly, that the image of $(1 - \zeta_m)^{(1+\tau)/2}$ in $\mathbb{Q} \cdot \mathcal{O}_{\mathbb{Q}(m)^+, S(m)}^\times$ is stable under the action of the idempotent $e_{(m)} := e_{(\mathbb{Q}(m)^+)}$ and then, secondly, that its image under the isomorphism (13) is equal to $e_{(m)} \cdot \theta_{\mathbb{Q}(m)^+, S(m)}^*(0) \cdot (w_j - w_0)$, where w_j is the Archimedean

place of $\mathbb{Q}(m)^+$ that corresponds to j and w_0 is any choice of place of $\mathbb{Q}(m)^+$ that lies above a prime divisor of m .

This latter fact then combines with the explicit definition (8) of the map $\Theta_{\mathbb{Q}(m)^+}$ to imply that

$$(14) \quad \Theta_{\mathbb{Q}(m)^+}(\mathfrak{z}_{\mathbb{Q}(m)^+}) = e_{(m)}((1 - \zeta_m)^{(1+\tau)/2}) = (1 - \zeta_m)^{(1+\tau)/2},$$

as required. □

Lemma 5.2. *The \mathcal{R}^s -module $\text{VS}(\mathbb{Q}^s/\mathbb{Q})$ is free of rank one, with basis $(\mathfrak{z}_E)_{E \in \Omega^+}$.*

Proof: At the outset, we fix E in Ω^+ and recall (from [3, Proposition 3.4]) that the equivariant Tamagawa number conjecture for the pair $(h^0(\text{Spec}(E)), \mathbb{Z}[\mathcal{G}_E])$ asserts that $d_{\mathcal{G}_E}(C_{E,S(E)})$ is a free $\mathbb{Z}[\mathcal{G}_E]$ -module with basis \mathfrak{z}_E .

We further recall that this conjecture is known to be valid by the work of the first named author and Greither in [2] and of Flach in [9].

Given the explicit definition (in Definition 3.6) of $\text{VS}(\mathbb{Q}^s/\mathbb{Q})$ as an inverse limit, the claimed result will therefore follow if we can show that for each pair of fields E and E' in Ω^+ with $E \subseteq E'$ one has $\nu_{E'/E}(\mathfrak{z}_{E'}) = \mathfrak{z}_E$.

To prove this we use Remark 3.3 to identify $\text{RI}(\kappa(v)_{\mathcal{W}}, \mathbb{Z}[\mathcal{G}_E])^*[-1]$ for each place v in $S(E') \setminus S(E)$ with the complex Ψ_v that is equal to $\mathbb{Z}[\mathcal{G}_E]$ in degrees zero and one and has the differential $x \mapsto (1 - \text{Fr}_v^{-1})x$.

We write Y_v for the free abelian group on the set of places of E above v and, fixing a place w_v of E above v , note there are isomorphisms $\iota_v^i : H^i(\Psi_v) \cong Y_v$ for $i \in \{0, 1\}$ with $\iota_v^0(x) = |\mathcal{G}_{E,v}|^{-1}x \cdot w_v$ and $\iota_v^1(x) = x \cdot w_v$ where, we recall, $\mathcal{G}_{E,v}$ denotes the decomposition subgroup of v in \mathcal{G}_E .

The key fact now is that the $\text{Gal}(E'/E)$ -invariants of $\vartheta_{E'}$ differ from the composite $\vartheta_E \circ \nu_{E'/E}$ only in that for each $v \in S(E') \setminus S(E)$ and each χ in $\widehat{\mathcal{G}}_E$ these maps respectively use the upper and lower composite homomorphisms in the following diagram:

$$\begin{array}{ccccc} (d_{\mathcal{G}_E}(\Psi_v))_{\chi} & \xrightarrow{\alpha_1} & d_{\mathbb{C}}(Y_{v,\chi}) \cdot d_{\mathbb{C}}(Y_{v,\chi})^{-1} & \xrightarrow{\alpha_2} & \mathbb{C} \\ \parallel & & & & \downarrow \epsilon_v^{\chi} \\ (d_{\mathcal{G}_E}(\Psi_v))_{\chi} & \xrightarrow{\alpha_3} & d_{\mathbb{C}}(\mathbb{Z}[\mathcal{G}_E]_{\chi}) \cdot d_{\mathbb{C}}(\mathbb{Z}[\mathcal{G}_E]_{\chi})^{-1} & \xrightarrow{\alpha_4} & \mathbb{C} \end{array}$$

