Publ. Mat. **67** (2023), 199–257
DOI: 10.5565/PUBLMAT6712305

LIMIT GROUPS OVER COHERENT RIGHT-ANGLED ARTIN GROUPS

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Abstract: A new class of groups $\mathcal{C}$, containing all coherent RAAGs and all toral relatively hyperbolic groups, is defined. It is shown that, for a group G in the class $\mathcal{C}$, the $\mathbb{Z}[t]$-exponential group $G^{\mathbb{Z}[t]}$ may be constructed as an iterated centraliser extension. Using this fact, it is proved that $G^{\mathbb{Z}[t]}$ is fully residually G (i.e. it has the same universal theory as G) and so its finitely generated subgroups are limit groups over G. If $\mathbb{G}$ is a coherent RAAG, then the converse also holds – any limit group over $\mathbb{G}$ embeds into $\mathbb{G}^{\mathbb{Z}[t]}$. Moreover, it is proved that limit groups over $\mathbb{G}$ are finitely presented, coherent and CAT(0), so in particular have solvable word and conjugacy problems.

2020 Mathematics Subject Classification: 20F65, 20F05, 20F36, 20F67, 20E06.

Key words: partially commutative group, right-angled Artin group, limit group, hyperbolic group.

1. Introduction

When studying problems in R-modules or R-algebras it is often convenient to extend the ring of coefficients, for instance from an integral domain to a field of fractions. The multiplicative notation employed in the study of groups naturally defines an action of the ring of integers on the group via exponentiation and, as in the case of modules and algebras, it is sometimes useful to extend this exponential action of $\mathbb{Z}$ to more general rings, bringing us to the notion of an exponential group over a ring A. Exponential groups were initially considered by P. Hall in [**26**], where he studied nilpotent groups with exponents in a binomial ring. G. Baumslag (see [**4**]) considered groups with unique p-th roots and their embedding into divisible groups, which are exponential groups over the ring $\mathbb{Q}$. Lyndon was the first to take an axiomatic point of view, introducing axioms for exponential groups over an associative ring A, or A-groups, while studying equations and the first-order theory of the free group, and describing the set of solutions of an equation in one variable over a free group F in terms of a free object F^A in the category of

A-groups, where $A = \mathbb{Z}[t_1, \dots, t_d]$, that is, groups with exponents in the ring of polynomials $\mathbb{Z}[t]$; see [**33, 34**].

When extending the exponents of a group G to a ring A, there is a natural "free" A-group, called the A-completion G^A of G, defined via the universal property that any group homomorphism from G to an A-group H, factors through G^A, via a canonical embedding from G into G^A and an A-homomorphism from G^A to H.

Although the existence of an A-completion or a free exponential group may follow from general results on varieties of algebras, their structure may be very complex and difficult to describe, and in particular a group may not embed into its A-completion. G. Baumslag described the $\mathbb{Q}$-completion of a group with unique roots as a direct limit of a directed system of A-groups, for A a subring of $\mathbb{Q}$. More precisely, Baumslag's directed system involves the idea of an "iterated centraliser extension" (or ICE), that is, a group built by repeated formation of free products with amalgamation, over centralisers of elements (see Definition 6.2). Indeed in this setting, G^A, built as an ICE, is a Fraïssé limit of extensions of centralisers; see [**31**]. It is proved that, for A a subring of $\mathbb{Q}$, the A-completion of the ordinary free group F is the free A-group F^A in the variety of A-groups (see [**4**, Sections V–VIII]).

Lyndon gave an explicit description of the structure of the free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$ as the limit of an ascending chain of subgroups and proved that $F^{\mathbb{Z}[t]}$ is fully residually free, or put another way, is universally equivalent to the free group F. In [**39, 40**], Myasnikov and Remeslennikov made a systematic study of A-groups and A-completions and described the structure of the A-completion of CSA groups, that is, groups whose maximal abelian subgroups are malnormal. It is shown that if A is an associative ring, with a torsion-free additive subgroup, and G is a torsion-free CSA group, then the A-completion G^A of G may be constructed as an ICE, whence many results on its structure may be deduced; for example that G embeds into G^A, which is itself a torsion-free CSA group. In this case too, a free A-group is the A-completion of a free group.

In the introduction to [**34**] Lyndon comments that "... although the connection is perhaps remote, my interest in the present problem derives from a question of A. Tarski, whether the 'elementary theory' of free groups is decidable". It is now clear that the connection is anything but remote. Indeed, crucial properties of Lyndon's free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$ are that it serves as a universe for the class of finitely generated models of the universal theory of the free group and that it has a very robust algebraic structure. More precisely, in [**27**] (see also [**17**]), it is shown that any finitely generated fully residually free group (equivalently, any finitely generated model of the universal theory of the free group, also known as a limit group; see Subsection 2.3) embeds into $F^{\mathbb{Z}[t]}$.

An immediate consequence of this result is the fact that limit groups split either as a free product or over an infinite cyclic subgroup and, since finitely generated subgroups of limit groups are limit groups, this endows them with a hierarchical structure. This fact allows one to use induction to prove a variety of powerful results on limit groups. For example, it readily follows that limit groups are finitely presented; in [**16**], the hierarchical structure of limit groups is used to prove that they are conjugacy separable, and in [**50**] it is established that limit groups are subgroup separable.

In a different direction, the knowledge that limit groups embed into $F^{\mathbb{Z}[t]}$ was key to establishing the fact that limit groups act freely on $\mathbb{Z}^n$-trees and made it possible to use the techniques of non-Archimedian words to establish the Howson property, analogues of Hall's and Greenberg–Stallings's theorems for free groups (see [**42, 50**]), and to address all the major algorithmic questions for limit groups; see [**30, 12**] and references therein.

This type of result, showing limit groups over a group G embedded in the $\mathbb{Z}[t]$-completion of G, has been established for other groups with hyperbolic features. The proofs divide into two main steps. The first one is to show that the $\mathbb{Z}[t]$-completion is a model of the universal theory of the base group G, i.e. is fully residually G. In their paper [**7**], Baumslag, Myasnikov, and Remeslennikov carried out this step in the case of a torsion-free hyperbolic group G and distilled the key ideas of the proof. As in [**27**], a class of groups that contains G and is closed under extensions of centralisers is required. In this case the group G is torsion-free hyperbolic and the class chosen is the class of CSA groups satisfying the big powers (BP) property. The BP property is an algebraic condition that asserts that big powers of non-commuting elements generate a free group (see Definition 3.2 below). It turns out that an extension of centralisers of a group G which is CSA and has the BP property is fully residually G, and so a model of its universal theory. The strategy is then to show that an extension of a centraliser of a group in the class of CSA and BP groups is again in this class, to use transfinite induction to show that an ICE of such a group remains in the class, and to conclude that $G^{\mathbb{Z}[t]}$ is fully residually G. These ideas were taken further in [**29**] in the study of the $\mathbb{Z}[t]$-completions of torsion-free toral relatively hyperbolic groups.

The second main step of the proof is to show that any limit group over G embeds into $G^{\mathbb{Z}[t]}$. In order to prove this statement, one needs to understand the structure of limit groups over G. In the case of free groups, this was achieved by Kharlampovich and Myasnikov in [**27**], using the Makanin–Razborov process, and by Sela in [**48**] using actions on real trees.

The aim of this paper is to describe $\mathbb{Z}[t]$-completions for (a class of) right-angled Artin groups, or RAAGs for short; see Definition 2.1. RAAGs form a prominent class of groups which contains both free and free abelian groups and is widely studied in different branches of mathematics and computer science. We refer the reader to [**18, 22**] for further details.

In general, the subgroup structure of RAAGs is very complex. For example, Bestvina and Brady ([**9**]) showed how to construct a subgroup of a RAAG to give the first example of a group which is homologically finite (of type FP) but not geometrically finite (in fact not of type F_2). Again, Mihaĭlova's example [**37**] of a group with the unsolvable subgroup membership problem is constructed from the RAAG $F_2 \times F_2$. More recently results of Wise and others (see e.g. [**51**]) led to Agol's proof of the virtual Haken conjecture: that every hyperbolic Haken 3-manifold is virtually fibred. An essential step in the argument uses the result that the fundamental groups of so-called special cube complexes embed into RAAGs. If one is to use the $\mathbb{Z}[t]$-completion as a universe for limit groups over a given RAAG and as a tool to study limit groups algorithmically, it is natural to restrict to a class of RAAGs with tame subgroup structure and good algorithmic behaviour – in our case, to the class of coherent RAAGs. A RAAG is coherent if and only if its underlying graph contains no full subgraph isomorphic to an n-cycle, for $n \geq 4$; see [**20**]. Furthermore, a RAAG is coherent if and only if it satisfies the BP property; see [**10**].

In this paper, we define a new class of groups $\mathcal{C}$ which contains all toral relatively hyperbolic groups and all coherent RAAGs and prove that it is closed under extensions of centralisers and direct limits. For groups G from this class, we show that the $\mathbb{Z}[t]$-completion may be built as an ICE, enabling us to prove it is fully residually G. Thus, we give a general framework for the results of Baumslag, Kharlampovich, Myasnikov, and Remeslennikov [**27, 7, 29**] and establish analogous results for coherent RAAGs. More precisely we prove the following theorem.

Theorem (see Theorem 6.3). *Let G be a group from class $\mathcal{C}$ satisfying condition* (R) *of Subsection 6.2. The $\mathbb{Z}[t]$-completion $G^{\mathbb{Z}[t]}$ of G can be built as an ICE. It is fully residually G and so it has the same universal and existential theory as G.*

In the case that G is a coherent RAAG, we use the fact that any limit group over a RAAG is a subgroup of a graph tower (see [**15**]) to conclude that $G^{\mathbb{Z}[t]}$ is a universe for the class of limit groups over G.

Corollary (see Theorem 8.1). *Let G be a coherent RAAG. Then $G^{\mathbb{Z}[t]}$ can be built as an ICE. It is fully residually G and so it has the same universal and existential theory as G.*

Furthermore, it is a universe for the class of limit groups over G, i.e. any finitely generated model of the universal theory of G embeds into $G^{\mathbb{Z}[t]}$.

To prove our main result, we follow the strategy of papers [**7**], [**29**], and [**27**] for hyperbolic groups, that we have sketched above. In these works the authors consider the class of BP and CSA groups. The CSA property has strong structural consequences, namely that in such groups commutation is transitive and so the centralisers (of non-trivial elements) are abelian. The fact that centralisers are abelian (in fact, in the cases considered, they are f.g. free abelian) is essential to showing that a countable sequence of extensions of a centraliser admits a $\mathbb{Z}[t]$-action. Then, using the malnormality of abelian subgroups, one extends the $\mathbb{Z}[t]$-action, in a consistent way, from the abelian subgroups to the whole group, thereby establishing that indeed the resulting limit is the $\mathbb{Z}[t]$-completion of G. The BP property is needed to show that extensions of centralisers are fully residually G.

In our case, we define the class $\mathcal{C}$ to contain groups that are torsion-free and BP, and such that the algebraic structure of centralisers of elements, and their intersections, has a clear description; see Definition 3.4. The main technical work is to show that the class $\mathcal{C}$ is indeed closed under extension of centralisers; see Theorem 4.2. We then prove that a group obtained as an ICE admits a $\mathbb{Z}[t]$-action and so does the group $G^{\mathbb{Z}[t]}$; see Theorem 6.3.

In order to show that the group $G^{\mathbb{Z}[t]}$ is a universe for the class of limit groups over G, we use the structure theorem for limit groups proved in [**15**], where it is shown that any limit group over a RAAG G is a subgroup of a *graph tower* over G. In the case of coherent RAAGs, we provide a neat description of graph towers, which is key to showing that they embed into the completion $G^{\mathbb{Z}[t]}$. More precisely, we prove that any graph tower can be obtained as a free product with amalgamation over a free abelian group, where one of the vertices is a tower of lower height and the other vertex group is either free abelian or the direct product of a free abelian group and a fundamental group of a non-exceptional surface, and where the free abelian factor is contained in the edge group. The base group is a coherent RAAG G' which is obtained from G by extending centralisers of canonical generators of G; see Lemma 7.4.

In Section 7, we show that graph towers over coherent RAAGs belong to the class $\mathcal{C}$ and use this fact to prove the following structural results:

Theorem (see Theorem 7.7). *Let $\mathbb{G}$ be a coherent RAAG and let $\mathfrak{T} = (G, \mathbb{H}, \pi)$ be a graph tower associated to a limit group L over $\mathbb{G}$. Then G has a graph of groups decomposition (in the same generating set as G) where:*

- (i) *the graph of the decomposition is a tree;*
- (ii) *edge groups are finitely generated free abelian;*
- (iii) *vertex groups are either graph towers of lower height, or a finitely generated free abelian group or the direct product of a finitely generated free abelian group and a non-exceptional surface group.*

This result is independent of the first theorem stated above (Theorem 6.3) and it is of interest in its own right. For instance we can deduce the following.

Corollary (see Corollary 7.9). *Limit groups over coherent RAAGs are coherent, and so, in particular, finitely presented.*

In Section 8, we use the structure of the graph tower over a coherent RAAG $\mathbb{G}$ to show that they embed into $\mathbb{G}^{\mathbb{Z}[t]}$ and obtain a new characterisation of limit groups over coherent RAAGs – they are precisely finitely generated subgroups of $\mathbb{G}^{\mathbb{Z}[t]}$.

Finally, in Section 9 we follow the approach of Alibegovič–Bestvina (see [**1**]) to prove the following corollary.

Corollary (see Corollary 9.7). *Limit groups over coherent RAAGs are* CAT(0) *groups.*

One may consider the relation with other generalisations of hyperbolicity such as acylindrical hyperbolicity or hierarchical hyperbolicity; see [**46**] and [**8**]. A RAAG is acylindrically hyperbolic if and only if it is directly indecomposable and not cyclic; see [**13, 46**]. From the structure of limit groups over coherent RAAGs (see Lemma 7.4 and results from [**38**]), it follows that limit groups over directly indecomposable, non-cyclic coherent RAAGs are acylindrically hyperbolic. On the other hand, all RAAGs are hierarchically hyperbolic groups (HHG for short) and one would expect this also to be true of towers over RAAGs. In general, finitely generated subgroups of RAAGs need not be HHGs but we expect subgroups of coherent RAAGs, and limit groups over them, to be HHGs. More precisely we ask the following question.

Question 1. Are limit groups over coherent RAAGs hierarchically hyperbolic?

In fact, D. T. Wise ([**51**]) showed that limit groups have a quasiconvex hierarchy, and thence deduced that they are virtually compact special. Similarly, we expect a positive answer to the following.

Question 2. Are limit groups over coherent RAAGs virtually (compact) special?

Notice that virtually compact special groups are hierarchically hyperbolic [**8**].

Since RAAGs are linear so are limit groups over them; see [**5**, Proposition 8] and [**36**]. In particular the word problem is decidable for limit groups over RAAGs [**47**]. Furthermore, from the fact that limit groups over coherent RAAGs are CAT(0), we deduce the following.

Corollary (see Corollary 9.7)**.** *Limit groups over coherent RAAGs have decidable word and conjugacy problems.*

As we mentioned, we expect that the embedding of limit groups in $\mathbb{Z}[t]$-completions will be useful in addressing algorithmic problems and establishing residual properties. For instance, we ask the following.

Question 3. Are the following statements true?

(i) Quasiconvex subgroups of graph towers over (coherent) RAAGs are subgroup separable. (Notice that we consider the metric induced by the canonical generators of the graph tower (instead of the CAT(0) metric). In particular, this metric induces the ℓ_1-metric on abelian subgroups. Algebraically, we consider subgroups H such that if $h = h_1 \cdots h_k \in H$ and h_i are pairwise commuting blocks, then $h_i^{k_i} \in H$ for some $k_i \in \mathbb{N}$ and for all $i = 1, \dots, k$.)

(ii) Cyclic subgroups of limit groups over (coherent) RAAGs are closed in the profinite topology.

(Since this paper was submitted, Fruchter ([**24**]) has shown, using the class $\mathcal{C}$ introduced below, that this statement is true.)

(iii) Limit groups over coherent RAAGs are conjugacy separable.

(iv) The subgroup membership problem in limit groups over coherent RAAGs is decidable.

(v) The isomorphism problem in the class of limit groups over coherent RAAGs is decidable.

Acknowledgements. Support for this work comes from a Basque Government grant (no. IT1483-22) and Spanish Government grants (nos. PID2019-107444GA-I00 and PID2020-117281GB-I00).

The authors are grateful to the anonymous referees for careful and constructive comments which prompted significant improvements to the exposition of results.

2. Preamble

In this section we recall some basic definitions and properties of RAAGs; see [**23, 14**] for further details.

2.1. Right-angled Artin groups.

Definition 2.1. Let Γ be a finite simple (no loops, no multiple edges) graph with vertices X and edge set E. The *right-angled Artin group* (RAAG) $\mathbb{G}(\Gamma)$ with *commutation graph* Γ is the group with presentation $\langle X \mid R\rangle$, where $R = \{[a,b] : \{a,b\} \in E\rangle$.

If Y is a subset of X, denote by $\Gamma(Y)$ the full subgraph of Γ with vertices Y. Then $\mathbb{G}(\Gamma(Y))$ is the RAAG with graph $\Gamma(Y)$. One can show that $\mathbb{G}(\Gamma(Y))$ is the subgroup $\langle Y\rangle$ of $\mathbb{G}(\Gamma)$ generated by Y. We call $\mathbb{G}(\Gamma(Y))$ a *canonical parabolic* subgroup of $\mathbb{G}(\Gamma)$ and, when no ambiguity arises, denote it by $\mathbb{G}(Y)$. The elements of Y are termed the *canonical* generators of $\mathbb{G}(Y)$.

A subgroup P of $\mathbb{G}$ is called *parabolic* if it is conjugate to a canonical parabolic subgroup $\mathbb{G}(Y)$ for some $Y \subseteq X$.

If w is a word in the free group on X, then we say that w is *reduced* in $\mathbb{G}(\Gamma)$ if w has minimal length amongst all words representing the same element of $\mathbb{G}(\Gamma)$ as w. Notice that reduced words correspond to geodesics in the corresponding Caley graph.

If w is reduced in $\mathbb{G}(\Gamma)$, then we define $\alpha(w)$ to be the set of elements of X such that x or x^{-1} occurs in w. It is well known that all reduced words representing a particular element have the same length, and that if $w = w'$ in $\mathbb{G}(\Gamma)$, then $\alpha(w) = \alpha(w')$. We say that elements u and v of a RAAG *disjointly* commute if $\alpha(u) \cap \alpha(v) = \emptyset$ and $[u,v] = 1$. As shown in [**3**], if u and v disjointly commute, then $[\alpha(u), \alpha(v)] = 1$.

A reduced word w is *cyclically reduced* in $\mathbb{G}(\Gamma)$ if the length of w^2 is twice the length of w.

An element $w \in \mathbb{G}$ is called a *root* of $v \in \mathbb{G}$ if there exists a positive integer $m \in \mathbb{N}$ such that $v = w^m$ and there does not exist $w' \in \mathbb{G}$, $1 \neq w'$, and $0, 1 \neq m' \in \mathbb{N}$ such that $w = {w'}^{m'}$. In this case we write $w = \sqrt{v}$. By a result from [**21**], RAAGs have least (or unique) roots, that is, the root element of v is defined uniquely.

The *link* of a vertex x of Γ is the set of vertices $\mathrm{lk}(x)$ of Γ adjacent to x, and the *star* of x is $\mathrm{star}(x) = \{x\} \cup \mathrm{lk}(x)$. For a set of vertices $Y \subseteq V(\Gamma)$ define the link $\mathrm{lk}(Y)$ and the star of Y, in symbols $\mathrm{star}(Y)$, to be intersections of the links and stars, respectively, of all vertices in Y. By the link and star of $w \in \mathbb{G}$ we mean the link and star, respectively, of $\alpha(w) \subseteq V(\Gamma)$.

The complement of a graph Γ is the graph $\overline{\Gamma}$ with the same vertex set as Γ and an edge joining different vertices x and x' if and only if there is no such edge in Γ. The RAAG $\mathbb{G}(\Gamma)$ is said to have *non-commutation graph* $\overline{\Gamma}$.

Consider cyclically reduced $w \in \mathbb{G}$ and the set $\alpha(w)$. For this set, consider the graph $\overline{\Gamma}(\alpha(w))$ (that is, the full subgraph of $\overline{\Gamma}$ induced by $\alpha(w)$). If this graph is connected, we call w a *block*.