Here α_1 denotes the morphism induced by the maps ι_v^0 and ι_v^1 , α_2 the morphism induced by multiplication by $\log(N(w_v))$, α_3 is the obvious identification, α_4 is induced by the identity map on $\mathbb{Z}[\mathcal{G}_E]_{\chi}$, and ϵ_v^{χ} is defined to be $1 - \chi(\text{Fr}_v)^{-1}$ if $\chi(\text{Fr}_v) \neq 1$ and to be $|\mathcal{G}_{E,v}|^{-1} \cdot \log(N(w_v)) = \log(N(v))$ otherwise.

The claimed result then follows from the fact that the argument of [1, Lemma 10] shows that the above diagram commutes, whilst an explicit computation using (10) shows that for each χ in $\widehat{\mathcal{G}}_E$ one has

$$L_{S(E')}^*(\chi, 0) = \left(\prod_{v \in S(E') \setminus S(E)} \epsilon_v^\chi \right) \cdot L_{S(E)}^*(\chi, 0). \quad \square$$

5.1.2. In the next two results we establish some useful properties of the ideal \mathcal{A}^s . We write $\mathcal{I}_{\mathbb{Q},(2)}$ for the kernel of the natural projection map $\mathcal{R}^s \rightarrow \mathbb{Z}/(2)$.

Lemma 5.3. *One has $\{r \in \mathcal{R}^s \mid r(\text{ES}_{\mathbb{Q}}^b) \subseteq \text{ES}_{\mathbb{Q}}\} = \mathcal{I}_{\mathbb{Q},(2)} = \mathcal{A}^s$.*

Proof: We set $X := \{r \in \mathcal{R}^s \mid r(\text{ES}_{\mathbb{Q}}^b) \subseteq \text{ES}_{\mathbb{Q}}\}$.

We also recall that the equality $\mathcal{I}_{\mathbb{Q},(2)} = \mathcal{A}^s$ was explained just before the statement of Theorem 2.8 and that Theorem 3.8(ii) implies X contains \mathcal{A}^s .

On the other hand, if r belongs to $\mathcal{R}^s \setminus \mathcal{I}_{\mathbb{Q},(2)}$, then it can be written as $r = r' + 1$ with $r' \in \mathcal{I}_{\mathbb{Q},(2)}$.

To deduce $\mathcal{I}_{\mathbb{Q},(2)} = X$ it is thus enough to show that the system ε that occurs in Theorem 5.1 does not belong to $\text{ES}_{\mathbb{Q}}$, or equivalently that there exists an m in \mathbb{N}^\dagger for which $\varepsilon_{\mathbb{Q}(m)^+}$ is not contained in $\mathcal{O}_{\mathbb{Q}(m)^+, S(m), \text{tf}}^\times$.

But, if $\varepsilon_{\mathbb{Q}(m)^+}$ belongs to $\mathcal{O}_{\mathbb{Q}(m)^+, S(m), \text{tf}}^\times$, then there exists an element u of $\mathbb{Q}(m)^+$ with $\varepsilon_{\mathbb{Q}(m)^+} = \pm u$ and hence also

$$u^2 = (\varepsilon_{\mathbb{Q}(m)^+})^2 = (1 - \zeta_m)^{1+\tau} = -\zeta_m^{-1}(1 - \zeta_m)^2.$$

This implies $-\zeta_m$ is a square in $\mathbb{Q}(m)$ and so is impossible if m is divisible by 4. \square

In the sequel we write μ_∞ for the union of μ_m over all m , set $\mu_\infty^* := \mu_\infty \setminus \{1\}$ and write \mathcal{F} for the multiplicative group of $G_{\mathbb{Q}}$ -equivariant maps from μ_∞^* to $\mathbb{Q}^{c, \times}$.