If $\overline{\Gamma}(\alpha(w))$ is not connected, then w is a product of commuting words

$$w = w_{j_1} \cdot w_{j_2} \cdots w_{j_t}; \quad j_1, \dots, j_t \in J, \tag{1}$$

where $|J|$ is the number of connected components of $\overline{\Gamma}(\alpha(w))$ and $\alpha(w_{j_i})$ is the set of vertices of the j_i-th connected component. Clearly, the words $\{w_{j_1}, \dots, w_{j_t}\}$ pairwise disjointly commute. Each word w_{j_i}, $i \in 1, \dots, t$, is a block and so we refer to presentation (1) as a block normal form of w. Notice that a block normal form is unique up to the order of the commuting blocks and the order of letters within the blocks. Therefore, abusing the terminology, we usually refer to “the” block normal form.

From [**3**], if the block normal form of a cyclically reduced element w of $\mathbb{G}$ is $w = w_1 \cdots w_t$ and $\sqrt{w_i} = v_i$, then the centraliser of w is

$$C(w) = \langle v_1 \rangle \times \cdots \times \langle v_t \rangle \times \langle \mathrm{lk}(w) \rangle. \tag{2}$$

2.2. Coherent RAAGs. Recall that a group is *coherent* if any finitely generated subgroup is finitely presented. In [**20**], Droms gives a characterisation of RAAGs that are coherent in terms of the defining commutation graph. We recall this characterisation in the following theorem. A graph is *chordal* if it has no full subgraph isomorphic to a cycle graph of more than three vertices.

Theorem 2.2 (Theorem 1, [**20**])**.** *A RAAG $\mathbb{G}(\Gamma)$ defined by the commutation graph Γ is coherent if and only Γ is chordal.*

Notice that among coherent RAAGs one finds free groups, free abelian groups, and all RAAGs which are fundamental groups of a 3-manifold (see [**20**, Theorem 2]).

To motivate the axioms introduced in Section 3 below we give some details of centralisers of elements of coherent RAAGs. Let Γ be a (simple) chordal graph and let $\overline{\Gamma}$ be its complement. Then, as Γ is chordal, $\overline{\Gamma}$ has at most one connected component with more than one vertex. Therefore the block normal form of an element g of the RAAG $\mathbb{G}(\Gamma)$ has at most one block which is not a power of a generator. Moreover, if g is an element which has a block containing more than one generator, then $\mathrm{lk}(g)$ is a clique (a set of vertices spanning a complete subgraph). Indeed, if $\mathrm{lk}(g)$

is not a clique, it contains two non-adjacent vertices, say u and v. Since we assume that g has a block w with more than one generator, then from the definition of a block we have that $\overline{\Gamma}(\alpha(w))$ is connected and so it follows that $\alpha(w)$ contains two generators whose associated vertices are not adjacent in Γ, say x and y. But then, since $u, v \in \mathrm{lk}(g) \subset \mathrm{lk}(w) \subset \mathrm{lk}(\{x, y\})$, we have that u and v are both adjacent to x and y and so the full subgraph defined by $\{u, v, x, y\}$ in Γ is a square, contradicting that the RAAG $\mathbb{G}$ is coherent.

Lemma 2.3. *Let $\mathbb{G}$ be a coherent RAAG, g a cyclically reduced element of $\mathbb{G}$, and $C(g)$ the centraliser of g in $\mathbb{G}$.*

(i) *If $C(g)$ is non-abelian, then g is the product of powers of generators that pairwise commute. In this case $C(g) = \prod_{x\in\alpha(g)}\langle x\rangle \times \langle \mathrm{lk}(g)\rangle$.*

(ii) *If $C(g)$ is abelian, then $\mathrm{lk}(g)$ is a clique (possibly empty) and either*
  - (a) *g is a product of powers of generators that pairwise commute, in which case $C(g) = \prod_{x\in\alpha(g)}\langle x\rangle \times \prod_{y\in \mathrm{lk}(g)}\langle y\rangle$; or*
  - (b) *$g = w^r v$, where w is a root block element of length greater than 1, r is a non-zero integer, v is a product of powers of generators that pairwise commute and belong to $\mathrm{lk}(w)$, and $C(g) = \langle w\rangle \times \prod_{x\in\alpha(v)}\langle x\rangle \times \prod_{y\in \mathrm{lk}(g)}\langle y\rangle$.*

Proof: From the remarks preceding the lemma and (2), if $C(g)$ is non-abelian, then the blocks of g are powers of generators, which commute by definition, giving (i). From (2), when $C(g)$ is abelian $\mathrm{lk}(g)$ must be a clique. As pointed out above, either g has a single block with more than one generator, in which case we have (ii)(b); or g is a product of powers of commuting generators, in which case (ii)(a) holds. ☐

Remark 2.4. Let $\mathbb{G}(\Gamma)$ be a coherent RAAG. Then if g_1, g_2, h_1, h_2 are elements of $\mathbb{G}$ so that $[h_1, h_2] \neq 1$, $[g_1, g_2] \neq 1$, there exists $i, j = 1, 2$ so that $[g_i, h_j] \neq 1$.

Indeed, assume to the contrary that $[h_1, h_2] \neq 1$, $[g_1, g_2] \neq 1$, and $[g_i, h_j] = 1$ for all $i, j = 1, 2$. Since $[h_1, h_2] \neq 1$ (correspondingly, $[g_1, g_2] \neq 1$), it follows that the centralisers of g_1 and g_2 (correspondingly, h_1 and h_2) are non-abelian as they contain h_1 and h_2 (correspondingly, g_1 and g_2). By Lemma 2.3, it follows that g_i is the product of powers of pairwise commuting generators, i.e. $g_i = \prod_{x\in\alpha(g_i)} x^{r(x)}$ and $C(g_i) = \prod_{x\in\alpha(g_i)}\langle x\rangle \times \langle \mathrm{lk}(g_i)\rangle$; similarly, for h_i, $i = 1, 2$.

Since $g_2 \notin C(g_1)$, it follows that there exist $x_2 \in \alpha(g_2)$ such that $x_2 \notin \mathrm{lk}(g_1)$. Since by definition $\mathrm{lk}(g_1) = \bigcap\limits_{x\in\alpha(g_1)} \mathrm{lk}(x)$ and $x_2 \notin \mathrm{lk}(g_1)$, it follows that there exists $x_1 \in \mathrm{lk}(g_1)$ such that $x_2 \notin \mathrm{lk}(x_1)$, that is, x_1 is not adjacent to x_2 and so $[x_1, x_2] \neq 1$. A symmetric argument shows that there exist $y_i \in \alpha(h_i)$, $i = 1, 2$, such that y_1 and y_2 are not adjacent and so $[y_1, y_2] \neq 1$.

Since $h_i \in C(g_j)$ for $i = 1, 2$, from the description of centralisers we have that y_i and x_j commute, for $i = 1, 2$. Furthermore, $y_i \neq x_j$ as $[x_1, x_2] \neq 1$ and $[y_i, x_j] = 1$ for $i, j = 1.2$. Therefore, x_1, x_2, y_1, y_2 are different and the full subgraph that they define is a square – a contradiction.

We shall later make use of the following property of centralisers of sets of generators of a RAAG.

Lemma 2.5. *Let $\mathbb{G}$ be a RAAG with canonical generating set X and let $Y \subseteq X$ be a finite set which generates a free abelian subgroup. Then there exists $g \in \langle Y \rangle$ such that $C(Y) = C(g)$.*

Proof: Let $Y = \{y_1, \ldots, y_m\}$ and set $g = y_1 \cdots y_m$. Then $C(Y) \subseteq C(g)$ and if $a \in C(g)$, then Lemma 2.3 implies $a = bc$, where $b \in \langle Y \rangle$ and $c \in \langle \mathrm{lk}(Y) \rangle$. As Y is a clique, $[b, y] = 1$, and by definition $[c, y] = 1$, for all $y \in Y$, so $a \in C(Y)$. ☐

Remark 2.6 (Representatives in a RAAG). A key element of the main construction of this paper is a choice of representatives of centralisers of a group; see $(\mathcal{C}5)$(i) below. As an initial example we describe how such a set of representatives may be chosen in a coherent RAAG with commutation graph Γ and generating set $X = V(\Gamma)$. To begin with let $\mathcal{K} = \{C \subset X : C \text{ is a clique}\}$, define an equivalence relation $\sim$ on $\mathcal{K}$ by $C \sim D$ if and only if $\mathrm{star}(C) = \mathrm{star}(D)$, and for each element $C = \{x_1, \ldots, x_k\} \in \mathcal{K}$ let $g_C = x_1 \cdots x_k$. Let $\overline{\mathcal{K}}$ denote the set of equivalence classes of $\sim$ and for each equivalence class $[C]$ of $\sim$ let $[C]_{\min}$ be the set of cliques of minimal cardinality in $[C]$. Let $\mathrm{star}([C]) = \mathrm{star}(C)$, where C is some (hence any) element of $[C]$. If $\mathrm{star}([C])$ is a clique, set $W_{[C]} = \{g_D\}$, for some $D \in [C]_{\min}$. If $\mathrm{star}([C])$ is not a clique, set $W_{[C]} = \{g_D : D \in [C]_{\min}\}$. Define $W_{\mathcal{K}}$ to be the union of the sets $W_{[C]}$, as $[C]$ ranges over $\overline{\mathcal{K}}$. Now let B be the set of cyclically reduced root block elements of length at least 2 in $\mathbb{G}$ and let $\sim_B$ be the equivalence relation on B given by $b \sim_B c$ if and only if c is a conjugate of b or b^{-1}. Let W_B be a set of representatives of equivalence classes of $\sim_B$ on B. Finally, let $W_{\mathbb{G}} = W_{\mathcal{K}} \cup W_B$.

Example 2.7. Let P_4 be the path graph on four vertices and $\mathbb{G} = \mathbb{G}(P_4) = \langle a, b, c, d \mid [a, b], [b, c], [c, d] \rangle$. The cliques of P_4 are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\mathrm{star}(a) = \mathrm{star}(a, b)$, $\mathrm{star}(d) = \mathrm{star}(c, d)$, whence $\{a\} \sim \{a, b\}$ and $\{d\} \sim \{c, d\}$, while all other equivalence classes are singletons. Thus $\overline{\mathcal{K}} = \{[\{a\}], [\{b\}], [\{c\}], [\{d\}], [\{b, c\}]\}$ and, with an obvious abuse of notation, $g_a = a$, $g_b = b$, $g_c = c$, $g_d = d$, $g_{bc} = bc$. As $C_{\min}$ is a singleton for each equivalence class, $W_{\mathcal{K}} = \{a, b, c, d, bc\}$.

Lemma 2.8. *Let $\mathbb{G}$ be the RAAG with commutation graph Γ and let g be an element of $\mathbb{G}$. Then there is $w \in W_{\mathbb{G}}$ such that $C(g)$ is conjugate to $C(w)$. If $C(g)$ is abelian, then w is unique.*

Proof: Without loss of generality we may assume g is cyclically reduced. If $C(g)$ is non-abelian, then from Lemma 2.3, g is the product of powers of generators that pairwise commute, and so $\alpha(g)$ is a clique. Let D be of minimal cardinality in $[\alpha(g)]$. Then $g_D \in W_{\mathcal{K}}$ and $C(g_D) = C(g)$, as required. Similarly, if $C(g)$ is abelian and canonical, $g_D \in W_{\mathcal{K}}$, for a unique D of minimal cardinality in $[\alpha(g)]$, and again $C(g_D) = C(g)$. From the definitions, $C(g_D) \neq C(w)$, for all other $w \in W$, so the lemma holds if $C(g)$ is canonical abelian.

If $C(g)$ is abelian and non-canonical, then $g = b^r x_1^{r_1} \cdots x_k^{r_k}$, for some cyclically reduced root block element b of length at least 2, $x_i \in \mathrm{lk}(b)$, and non-zero integers r, r_i. From Lemma 2.3, $C(g) \leq C(b)$. On the other hand if $y \in C(b)$, since Lemma 2.3 implies $C(b) = \langle b \rangle \times \langle \mathrm{lk}(b) \rangle$ and $\mathrm{lk}(b)$ is a clique, we have $[y, x_i] = 1$, for all i, so $y \in C(g)$. Hence $C(g) = C(b)$. There is a unique element $d \in W_B$ such that d is a conjugate of b or b^{-1} and so $C(g) = C(b) = C(d)^h$, for some $h \in \mathbb{G}$. If $w \neq d$ is an element of W and $C(w)$ is conjugate to $C(d)$, then $w \in W_B$, as $C(d)$ is non-canonical, so some conjugate w' of w belongs to $C(d)$, and therefore $w' = d^r v$, for some $v \in \langle \mathrm{lk}(d) \rangle$. It follows, as both w and d are cyclically reduced root block elements, that $w' = d^{\pm 1}$, so $w \sim_B d$, and by definition of W_B, we have $w = d$. Hence the lemma holds in all cases. $\square$

The following lemma is well known.

Lemma 2.9. *Let C be a coherent group and let A be a finitely generated abelian group. Then $C \times A$ is coherent.*

Proof: Since A is finitely generated, using induction, it suffices to consider the case when A is cyclic. Let π be the natural epimorphism from $C \times A$ onto C. The group $\pi(H)$ is finitely presented by coherence of C, and the kernel K of $\pi_{|H}$ is cyclic (possibly finite or even trivial) as a subgroup of the cyclic group A. Hence there is an exact sequence $1 \to K \to H \to \pi(H) \to 1$, which shows that H is finitely presented. $\square$

2.3. Limit groups. Limit groups have played an important role in the classification of finitely generated groups elementarily equivalent to a free group; see [**28, 49**]. They can be characterised from many different points of view. In this subsection, we briefly recall some of these equivalences; see [**19**] for further details.

Let $F(X)$ be a free group with basis X and denote by $G[X]$ the free product $G*F(X)$. A *system of equations with coefficients in the group* G is defined as a set of formal equalities $\{s(X)=1 \mid s(X)\in S(X)\}$, where $S(X)\subset G[X]$ is a (possibly infinite) subset. A *solution* of the system of equations is a tuple of elements $\bar{g}\in G^{|X|}$ such that $S(\bar{g})=1$ in G, or equivalently, a solution is a homomorphism from $G[X]$ to G (defined by $X\to\bar{g}$) such that $S(X)$ is contained in the kernel. The set of solutions of a system of equations is called an *algebraic set*.

The group-theoretic counterpart to the notion of a Noetherian ring is the notion of an equationally Noetherian group: a group G is called *equationally Noetherian* if every system of equations $S(X)=1$ with coefficients in G is equivalent to a finite subsystem $S_0(X)=1$, where $S_0(X)\subset S(X)$, i.e. the algebraic set defined by S coincides with the one defined by S_0. It is known (see [**5**]) that all linear groups are equationally Noetherian.

Let G and H be groups. We say that H is *discriminated* by G if for every finite set of non-trivial elements $H_0\subset H$ there exists a homomorphism $\phi\colon H\to G$ injective on H_0, that is, $h^{\phi}\neq 1$ for every $h\in H_0$. In this case we also sometimes say that H is *fully residually* G. Following the case of free groups, finitely generated fully residually G groups are termed *limit groups over* G.

The universal theory of a group G is the set of all universal first-order sentences (that is, sentences with only $\forall$ quantifiers in prenex normal form) satisfied by G. We say that H is a *model of the universal theory* of G if H satisfies all universal first-order sentences satisfied by G.

One can prove (see [**41**]) that if G is equationally Noetherian and H is a G-group, then the following statements are equivalent:

- H is a limit group;
- H is a finitely generated model of the universal theory of G.

Limit groups over RAAGs, and their actions on higher dimensional analogues of real trees, have been studied in [**15**].

3. The class $\mathcal{C}$

In this section we define the class of groups $\mathcal{C}$. Our general aim is to give a combinatorial description of a class of groups that contains coherent RAAGs and is closed under extensions of centralisers (of some elements) and direct limits; see Definition 4.1. These closure properties ensure that an appropriate iterated centraliser extension (ICE for short) of a group in the class remains in the class. One also needs to ensure that the ICE over G admits a $\mathbb{Z}[t]$-action, that it coincides with the exponential group $G^{\mathbb{Z}[t]}$, and that it is fully residually G.

The definition of the class $\mathcal{C}$ (see Definition 3.4) is rather technical. As RAAGs are described via a specific presentation, the definition of the groups in the class depends on the existence of a generating set for which some properties are satisfied. In particular, condition ($\mathcal{C}4$) asks for some minimality of the generating set.

As in the case of hyperbolic groups, we require the groups in the class to be torsion-free; see condition ($\mathcal{C}1$). Furthermore, to ensure that extensions of centralisers of elements in a group G from the class are fully residually G, we require the groups to satisfy the BP property; see condition ($\mathcal{C}2$). This condition also implies that centralisers are isolated, which means, in the presence of condition ($\mathcal{C}3$), that only extensions of centralisers of root elements need be considered.

As we mentioned in the introduction, for our approach, it is essential that centralisers have a tractable structure. We extract conditions on the structure of centralisers of elements in a RAAG so that, on the one hand, this structure is preserved under iterated centraliser extensions and, on the other hand, after countably many extensions of centralisers, the centre of a centraliser admits a $\mathbb{Z}[t]$-action; see conditions ($\mathcal{C}5$) and ($\mathcal{C}7$) in Definition 3.4.

To ensure that the $\mathbb{Z}[t]$-action is well defined in the group, we require a weak form of malnormality on centralisers, namely condition ($\mathcal{C}6$).

In some sense, conditions ($\mathcal{C}5$), ($\mathcal{C}6$), and ($\mathcal{C}7$) generalise the CSA property of torsion-free hyperbolic groups (see Lemma 3.10).

We next define the properties required from the groups in the class.

Definition 3.1. The centraliser of an element $g \in G$ is *isolated* if for all $w^n \in C(g)$, $n \in \mathbb{Z} \setminus \{0\}$, it follows that $w \in C(g)$.

Definition 3.2. We say that a k-tuple $u = (u_1, \dots, u_k)$ of elements of a group is *generic* if

$$[u_i, u_{i+1}] \neq 1 \quad \text{for} \quad i = 1, \dots, k-1.$$

A group G is said to have the *big powers* (BP) property if, for any positive integer k and any generic k-tuple $u = (u_1, \dots, u_k)$ of non-trivial elements of G, there exists an integer $n = n(u)$ such that, for positive integers $\alpha_1, \dots, \alpha_k$,

$$u_1^{\alpha_1} \cdots u_k^{\alpha_k} = 1$$

implies $\alpha_i < n$, for some i.

All torsion-free abelian groups are BP, as is the direct product of a BP group with a torsion-free abelian group. From [**32**], groups G and H are BP if and only if the free product $G * H$ is BP. However, if G and H are non-abelian groups, then the direct product $G \times H$ is not a BP group. From [**10**], a RAAG is BP if and only if its commutation graph is chordal.

If G is a BP group and $g \in G$, then the centraliser of g is isolated [**32**]. (In fact the centralisers of sets of elements are "strongly isolated", but we don't use this.)

Definition 3.3. Let $G = \langle X \rangle$ be a group. We say that a subgroup K is *canonical* (with respect to the generating set X) if $K = \langle X' \rangle$, for some $X' \subset X$.

Definition 3.4. We define $\mathcal{C}$ to be the class of groups satisfying the following. A group G belongs to $\mathcal{C}$ if and only if G has a presentation with generating set X such that properties ($\mathcal{C}1$)–($\mathcal{C}7$) below hold.

($\mathcal{C}1$) G is torsion-free.
($\mathcal{C}2$) G satisfies the BP property.
($\mathcal{C}3$) G has unique roots.
($\mathcal{C}4$) (i) Let Y and Y' be subsets of X such that $[y, y'] = 1$ for all $y \in Y$, $y' \in Y'$, and $Y \cap Y' = \emptyset$. Then $\langle Y, Y' \rangle = \langle Y \rangle \times \langle Y' \rangle$.
(ii) If $x \in X$, then $x \in \langle Y \rangle$ implies $x \in Y$, for all $Y \subseteq X$. (In particular, with $Y = \emptyset$ we have $x \neq 1_G$, for all $x \in X$, that is, $1_G \notin X$.)

There exists a subset W of G such that the following hold.

($\mathcal{C}5$) (i) For every $g \in G$, there exist elements $w_1, \dots, w_k \in W$, $k \geq 1$, and $h \in G$ such that $C(g) = h^{-1}C(w_i)h$, for $1 \leq i \leq k$. If $C(g)$ is abelian, then $k = 1$.
(ii) For all $w \in W$, $C(w)$ can be written as a direct product $Z(w) \times O(w)$, where $Z(w)$ and $O(w)$ are defined as follows.
(a) If $C(w)$ is a canonical abelian subgroup, then $Z(w) = 1$ and $O(w) = C(w)$.
(b) If $C(w)$ is abelian and non-canonical, then $Z(w)$ is cyclic and not canonical and $O(w)$ is the maximal canonical subgroup of G satisfying, for each minimal (by inclusion) subset $Y \subseteq X$ such that $Z(w) \subseteq \langle Y \rangle$:
- every generator of $O(w)$ commutes with each generator of Y and
- no generator of $O(w)$ belongs to Y.

[Although X may be infinite, maximal subgroups always exist.]

(c) If $C(w)$ is non-abelian, then $Z(w)$ is defined in $(\mathcal{C}7)$ below and $O(w)$ is the maximal canonical subgroup of G satisfying, for each minimal (by inclusion) subset $Y \subseteq X$ such that $w \in \langle Y \rangle$:
- every generator of $O(w)$ commutes with each generator of Y and
- no generator of $O(w)$ belongs to Y.

(iii) If $w \in W$, $g \in O(w)$, and $C(g)$ is not conjugate to $C(w)$, then there exist $h \in G$, $w_0 \in W$, such that $C(g) = h^{-1}C(w_0)h$ and $h, w_0 \in O(w)$.

There may be several minimal canonical subgroups containing an element of G, but for fixed $w \in W$ the maximal canonical subgroup $O(w)$ satisfying the properties of $(\mathcal{C}5)$(ii)(b) or $(\mathcal{C}5)$(ii)(c) is unique (indeed, if there were two maximal subgroups $O_1(w)$, $O_2(w)$, then the group $O(w) = \langle O_1(w), O_2(w) \rangle$ would also satisfy the required properties).

$(\mathcal{C}6)$ If $w \in W$ and $C(w)$ is abelian, then $C(w)$ satisfies the property that if $a, a^h \in C(w)$ and $h \notin C(w)$, then $a \in O(w)$ and $[h, a] = 1$.

It follows that for all $g \in G$, if $C(g)$ is abelian, $a, a^h \in C(g)$ and $h \notin C(g)$, then $h \in C(a)$ and $a \in O(g)$.

$(\mathcal{C}7)$ If $w \in W$ and $C(w)$ is non-abelian, then the following hold.

(i) $C(w)$ is a canonical subgroup: $C(w) = \langle Y(w) \rangle \times O(w)$, where $Y(w)$ is a minimal subset of X such that $w \in \langle Y(w) \rangle$ and, by definition, $\mathrm{Z}(w) = \langle Y(w) \rangle$.

(ii) The centre $Z(C(w))$ of $C(w)$ is a canonical subgroup.

In this case $Y(w)$ is the unique minimal subset of X such that $w \in \langle Y(w) \rangle \leq C(w)$. Indeed, suppose that Y_1 is another such subset. If $y_1 \in Y_1$, then $y_1 \in C(w)$ implies $y_1 \in Y(w) \cup K$ (using $(\mathcal{C}4)$(ii)), where K is a set of canonical generators of $O(w)$, and so $y_1 \in Y(w)$. As $Y(w)$ is minimal it follows that $Y_1 = Y(w)$.

If $C(w) = \langle V \rangle$, for some subset V of X, then let V' be a minimal subset of V such that $w \in \langle V' \rangle$. Then $V' \subseteq Y(w)$ (using $(\mathcal{C}4)$(ii) and $(\mathcal{C}5)$(ii)) and the minimality of $Y(w)$ implies $V' = Y(w)$. Hence $\mathrm{Z}(w) \subseteq Z(C(w))$. If $Z(C(w))$ is finitely generated, then w can be chosen such that $\mathrm{Z}(w) = Z(C(w))$.

Remark 3.5.

(1) The set W is not uniquely determined by the conditions above, but for fixed W satisfying the conditions, $Z(w)$ and $O(w)$ are uniquely determined, for all $w \in W$. This follows directly from the definitions for $O(w)$ and from $(\mathcal{C}5)$(ii)(a) and $(\mathcal{C}7)$, if $C(w)$ is canon-

ical abelian or $C(w)$ is non-abelian. If $C(w)$ is abelian and non-canonical, then let Y be a minimal subset of X such that $Z(w) \subseteq \langle Y\rangle$ and let $Z(w) = \langle z\rangle$. Suppose that $Z'(w)$ also satisfies the conditions of $(\mathcal{C}5)$(ii)(b) with $C(w) = Z'(w) \times O(w)$, let $Z'(w) = \langle z'\rangle$, and let Y' be a minimal subset of X such that $Z'(w) \subseteq \langle Y'\rangle$. From $(\mathcal{C}4)$(ii), $\langle Y \cup Y' \cup O(w)\rangle = \langle Y \cup Y'\rangle \times O(w)$. It follows that $z = z'o$, for some $o \in O(w)$. Then $z'^{-1}z = o \in \langle Y \cup Y'\rangle \cap O(w)$, so $z = z'$ and $Z(w) = Z'(w)$, as claimed. We may then choose $w = z$ in this case.

(2) For $g \in G$ let $w \in W$ be such that $C_G(g) = C_G(w)^h$, for some $h \in G$. If $C(w)$ is abelian and $C_G(g)^{h_1} = C_G(w)^{h_2}$, then w and $w^{h_1h_2^{-1}}$ belong to $C_G(w)$, so $(\mathcal{C}6)$ implies that $h_1h_2^{-1} \in C_G(w)$, and so $w^{h_1} = w^{h_2}$. In this case we define $Z(g) = Z(w)^h$ and $O(g) = O(w)^h$. Then $h_1 = zoh_2$, where $o \in O(w)$ and $z \in Z(w)$, so $Z(w)^{h_1} = Z(w)^{h_2}$ and $O(w)^{h_1} = O(w)^{h_2}$, so $Z(g)$ and $O(g)$ are well defined.

If $C(w)$ is non-abelian, consider first $g \in G$ such that $C(g) = C(w)$. Then $g \in Z(C(w))$, which is canonical, so a minimal subset $Y(g)$ of X such that $g \in \langle Y(g)\rangle$ and $Y(g) \subseteq Z(C(w))$ may be chosen. Define $Z(g) = \langle Y(g)\rangle$ and define $O(g)$ to be the subgroup of $C(w)$ generated by $(Y(w)\cup K(w))\setminus Y(g)$, where $K(w)$ is a canonical generating set for $O(w)$. Then $C(w) = Z(g)\times O(g)$. In general let $T(w)$ be a transversal (containing 1) for right cosets of the subgroup $U(w) = \{u \in G : C_G(w)^u = C_G(w)\}$ in G. When $C_G(g) = C_G(w)^h$, let $h = ut$, for $t \in T(w)$, $u \in U(w)$ so $C_G(g) = C_G(w)^t$. Then $C_G(g^{t^{-1}}) = C_G(w)$ and we may define $\mathrm{Z}(g) = \mathrm{Z}(g^{t^{-1}})^t$ and $O(g) = O(g^{t^{-1}})^t$.

If G belongs to the class $\mathcal{C}$, has generating set X and subset W satisfying $(\mathcal{C}1)$–$(\mathcal{C}7)$ above, we say that G is in $\mathcal{C}(X, W)$.

The following lemmas will be useful in Sections 4 and 7.

Lemma 3.6. *Let G be a group in class $\mathcal{C}$ and let w and g be elements of G such that $w \in W$, $C(w)$ is abelian, and $g \in C(w)$. If $C(g)$ is non-abelian, then $g \in O(w)$ and there exists $w_0 \in W\cap O(w)$ such that $C(g) = C(w_0)$ and, using the notation of Remark 3.5(2), $C(g) = \mathrm{Z}(g) \times O(g)$ is canonical with $\mathrm{Z}(g) \le O(w)$.*

Proof: As $C(g)$ is non-abelian there exists $x \in C(g)$, $x \notin C(w)$, so g and g^x belong to $C(w)$ and, from $(\mathcal{C}6)$, then $g \in O(w)$. From $(\mathcal{C}5)$(iii), there exists $w_0 \in O(w)\cap W$ and $h \in O(w)$, such that $C(g) = h^{-1}C(w_0)h$, and as $C(w)$ is abelian, $C(g) = C(w_0)$. From Remark 3.5(2) we have $\mathrm{Z}(g) \le Z(C(w_0))$. This implies that if $s \in \mathrm{Z}(g)$, then $w \in C(g) \le C(s)$

and so $C(s)$ is non-abelian. Hence s and s^x belong to $C(w)$ and $s \in O(w)$, as claimed. $\square$

In the rest of this section, we give examples of groups that belong to the class $\mathcal{C}$, namely, RAAGs and toral relatively hyperbolic groups. Recall that a toral relatively hyperbolic group is a torsion-free group which is hyperbolic relative to a finite family $\{A_\lambda : \lambda \in \Lambda\}$ of finitely generated free abelian groups.

Example 3.7. Free abelian groups are in $\mathcal{C}(X, W)$, where X is a free basis and $W = \{0\}$. In this case $Z(a) = \{0\}$ and $O(a)$ is the entire group, for all group elements a.

Lemma 3.8. *A coherent RAAG belongs to $\mathcal{C}(X, W)$, where X is the vertex set of the commutation graph of the group and W is the set defined in Remark 2.6.*

Proof: Properties ($\mathcal{C}1$), ($\mathcal{C}3$), and ($\mathcal{C}4$) hold for all RAAGs [**3**]. Blatherwick ([**10**]) proves that a RAAG satisfies the BP property if and only if it is coherent. Property ($\mathcal{C}5$)(i) follows directly from Lemma 2.8. In the terminology of the preamble to Lemma 2.8, if $w \in W_{\mathcal{K}}$, then, if $C(w)$ is canonical abelian, then $O(w) = C(w)$; if $C(w)$ is abelian and non-canonical, set $Z(w)$ to be cyclic (and not canonically) generated by the root of w and $O(w) = \langle \mathrm{lk}(g) \rangle$; and if $C(w)$ is non-abelian, set $Z(w) = \langle \alpha(w) \rangle$ and $O(w) = \langle \mathrm{lk}(w) \rangle$. Then ($\mathcal{C}5$)(ii)(a), ($\mathcal{C}5$)(ii)(c), ($\mathcal{C}6$), and ($\mathcal{C}7$) follow immediately from Lemma 2.8.

To see that ($\mathcal{C}5$)(iii) holds, when $w \in W_{\mathcal{K}}$, assume that $g \in O(w)$ and $C(g)$ is not conjugate to $C(w)$. Then $g = g_1^{-1} g_0 g_1$, for some $g_i \in O(w)$ such that g_0 is cyclically reduced. There is $w_0 \in W$ such that $C(g_0)$ is conjugate to $C(w_0)$ and, as both g_0 and w_0 are cyclically reduced, it follows from Lemma 2.3 that either $C(w_0) = C(g_0)$ or (as words) w_0 is a cyclic permutation of g_0, and in both cases this implies $w_0 \in C(w)$. In the case where $C(w)$ is abelian, by definition, $w_0 \in O(w)$. Assume then that $C(w)$ is non-abelian. If $\alpha(g_0)$ is not a clique, then $w_0 \in W_B$ and so by definition $\alpha(w_0) \subseteq \alpha(g_0)$, which implies $w_0 \in O(w)$. If $\alpha(g_0)$ is a clique, then there exists a minimal element D of $[\alpha(g_0)]$ such that $D \subseteq \alpha(g_0)$ and by definition $g_D \in W_{\mathcal{K}}$. Taking $w_0 = g_D$, we have $w_0 \in O(w)$ and $C(g_0) = C(w_0)$, as required. Hence $w_0 \in O(w)$ in all cases and ($\mathcal{C}5$)(iii) holds when $w \in W_{\mathcal{K}}$.

If $w \in W_B$, then set $Z(w) = \langle w \rangle$ and $O(w) = \langle \mathrm{lk}(w) \rangle$, and ($\mathcal{C}5$)(ii)(c) follows. For ($\mathcal{C}5$)(iii) assume that $g \in O(w)$, which in this case is torsion-free abelian so g is cyclically reduced. Then $\alpha(w) \subseteq \mathrm{star}(g)$, so $\alpha(g)$ is a clique and $\mathrm{star}(g)$ is not a clique; and ($\mathcal{C}5$)(iii) follows as in the previous case. $\square$

Example 3.9. To see that in Lemma 3.8 the set W cannot be simplified in such a way that every centraliser (of an element) is conjugate to the centraliser of a unique element of W, consider the graph Γ of Figure 1 and the group $\mathbb{G} = \mathbb{G}(\Gamma)$. There are non-abelian centralisers $C(d_1d_2) = \langle a, c_1, c_2, d_1, d_2\rangle$, $C(ad_2) = \langle a, b_2, c_1, c_2, d_1, d_2\rangle$, with $O(d_1d_2) = \langle a, c_1, c_2\rangle$ and $O(ad_2) = \langle b_2, c_1, c_2, d_1\rangle$. Then $ac_1 \in O(d_1d_2)$ and $d_1c_1 \in O(ad_2)$ and $C(ac_1) = C(d_1c_1) = \langle a, b_1, c_1, d_1, d_2\rangle$. The elements of $\mathbb{G}$ with centraliser equal to $C(d_1d_2)$ are the elements of $\langle d_1, d_2\rangle$. Indeed, for v a vertex of Γ, $\mathrm{star}(d_1, d_2) \subseteq \mathrm{star}(v)$ if and only if $v = a$, d_1, or d_2. As $\mathrm{star}(a, d_i) \neq \mathrm{star}(d_1, d_2)$ the claim follows from Lemma 2.3. Similarly, the elements of $\mathbb{G}$ with centraliser equal to $C(ad_2)$ are the elements of $\langle a, d_2\rangle$. For all elements $g \in \langle d_1, d_2\rangle$ we have $O(g) = O(d_1d_2)$ and for all $h \in \langle a, d_2\rangle$ we have $O(h) = O(ad_2)$. To satisfy $(\mathcal{C}5)$(iii) the set W must then contain an element w_1 conjugate to $ac_1 \in O(g)$ and an element w_2 conjugate to ac_1 in $O(h)$, forcing $C(w_1)$ and $C(w_2)$ to be conjugate.

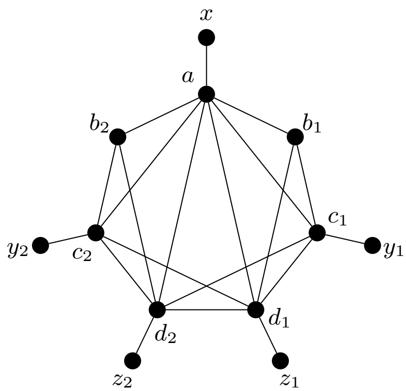

FIGURE 1. Example 3.9.

Lemma 3.10. *Let G be a non-abelian group, generated by a finite set X, satisfying $(\mathcal{C}1)$, $(\mathcal{C}2)$, $(\mathcal{C}4)$, and the condition that, for all $g \in G$, either $C(g)$ is conjugate to a canonical subgroup or $C(g) = \langle g_0\rangle$, where g_0 is not a proper power.*

If G is a CSA group, then G belongs to $\mathcal{C}$.

Proof: From [**40**], as G is CSA and torsion-free it satisfies $(\mathcal{C}3)$ and, for all $g \in G$, the centraliser $C(g)$ is maximal abelian and malnormal. If $C(g)$ is conjugate to a canonical centraliser $C(h)$, then, as G is a CSA group, there exists $x \in X \cap C(h)$ such that $C(h) = C(x)$. Let W_0 be a subset of X such that if $y, z \in W_0$, then $C(y)$ is not conjugate to $C(z)$ and if $x \in X$, then $C(x)$ is conjugate to $C(w)$ for some $w \in W_0$. Then every canonical centraliser $C(g)$ is conjugate to $C(w)$, for precisely one $w \in W_0$.

Now let $C(g)$ be a non-canonical centraliser, so from the hypothesis $C(g) = \langle g_0 \rangle$, where g_0 is the root of g. Choose a subset Y of X which is minimal with the property that there exists a root element $g_1 \in \langle Y \rangle$ with $C(g_1)$ conjugate to $C(g_0)$, and define $W'(g) = g_1$. Now let W' be a subset consisting of elements w of G such that $w = W'(g)$, for some $g \in G$ and such that no two elements of W' have conjugate centralisers. Using Zorn's lemma the set of all such W' has maximal elements. If W_1 is a maximal subset of this form, then it follows that, for all $g \in G$, if $C(g)$ is non-canonical, then $C(g)$ is conjugate to $C(w)$, for a unique element of W_1.

Set $W = W_0 \cup W_1$, if $w \in W_0$, set $Z(w) = \{1\}$ and $O(w) = C(x)$, and if $w \in W_1$, set $Z(w) = \langle w \rangle$ and $O(w) = 1$. Then $(\mathcal{C}5)$(i) and (ii) follow directly. If $g \in O(w)$, for some non-trivial $g \in G$ and $w \in W$, then $w \in W_0$ and as G is a CSA group $C(g) = C(w)$, so $(\mathcal{C}5)$(iii) holds. Finally, $(\mathcal{C}6)$ follows immediately from the CSA property. □

Lemma 3.11. *Let G be a torsion-free group which is hyperbolic relative to a finite family $\{A_\lambda : \lambda \in \Lambda\}$ of finitely generated free abelian groups (G is toral relatively hyperbolic). Then G is a BP group.*

Proof: Let k be a positive integer and let $u = u_1, \dots, u_k$ be a generic sequence of elements of G. Define $T = \{u_1, \dots, u_k, [u_1, u_2], \dots, [u_{k-1}, u_k]\}$. From [**44**], if u_i is hyperbolic, then u_i is contained in a maximal elementary subgroup $E(u_i)$ and G is hyperbolic relative to $\{A_\lambda\} \cup \{E(u_i)\}$. Moreover, as G is torsion-free hyperbolic, $E(u_i)$ is cyclic. Adding these $E(u_i)$ to the family of peripheral subgroups, we may assume that u_i is contained in a peripheral subgroup A_i, $i \in \Lambda$, for all i. Theorem 1.1 of [**45**] states that there exists a finite subset $\mathfrak{F}$ of non-trivial elements of G with the following property. Let $\mathfrak{N} = \{N_\lambda\}_{\lambda \in \Lambda}$ be a collection of subgroups $N_\lambda \triangleleft A_\lambda$ such that $N_\lambda \cap \mathfrak{F} = \emptyset$ for all $\lambda \in \Lambda$. Write $G(\mathfrak{N})$ for the quotient $G/\operatorname{ncl}\langle \bigcup_\lambda N_\lambda \rangle$ (where ncl denotes the normal closure in G). Then, for each $\lambda \in \Lambda$, the natural map from A_λ/N_λ to $G(\mathfrak{N})$ is injective and $G(\mathfrak{N})$ is hyperbolic relative to the collection $\{A_\lambda/N_\lambda\}_{\lambda \in \Lambda}$. Moreover, for any finite subset $S \subset G$, there exists a finite subset $\mathfrak{F}(S)$ of non-trivial elements of G such that the restriction of the natural homomorphism $G \to G(\mathfrak{N})$ to S is injective whenever $N_\lambda \cap \mathfrak{F}(S) = \emptyset$ for $\lambda \in \Lambda$. Let $T \subset G$ be the finite set defined above and $\mathfrak{F}(T)$ be the set given by the aforementioned theorem. For each $\lambda \in \Lambda$ let $T_\lambda = (\mathcal{F} \cup \mathcal{F}(T) \cup T) \cap A_\lambda$. From [**6**] it follows that free abelian groups are discriminated by cyclic groups, so for all λ, there exists a homomorphism ϕ_λ from A_λ to a cyclic group C_λ such that ϕ_λ restricted to T_λ is injective. Let N_λ be the kernel of ϕ_λ and let $H = G/\operatorname{ncl}\langle\bigcup_\lambda N_\lambda\rangle$. Then $N_\lambda \cap \mathcal{F}(T) = \emptyset$ and from [**45**, Theorem 1.1], the canonical map ϕ from G to H induces an embedding from A_λ/N_λ to H. It follows that $\phi(u_i)$ is of infinite order in H, for all i, and that $\phi(u_1), \dots, \phi(u_k)$ is a generic sequence of elements of H. From [**45**, Corollary 1.2], the group H is hyperbolic and from [**43**] hyperbolic groups have the big powers property for tuples of elements of infinite order; so there exists $n(u)$ such that, whenever $\alpha_i \geq n(u)$, for all i, we have $\phi(u_1)^{\alpha_1} \cdots \phi(u_k)^{\alpha_k} \neq 1$. It follows that $u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq 1$, and therefore G is a BP group. $\square$

Corollary 3.12. *Let G be a torsion-free group which is hyperbolic relative to a finite family $\{A_\lambda : \lambda \in \Lambda\}$ of finitely generated free abelian groups. Then G is in $\mathcal{C}$.*

Proof: From Lemma 3.11, the group G is in BP and ($\mathcal{C}1$) and ($\mathcal{C}4$) hold from the definitions. The centraliser of a non-trivial element g satisfies that either $C(g) = A_\lambda$ for some $\lambda \in \Lambda$ or $C(g) = \langle g_0 \rangle$, where g_0 is the root of g, and from [**25**, Lemma 2.5], such groups are CSA. Hence we may apply Lemma 3.10. $\square$

4. Preservation operations

Definition 4.1 (Extension of centralisers)**.** Let G and H be groups, $u \in G$, $C = C_G(u)$, and $\phi\colon C \to H$ a monomorphism such that $\phi(u) \in Z(H)$. The *extension of the centraliser* C by H (*with respect to* ϕ) is

$$G(u, H) = G *_\phi H,$$

the group with relative presentation $\langle G, H \mid \phi(g) = g, \forall g \in C\rangle$.