We note that \mathcal{F} is naturally an \mathcal{R} -module and that it contains the function Φ that sends each element ζ of μ_∞^* to $1 - \zeta$.

Lemma 5.4. *The image of the set $\{r \in \mathcal{R} \mid (1 - \zeta_m)^{r(\tau-1)} = 1 \text{ for all } m\}$ under the projection map $\mathcal{R} \rightarrow \mathcal{R}^s$ is \mathcal{A}^s .*

Proof: Write $\widehat{\mathbb{Z}}(1)$ for the inverse limit of the groups μ_m with respect to the transition morphisms $\mu_m \rightarrow \mu_{m'}$ for each divisor m' of m that are given by raising to the power m/m' .

Then $\widehat{\mathbb{Z}}(1)$ is naturally an \mathcal{R} -module and the result of [6, Theorem 1.2] proves that it lies in an exact sequence of \mathcal{R} -modules

$$0 \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow \mathcal{R} \cdot \Phi \xrightarrow{\pi} \mathcal{R}^s \longrightarrow 0$$

in which π sends each element $r \cdot \Phi$ to the projection of r in \mathcal{R}^s .

Set $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \langle \tau \rangle$. Then, since $H^0(\Gamma, \widehat{\mathbb{Z}}(1))$ vanishes, the long exact sequence of Tate cohomology of this sequence gives an exact sequence

$$(15) \quad 0 \longrightarrow \widehat{H}^0(\Gamma, \mathcal{R} \cdot \Phi) \longrightarrow \widehat{H}^0(\Gamma, \mathcal{R}^s) \xrightarrow{\pi'} \widehat{H}^{-1}(\Gamma, \widehat{\mathbb{Z}}(1)).$$

Here the map π' is the composite of the connecting homomorphism

$$\widehat{H}^0(\Gamma, \mathcal{R}^s) \longrightarrow \widehat{H}^1(\Gamma, \widehat{\mathbb{Z}}(1))$$

that is induced by the above exact sequence and the canonical isomorphism $\widehat{H}^1(\Gamma, \widehat{\mathbb{Z}}(1)) \cong \widehat{H}^{-1}(\Gamma, \widehat{\mathbb{Z}}(1))$ that results from the fact that Tate cohomology over Γ is periodic of order 2.

Now it is clear that the groups $\widehat{H}^0(\Gamma, \mathcal{R}^s)$ and $\widehat{H}^{-1}(\Gamma, \widehat{\mathbb{Z}}(1))$ respectively identify with $\mathcal{R}^s/(2 \cdot \mathcal{R}^s)$ and $\mathbb{Z}/(2 \cdot \mathbb{Z})$ and an explicit computation shows that, with respect to these identifications, the map π' in (15) is induced by the natural projection map $\mathcal{R}^s \rightarrow \mathbb{Z}$.

Given these facts, the exact sequence (15) implies that the image under π of $H^0(\Gamma, \mathcal{R} \cdot \Phi)$ is equal to $\mathcal{I}_{\mathbb{Q},(2)}$, and hence also to \mathcal{A}^s by Lemma 5.3.

It therefore suffices to note that an element r of \mathcal{R} is such that $r \cdot \Phi$ belongs to $H^0(\Gamma, \mathcal{R} \cdot \Phi)$ if and only if one has $(1 - \zeta_m)^{r(\tau-1)} = 1$ for all m . \square

5.2. Circular distributions and the proof of Theorem 2.8. In this section we quickly review Coleman's conjecture concerning circular distributions and then prove Theorem 2.8.