If $H = \phi(C) \times A$, for some subgroup A of H, then the extension is said to be *direct*.

An element of an amalgamated free product is said to be cyclically reduced if it has no reduced form which begins and ends with an element from the same factor. Every element of an amalgam is conjugate to a cyclically reduced element ([**35**, Theorem 4.6]).

Theorem 4.2. *Let $G \in \mathcal{C}$ (with respect to the generating set X_G and subset W_G satisfying ($\mathcal{C}1$)–($\mathcal{C}7$) above) and let $u \in W_G$ be such that $C_G(u)$ is abelian. Let B be a free abelian group, let $\phi\colon C_G(u) \to B$ be a monomorphism such that $B = \phi(C_G(u)) \times A$, for some $A \leq B$, and let $G(u, B) = G *_\phi B$ be the direct centraliser extension of $C = C_G(u)$ by B. Then the following hold.*

(i) *$G(u,B)\in\mathcal{C}$, with respect to the generating set $X(u,B)=X_G\cup X_A$, where X_A is a free generating set for A, and cyclically reduced elements $W(u,B)=W_G\cup W_*$, where W_* is the set of elements $z\in G(u,B)$ that satisfy the following conditions:*
  (a) *z is a cyclically reduced root element with a factorisation $z=g_1a_1\cdots g_ra_r$, where $r\geq 1$, $g_i\notin C$, $a_i\in A$, $a_i\neq 0$, for all i, such that*
  (b) *If $Y(z)$ is a minimal subset of X_G such that $g_i\in\langle Y(z)\rangle$, for all i, and $O'(z)=\bigcap_{i=1}^r C_G(g_i)\cap C_G(u)$, then $\langle Y(z),O'(z)\rangle=\langle Y(z)\rangle\times O'(z)$.*
  (c) *If z satisfies (a) and (b) above, then exactly one element of the set of elements $v\in G(u,B)$ such that v is conjugate to z or z^{-1} and satisfies (a) and (b) belongs to W_*.*

(ii) *Assume $\phi(u)=v\in B$ and identify the cyclic subgroup $\langle v\rangle$ of B with $\mathbb{Z}$. Let $\psi_i\colon B\to\mathbb{Z}$, $i\in I$, be a discriminating family of (additive group) homomorphisms of B by its subgroup $\mathbb{Z}$, indexed by a set I. For $(i,m)\in I\times\mathbb{N}$, define $\lambda_{i,m}\colon G(u,B)\to G$ to be the homomorphism induced by the identity homomorphism on G and the composition of the inverse image of ϕ with the m-th scalar multiple $m\psi_i$ of the homomorphism ψ_i on B: that is, $\lambda_{i,m}(g)=g$ and $\lambda_{i,m}(b)=\phi^{-1}(m\psi_i(b))=u^{m\psi_i(b)}$, for $g\in G$ and $b\in B$. Then $G(u,B)$ is discriminated by G via the family $\lambda_{i,m}$, $(i,m)\in I\times\mathbb{N}$.*

Proof: Note that, although there may be more than one choice of $w\in W_G$ such that $C_G(w)=C$, the group $G(u,B)$ and subsets $X(u,B)$, $W(u,B)$, $Z_G(u)$, and $O_g(u)$ are independent of the choice $w=u$.

(ii). Let $\mathbf{w}=(w_1,\dots,w_{n+1})$ be an $n+1$ tuple of elements of G, such that $w_i\notin C$, for $i\geq 2$, and let $\mathbf{m}=(m_1,\dots,m_n)$ be an n-tuple of non-zero integers. Then, setting $u_i=u^{m_i}$, the tuples $(w_1,\dots,w_{n+1})$ and $(u_1,\dots,u_n)$ satisfy the condition that $[u_i^{w_{i+1}},u_{i+1}]\neq 1$ in G, since the fact that centralisers in G are isolated implies $[u_i^{w_{i+1}},u_{i+1}]=1$ if and only if $(u^{m_i})^{w_{i+1}}\in C(u_{i+1})=C$ if and only if $u^{w_{i+1}}$, $u\in C$, if and only if $[w_{i+1},u]=1$. Since G satisfies the BP property there exists an integer $K(\mathbf{w},\mathbf{m})$ such that $w_1u_1^m\cdots u_n^m w_{n+1}\neq 1$, for all $m\geq K(\mathbf{w},\mathbf{m})$.

Let g be a non-trivial element in $G(u,B)$. One can write it in a reduced form

$$g=w_1a_1w_2\cdots a_nw_{n+1},$$

where $1\neq a_i\in A$ and $w_i\in G$, for all i, and $w_i\notin C_G(u)$ for $i=2,\dots,n$, and either $w_{n+1}=1$ or $w_{n+1}\notin C_G(u)$. Let ψ_j discriminate the elements $a_1,\dots,a_n$ in $\mathbb{Z}$. Define $m_i=\psi_j(a_i)$, for $i=1,\dots,n$. By definition,

$$\lambda_{j,m}(a_i)=u_i^m=u^{mm_i}\in C,$$

so, with $\mathbf{w}$, $\mathbf{m}$, and $K(\mathbf{w},\mathbf{m})$ as above, $\lambda_{j,m}(g) \neq 1$, for all $m \geq K(\mathbf{w},\mathbf{m})$. (If $w_{n+1} = 1$, then we replace g with $a_n g a_n^{-1}$ and obtain $\lambda_{j,m}(a_n g a_n^{-1}) \neq 1$, so again $\lambda_{j,m}(g) \neq 1$.) Thus, we can separate any given non-trivial element $g \in G(u,B)$ by $\lambda_{j,m}$, for all $m \geq K(\mathbf{w},\mathbf{m})$.

Consequently, if we have a finite number of elements $g_1,\dots,g_k \in G(u,B)$, say

$$g_i = w_{i,1}a_{i,1}w_{i,2}\cdots a_{i,n_i}w_{i,n_i+1},$$

then, choosing ψ_j to discriminate the elements $a_{1,1},\dots,a_{k,n_k}$ in $\mathbb{Z}$, it follows that $\lambda_{j,m}$ discriminates $g_1,\dots,g_k$, as long as $m \geq \max\{K(\mathbf{w}_1,\mathbf{m}_1), \dots, K(\mathbf{w}_k,\mathbf{m}_k)\}$, where $\mathbf{w}_i$ and $\mathbf{m}_i$ have the obvious definitions. Hence $G(u,B)$ is discriminated by G.

(i): $(\mathcal{C}1)$ and $(\mathcal{C}2)$. Since $G(u,B)$ is discriminated by G, it follows that $G(u,B)$ is torsion-free and since G satisfies the BP property, so does $G(u,B)$ ([**7**, Lemma 11]), so $(\mathcal{C}1)$ and $(\mathcal{C}2)$ hold.

$(\mathcal{C}4)$(i). Let Y and Y' be disjoint and commuting subsets of $X(u,B)$. Then $Y = L_1 \cup L_2$ and $Y' = L_1' \cup L_2'$, where $L_1, L_1' \subset X_G$ and $L_2, L_2' \subset X_A$. If $Y, Y' \subseteq G \cup B$, then immediately from the definitions $(\mathcal{C}4)$(i) holds. Assume then that L_1 and L_2 are both non-empty and that $a \in L_2$ $(a \neq 1)$. Then, $[a,y'] = 1$ and $a \in A$, so $y' \in C$ for all $y' \in L_1'$. Hence $Y' \subseteq C \times A = B$. If also $L_2' \neq \emptyset$, then $Y \subseteq B$, a contradiction. Hence, without loss of generality, we have $Y' \subseteq C$. In this case if $w \in \langle Y\rangle \cap \langle Y'\rangle$, then $w \in \langle Y'\rangle < C$ implies that w has syllable length 0, and so $w \in \langle Y\rangle \cap G = \langle L_1\rangle$, so $w \in \langle L_1\rangle \cap \langle Y'\rangle$. Since $L_1 \cap Y' = \emptyset$ and $L_1 \cup Y' \subseteq X_G$, it follows from $(\mathcal{C}4)$(i) in G that $w = 1$ and so $\langle Y, Y'\rangle = \langle Y\rangle \times \langle Y'\rangle$.

$(\mathcal{C}4)$(ii). Let $Y \subset X(u,B)$ and let $x \in X(u,B) \cap \langle Y\rangle$. Write $Y_G = Y \cap X_G$ and $Y_A = Y \cap X_A$ and let $x = g_0a_0\cdots g_na_n$ be an expression for x with $a_i \in \langle Y_A\rangle$ and $g_i \in \langle Y_G\rangle$. If $n > 0$, then $a_n = 1_B$ and $g_n \in C$, so $x = g_0a_0\cdots g_{n-1}g_na_{n-1}$; and continuing this way we see $x = g_0'a_0$, with $g_0' \in \langle Y_G\rangle$. If $x \in X_A$, then $g_0' \in C$ and $x = \phi(g_0')a_0$, from which it follows that $\phi(g_0') = 1_B$ and then that $x \in Y_A$. If $x \in X_G$, then $a_0 = 1_B$ and $x \in Y_G$, by $(\mathcal{C}4)$(ii) in G.

$(\mathcal{C}3)$. Let $g \in G(u,B)$ be in a conjugate of a factor. Then, without loss of generality, we may assume that g belongs to a factor; and it follows that g has a unique root, using $(\mathcal{C}3)$ in the case $g \in G$. Hence we may assume that g is not in a conjugate of a factor. First note that for all $h, f \in G(u,B)$ and $s \geq 1$, if $h^s = f^s$, then $[f^s,h^s] = 1$, so $[f,h] = 1$, as centralisers in $G(u,B)$ are isolated. Hence $(fh^{-1})^s = 1$ and, as $G(u,B)$ is torsion-free, $h = f$. Now if g is not in a conjugate of a factor, then without loss of generality we may assume that g is cyclically reduced so, for all h and r such that $g = h^r$, the syllable length of g is $k_0|r|$, where k_0 is the syllable length of h_0. Therefore there is a unique maximal positive integer $r(g)$, such that $g = h^{r(g)}$, for some $h \in G(u, B)$. Suppose that $g = h^r = f^s$, for some $h, f \in G(u, B)$, where $r = r(g)$ and $1 \le s \le r$. Let $d = \gcd(r, s)$ and $a, b \in \mathbb{Z}$ be such that $d = ar + bs$. Then $h^d = h^{ar}h^{bs} = g^a h^{bs} = (f^a h^b)^s$ and, setting $h_1 = (f^a h^b)^{s/d}$, we have $h^d = h_1^d$, so $h = h_1$. Thus $h = (f^a h^b)^{s/d}$ and, by the maximality of $r(g)$, we have $s = d$, so $s|r$ and $f^s = (h^{r/s})^s$ implies $f = h^{r/s}$. Therefore g has a unique root $\sqrt{g} = h$.

To verify that ($\mathcal{C}5$)–($\mathcal{C}7$) hold, a description of centralisers in $G(u, B)$ is needed. We will use the following description of commuting elements in free products with amalgamation (see [**35**]): if $[x, y] = 1$, then one of the following conditions holds:

(I) x or y belongs to some conjugate of the amalgamated subgroup $C = C_G(u)$;

(II) neither x nor y is in a conjugate of C, but x is in a conjugate of a factor (G or B), in which case y is in the same conjugate of the factor;

(III) neither x nor y is in a conjugate of a factor, in which case $x = g^{-1}cgz^n$ and $y = g^{-1}c'gz^m$, where $c, c' \in C$, and $g^{-1}cg, g^{-1}c'g$ and z commute pairwise.

Considering each of these three possible cases in turn we shall prove the following lemma and show that ($\mathcal{C}5$)–($\mathcal{C}7$) hold in $G(u, B)$.

Lemma 4.3. *Let v be a cyclically reduced element of $G(u, B)$ and let $C(v)$ denote the centraliser of v in $G(u, B)$.*

(i) *If $v \in C$, then either*
  (a) *$v \notin O_G(u)$, in which case $C(v) = C \times A = B = C(u)$; or*
  (b) *$v \in O_G(u)$, in which case $C(v) = C(w)$, for some $w \in W_G$, and $C(v) = \mathrm{Z}_G(v) \times \langle O_G(v), A\rangle$.*

(ii) *If $v \in G \backslash C$, then $C(v) = C_G(v)$ and if $v \in A$, and is not the identity, then $C(v) = B = C(u)$.*

(iii) *If v is not in $G \cup B$, then there exists $z \in W_*$ such that $C(v) = \langle z\rangle \times O'(z)$ (as defined in Theorem 4.2(i)(b) above).*

In case (i)(b) $C(v)$ is non-abelian and canonical, while in all other cases $C(v)$ is abelian.

Case (I). If x belongs to some conjugate of the amalgamated subgroup $C = C_G(u) = Z_G(u) \times O_G(u) = \phi(C) \le B$, without loss of generality we may assume that $x \in C$. In this case, as C is abelian $B = C \times A \subseteq C(x)$.

Let $[x, y] = 1$ and let $y = g_1 a_1 \cdots g_n a_n$ be any reduced form of y, that is, $a_i \in A$, $g_i \in G$, $a_i \neq 1_A$, for $i < n$, and for $i \geq 2$, $g_i \notin C$. From $yx = xy$ and the theory of amalgamated products, it follows that either ($n = 1$ and $g_1 \in C$) or ($n \geq 1$, $g_n \notin C$, and $x^{g_n^{-1}} \in C$). In the latter case, as $(\mathcal{C}6)$ implies that $g_n \in C_G(x)$, the centraliser of x in G is non-abelian: $g_n \in C_G(x)$, $u \in C_G(x)$, but $[g_n, u] \neq 1$. Hence, if $C_G(x)$ is abelian,

$$y \in C \times A, \text{ and } C(x) = C \times A = C(u). \tag{3}$$

Now assume that $C_G(x)$ is non-abelian, so contains y as above where $n \geq 1$ and $g_n \notin C_G(u)$, so $g_n \in C_G(x)$. From Lemma 3.6, $C_G(x) = \mathrm{Z}_G(x) \times O_G(x)$ is canonical, there exists $x_0 \in O_G(u)$ such that $C_G(x) = C_G(x_0)$ and $\mathrm{Z}_G(x) \leq O_G(u)$.

Since $g_n \in \mathrm{Z}_G(x) \times O_G(x)$, we have $g_n = x_n o_n$, where $x_n \in \mathrm{Z}_G(x) < O_G(u)$ and $o_n \in O_G(x)$. Then $y = y' x_n o_n a_n$ and $[x, y'] = 1$. By induction on syllable length we have that $y' = K_{n-1} O_{n-1}$, where $K_{n-1} \in \mathrm{Z}_G(x)$, $O_{n-1} \in \langle O_G(x), A\rangle$. Then

$$y = K_{n-1} O_{n-1} x_n o_n a_n = K_n O_n,$$

where $K_n = K_{n-1} x_n \in \mathrm{Z}_G(x)$ and $O_n = O_{n-1} o_n a_n \in \langle O_G(x), A\rangle$.

Hence, if $C_G(x)$ is non-abelian, then there exists $x_0 \in W_G$ such that

$$C(x) = \mathrm{Z}_G(x) \times \langle O_G(x), A\rangle = C(x_0) \text{ with } x, x_0 \in O_G(u). \tag{4}$$

In particular, for any $x \in C$, $C_G(x)$ is abelian if and only if $C(x)$ is abelian, and if $C_G(x)$ is canonical, so is $C(x)$.

Now let us verify that conditions $(\mathcal{C}5)$ to $(\mathcal{C}7)$ hold for x in Case (I). If $C_G(x)$ is abelian, so $C(x) = C(u)$ is given in (3), define $Z(u) = Z_G(u)$ and $O(u) = O_G(u) \times A$. As $C_G(u) \neq C_G(v)$, for all $v \in W$, $v \neq u$, it follows that if $w \in W$ and $w \neq u$, then $C(w) \neq C(u)$. Hence $(\mathcal{C}5)$(i) and $(\mathcal{C}5)$(ii) hold.

For $(\mathcal{C}5)$(iii), if $g \in O(u)$, let $g = oa$, where $o \in O_G(u)$ and $a \in A$. From $(\mathcal{C}5)$(iii) in G and the fact that $C_G(u)$ is abelian it follows that $C_G(o) = C_G(w_0)$, for some $w_0 \in W_G \cap O_G(u)$ and from (4), if a is trivial, then $C(g) = C(w_0) = \mathrm{Z}_G(w_0) \times \langle O_G(w_0), A\rangle$, and $w_0 \in W$; so $(\mathcal{C}5)$(iii) holds. Otherwise a is non-trivial and $g \in B$ so $C(g) = B = C(u)$ and $(\mathcal{C}5)$(iii) follows immediately.

To see that $(\mathcal{C}6)$ holds, when $C(x)$ is abelian, assume that $w \in C(u)$, $w^h \in C(u)$, and $h \notin C(u)$. Then $w = w_G a$, where $a \in A$ and $w_G \in C$. Also, $h \notin C(u)$ implies $h = g_1 a_1 \cdots g_n a_n$, where $g_i \in G$, $a_i \in A$, and either $n = 1$ and $g_1 \notin C$; or $n > 1$ and $g_i \notin C$, for $i \geq 2$. Then

$$w^h = a_n^{-1} g_n^{-1} \cdots a_1^{-1} g_1^{-1} w_G a g_1 a_1 \cdots g_n a_n \in C(u) = C \times A$$

so $g_1^{-1}w_Gag_1 \in C$, whence $a = 1$ or $g_1 \in C$. If $n = 1$, then $g_1 \notin C$ so $a = 1$, $w = w_G$, and $w_G^{g_1} \in C$, so $w_G \in O_G(u) \subseteq O(u)$ and $[g_1, w_G] = 1$; so $[h, w] = 1$. If $n > 1$ and $g_1 \in C$, then

$$g_n^{-1} \cdots a_1^{-1} w_G a a_1 \cdots g_n = g_n^{-1} \cdots g_2^{-1} w_G a g_2 \cdots g_n \in C \times A$$

and, by induction on syllable length, $w \in O(u)$ and $[w, g_2 \cdots g_n] = 1$; so $[w, h] = 1$.

On the other hand, if $g_1 \notin C$, as before $a = 1$ and $[g_1, w_G] = 1$ so

$$g_n^{-1} \cdots g_2^{-1} w_G g_2 \cdots g_n \in C \times A.$$

As $g_2 \cdots g_n \notin C(u)$, by induction $[w_G, g_2 \cdots g_n] = 1$. It follows that in all cases $[w, h] = 1$.

Thus $(\mathcal{C}5)$ and $(\mathcal{C}6)$ hold in Case (I) when $C_G(x)$ is abelian.