5.2.1. Coleman's conjecture. In the 1980s Coleman defined a 'circular distribution', or 'distribution' for short in the sequel, to be a function f in \mathcal{F} that satisfies the relation

$$\prod_{\zeta^a = \varepsilon} f(\zeta) = f(\varepsilon)$$

for all natural numbers a and all ε in μ_{∞}^* . (A similar notion was also subsequently introduced by Coates in [7] in the context of abelian extensions of imaginary quadratic fields.)

It is clear that the subset \mathcal{F}^d of \mathcal{F} comprising all such distributions is naturally an \mathcal{R} -module.

Further, recalling the module $\text{ES}_{\mathbb{Q}}^{\text{cl}} = \text{ES}^{\text{cl}}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ of classical Euler systems of rank one for \mathbb{G}_m over $\mathbb{Q}^{\text{ab}}/\mathbb{Q}$, as discussed in Remark 2.4, there exists a canonical isomorphism of \mathcal{R} -modules

$$(16) \quad \mathcal{F}^d \cong \text{ES}_{\mathbb{Q}}^{\text{cl}}.$$

This map sends each f in \mathcal{F}^d to the unique element c_f of $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ with $c_{f, \mathbb{Q}(m)} = f(\zeta_m)$ for all $m \in \mathbb{N}^{\dagger}$ and its inverse sends each c in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ to the unique function f_c in \mathcal{F}^d that satisfies

$$f_c(\zeta_m) := \begin{cases} c_{\mathbb{Q}(m)}, & \text{if } m \in \mathbb{N}^{\dagger}, \\ (c_{\mathbb{Q}(m')})^{1 - \text{Fr}_2^{-1}}, & \text{if } m = 2m' \text{ with } m' \text{ odd and } m' > 1, \\ \mathbb{N}_{\mathbb{Q}(4)/\mathbb{Q}}(c_{\mathbb{Q}(4)}), & \text{if } m = 2. \end{cases}$$

(In this regard, we point out that our definition of the set of fields $\Omega(\mathbb{Q}^c/\mathbb{Q})$ means that it does not contain \mathbb{Q} and hence that a system in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ has no component at the field \mathbb{Q} .)

The function Φ that is defined just prior to Lemma 5.4 clearly belongs to \mathcal{F}^d and Coleman's conjecture predicts that

$$(17) \quad \mathcal{F}^d = \mathcal{F}_{\text{tor}}^d + \mathcal{R} \cdot \Phi.$$

Since [16, Theorem B] implies that every distribution of finite order has order dividing two, the isomorphism (16) implies this conjecture is equivalent to asserting that for every system $c = (c_F)_F$ in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ there exists an element r'_c of \mathcal{R} such that for every m in \mathbb{N}^{\dagger} one has

$$(18) \quad c_{\mathbb{Q}(m)} = \pm \Phi(\zeta_m)^{r'_c} = \pm (1 - \zeta_m)^{r'_c}$$

in $\mathbb{Q}(m)^{\times}$.

We further recall that the recent result of [6, Theorem 1.2] implies both that the quotient group $\mathcal{F}^d/(\mathcal{F}_{\text{tor}}^d + \mathcal{R} \cdot \Phi)$ is torsion-free and also that the homomorphism of \mathcal{R} -modules

$$\mathcal{F}^d/(\mathcal{F}_{\text{tor}}^d + \mathcal{R} \cdot \Phi) \rightarrow (1 + \tau)\mathcal{F}^d/(\mathcal{R} \cdot \Phi^{1+\tau})$$

that sends the class of each f in \mathcal{F}^d to the class of $f^{1+\tau}$ is bijective.

In addition, if f belongs to \mathcal{F}^d , then [6, Theorem 5.1 and Remark 5.2] implies that to show $f^{1+\tau}$ belongs to $\mathcal{R} \cdot \Phi^{1+\tau}$ it is enough to show the existence of a natural number t in \mathbb{N}^* and an element $r = r_f$ of \mathcal{R} for which one has $f(\zeta_m)^{1+\tau} = (1 - \zeta_m)^{(1+\tau)r}$ for all m in \mathbb{N}^{\dagger} that are divisible by t .