Assume then that $x \in C$ and $C_G(x)$ is non-abelian, so (4) describes $C(x) = C(x_0)$. If $w \in W$ such that $C_G(w) = C_G(x_0)$, then $w \in W_G$ since, as we shall see in the final case of this proof, $v \in W_*$ implies that $C(v)$ is abelian. Hence for all such w we have $w \in W_G$ and we may set $Z(w) = \mathrm{Z}_G(w)$ and $O(w) = \langle O_G(w), A\rangle$. If $w \neq x_0$, note that $\mathrm{Z}_G(w) \leq Z(C_G(w)) = Z(C_G(x_0)) \leq C_G(u)$, so $C_G(w) = \mathrm{Z}_G(w) \times \langle O_G(w), A\rangle$. Hence, in this case $(\mathcal{C}5)$(i), $(\mathcal{C}5)$(ii), and $(\mathcal{C}7)$ follow directly from the definitions and we defer consideration of $(\mathcal{C}5)$(iii) until we have completed our description of centralisers of elements of $G(u, B)$, below.

Case (II). If x is in a conjugate of G but not in a conjugate of C, then $x \in G^g$, for some $g \in G(u, B)$. In this case $C(x^{g^{-1}}) = C_G(x^{g^{-1}})$ and $(\mathcal{C}5)$ and $(\mathcal{C}7)$ hold for $C(x)$, as they hold in G. To see that $(\mathcal{C}6)$ holds assume that $C(x)$ is abelian and conjugate to $C(w) = C_G(w)$, for some $w \in W_G$, and that b and b^h are in $C(w)$, but $h \notin C(w)$. If $h \in G$, then $(\mathcal{C}6)$ implies $b \in O_G(w) = O(w)$ and $[h, b] = 1$, as required. Assume then that $h = g_1a_1 \cdots g_na_n$ in reduced form, with $a_i \in A$ and $g_i \in G$, and $a_1 \neq 1$. Then $b^h \in C(w) \leq G$ implies that $b^{g_1} \in C$ and so $[a_1, b^{g_1}] = 1$ and then $b^h = (b^{g_1})^{h_1} \in C(w)$, where $h_1 = g_2a_2 \cdots g_na_n$. Hence $(b^{g_1})^{g_2} \in C$ and $g_2 \notin C$, so $[g_2, b^{g_1}] = 1$. Continuing this way we obtain $[g_i, b^{g_1}] = 1$, $2 \leq i \leq n$, so $b^h = b^{g_1} \in C(w)$. If $g_1 \notin C(w)$, then, from $(\mathcal{C}6)$ in G, we have $[b, g_1] = 1$, so $[b, h] = 1$, and $b \in O_G(w) = O(w)$. Otherwise $g_1 \in C(w)$ so $b = b^{g_1} = b^h \in C$ and then $b, b^u \in C_G(w)$, but $u \notin C(w)$, as $w \notin C$, so again $b \in O(w)$ and $[b, h] = 1$. Therefore, $(\mathcal{C}6)$ holds for w in $G(u, B)$.

If x is in a conjugate of B, but not in C, then $C(x)$ is conjugate to $C(u)$ and we have Case (I) again.

Case (III). If x does not belong to any conjugate of a factor, without loss of generality we may assume that x is cyclically reduced (as an element of the amalgamated free product $G(u,B)$) and its reduced factorisations begin with an element of $G\backslash C$. Let $x = g_1a_1\cdots g_na_n$ be any reduced form of x, that is, $n\geq 1$, $g_i\in G\backslash C$, and $a_i\in A\backslash\{1_A\}$, for $i=1,\dots,n$.

Let $O'(x) = \bigcap_{i=1,\dots,n} C_G(g_i)\cap O_G(u)$. We claim that $O'(x)$ is a canonical subgroup of G with canonical generating set $U_x = X_G\cap O'(x)$, and that $C(x) = O'(x)\times\langle z_x\rangle$, where z_x is the unique root element ($\sqrt{z_x}=z_x$) such that

- $x = dz_x^m$, with $m\geq 1$, $d\in O'(x)$; and
- there is a minimal subset Z of X_G such that $z_x\in\langle Z, X_A\rangle$, and $\langle U_x, Z, X_A\rangle = O'(x)\times\langle Z, X_A\rangle$.

By the description of centralisers in amalgamated products, if $[x,y]=1$, then $x=dv^l$ and $y=d'v^m$, where $d,d'\in C$ and d, d', v pairwise commute. As x is cyclically reduced, so is v and so both l and m divide n. As $[d,x]=[d',x]=1=[d,u]=[d',u]$ but $[x,u]\neq 1$ (as $C(u) = C\times A$), both $C(d)$ and $C(d')$ are non-abelian, so from Lemma 3.6, $\mathrm{Z}_G(d)\cup \mathrm{Z}_G(d')\subseteq O_G(u)$. Since $v\in C(d)$, it follows from (4) that $v=d_1w$, where $d_1\in\mathrm{Z}_G(d)\leq O_G(u)$ and $w=g_1'a_1'\cdots g_r'a_r'\in O(d)=\langle O_G(d),A\rangle$; that is, $g_i'\in O_G(d)$ and $a_i'\in A$. Since $v=d_1w$, we have $xv^{-l}d^{-1} = xw^{-l}d_1^{-l}d^{-1}=1$, so $a_r'=a_n$ and $g_r'g_n^{-1}\in C$. Since $d\in C$ and C is abelian, we deduce that $[g_r'g_n^{-1},d]=1$. Now since $[g_r',d]=1$, we have that $[g_n,d]=1$ and so $d\in C_G(g_n)$. Continuing this process, we conclude that $d\in C_G(g_i)$ and, similarly, $d'\in C_G(g_i)$, for $i=1,\dots,n$. Hence $d,d'\in O'(x)$.

Now, let p be any element of $O'(x)$. Then $p\in O_G(u)$ so Lemma 3.6 gives $C_G(p)=\mathrm{Z}_G(p)\times O_G(p)$ and $\mathrm{Z}_G(p)\leq O_G(u)$. Let $s\in\mathrm{Z}_G(p)$. Then $s\in O_G(u)$ and, by definition, $\mathrm{Z}_G(p)\leq Z(C_G(p))$ so $[p,g_i]=1$ implies $[s,g_i]=1$. Hence $s\in O'(x)$ and, as $\mathrm{Z}_G(p)$ is canonical, it follows that $O'(x)$ is a canonical subgroup of G.

Let $U_x = O'(x)\cap X_G$, the canonical generating set for $O'(x)$, and let $x=dv^l$, where $d\in C$ and $[d,v]=1$, so as above $d\in O'(x)$ and $v=d_1w$, where $d_1\in\mathrm{Z}_G(d)$ and $w\in\langle O_G(d),A\rangle$. Using the description of $C(d)$ again, and the uniqueness of roots in $G(u,B)$, if $r(w)=q$ and $\sqrt{w}=w_0$, so $w=w_0^q$, then we may factorise $w_0=g_1'a_1'\cdots g_r'a_r'$, with $g_i'\in O_G(d)$ and $a_i'\in A$. Let Y be a minimal subset of X_G such that $g_i'\in\langle Y\rangle$, for $i=1,\dots,r$, let $Y_1=U_x\cap Y$ and let $Y_2=Y\backslash Y_1$. Suppose $s\in U_x$ and $t\in Y_2$. As $s\in O'(x)$ we have, as above, $[s,g_i']=1$, so $g_i'=x_i'o_i'$, with $x_i'\in\mathrm{Z}_G(s)$ and $o_i'\in O_G(s)$, for $i=1,\dots,r$. By the minimality of Y, there is i such that $g_i'\in\langle Y\rangle$ but $g_i'\notin\langle Y\backslash\{t\}\rangle$. For such i we have $g'_i = x'_i o'_i$, with $x'_i \in \mathrm{Z}_G(s)$ and $o'_i \in O_G(s)$, and as $t \notin \mathrm{Z}_G(s)$ (since $t \in Y_2$) it follows that $t \in O_G(s)$; so $[s,t]=1$. Therefore $\langle U_x, Y_2\rangle = O'(x) \times \langle Y_2\rangle$. For all i, by definition $g'_i \in \langle Y\rangle \leq \langle U_x, Y_2\rangle$ so we have $g'_i = x_i o_i$, $x_i \in O'(x)$, $o_i \in \langle Y_2\rangle$, and we may further factorise $w_0 = d_2 w_1$, where $d_2 = x_1 \cdots x_r$ and $w_1 = o_1 a'_1 \cdots o_r a'_r$. Thus $x = dv^l = d_3 w_1^{lq}$, where $d_3 = dd_1^l d_2^{lq} \in O'(x)$ (as $d_1 \in \mathrm{Z}_G(d) \leq C$) and $w_1 \in \langle Y_2, X_A\rangle$. Replacing w_1 by w_1^{-1} if necessary we may also assume $lq \geq 1$. Hence $C(x) = \langle w_1\rangle \times O'(x)$, which is abelian.

This implies that $C(x) \leq C(w_1)$. Applying this argument to the element w_1 in place of x, with initial reduced factorisation $w_1 = o_1 a'_1 \cdots o_r a'_r$ and $Y = Y_2$, gives $C(w_1) = \langle w_1\rangle \times O'(w_1)$, where $O'(w_1) = \bigcap_{i=1,\dots,r} C_G(o_i) \cap O_G(u)$. If $b \in O'(w_1)$, then, for all i, we have $[b, o_i] = 1$ and $g'_i = x_i o_i$, with $x_i \in O_G(u)$, which is abelian, so $[b, g'_i] = 1$ and $b \in C(x)$. As before, as $b \in C$ we obtain $[b, g_i] = 1$, so $b \in O'(x)$. Conversely, if $b \in O'(x)$, then by definition of o_i, we have $[b, o_i] = 1$, so we conclude that $O'(x) = O'(w_1)$. Therefore $C(x) = C(w_1) = \langle w_1\rangle \times O'(w_1)$, with $O'(x) = O'(w_1)$.

Now suppose $x = d' z^m$, where $m \geq 1$, $d' \in O'(x)$, z is a root element, $z \in \langle Y', X_A\rangle$ for some minimal subset Y' of X_G, such that $Y' \cap U_x = \emptyset$ and $[y', y''] = 1$, for all $y' \in Y'$ and $y'' \in U_x$. Then $d_3^{-1} d' = w_1^{lq} z^{-m}$; so from $(\mathcal{C}4)$(i) and the disjointness of U_x and $Y_2 \cup Y' \cup X_A$ we have $d_3 = d'$ and $w_1^{lq} = z^m$. As w_1 and z are root elements and both m and lq are positive we have $w_1 = z$, completing our claim.

Given the validity of the claim, to show in addition that w_1 or w_1^{-1} is in W_* it is enough to show that given a minimal subset Y' of X_G such that $o_i \in \langle Y'\rangle$, for all i, we have $\langle Y' \cup U_x\rangle = \langle Y'\rangle \times \langle U_x\rangle$. To see this, given such a subset Y' let $Y'_2 = Y' \backslash (Y' \cap U_x)$, so as above, $\langle Y'_2 \cup U_x\rangle = \langle Y'_2\rangle \times \langle U_x\rangle$ and we may write $o_i = x'_i o''_i$, with $x'_i \in O'(x)$ and $o''_i \in \langle Y'_2\rangle$. Therefore $x = d'_3 w_1'^{lq}$, where $d'_3 = (x'_1 \cdots x'_r)^{lq} d_3 \in O'(x)$ and $w'_1 = o''_1 \cdots o''_r \in \langle Y'_2\rangle$. From the previous paragraph it follows that $d'_3 = d_3$ and $w'_1 = w_1$, so $Y'_2 = Y'$ and $\langle Y' \cup U_x\rangle = \langle Y'\rangle \times \langle U_x\rangle$, as required. Hence either w_1 or $w_1^{-1} \in W_*$. This completes the proof of Lemma 4.3.

Now we are in a position to verify properties $(\mathcal{C}5)$ and $(\mathcal{C}6)$. From the previous two paragraphs it follows that $C(x)$ is conjugate to $C(w_1)$ for a unique element w_1 of W_*, so $(\mathcal{C}5)$(i) holds. To simplify notation we may now assume $x = w_1 = o_1 a_1 \cdots o_r a_r \in W$, with $o_i \in \langle Y_2\rangle$, $a_i \in A$, so $x \in W_*$, and let U_x be a canonical generating set for $O'(x)$ as before. Then $C(x)$ is abelian and non-canonical and we set $Z(x) = \langle x\rangle$

and $O(x) = O'(x)$. For $(\mathcal{C}5)$(ii)(b) let V be a minimal subset of X such that $x \in \langle V \rangle$, so x has a reduced factorisation $x = h_1 b_1 \cdots h_r b_r$, with $h_i \in \langle V \cap X_G \rangle$ and $b_i \in \langle V \cap X_A \rangle$. Setting $V_G = X_G \cap V$, it is enough to show that $\langle V_G \cup U_x \rangle = \langle V_G \rangle \times \langle U_x \rangle$. Let $V_1 = V_G \cap U_x$ and $V_2 = V_G \backslash V_1$. Since, from the above, $O'(x) = O_G(u) \cap \bigcap_{i=1}^r C(h_i) = \langle U_x \rangle$ we have $h_i = y_i p_i$, $y_i \in \langle V_1 \rangle$, $p_i \in \langle V_2 \rangle$, and may rewrite x as $x = d_2' w_1'$, where $d_2' = y_1 \cdots y_r \in O'(x)$ and $w_1' = p_1 \cdots p_r$. As in the proof of the final part of the claim above, we have $d_2' = 1$, $w_1' = x$, and $V_2 = V_G$, and $(\mathcal{C}5)$(ii)(b) follows. If $g \in O(x)$, then $g \in O_G(u)$, so $C(g)$ is given by (4), and as $(\mathcal{C}5)$(iii) is satisfied by the centraliser $C_G(g)$ in G it holds also for $C(g)$.

To see that $C(x)$ satisfies property $(\mathcal{C}6)$, assume that $h \notin C(x)$, $v \in C(x)$, and $v^h \in C(x)$. Then, for some integers p, q, and elements $b, c \in O'(x)$ we have $v = bx^p$ and $h^{-1}vh = cx^q$. This means that bx^p is conjugate to cx^q and both are cyclically reduced. From the conjugacy criterion for free products with amalgamation, $p = q$ and bx^p is obtained from cx^p by cyclic permutation, followed by conjugation by an element of C. As $b \in O'(x)$, it follows that x has a factorisation $x = x_1 x_2$, where x_1 and x_2 are both reduced and begin with elements of G, such that $cx^p = d^{-1}b(x_2x_1)^p d$, for some $d \in C$. As C is abelian, we have $b^{-1}cx^p = d^{-1}(x_2x_1)^p d$. As $O_G(u)$ and $O'(x)$ are canonical we may choose a minimal subset D of $X \cap O_G(u)$ such that $d \in \langle D \rangle$, and write $D = D_1 \cup D_2$, where $D_2 = D \cap O'(x)$ and $D_1 \cap D \backslash D_2$. Then $d^{-1}(x_2x_1)^p d = d_1^{-1}(x_2x_1)^p d_1^{-1}$, so we may assume $d = d_1$ and $D = D_1$. We have $x \in \langle Y(x) \cup X_A \rangle$ and it follows that $d^{-1}(x_2x_1)^p d \in \langle D \cup Y(x) \cup X_A \rangle$ and, by definition of $Y(x)$, that $\langle O'(x) \cup D \cup Y(x) \cup X_A \rangle = O'(x) \times \langle D \cup Y(x) \cup X_A \rangle$ (using $(\mathcal{C}4)$(i)). Therefore $b^{-1}c = 1$, and we have $h^{-1}bx^p h = bx^p$. However, $h \notin C(x)$, so this implies that $p = 0$, so $v = x^p b = b \in O(x)$ and $[h, v] = 1$. That is, $(\mathcal{C}6)$ holds for $C(x)$.

Finally, we show that $(\mathcal{C}5)$(iii) holds when $x \in W_G \cap C$ and $C(x)$ is non-abelian, in which case (4) implies $C(x) = \mathrm{Z}_G(x) \times \langle O_G(x), A \rangle$. Assume $v \in O(x)$ and that $C(v)$ is not conjugate to $C(x)$. Either $v \in O_G(x)$, $v \in A$, or v has reduced factorisation of length at least 2. In the case $v \in O_G(x)$, from $(\mathcal{C}5)$(iii) in G, there exist v_0, v_1 in $O_G(x)$, such that $v_0 \in W_G$ and $C_G(v) = C_G(v_0)^{v_1}$. If $v_0 \in W_G \backslash C$, then $C(v_0) = C_G(v_0)$ and so $C(v) = C(v_0)^{v_1}$, as required. If $v_0 \in C$ and $C_G(v_0)$ is non-abelian, then $v_0 \in O_G(u)$ and, from (4), $C(v^{v_1^{-1}}) = C(v_0) = \mathrm{Z}_G(v^{v_1^{-1}}) \times \langle O_G(v^{v_1^{-1}}), A \rangle$, so $C(v) = C(v_0)^{v_1}$.

If $v_0 \in C$ and $C_G(v_0)$ is abelian, or if $v \in A$, then $C(v_0) = C \times A = C(u)$ and so to show that $(\mathcal{C}5)$(iii) holds it suffices to show that $u \in O(x)$. From (4), we have $u \in C(x) = \mathrm{Z}_G(x) \times O(x)$ and $\mathrm{Z}_G(x) \leq O_G(u)$. If Y is a minimal subset of X such that $u \in \langle Y \rangle$, then $Y \cap O_G(u) = \emptyset$, so $Y \cap \mathrm{Z}_G(x) = \emptyset$, whence $u \in O(x)$, as required.

Otherwise v has a reduced factorisation of length at least 2 and we choose U to be a minimal subset of X such that $v \in \langle U \rangle \leq O(x)$. Then we may write $v = v_1^{-1} v_0 v_1$, where $v_0, v_1 \in \langle U \rangle$ and v_0 is cyclically reduced (as an element of the free product with amalgamation $G(u, B)$) and belongs to $O(x)$. Thus $C(v) = C(v_0)^{v_1}$ and there is an element $z \in W_*$ such that $C(v_0) = C(z)$, with $v_0 = dz^p$, for some $d \in O(z) = O'(v_0)$ and non-zero $p \in \mathbb{Z}$. As $x, d \in C$ we have $[x, d] = 1$, so $z \in C(x)$ (as centralisers are isolated). To demonstrate that $(\mathcal{C}5)$(iii) holds it suffices to show that $z \in O(x)$. As $z \in W_*$, we may assume that z has a factorisation $z = g_1 a_1 \cdots g_r a_r$ and there is a minimal subset $Y(z)$ of X_G satisfying conditions (a) and (b) of Theorem 4.2(i). As v_0 belongs to $O(x) = \langle O_G(x), A \rangle$ it follows that $g_i \in C_G(x)$, for all i, and as $C_G(x)$ is canonical we may assume that $Y(z) \subseteq C_G(x)$. As $\mathrm{Z}_G(x) \leq Z(C(x))$, if $s \in \mathrm{Z}_G(x)$, then $[s, g_i] = 1$, as $g_i \in C_G(x)$, and $[s, u] = 1$, as $\mathrm{Z}(x) \leq O_G(u)$; so $s \in O'(z) = O(z)$. Hence $\mathrm{Z}_G(x) \leq O'(z)$ and $Y(z) \cap \mathrm{Z}_G(x) = \emptyset$. Therefore $(\mathcal{C}4)$(ii) implies $Y(z) \subseteq O_G(x)$, and so $z \in O(x)$. $\square$

Corollary 4.4. *Let G be a non-abelian group in $\mathcal{C}(X_G, W_G)$ and let $u \in W_G$ such that $C_G(u) = \langle u \rangle \times O_G(u)$ is abelian and non-canonical. Let $\mathbf{x} = \{x_1, \ldots, x_m\}$ be a set disjoint from G, either let $D = \prod_{i=1}^{g} [x_{2i-1}, x_{2i}]$, where m is even, $m \geq 4$, and $g = m/2$, or let $D = \prod_{i=1}^{m} x_i^2$, where $m \geq 3$, and let S be the surface group with presentation $\langle \mathbf{x}, u \mid D = u \rangle$. Write $C = C_G(u)$ and define T to be the free product with amalgamation*

$$T = G *_C (S \times O_G(u)),$$

identifying $u \in G$ with $u \in S$. Let $Z = X \cap O_G(u)$ be the canonical generating set for $O_G(u)$ and let $z \in Z$. Then

(i) *$S \times O_G(u)$ is in $\mathcal{C}(\mathbf{x} \cup Z, W_S \cup \{z\})$, where W_S is a subset of the free group on $\mathbf{x}$ and $z \in Z$, and*

(ii) *T is in $\mathcal{C}(X_G \cup \mathbf{x}, W_T)$, where $W_T = W_G \cup W_S \cup W'$, for some subset W' of the set of elements of T of reduced length at least 2.*

Proof: For the first statement note that S is free on $\mathbf{x}$ and $O_G(u)$ is free abelian with basis Z, whence $S \times O_G(u)$ is a coherent RAAG. From Lemma 3.8, $S \times O_G(u)$ is then in $\mathcal{C}$, with respect to the generating set $\mathbf{x} \cup Z$ and the set, W'_S say, defined as above in Lemma 2.8. From the definition, for some $z \in Z$, the set W'_S is the union of $W_{\mathcal{K}} = \mathbf{x} \cup \{z\}$ with a set W_B of cyclically reduced root elements of the free group on $\mathbf{x}$, of length at least 2. Taking $W_S = \mathbf{x} \cup W_B$, we have the first statement.