These facts combine to imply that the conjectural equality (17) is valid if for each system c in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ there exist elements t_1 and t_2 in \mathbb{N}^* and an element $r = r_{c,t_1}$ of \mathcal{R} for which one has

$$(19) \quad (c_{\mathbb{Q}(m)})^{t_1(1+\tau)} = (1 - \zeta_m)^{(1+\tau)r}$$

for all m in \mathbb{N} that are divisible by t_2 .

Remark 5.5. For more details concerning Coleman's conjecture, the reader can consult [6] or the earlier articles [14, 15, 16, 17] of the fourth author and the associated work of Saikia in [12].

5.2.2. The proof of Theorem 2.8. We first assume the validity of Coleman's conjecture and use it to deduce the validity of Conjecture 2.5 in the case $K = \mathbb{Q}$.

To do this we fix an element a of \mathcal{A}^s and a system $c = (c_F)_F$ in $\text{ES}_{\mathbb{Q}} = \text{ES}_1(\mathbb{Q}^s/\mathbb{Q})$. Then, by the discussion in Remark 2.4, there exists a canonical system c' in $\text{ES}^{\text{cl}}(\mathbb{Q}^s/\mathbb{Q})$ that projects to c^a in $\text{ES}_{\mathbb{Q}}$.

For this system c' one has $c'_{\mathbb{Q}(m)^+} > 0$ for every m in \mathbb{N}^{\ddagger} . In particular, since the argument of [14, Lemma 2.2] implies $c'_{\mathbb{Q}(m)^+}$ is a global unit for every m in \mathbb{N}^{\ddagger} that is divisible by two distinct primes, one has $N_{\mathbb{Q}}^{\mathbb{Q}(m)^+}(c'_{\mathbb{Q}(m)^+}) = 1$ for all such m .

As a result, there exists a unique system c'' in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ for which at each m in \mathbb{N}^{\ddagger} one has

$$c''_{\mathbb{Q}(m)} := \begin{cases} c'_{\mathbb{Q}(m)^+}, & \text{if } m \in \mathbb{N}^{\ddagger}, \\ N_{\mathbb{Q}}^{\mathbb{Q}(9)^+}(c_{\mathbb{Q}(9)^+}), & \text{if } m = 3, \\ N_{\mathbb{Q}}^{\mathbb{Q}(8)^+}(c_{\mathbb{Q}(8)^+}), & \text{if } m = 4. \end{cases}$$

For this system one has $(c'')^{\tau-1} = 1$. Thus, if Coleman's conjecture (in the form of (18)) is valid, then Lemma 5.4 implies the existence of an element r of \mathcal{R} which projects to \mathcal{R}^s to give an element a' of \mathcal{A}^s and is such that for every m in \mathbb{N}^{\ddagger} one has

$$c'_{\mathbb{Q}(m)^+} = c''_{\mathbb{Q}(m)} = \pm(1 - \zeta_m)^r$$

and hence also

$$(c'_{\mathbb{Q}(m)^+})^{1+\tau} = (1 - \zeta_m)^{(1+\tau)r} = (1 - \zeta_m)^{(1+\tau)a'} = (\varepsilon_{\mathbb{Q}(m)^+})^{2a'}.$$

But then in $\text{RES}_{\mathbb{Q}}$ one has

$$c^a = c' = ((c')^2)^{1/2} = ((c')^{1+\tau})^{1/2} = (\varepsilon^{2a'})^{1/2} = \varepsilon^{a'}.$$

Since a is an arbitrary element of \mathcal{A}^s , and a' belongs to \mathcal{A}^s , this proves the claimed inclusion $\mathcal{A}^s \cdot \text{ES}_{\mathbb{Q}} \subseteq \mathcal{A}^s \cdot \text{ES}_{\mathbb{Q}}^{\text{b}}$.

To prove the converse it suffices to show the latter inclusion implies that every system in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$ satisfies the condition (19).

To do this we fix c in $\text{ES}_{\mathbb{Q}}^{\text{cl}}$. Then by first restricting c to the subset Ω^+ of $\Omega(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ and then reducing its values modulo torsion one obtains a system \tilde{c} in $\text{ES}_{\mathbb{Q}}$.