To prove the second statement we shall show that T embeds in an extension of centralisers of G, and use this together with Theorem 4.2. From the hypotheses, C is a maximal abelian subgroup of G; so the assumptions of [**29**, Lemma 3.2] hold. Let $\rho\colon T \to G$ be the natural retraction with $\rho(S)$ being non-abelian; see [**15**]. It is shown in [**15**, Section 7] that ρ is injective on both vertex groups of T, G, and $S \times O_G(u)$. Set $T^* = \langle G, y \mid [C_G(\rho(C_S(u))), y]\rangle$. Then, by [**29**, Lemma 3.2(1)] the group T embeds into T^* via the map ψ given by $\psi(g) = g$, for all $g \in G$ and $\psi(s) = (\rho(s))^y$, for all $s \in \langle \mathbf{x}\rangle$. Moreover, as $C_S(u) = C \subseteq G$, we have

$$T^* = \langle G, y \mid [C, y]\rangle = G *_C (C \times \langle y\rangle). \tag{5}$$

From Theorem 4.2, $T^* \in \mathcal{C}(X, W)$, where set $X = X_G \cup \{y\}$, $W = (W_G \backslash W_{G,C}) \cup \{y\} \cup W_*$, and $W_{G,C}$ and W_* are defined in the statement of the theorem. Also, T^* is discriminated by G, whence T is discriminated by G and it remains to show that T is in $\mathcal{C}$. Properties ($\mathcal{C}1$) and ($\mathcal{C}2$) hold in T as they hold in T^*. To see that property ($\mathcal{C}3$) holds in $\psi(T)$ it suffices to show that if w is an element of $\psi(T)$, then the unique root of w in T^* belongs to $\psi(T)$. Suppose then that $w = \psi(t)$, for some $t \in T$, say $t = g_1 s_1 \cdots g_m s_m$ in reduced form, with $g_i \in T$ and $s_i \in S \times O_G(u)$. Then $w = \psi(t) = g_1 t_1^y \cdots g_m t_m^y$, where $t_i = \rho(s_i)$. Moreover this is a reduced factorisation of w, since the given factorisation of t is reduced. In this case, for all $c \in C$ and j such that $1 \le j \le m$, $g_j c$, $c^{-1} g_j$, $(c^{-1} t_j)^y$ and $(t_j c)^y$ are in $\psi(T)$, and it follows that if $w = g_1'(t_1')^y \cdots g_j c (c^{-1} t)^y g_m' (t_m')^y$ is any reduced factorisation of w in T^*, then $g_i' \in G$ and $t_i' \in \psi(S \times O_G(u))$. Now suppose that t is cyclically reduced, in which case so is w, that the root of w in T^* is w_0 and that $w_0^d = w$. If w_0 has reduced factorisation $w_0 = h_1 y_1 \cdots h_k y_k$, with $h_i \in G$ and $y_k \in \langle y\rangle \times C$, then w has reduced factorisation $(h_1 y_1 \cdots h_k y_k)^d$. As w is in $\psi(T)$ it follows that the y_i are of the form $c_i y^{-1}$ when i is odd and $c_i y$ when i is even, with $c_i \in C$. It follows that $w_0 = h_1' y^{-1} h_2' y \cdots h_{2l-1}' y^{-1} h_{2l}' y$, where $h_i' \in G$ and $h_i' \in \rho(S \times O_G(u))$, whenever i is even. That is, w_0 is in $\psi(T)$. As roots in T^* are unique, $\psi^{-1}(w_0)$ is the unique root of t in T. That is, cyclically reduced elements of T have unique roots, whence all elements of T have unique roots, confirming that ($\mathcal{C}3$) holds.

To see that ($\mathcal{C}4$)(i) holds assume that Y and Y' are disjoint commuting subsets of $X_G \cup \mathbf{x}$. If $Y \cap \mathbf{x} \neq \emptyset$, then $Y' \subseteq O_G(u)$ and $\langle Y\rangle \cap \langle Y'\rangle \subseteq O_G(u)$. Otherwise Y and Y' are contained in X_G. In both cases we have

$\langle Y\rangle \cap \langle Y'\rangle = \{1\}$, using $(\mathcal{C}4)$(i) in G. Again, $(\mathcal{C}4)$(ii) follows from the fact that it holds in both G and $S \times O_G(u)$.

To verify that $(\mathcal{C}5)$, $(\mathcal{C}6)$, and $(\mathcal{C}7)$ hold we consider the centraliser of an element $v \in T$. If v is in G but is not conjugate to an element of C, then $C_T(v) = C_G(v)$ and it follows that there are elements $w \in W_G$ and $h \in G$ such that $C_G(v) = C_G(w)^h$. Also, w cannot be in C so $C_T(w) = C_G(w)$ and so $C_T(v)$ is conjugate to $C_T(w)$, as required. Moreover, $(\mathcal{C}5)$, $(\mathcal{C}6)$, and $(\mathcal{C}7)$ hold for w in T, as they hold for w in G. If $v \in S \times O_G(u)$ but is not conjugate to an element of C, then $C_{S\times O_G(u)}(v)$ is conjugate to $C_{S\times O_G(u)}(w)$, for some $w \in W_S \cup \{z\}$, and $w \neq z$, so $w \in W_S$.

Suppose now that v is conjugate to an element of C. Without loss of generality we may assume $v \in C$, so $C_G(v) \geq C$. In this case, if $b \in C_T(v)$ and has reduced factorisation $b = s_1g_1 \cdots s_mg_m$, with $g_i \in G$ and $s_i \in \langle \mathbf{x}\rangle$, then either $m = 1$ and $s_1 \in C$; or $m \geq 1$ and $s_m \notin C$. In the first case $g_1s_1 \in C_G(v)$, so $g_1 \in C_G(v)$. In the second case $v^{g_m} \in C$, so either $g_m \in C \leq C_G(v)$ or $g_m \notin C$, and then $(\mathcal{C}6)$ in G implies $v \in O_G(u)$ and $g_m \in C_G(v)$. In all cases then $v^{s_m} \in C$, and $(\mathcal{C}6)$ in $S \times O_G(u)$ implies $v \in O_G(u)$ and $C_G(v)$ is non-abelian. On the other hand if $v \notin O_G(u)$, a similar argument shows that $C_T(v) = C_G(v)$ and from $(\mathcal{C}6)$ and $(\mathcal{C}7)$ in G and the fact that C is abelian, it follows that $C_G(v)$ is abelian, so equal to C. Now if $C_G(v)$ is abelian, then $C_T(v) = C_T(u) = C$, and $u \in W_T = W_G \cup W_S \cup W'$. In this case $(\mathcal{C}5)$ holds for u in T, as it holds in G. For $(\mathcal{C}6)$, suppose $g \in C$ and $b \in T$ such that $g^b \in C$ and $b \notin C$. If $b \in G$ or $S \times O_G(u)$, then we have $b \in C_G(g)$ and $g \in O_G(u)$, from $(\mathcal{C}6)$ in these groups. Assuming then that b has a reduced factorisation $b = s_1g_1 \cdots s_mg_m$, as above with $m \geq 1$, it follows again that $g \in O_G(u)$ and $g_i \in C_G(g)$, so $b \in C_G(g)$, whence $(\mathcal{C}6)$ holds for u in T.

If $v \in O_G(u)$, then $C_T(v) \geq \langle C_G(v), \mathbf{x}\rangle$. Repeating the argument with a reduced factorisation of an element $b \in C_T(v)$ we see that $C_T(v) = \langle C_G(v), \mathbf{x}\rangle$. We have $C_G(v) = \mathrm{Z}_G(v) \times O_G(v)$ and from the definitions $\mathrm{Z}_T(v) \cap \langle O_G(v), \mathbf{x}\rangle = \{1\}$ and $[\mathrm{Z}_T(v), \langle O_G(v), \mathbf{x}\rangle] = 1$, so setting $\mathrm{Z}_T(v) = \mathrm{Z}_G(v)$ and $O_T(v) = \langle O_G(v), \mathbf{x}\rangle$ we have $C_T(v) = \mathrm{Z}_T(v) \times O_T(v)$. As the only generators of T not in X_G are those in $\mathbf{x}$, both $(\mathcal{C}5)$(i) and (ii) follow. We defer consideration of $(\mathcal{C}5)$(iii) till the end of the proof. To see $(\mathcal{C}7)$ holds, note that $Z(C_T(v)) = O_G(u)$, which is canonical.

Finally, suppose that $v \notin G$ or $S \times O_G(u)$, and let $v = g_1s_1 \cdots g_ms_m$, with $g_i \in G$, and $s_i \in \langle \mathbf{x}\rangle$ be a reduced factorisation of v. Without loss of generality we may assume that v is cyclically reduced. Then, setting $t_i = \rho(s_i)$, it follows that $\psi(v) = g_1t_1^y \cdots g_mt_m^y$ is cyclically reduced in T^*. Let W_* be the subset of T^* defined in Theorem 4.2 and define

$W' = \psi^{-1}(W_*)$. From the proof of Lemma 4.3, $\psi(v)$ is conjugate to an element dz^m, say $\psi(v)^h = dz^m$, for some $z \in W_*$, $d \in C$, $h \in T^*$, with $C_{T^*}(\psi(v))^h = C_{T^*}(z) = \langle z \rangle \times O_{T^*}(z)$ and $O_{T^*}(z) \leq C$. As both $\psi(v)$ and dz^m are cyclically reduced, dz^m may be obtained from $\psi(v)$ by cyclic permutation followed by conjugation by an element of C. Hence h and dz^m are in $\psi(T)$, and as $d \in C$ we have $z^m \in \psi(T)$ whence, from the argument verifying $(\mathcal{C}3)$, $z \in \psi(T)$. Therefore $C_{T^*}(z) \leq \psi(T)$ and $C_{\psi(T)}(z) = C_{T^*}(z) \cap \psi(T) = C_{\psi(T)}(z)$. As $T^* \in \mathcal{C}$ both $(\mathcal{C}5)$ and $(\mathcal{C}6)$ hold for $z \in T^*$ and hence in $\psi(T)$. Setting $w = \psi^{-1}(z)$ and $g = \psi^{-1}(h)$ we have $C_T(v)^g = C_T(w)$, with $w \in W'$, and $(\mathcal{C}5)$ and $(\mathcal{C}6)$ hold for $w \in T$, as ψ is injective.

To complete the proof we show that $(\mathcal{C}5)$(iii) holds when $v \in O_G(u)$, so $C_T(v)$ is non-abelian. Then $\psi(v) = v$ so $C_{T^*}(v)$ is non-abelian and $C_{T^*}(v) = \mathrm{Z}_G(v) \times O_{T^*}(v)$, while $C_{\psi(T)}(v) = \psi(\mathrm{Z}_G(v) \times \langle O_G(u), \mathbf{x}\rangle) = \mathrm{Z}_G(v) \times \langle O_G(u), \rho(\mathbf{x})\rangle$. As $C_{\psi(T)}(v) = C_{T^*}(v) \cap \psi(T)$ and $\mathrm{Z}_G(u) \leq \psi(T)$ this implies $\langle O_G(u), \rho(\mathbf{x})\rangle = O_{T^*}(v) \cap \psi(T)$. If $a \in O_T(v)$, then $b = \psi(a) \in O_{T^*}(v) \cap \psi(T)$, so there exists $w_0 \in W$ and $h \in T^*$, both in $O_{T^*}(v)$, such that $C_{T^*}(b) = C_{T^*}(w_0)^h$. From the above descriptions of centralisers, we also have w_0 and h in $\psi(T)$, whence $w_0, h \in \langle O_G(u), \rho(\mathbf{x})\rangle$. Therefore $w_1 = \psi^{-1}(w_0)$ is in W_T and $h_1 = \psi^{-1}(h)$ is in T, both w_1 and h_1 belong to $O_T(v)$ and $C_T(a) = C_T(w_1)^{h_1}$. ☐

Theorem 4.5. *If G_i in $\mathcal{C}(X_i, W_i)$ is a family of groups, then the free product $*G_i$ is in $\mathcal{C}(X, W)$, where $X = \bigcup_i X_i$ and $W = \bigcup_i W_i \cup W_*$, where W_* is the set of all cyclically reduced root elements of $*G_i$.*

If G is in $\mathcal{C}(X_G, W_G)$ and A is a free abelian group with basis X_A, then the direct product $H = G \times A$ is in $\mathcal{C}(X_G \cup X_A, W_G)$.

If $\{A_v\}_{v \in V(\Gamma)}$ is a family of free abelian groups and Δ is a chordal graph, then the graph product $\mathbb{G}(\Delta, \{A_v\})$ is in $\mathcal{C}$.

Proof: Let G_i be in $\mathcal{C}$ and let $G = *G_i$. Then $(\mathcal{C}1)$ and $(\mathcal{C}3)$ follow immediately. For finite families G_i, property $(\mathcal{C}2)$ follows from [**32**] and induction, and in general, if u and w are a pair of tuples, as in the definition of BP groups above, then the support of u and v is a finite subfamily of G_i, and the BP property follows since it holds for this finite subfamily. If $g \in G$ and g is in a conjugate of some G_i, then $(\mathcal{C}5)$–$(\mathcal{C}7)$ hold because they hold in G_i. If g is not in a conjugate of a factor, then g is conjugate to a unique cyclically reduced element g_1 and the centraliser of g is conjugate to $C_G(g_1) = \langle g_0 \rangle$, where $g_0 = \sqrt{g_1}$, so is abelian, and setting $Z(g_0) = \langle g_0 \rangle$ and $O(g_0) = \{1\}$, we see that $g_0 \in W$ and $(\mathcal{C}5)$–$(\mathcal{C}7)$ hold.

Conditions ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3$) for H follow immediately. As the generating set for $G \times A$ is $X_G \cup X_A$, property ($\mathcal{C}4$) lifts from G and A to H. For $g \in G$ and $a \in A$, we have $C_H(ga) = C_G(g) \times A = C_H(g)$. If $g \in W_G = W_H$, we set $Z_H(g) = Z_G(g)$ and $O_H(g) = O_G(g) \times A$ and ($\mathcal{C}5$), ($\mathcal{C}6$), and ($\mathcal{C}7$) follow.

Let X_v be a basis for A_v (not necessarily finite), for each vertex v. Expanding each vertex v of Δ to a complete subgraph on $|X_v|$ we may regard G as a RAAG defined by a chordal graph. (In more detail, for each vertex v of Δ let K_v be a complete graph with vertex set X_v. Let Δ' be the graph obtained from the disjoint union of the K_v, over $v \in V(\Delta)$, by adding an edge joining $x \in X_u$ to $y \in X_v$ for all vertices $u \neq v$ of Δ and all $x \in X_u$, $y \in X_v$.) From Lemma 3.8, G is then in $\mathcal{C}$. □

Proposition 4.6. *Let $\{G_i, \lambda_j^i\}$ be a direct system of groups and monomorphisms, indexed by a set I, where λ_j^i maps G_i to G_j, for $i \leq j$, and let $\overline{G}$ be the direct limit of this system. Assume that*

(i) *G_i is in $\mathcal{C}(X_i, W_i)$ and $\lambda_j^i(X_i) \subseteq X_j$, for all $i, j \in I$; and*
(ii) *for all $j \in I$ and $g \in G_j$ there exists $i \geq j$ such that, for all $k \geq i$, the centraliser $C_{G_k}(\lambda_k^j(g)) = \lambda_k^i(C_{G_i}(\lambda_i^j(g))$. In other words, centralisers of elements of the G_i are eventually stable.*

Let ψ_i be the canonical homomorphism from G_i to $\overline{G}$, let $\overline{X} = \bigcup_{i \in I} \psi_i(X_i) \subseteq \overline{G}$ and let $\overline{W} = \bigcup_{i \in I} \psi_i(W_i) \subseteq \overline{G}$. Then $\overline{G}$ is in $\mathcal{C}(\overline{X}, \overline{W})$.

Proof: From the definition, $\overline{G}$ is generated by $\overline{X}$. If $g \in \overline{G}$, then (taking a representative on its equivalence class we may assume that) $g \in G_j$, for some $j \in I$. Moreover, j may be chosen large enough so that every element involved in ($\mathcal{C}1$) and ($\mathcal{C}2$) is also in G_j. As ($\mathcal{C}1$) and ($\mathcal{C}2$) hold in $\lambda_j^i(G_j)$, for all $i \geq j$, they also hold in $\overline{G}$. Similarly, if $Y, Y' \subseteq \overline{X}$ satisfy the conditions of ($\mathcal{C}4$)(i) and $y \in \langle Y \rangle \times \langle Y' \rangle$, then there exist finite subsets $Y_0 \subseteq Y$ and $Y_0' \subseteq Y'$ such that $y \in \langle Y_0 \rangle \times \langle Y_0' \rangle$. Choosing j large enough to contain Y_0 and Y_0' it follows, from the fact that G_j is in $\mathcal{C}$, that $y = 1$ in G_j. Hence ($\mathcal{C}4$)(i) holds. A similar argument shows that ($\mathcal{C}4$)(ii) and ($\mathcal{C}5$)(i) hold. Moreover if $x \in G_j$ and $\psi_i(x) = \psi_j(x')$, for some j and $x' \in G_j$, then there exists k in I such that $k \geq i, j$ and so $\lambda_k^i(x) = \lambda_k^j(x')$. Hence if x has a unique root y in G_j, then $\lambda_k^i(x)$ and $\lambda_k^j(x')$ both have a unique root $\lambda_k^i(y)$ in G_k; and $\psi(y)$ is the unique root of $\psi(x)$ in $\overline{G}$; giving ($\mathcal{C}3$).

If $g \in G_i$, then we may assume that i has been chosen large enough so that, for all $k \geq i$, we have $C_{G_k}(\lambda_k^i(g)) = \lambda_k^i(C_{G_i}(g))$. As ($\mathcal{C}5$)(ii), ($\mathcal{C}5$)(iii), ($\mathcal{C}6$), and ($\mathcal{C}7$) hold in G_i it follows that they hold in $\overline{G}$. Therefore $\overline{G} \in \mathcal{C}$. □

The following definition is taken from [**40**], to which the reader is referred for further details.

Definition 4.7. Let $\mathfrak{C} = \{C_i = C_G(g_i)\}_{i\in I}$ be a set of centralisers in the group G and let $\phi_i\colon C_i \to H_i$ be a monomorphism from C_i to a group H_i, such that $\phi_i(g_i) \le Z(H_i)$, for all $i \in I$. Let T be the tree with vertices v, v_i, and directed edges $e_i = (v, v_i)$, from v to v_i, for $i \in I$. Then there is a graph of groups with vertex groups $G(v) = G$, $G(v_i) = H_i$; edge groups $G(e_i) = C_i$; and with edge maps the inclusions of C_i into G, and ϕ_i from C_i into H_i, for all $i \in I$. The fundamental group of this graph of groups is called a *tree extension of centralisers* and is denoted by $G(\mathfrak{C}, \mathcal{H}, \Phi)$, where $\mathcal{H} = \{H_i : i \in I\}$ and $\Phi = \{\phi_i : i \in I\}$.

Proposition 4.8. *Let G be in $\mathcal{C}(X, W)$ and let $\mathfrak{C} = \{C_i : i \in I\}$ be a set of abelian centralisers of G satisfying the following conditions.*

(i) *For all $i \in I$, there is $g_i \in W$ such that $C_i = C_G(g_i)$; and if $g \in W$, $C_G(g) = C_i$ and $i \ne j$, then $g \notin C_j$; and*
(ii) *no two centralisers of $\mathfrak{C}$ are conjugate.*

For each i in I, let H_i be a free abelian group and $\phi_i\colon C_i \to H_i$ be a monomorphism from C_i to H_i, such that $H_i = \phi(C_i) \times K_i$, for some subgroup K_i of H_i. Then, in the notation of Definition 4.7, the group $G(\mathfrak{C}, \mathcal{H}, \Phi)$ is in $\mathcal{C}$. Moreover, $G(\mathfrak{C}, \mathcal{H}, \Phi)$ is discriminated by G.

Proof: Choose a well-ordering of the set I, let $G_0 = G$ (assuming $0 \notin I$ and is taken to be less than the least element of I), and recursively define groups $G_i \in \mathcal{C}$ and monomorphisms λ_i^j, for all $i \in I$ and $j < i$, as follows. For a fixed successor ordinal $i \in I$ and all $j \le i$ assume that groups G_j in $\mathcal{C}(X_j, W_j)$ have been defined, and for $j \le k \le i$, monomorphisms $\lambda_k^j\colon G_j \to G_k$ have been defined, such that $\{\lambda_k^j : j < k < i\}$ forms a direct system of monomorphisms (that is, $\lambda_l^k \circ \lambda_k^j = \lambda_l^j$, with λ_j^j the identity). Assume further that, for all $j \le i$ and $m > j$, the centraliser $C_{G_j}(\lambda_j^0(g_m)) = \lambda_j^0(C_m)$; and that conditions (i) and (ii) hold for $\mathfrak{C}_j = \{\lambda_j^0(C_m) : m > j\}$, with $\mathcal{C}(X_j, W_j)$ in place of $\mathcal{C}(X, W)$. Finally, we assume that for $g \in W$ with $C_G(g) = C_m$, we have $\lambda_j^0(g) \in W_j$, for all $j \le i$ and $j < m$. By abuse of notation we let ϕ_m denote the map from $\lambda_j^0(C_m)$ to H_m, given by mapping $\lambda_j^0(c)$ to $\phi_m(c)$, for all $c \in \lambda_j^0(C_m)$, where $j \le i$ and $j < m$.

Now define $G_{i+1} = G_i *_{\phi_{i+1}} H_i$, noting that $\lambda_i^0(C_{i+1}) \le \lambda_i^0(G_0) \le G_i$ and that $\lambda_i^0(C_{i+1}) = C_{G_i}(\lambda_i^0(g_{i+1}))$. Also, from the definitions, $H_{i+1} = \phi_{i+1}(\lambda_i^0(C_{i+1})) \times K_{i+1}$. Define λ_{i+1}^i to be the canonical embedding of G_i in G_{i+1}. From Theorem 4.2, G_{i+1} is in $\mathcal{C}(X_{i+1}, W_{i+1})$, where $X_{i+1} =$

$\lambda^i_{i+1}(X_i) \cup X'_{i+1}$ and $W_{i+1} = \lambda^i_{i+1}(W_i) \cup K_{i+1} \cup W'_{i+1}$, with K_{i+1} as in Definition 4.7 and X'_{i+1} and W'_{i+1} defined as X_A and W_*, respectively, in the statement of Theorem 4.2. For $j < i$ define $\lambda^j_{i+1} = \lambda^j_i \circ \lambda^i_{i+1}$. For $m > i+1$, as $\lambda^0_i(g_m) \in G_i \backslash \lambda^0_i(C_i)$ it follows, as in the proof of Theorem 4.2, that

$$C_{G_{i+1}}(\lambda^0_i(g_m)) = \lambda^i_{i+1}(C_{G_i}(\lambda^0_i(g_m))) = \lambda^i_{i+1}(\lambda^0_i(C_m)) = \lambda^0_{i+1}(C_m).$$

Also, $\lambda^0_{i+1}(g_m) = \lambda^i_{i+1} \circ \lambda^0_i(g_m) \in \lambda^i_{i+1}(W_i) \subseteq W_{i+1}$. Therefore all assumptions on i above hold for $i+1$.

If $l \in I$ is a limit ordinal, assume that groups G_j in $\mathcal{C}(X_j, W_j)$ and monomorphisms λ^j_k have been defined, and satisfy the properties above, for all $j \le k < l$. Let $L = \{j \in I : j < l\}$, let $\overline{G}_l = \varinjlim G_j$, where the limit is over the direct system L, and let ψ_j be the canonical map from G_j to $\overline{G}_l$, for all $j \in L$. As before, we may assume that ϕ_l maps $\psi_0(c)$ to $\phi_l(c) \in H_l$, for all $c \in C_l$, and define $G_l = \overline{G}_l *_{\phi_l} H_l$. We must check that G_l is in $\mathcal{C}$. As G_l is formed from $\overline{G}_l$ by extension of centralisers, in the light of Theorem 4.2, it suffices to show that $\overline{G}_l$ is in $\mathcal{C}$. In fact, from Proposition 4.6, $\overline{G}_l = G(\overline{X}_l, \overline{W}_l)$, where $\overline{X}_l$ and $\overline{W}_l$ are the subsets $\overline{X}_l = \bigcup_{j<l} \psi_j(X_j)$ and $\overline{W}_l = \bigcup_{j<l} \psi_j(W_j)$ of $\overline{G}_l$.

Set $X_l = \overline{X}_l \cup X'_l$ and $W_l = \overline{W}_l \cup K_l \cup W'_l$, both subsets of $G'_l * H_l$. From Theorem 4.2, G_l is in $\mathcal{C}(X_l, W_l)$. As in the case of successor ordinals above, the assumptions made for $j < l$ are now seen to hold also for l. Therefore, by induction the direct limit $\varinjlim\{G_i : i \in I\} = G(\mathfrak{C}, \mathcal{H}, \Phi)$ is in $\mathcal{C}$.

For the final statement, assume that $\{\partial_d : d \in D\}$ is a discriminating family of homomorphisms for G_i by G_0. From Theorem 4.2, G_{i+1} is discriminated by G_i via a family $\{\lambda_{i,m} : (i,m) \in I \times \mathbb{N}\}$. Then G_{i+1} is discriminated by G_0 via the family $\{\partial_d \circ \lambda_{i,m} : d \in D, (i,m) \in I \times \mathbb{N}\}$.

Now suppose l is a limit ordinal and, for all $j < l$, the family $\{\partial_{j,d} : d \in D_j\}$ is discriminating for G_j by G_0. Any finite set of elements of the direct limit $\overline{G}$ of the G_j may be represented by elements of G_j, for some fixed $j < l$. It follows that $\overline{G}$ is discriminated by G_0 via the family $\bigcup_{j<l}\{\partial_{j,d} : d \in D_j\}$. Now it follows from the first part of the proof that G_l is discriminated by G_0. Therefore $G(\mathfrak{C}, \mathcal{H}, \Phi)$ is discriminated by $G = G_0$, as required. □

Remark 4.9. From the proof, given a well-ordering of I, there is a direct system $\{G_i\}_{i \in I}$ of groups such that $G_i \in \mathcal{C}$, for all $i \in I$, and $G(\mathfrak{C}, \mathcal{H}, \Phi) = \varinjlim\{G_i : i \in I\}$. Consequently, the properties of tree extensions of centralisers are similar to those of ordinary extensions of centralisers. That is, there exist canonical embeddings of G and H_i

into $G(\mathfrak{C}, H, \Phi)$; the centraliser of g_i in $G(\mathfrak{C}, H, \Phi)$ contains the group H_i and the group $G(\mathfrak{C}, H, \Phi)$ has the corresponding universal property. Moreover, groups $G(\mathfrak{C}, H, \Phi)$ for different well-orderings of I have the same universal property, so are isomorphic.

5. Exponential groups

Following [**39, 40**], where further detail may be found, we define exponential groups as follows.

Definition 5.1. Let A be an arbitrary associative ring with identity and let G be a group. Fix a map from $G \times A$ to G and write the image of (g, α) as g^α. Consider the following axioms:

(i) $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;
(ii) $g^{\alpha+\beta} = g^\alpha g^\beta$; $g^{\alpha\beta} = (g^\alpha)^\beta$;
(iii) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
(iv) $[g, h] = 1 \Rightarrow (gh)^\alpha = g^\alpha h^\alpha$.

Groups that admit a map $G \times A \to G$ satisfying axioms (i) to (iv) are called A-groups.

The class of A-groups over arbitrary associative rings is referred to as the class of *exponential* groups. Every group is a $\mathbb{Z}$-group, and an abelian group which is an A-group is, by definition, an A-module.

Given A-groups G and H, a homomorphism $f\colon G \to H$ is called an A-*homomorphism* if $f(g^\alpha) = (f(g))^\alpha$, for all $g \in G$ and $\alpha \in A$. A subgroup $H \leq G$ is called an A-*subgroup* if $h^\alpha \in H$, for all $h \in H$ and $\alpha \in A$. It follows that an intersection of A-subgroups is an A-subgroup. The A-subgroup A-*generated* by a subset X of G, written $\langle X\rangle_A$, is defined as the smallest A-subgroup of G containing X.

A basic operation in the class of exponential groups is that of A-completion. Here we give a particular case of this construction (see [**39**] for the general definition). Later on we always assume that the ring A and its subring A_0 have a common identity element.

Definition 5.2. Let A be a ring, A_0 a subring of A, and G an A_0-group. Then an A-group G^A is called an (A_0, A)-*completion* of the group G if G^A satisfies the following universal property.

(i) There exists an A_0-homomorphism $\tau\colon G \to G^A$ such that $\tau(G)$ A-generates G^A: that is, $\langle \tau(G)\rangle_A = G^A$; and
(ii) for any A-group H and an A_0-homomorphism $\varphi\colon G \to H$ there exists a unique A-homomorphism $\psi\colon G^A \to H$ such that $\psi \circ \tau = \varphi$.

As every group is a $\mathbb{Z}$-group, one can consider $(\mathbb{Z}, A)$-completions of arbitrary groups for each ring of characteristic zero, i.e. $\mathbb{Z} \leq A$. In practice, our use of (A_0, A)-completions will be restricted to the case where $A_0 = \mathbb{Z}$, in which case we omit A_0 from the notation and refer to the $(\mathbb{Z}, A)$-completion simply as the A-completion of G.

If G is an abelian A_0-group, then the group G^A is also abelian, i.e. it is an A-module. In this case G^A satisfies the same universal property as the tensor product $G \otimes_{A_0} A$ of the A_0-module G and the ring A. Therefore $G^A \simeq G \otimes_{A_0} A$. In particular, if M is an A-module, then $M^A \cong M$, as A-modules.

The A-completion of a coherent RAAG G will be constructed by defining an A-action on successively larger subsets of G, and necessarily involves groups in which A acts on some, but not all, elements. This brings us to the following definition.

Definition 5.3. A group G is called a *partial A-group*, for an associative ring A (with identity 1) if there exists a subset P of $G \times A$, such that g^α is defined whenever $(g, \alpha) \in P$, and axioms (i) to (iv) in Definition 5.1 hold whenever the arguments belong to P.

In this case we say the partial A-action is *defined on* P. Let H, G be partial A-groups with A-actions defined on subsets P_H and P_G, respectively. A homomorphism of groups $\phi\colon G \to H$ is called a partial A-homomorphism if $(g^a)^\phi$ is in P_H and $(g^a)^\phi = (g^\phi)^a$ for all pairs $(g, a) \in P_G$. We say that X is a *partial A-generating* set for the partial A-group G if G is generated (as a group) by $\{x^a \mid (x, a) \in P_G\}$.

When the sets on which A-actions are defined are clear from the context, no explicit reference to the sets P_G and P_H will be made. In particular, when H is an A-group (the A-action is defined everywhere) it is always assumed that a partial A-homomorphism from H to G is defined with respect to $P_H = H \times A$.

Definition 5.4. Let G be a partial A-group with action defined on $P \subset G \times A$. We say that an A-group G^A is an *A-completion* of G, with respect to P, if G^A satisfies the following universal property:

(i) there exists a partial A-homomorphism $\tau\colon G \to G^A$ such that $\tau(G)$ is an A-generating set for G^A; and
(ii) given an A-group H and a partial A-homomorphism $\varphi\colon G \to H$ there exists a A-homomorphism $\psi\colon G^A \to H$ such that $\tau \circ \psi = \varphi$.

In particular, if A_0 is a subring of A, then any A_0-group is also a partial A-group and an (A_0, A)-completion is a partial A-completion of G, with respect to (G, A_0).

Theorem 5.5 ([**39**]). *Let G be a partial A-group, with action defined on the set P. Then there exists an A-completion G^A of G, with respect to P, and it is unique up to A-isomorphism.*

(The version of this theorem proved in [**39**] is restricted to the case of an (A_0, A)-completion of G, while the proof of the general case, which follows through almost word for word, is left to the reader.)

It is shown in [**2**] that the operation of A-completion commutes with taking direct sums and forming direct limits of directed systems of partial A-groups.

Definition 5.6. A partial A-group G is *faithful* (over A) if the partial A-homomorphism $\tau\colon G \to G^A$ is injective.

From the definition, it follows that a partial A-group G is faithful over A if and only if there is an injective partial A-homomorphism from G into an A-group.

Given a partial A-group G and a subgroup H of G, we say that H is a *full* A-subgroup when H is an A-subgroup: that is, h^α is defined and belongs to H, for all $h \in H$ and $\alpha \in A$. If H is a full A-subgroup of a partial A-group G, then H is necessarily faithful and $H^A = \tau(H)$.

6. A-completion of groups in $\mathcal{C}$

6.1. A-completion of abelian groups in $\mathcal{C}$. In general, the construction of the A-completion of an abelian partial A-group M is very similar to tensor multiplication by the ring A. (As we shall see, it is only necessary to make the existing partial action of A on M agree with the right action of A on $M \otimes_{\mathbb{Z}} A$.)

Let M be a partial right A-module (i.e. a partial abelian A-group with action defined on $P \subset G \times A$). Consider the tensor product $M \otimes_{\mathbb{Z}} A$ of the abelian group M by the ring A over $\mathbb{Z}$. The A-completion M^A of M may be obtained by factorising $M \otimes_{\mathbb{Z}} A$ by the right A-submodule generated by the set of all elements $(x^\alpha \otimes 1) - (x \otimes \alpha)$, for $(x, \alpha) \in P$, and defining $\tau(x)$ to be the image of $x \otimes 1$ in the quotient. The following proposition is a copy of the corresponding result from the theory of modules.

Proposition 6.1. *Let M be a torsion-free abelian partial A-group, with action defined on the set P, where A is a unitary associative ring with a torsion-free additive group. Assume that for all $x \in M$, $\alpha \in A$, and non-zero $n \in \mathbb{Z}$,*

$$\text{if } (x^n, \alpha) \in P, \text{ then } (x, \alpha) \in P. \tag{6}$$

Then M is A-faithful, M^A is a torsion-free abelian A-group, and $\tau(M)$ is a direct summand of M^A.

Proof: As a torsion-free abelian group is a direct limit of finitely generated subgroups, we may restrict attention to finitely generated free abelian groups, and so to infinite cyclic groups. If $M = \langle x \rangle$ is infinite cyclic, then the condition on P implies that either $P = \langle x \rangle \times A$ or $P = \emptyset$. In the first case $M = M^A$ and in the second case $M^A = M \otimes_{\mathbb{Z}} A$. $\square$

6.2. *A*-completion of non-abelian groups in $\mathcal{C}$. From now on all rings are associative, with a free abelian additive subgroup and a multiplicative identity 1. For such a ring A, by $\mathbb{Z} \subseteq A$ we mean the characteristic subring of A, and we always assume that a basis contains the element 1; so $A = \mathbb{Z} \times A'$, for some subring A'.

Let A be a ring and let G be a group satisfying condition (R), namely:

(R) $G \in \mathcal{C}$ is a partial A-group, such that for all $g \in G$;
- (i) if $C_G(g)$ is non abelian, and $C(g) = \mathrm{Z}(g) \times O(g)$, then the centre $Z(C(g))$ of $C(g)$ is a full A-subgroup (so A-faithful) and
- (ii) if $C_G(g)$ is abelian, then condition (6) holds for all $x \in C_G(g)$, $\alpha \in A$, and $n \in \mathbb{Z}$ $(n \neq 0)$.

In this subsection we describe the A-completion of such a group G. Assume then that G satisfies condition (R). Zorn's lemma guarantees the existence of a set $\mathfrak{C} = \{C_i : i \in I\}$, of centralisers C_i of elements of G, which satisfies the following conditions (S).

(S1) Any $C \in \mathfrak{C}$ is abelian but not a full A-subgroup.

(S2) No two centralisers from $\mathfrak{C}$ are conjugate.

Note that if $C \neq C' \in \mathfrak{C}$, then $C \cap C'^g$ is an A-module, for all $g \in G$. Indeed, assume that $C = C(a)$, $C'^g = C(b)$, and $x \in C(a) \cap C(b)$. If $[a, b] = 1$, that is, $a \in C(b)$ and $b \in C(a)$, then since $C(a)$ and $C(b)$ are abelian, we have that $C(a) < C(b)$ and $C(b) < C(a)$ and so $C(a) = C(b)$. If $[a, b] \neq 1$, then $C(x)$ is non-abelian and so by assumption $Z(C(x))$ is an A-subgroup. In particular x^α is defined and in $Z(C(x))$ and $Z(C(x)) \leq C(a) \cap C(b)$, for all $\alpha \in A$.

(S3) Any abelian centraliser in G which is not a full A-subgroup is conjugate to a centraliser in $\mathfrak{C}$.

Given $C_i \in \mathfrak{C}$, since by assumption C_i is abelian, from condition (R)(ii) and Proposition 6.1, we have $C_i^A = C_i \otimes_{\mathbb{Z}} A$, for all $i \in I$, and we can form the set $\mathcal{H}_A = \{C_i \otimes_{\mathbb{Z}} A \mid i \in I\}$ and the set of canonical embeddings $\Phi_A = \{\varphi_i : C_i \hookrightarrow C_i \otimes_{\mathbb{Z}} A \mid i \in I\}$. As C_i is a direct summand of $C_i \otimes_{\mathbb{Z}} A$, we may consider the tree extension of centralisers G^*, of this special type:

$$G^* = G(\mathfrak{C}, \mathcal{H}_A, \Phi_A).$$

If G^* satisfies condition (R), then we can iterate this construction up to level ω:

$$G = G^{(0)} < G^{(1)} < G^{(2)} < \cdots < G^{(n)} < \cdots, \tag{7}$$

where $G^{(n+1)} = G^{(n)}(\mathfrak{C}_n, \mathcal{H}_{A,n}, \Phi_{A,n})$, and the set $\mathfrak{C}_n$ of centralisers in the group $G^{(n)}$ satisfies conditions (S).

Definition 6.2. The union $\bigcup_{n\in\omega} G^{(n)}$ of the chain (7) is called an *iterated centraliser extension* (ICE for short) of G by the ring A. (If $A = \mathbb{Z}[t]$, we refer simply to an *ICE* of G.)

Theorem 6.3 (*cf.* [**40**, Theorem 8])**.** *Let A be a ring (as at the beginning of this subsection) and let G be a non-abelian partial A-group satisfying condition* (R)*. If all abelian centralisers of G are faithful over A, then the A-completion G^A of G by A is an ICE of G by A. Furthermore G^A is in $\mathcal{C}$ and is discriminated by G.*

To prove this theorem, we shall show that the tree extension of centralisers $G^* = G(\mathfrak{C}, A)$, defined above, satisfies condition (R) and has an appropriate universal property. This will allow us to conclude the same for all the groups $G^{(n)}$ from the chain (7) and then in turn to prove the theorem.