Further, since 2 belongs to \mathcal{A}^{s} , the assumed inclusion $\mathcal{A}^{\text{s}} \cdot \text{ES}_{\mathbb{Q}} \subseteq \mathcal{A}^{\text{s}} \cdot \text{ES}_{\mathbb{Q}}^{\text{b}}$ combines with Theorem 5.1 to imply the existence of an element r of \mathcal{R} such that $\tilde{c}^4 = \varepsilon^{2r}$ in $\text{RES}_{\mathbb{Q}}$.

This equality of functions implies that for each m in \mathbb{N}^{\dagger} one has

$$(c_{\mathbb{Q}(m)})^{4(1+\tau)} = (\tilde{c}_{\mathbb{Q}(m)^+})^4 = (\varepsilon_{\mathbb{Q}(m)^+})^{2r} = (1 - \zeta_m)^{(1+\tau)r}.$$

This shows that c verifies the condition (19) with $t_1 = t_2 = 4$ and hence implies Coleman’s conjecture.

This completes the proof of Theorem 2.8.

5.3. Norm-coherent sequences in p -power conductor cyclotomic fields. In this final section we shall give a proof of Theorem 2.9. To do this we fix an odd prime p and for each natural number n set $G_n := \text{Gal}(\mathbb{Q}(p^n)/\mathbb{Q})$ and $G_n^+ := \text{Gal}(\mathbb{Q}(p^n)^+/\mathbb{Q})$.

At the outset we recall that the discussion in [17, §3] shows that if $(a_n)_n$ is a norm-coherent sequence in $\bigcup_n \mathbb{Q}(p^n)$ such that each a_n belongs to $\mathbb{Z}[G_n] \cdot (1 - \zeta_{p^n})$, then there exists a circular distribution f such that $f(\zeta_{p^n}) = a_n$ for all n .

To prove Theorem 2.9, it is thus enough to fix f in \mathcal{F}^{d} and show that there exists a natural number t (possibly depending on f) for which $f(\zeta_{p^n})^{2^t}$ belongs to $\mathbb{Z}[G_n] \cdot (1 - \zeta_{p^n})$ for all n .

Hence, since [15, Theorem B] implies the existence of a natural number c for which $f(\zeta_{p^n})^c$ belongs to $\mathbb{Z}[G_n] \cdot (1 - \zeta_{p^n})$ for all n , it is enough to show for each n that there exists a natural number t_n with $f(\zeta_{p^n})^{2^{t_n}} \in \mathbb{Z}[G_n] \cdot (1 - \zeta_{p^n})$.

In addition, for any t in \mathbb{N}^* one has $f^{2^t} = f^{(1-\tau)2^{t-1}} f^{(1+\tau)2^{t-1}}$ and, by [6, Theorem 4.1], one knows that $f(\zeta_{p^n})^{1-\tau} \in \mathbb{Z}[G_n] \cdot (1 - \zeta_{p^n})$ for all n .

It therefore suffices to show that for each n , and for every odd prime ℓ , one has

$$f(\zeta_{p^n})^{1+\tau} \in \mathbb{Z}_{\ell}[G_n] \cdot \varepsilon_n,$$

where we set

$$\varepsilon_n := (1 - \zeta_{p^n})^{1+\tau}.$$

For the prime $\ell = p$ the above containment is proved in [6, Theorem 3.1] and so we assume in the sequel that $\ell \neq p$.

We decompose G_n as a direct product $\Lambda_n \times \Delta_n$ with Λ_n the Sylow ℓ -subgroup of G_n . We set $E := \mathbb{Q}(p^n)$ and $E^+ := \mathbb{Q}(p^n)^+$, write F and L for the fixed fields of E by Δ_n and Λ_n respectively and write L^+ for the maximal real subfield $L \cap E^+$ of L .

Then it suffices to prove that for each homomorphism $\phi: \Delta_n \rightarrow \mathbb{Q}_\ell^{c,\times}$ that is trivial on τ one has

$$(20) \quad f(\zeta_{p^n})^\phi \in \mathbb{Z}[G_n]_\phi \cdot \varepsilon_n,$$

where we use the same notation for ϕ -components as in §4.1.1.

We assume first that ϕ is trivial. In this case we can identify X_ϕ for each Δ_n -module X as a submodule of $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} X$ by sending each element x^ϕ to $\sum_{h \in \Delta_n} h(x)$.

Then, with respect to this identification, one has $\mathbb{Z}[G_n]_\phi \cdot \varepsilon_n = \mathbb{Z}_\ell[G_n] \cdot N_{E/F}(\varepsilon_n)$. In addition, since F/\mathbb{Q} is an ℓ -power degree extension that is only ramified at the prime p , one knows that the class number of F is not divisible by ℓ (by [20, Theorem 10.4]).

Upon combining the latter fact with the formulas proved by Sinnott in [18, Theorems 4.1 and 5.1] one finds that $N_{E/F}(\varepsilon_n)$ generates over $\mathbb{Z}_\ell[G_n]$ the ℓ -completion of the full group of p -units of F .

The required containment (20) is thus true in this case since $f(\zeta_{p^n})^\phi = N_{E/F}(f(\zeta_{p^n}))$ is a p -unit in F (by [14, Lemma 2.2]).

We now assume ϕ is non-trivial and regard it as a homomorphism $\text{Gal}(L^+/\mathbb{Q}) \rightarrow \mathbb{Q}_\ell^{c,\times}$. We also note that L^+ is not contained in $\mathbb{Q}(\mu_\ell)$, that ϕ^2 is not equal to the Teichmüller character at ℓ if $\ell = 3$, that p is not split in L^+ , and that the Archimedean place of \mathbb{Q} splits in L^+ .

This shows that the pair F and ϕ satisfies all of the hypotheses (H₁), (H₂), (H₃), (H₄), and (H₅) (with p replaced by ℓ) that are listed in §4.1.1.

We may therefore apply Theorem 4.1 to the system \tilde{c}_f in $\text{ES}_\mathbb{Q}$ that is obtained from f by the method used in the latter part of §5.2.2.

Noting that $\mathcal{A}_{E,\phi} = \mathbb{Z}[G_n]_\phi$ (since ℓ is odd and ϕ is even), we deduce in this way that the element $(f(\zeta_{p^n})^\phi)^2 = (f(\zeta_{p^n})^{1+\tau})^\phi = (\tilde{c}_{f,E^+})^\phi$ belongs to $\Theta_{E^+}(\mathfrak{d}_{G_n^+}(C_{E^+,S(E)}))_\phi$.

The required containment (20) is therefore true in this case since the equality (14) (with $m = p^n$) combines with the argument of Lemma 5.2 to imply that $\Theta_{E^+}(\mathfrak{d}_{G_n^+}(C_{E^+,S(E)}))_\phi = \mathbb{Z}[G_n^+]_\phi \cdot \varepsilon_n$.

This completes the proof of Theorem 2.9.

Remark 5.6. The result of Theorem 2.9 does not extend in a straightforward way to more general towers of cyclotomic fields. To be specific, if p is an odd prime and m is a multiple of p that is not a power of p , then there can exist norm-compatible sequences of cyclotomic units in

the tower $\bigcup_n \mathbb{Q}(mp^n)$ that are not equal to the restriction of any circular distribution. For an explicit example of this phenomenon see [6, Lemma 2.10].

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David Burns and Alexandre Daoud

King’s College London, Department of Mathematics, London WC2R 2LS, U.K.

E-mail address: david.burns@kcl.ac.uk

E-mail address: alexandre.daoud@kcl.ac.uk

Takamichi Sano

Osaka Metropolitan University, Department of Mathematics, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

E-mail address: tsano@omu.ac.jp

Soogil Seo

Yonsei University, Department of Mathematics, Seoul, Korea

E-mail address: sgseo@yonsei.ac.kr

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