Lemma 6.4. *Assume that G satisfies condition* (R)*, that $\mathfrak{C} = \{C_i : i \in I\}$ is a set of centralisers of elements of G which satisfies* (S)*, and that $G^* = G(\mathfrak{C}, \mathcal{H}_A, \Phi_A)$. For any $g \in G$,*

- (i) *if the centraliser of $g \in G$ is abelian, then either*
  - (a) *$C_G(g)$ is not conjugate to an element of $\mathfrak{C}$, in which case $C_{G^*}(g) \cong C_G(g)$, or*
  - (b) *$C_G(g)$ is conjugate to an element of $\mathfrak{C}$, in which case $C_{G^*}(g) \cong C_G(g) \otimes A = C_G(g)^A$.*

  In both cases $C_{G^*}(g)$ is an A-module. Otherwise,
- (ii) *the centraliser of g is non-abelian and $C_{G^*}(g) = \mathrm{Z}_G(g) \times O_G(g)$, $\mathrm{Z}(g) = \mathrm{Z}_G(g)$, $g \in \mathrm{Z}(g) \le Z(C(g))$, and $Z(C(g))$ is a full A-subgroup.*

Proof: As in the remark following the proof of Proposition 4.8, there is a directed system $\{G_i\}_{i\in I}$ of groups, with $G = G_0$, such that $G_i \in \mathcal{C}$, for all $i \in I$, and $G(\mathfrak{C}, \mathcal{H}_A, \Phi_A) = \varinjlim\{G_i : i \in I\}$. If the centraliser $C_{G_0}(g)$ is not conjugate to an element of $\mathfrak{C}$, assume that, for some $i \ge 0$, $C_{G_i}(g) \cong C_{G_0}(g)$. Then, as in the proof of Proposition 4.8, $C_{G_{i+1}}(g) \cong C_{G_0}(g)$. If l is a limit ordinal, then the standard argument shows that, if $C_{G_j}(g) \cong C_{G_0}(g)$, for all $j < l$, then again $C_{G_l}(g) \cong C_{G_0}(g)$. Hence, if $C_G(g)$ is not conjugate to an element of $\mathfrak{C}$, then, for all $i \in I$, $C_{G_i}(g) \cong C_{G_0}(g)$, so $C_{G^*}(g) \cong C_G(g)$. In this case if $C_G(g)$ is abelian, so by definition of $\mathfrak{C}$ a full A-subgroup, then this isomorphism induces an A-action on $C_{G^*}(g)$, making it into an A-subgroup. If $C_G(g)$ is non-abelian, then $C_G(g) = \mathrm{Z}_G(g) \times O_G(g)$ and as $Z(C_G(g))$ is a full A-subgroup, so is its isomorphic image $Z(C_{G^*}(g))$.

On the other hand suppose $C_G(g)$ is conjugate to an element C_i of $\mathfrak{C}$. Then, $C_{G_i}(g)$ is conjugate to $C_i \otimes A$ and as no two elements of $\mathfrak{C}$ are conjugate we may unambiguously extend the action of A on $C_i \otimes A$ to $C_{G_i}(g)$. As above, for all $m > i$, we have $C_{G_m}(g)$ isomorphic to $G_{G_i}(g)$ under the map λ_m^i, which from the definitions is an A-homomorphism. Hence $C_{G^*}(g) \cong G_{G_i}(g)$, which is an abelian A-subgroup. $\square$

Lemma 6.5. *G^* is a partial A-group and G is a full A-subgroup of G^*. Moreover, G^* is A-generated by G.*

Proof: By construction G^* is a partial A-group. Let $g \in G \leq G^*$. If $C_G(g)$ is abelian, then, from Lemma 6.4, $C(g)$ is a full A-subgroup. If $C_G(g)$ is non-abelian, then $g \in Z(C(g))$, which is also a full A-subgroup. In both cases the action of A on g is defined. As C_i^A is A-generated by C_i, for all $i \in I$, it follows that G^* is A-generated by G. $\square$

Lemma 6.6. *G^* satisfies condition* (R). *In particular G^* is in $\mathcal{C}$.*

Proof: That G^* is in $\mathcal{C}$ follows from Proposition 4.8. To see that the second condition of Proposition 4.8(i) holds note that if $w \in W$ (where G is in $\mathcal{C}(X, W)$) and $C_G(w)$ is abelian, then $C(w)$ is not conjugate to $C(w')$, for all other $w' \in W$. Hence if $w' \in W$ and $w' \in C_G(w)$, then $w = w'$. That G^* is a partial A-group satisfying the appropriate conditions on centralisers follows from Lemmas 6.4 and 6.5. $\square$

Lemma 6.7. *The group G^* has the following universal property with respect to the canonical embedding $\tau \colon G \hookrightarrow G^*$. For any A-group H and any partial A-homomorphism $f \colon G \to H$ there exists a partial A-homomorphism $f^* \colon G^* \to H$ such that $f = f^* \circ \tau$.*

Proof: Write $f = f_0$ and $G = G_0$ and, in the notation of the proof of Proposition 4.8, let $0 \leq i \in I$ and assume that for all $0 \leq j \leq i$, we have A-homomorphisms $f_j \colon G_j \to H$ such that $f_0 = f_j \circ \lambda_j^0$, where $f = f_0$. Assume in addition that, for all $j \leq k \leq i$, we have $f_j = f_k \circ \lambda_k^j$. Then $f_i(C_{i+1})$ is an abelian A-subgroup N_{i+1} of H. Using the universal property of A-completion for abelian groups, we can find a homomorphism $\Psi_{i+1} \colon C_{i+1} \otimes A \to N_{i+1}$ such that $f_i = \Psi_{i+1} \circ \tau_{i+1}$, where $\tau_{i+1} \colon C_{i+1} \hookrightarrow C_{i+1} \otimes A$ is the canonical embedding. By [**40**, Proposition 5], there exists an A-homomorphism $f_{i+1} \colon G_{i+1} \to H$ with the property $f_i = f_{i+1} \circ \lambda^i_{i+1}$. It then follows directly from the definitions that $f_j = f_k \circ \lambda^j_k$, for all $0 \le j \le k \le i+1$.

If l is a limit ordinal, the universal property of direct limits gives a homomorphism $\bar{f}_l \colon \overline{G}_l \to H$ such that $f_j = \bar{f}_l \circ \lambda^j_l$. Then using $\overline{G}_l$ and $\bar{f}_l$ in place of G_i and f_i, and G_l in place of G_{i+1}, the previous argument gives an A-homomorphism $f_l \colon G_l \to H$, making the necessary diagrams commute. The result now follows by induction. $\square$

Lemma 6.8. *G^* is discriminated by G.*

Proof: This follows from Proposition 4.8. $\square$

Proof of Theorem 6.3: Let $G^{(0)} = G$ and $G^{(n+1)} = (G^{(n)})^*$, as in the chain (7), and let $\overline{G} = \bigcup_{n \in \omega} G^{(n)}$. The claim is that the ICE $\overline{G}$ coincides with the A-completion G^A of G. Indeed, as a union of partial A-groups, $\overline{G}$ is a partial A-group, but an action of A on $\overline{G}$ is in fact defined everywhere: if $x \in \overline{G}$, then $x \in G^{(n)}$, and hence by Lemma 6.4 the action of A on x is defined in $G^{(n+1)}$. So $\overline{G}$ is an A-group. Similarly, from Lemma 6.5, $\overline{G}$ is A-generated by G.

To prove that $\overline{G}$ is the A-completion of the partial A-group G, it remains to verify the corresponding universal property. Let H be an A-group and $f \colon G \to H$ a partial A-homomorphism. Using Lemma 6.7, we can extend f to a partial A-homomorphism $f_n \colon G^{(n)} \to H$. By construction these homomorphisms commute with the canonical maps from $G^{(n)}$ into $G^{(m)}$, for $m \ge n$: that is, $f_n = f_m|_{G^{(n)}}$. Since $\overline{G}$ is the direct limit of the $G^{(n)}$ (with these inclusion maps) there exists a unique homomorphism $\bar{f} \colon \overline{G} \to H$ such that $f_n = \bar{f} \circ \psi_n$, where ψ_n is the canonical embedding of $G^{(n)}$ into $\overline{G}$. In particular $f = f_0 = \bar{f} \circ \psi_0$. As all the partial A-homomorphisms f_n restrict to A-homomorphisms on $G^{(n-1)}$, it follows that $\bar{f}$ is an A-homomorphism, so $\overline{G} = G^A$, as claimed.

Lemma 6.6 implies that $G^{(n)}$ is in $\mathcal{C}$, for all n. A given centraliser is extended at most once in the construction of chain (7), so condition (ii) of Proposition 4.6 holds, whence G^A is in $\mathcal{C}$. That G^A is discriminated by G follows using Lemma 6.8 to see that $G^{(n)}$ is discriminated by G, for all $n > 0$, and then by an argument similar to the last part of the proof of Lemma 6.8 to see that $\overline{G} = G^A$ is discriminated by G. $\square$

Applications. Toral relatively hyperbolic groups belong to the class $\mathcal{C}$ and trivially satisfy condition (R) (as partial A-groups with action defined on $G \times \mathbb{Z}$). Therefore, our results recover the results for torsion-free hyperbolic groups (as long as A is as at the beginning of Subsection 6.2), see [**7**], and for toral relatively hyperbolic groups; see [**29**].

Corollary 6.9 ([**29**]). *Let A be a ring (as at the beginning of Subsection 6.2) and let G be a torsion-free toral relatively hyperbolic group. Then the A-completion G^A of G by A is an ICE of G by A. Furthermore, G^A is in $\mathcal{C}$ and is discriminated by G.*

Our results can also be applied to coherent RAAGs. Given a coherent RAAG $\mathbb{G}$, we can view it as the graph product $\mathcal{G}(\Delta, \mathbb{Z})$, where Δ is a chordal graph. Let $G = \mathcal{G}(\Delta, A)$, where A is a ring of the type above. From Theorem 4.5 G is in $\mathcal{C}$ and it satisfies condition (R).

Lemma 6.10. *Let $G = \mathcal{G}(\Delta, A)$, where A is a ring (with the usual restrictions) and Δ a chordal graph. For all $g \in G$ such that $C(g)$ is non-abelian, we have that $C(g) = \mathrm{Z}(g) \times O(g)$, $\mathrm{Z}(g) \leq Z(C(g))$, and $Z(C(g))$ is an A-module.*

Proof: From Lemma 3.8, $C(g)$ is conjugate to $C(w) = \mathrm{Z}(w) \times O(w)$, for some cyclically reduced element w of the RAAG G. If $x \in Z(C(w)) \cap X$, then x belongs to X_v, for some vertex v of Γ, and basis X_v for the copy A_v of A associated to v. As $\langle X_v \rangle = A_v \cong \langle v \rangle \otimes A$ the subgroup A_v is an A-module, and from Lemma 2.3, $A_v \leq Z(C(w))$, so A acts on $Z(C(w))$, hence on $Z(C(g))$. □

Theorem 6.11. *In the notation above, the A-completion G^A of G by the ring A is an ICE of G over the ring A.*

Corollary 6.12. *Let $\mathbb{G}$ be a coherent RAAG. Then the A-completion $\mathbb{G}^A$ of $\mathbb{G}$ by the ring A is an ICE of the group G over the ring A, belongs to the class $\mathcal{C}$ and is discriminated by $\mathbb{G}$.*

Formally, the A-completion $\mathbb{G}^A$ of $\mathbb{G}$ is obtained in two steps: we first embed $\mathbb{G}$ into the graph product G to ensure property (R) and then we take an ICE of G over A. Abusing the terminology, we say that $\mathbb{G}^A$ is an ICE of $\mathbb{G}$ over A.

7. Coherence of limit groups over coherent RAAGs

In this section, we introduce graph towers over coherent RAAGs and prove that they are coherent; see Corollary 7.8. As a main corollary, we obtain that limit groups over coherent RAAGs are finitely presented; see Corollary 7.9. The proofs of this section are (almost) independent of the previous sections. More precisely, we only use (and prove) that graph towers belong to the class $\mathcal{C}$, in order to deduce a hierarchical decomposition of the tower as a graph of groups with abelian vertex groups; see Theorem 7.7. Coherence of graph towers is then deduced from that structural decomposition.

As we mentioned, we shall define (below) a graph tower $\mathfrak{T}$ of height l, over a RAAG $\mathbb{G} = \mathbb{G}(\Gamma)$, to be a sequence $\mathfrak{T}_0, \dots, \mathfrak{T}_l$ of triples: $\mathfrak{T}_l = (G_l, \mathbb{H}_l, \pi_l)$, where G_l is a group, $\mathbb{H}_l = \mathbb{G}(\Gamma_l)$, Γ_l, a simple graph, is the l-th RAAG of $\mathfrak{T}$, and π_l is an epimorphism of $\mathbb{H}_l$ onto G_l. The group G_l is called *the* group of $\mathfrak{T}_l$ and by abuse of notation is itself referred to as a graph tower over $\mathbb{G}$. The definition depends on a partition of the edges of Γ_l into disjoint subsets $E_c(\Gamma_l)$ and $E_d(\Gamma_l)$. Given this partition, we define a *co-irreducible* subgroup $\mathbb{K}$ of $\mathbb{H}_l$ to be a canonical parabolic subgroup which is closed (i.e. $\mathbb{K}^{\perp\perp} = \mathbb{K}$) and such that $\mathbb{K}^\perp$ is $E_d(\Gamma_l)$-directly indecomposable. As in [**15**], where more details are given, we make the following definition.

Definition 7.1. A *graph tower* $\mathfrak{T}$ of height l is a sequence of triples $\mathfrak{T}_0, \dots, \mathfrak{T}_l$, defined recursively, setting $\mathfrak{T}_0 = (G_0, \mathbb{H}_0, \pi_0)$, where $G_0 = \mathbb{H}_0 = \mathbb{G}$, $\Gamma_0 = \Gamma$, $\pi_0 = \mathrm{Id}_{\mathbb{G}}$, and $E_d(\Gamma) = E(\Gamma)$. In addition define the subset $S_0 = \emptyset$ of $\mathbb{H}_0 = \mathbb{G}(\Gamma_0)$. To define $\mathfrak{T}_l$ assume that $\mathfrak{T}_{l-1} = (G_{l-1}, \mathbb{H}_{l-1}, \pi_{l-1})$, Γ_{l-1}, $E_d(\Gamma_{l-1})$, $E_c(\Gamma_{l-1})$, and $S_{l-1} \subseteq \mathbb{H}_{l-1} = \mathbb{G}(\Gamma_{l-1})$ are defined, and choose a co-irreducible subgroup $\mathbb{K}_l$ of $\mathbb{H}_{l-1}$, canonically generated by a set Y_l; and a positive integer m_l. Let Γ_l be the graph with vertex set $V(\Gamma_{l-1}) \cup \{x_1^l, \dots, x_{m_l}^l\}$ (where the x_i^l are new symbols). The set of *d-edges* of Γ_l is $E_d(\Gamma_l) = E_d(\Gamma_{l-1}) \cup \{(x_i^l, y) : 1 \leq i \leq m_l,\ y \in Y_l\}$. The set of *c-edges* of Γ_l is $E_c(\Gamma_l) = E_c(\Gamma_{l-1}) \cup E_c'(\Gamma_l)$ and the set $S_l = S_{l-1} \cup S_l'$, where $E_c'(\Gamma_l)$ and S_l' are chosen according to one of the alternatives described below: *basic floor*, *abelian floor*, or *quadratic floor*. In all cases $\mathbb{H}_l = \mathbb{G}(\Gamma_l)$, $G_l = \mathbb{H}_l / \operatorname{ncl}\langle S_l \rangle$, and π_l is the canonical map from $\mathbb{H}_l$ to G_l.

Basic floor.

- $E_c'(\Gamma_l) = \emptyset$ if $\mathbb{K}_l^\perp$ is a directly indecomposable subgroup of $\mathbb{H}_{l-1}$; and
- $E_c'(\Gamma_l) = \{(x_i^l, x_j^l) : 1 \leq i < j \leq m_l\} \cup \{(x_i^l, y) : 1 \leq i \leq m_l,\ y \in Y_l^{\perp}\}$, if $\mathbb{K}_l^\perp$ is directly decomposable.

S_l' is the set of *basic relators*:

$$S_l' = [C, x_i^l], \quad 1 \leq i \leq m_l,$$

where $C = \pi_{l-1}^{-1}(C_{G_{l-1}}(\mathbb{K}_l^\perp)$.

Abelian floor.

- $E_c'(\Gamma_l) = \{(x_i^l, x_j^l) : 1 \leq i < j \leq m_l\}$, if $\mathbb{K}_l^\perp$ is a directly indecomposable subgroup of $\mathbb{H}_{l-1}$; and
- $E_c'(\Gamma_l) = \{(x_i^l, x_j^l) : 1 \leq i < j \leq m_l\} \cup \{(x_i^l, y) : 1 \leq i \leq m_l,\ y \in Y_l^{\perp}\}$, if $\mathbb{K}_l^\perp$ is directly decomposable.

Either

- $S'_l = [C, x_i^l]$, $1 \le i \le m_l$, where $C = \pi_{l-1}^{-1}(C_{G_{l-1}}(u))$ with u is a non-trivial cyclically reduced block root element of $\mathbb{K}_l^{\perp}$; or
- S'_l is the set of basic relators.

Quadratic floor. $E'_c(\Gamma_l)$ is defined as in the basic floor case.

S'_l consists of the set of basic relators and a relator W of the form either

- $W = \prod_{i=1}^{g}[x_{2i-1}, x_{2i}]\big(\prod_{i=2g+1}^{m} u_i^{x_i}\big)u_{m+1}$,
  with $u_{m+1}^{-1} = \prod_{i=1}^{g}[v_{2i-1}, v_{2i}]\prod_{i=2g+1}^{m} u_i^{w_i}$ or
- $W = \prod_{i=1}^{g} x_i^2\big(\prod_{i=g+1}^{m} u_i^{x_i}\big)u_{m+1}$, with $u_{m+1}^{-1} = \prod_{i=1}^{g} v_i^2 \prod_{i=g+1}^{m} u_i^{w_i}$,

for some u_i, v_j, w_k in $\mathbb{K}^{\perp}$ such that the following condition $\circledast$ holds.

$\circledast$ The Euler characteristic of W is at most -2 or $W = [x_1, x_2]u_3$, and in both cases the subgroup of G_{l-1} generated by the set of all u_i, v_j, w_k is non-abelian.

(In [**15**] in the definition of the quadratic floor the condition $\circledast$ is slightly less restrictive. However, with the simpler definition given here, all the properties of graph towers in [**15**] hold as before.)

We next recall some of the properties of the graph towers proved in [**15**].

Remark 7.2. From the definitions it follows that $\mathbb{H}_{l-1}$ is a retraction of $\mathbb{H}_l$ and that G_{l-1} is a retraction of $H_l/\operatorname{ncl}(S_{l-1})$. In [**15**, Theorem 7.1 and Theorem 8.1] it is shown that, given a limit group G over $\mathbb{G}$, there exists a graph tower $\mathfrak{T}_l$ over $\mathbb{G}$ and embedding of G into G_l (both of which may be effectively constructed if G is given by its presentation as the coordinate group of a system of equations) such that

1. G_l and $\mathbb{H}_l$ are discriminated by $\mathbb{G}$.
2. If $\mathbb{K}$ is a co-irreducible subgroup of $\mathbb{H}_l$, then $\mathbb{K}^{\perp}$ is either a directly indecomposable subgroup of $\mathbb{H}_l$ or $E_c(\Gamma_l)$-abelian ([**15**, Lemma 6.2]).
3. There is a discriminating family $\{\varphi_i\}$ of G_l by $\mathbb{G}$ such that, writing φ'_i for $\varphi_i\pi_l$, the following hold; see [**15**, Theorem 7.1].
   - For any edge $e = (x, y)$ of Γ_l, either x and y are sent to the same cyclic subgroup of $\mathbb{G}$, by all homomorphisms φ'_i, in which case e belongs to $E_c(\Gamma_l)$; or the images of x and y, under all homomorphisms φ'_i, disjointly commute, in which case e belongs to $E_d(\Gamma)$.
   - If $\mathbb{K}$ is a canonical parabolic subgroup of $\mathbb{H}_l$, then there exists a subgroup $\mathbb{G}_{\mathbb{K}}$ of $\mathbb{G}$ such that $\mathbb{G}_{\mathbb{K}} < \mathbb{K}$ and $\bigcup_i \varphi'_i(\mathbb{K}) = \mathbb{G}_{\mathbb{K}}$.

In particular, if $\mathbb{K}$ is a co-irreducible subgroup of $\mathbb{H}_l$, then $\mathbb{G}_{\mathbb{K}}$ is co-irreducible in $\mathbb{G}$ (here $E_d(\Gamma) = E(\Gamma)$). Furthermore, in this case, $\mathbb{G}_{\mathbb{K}}^{\perp} = \mathbb{G}_{\mathbb{K}^{\perp}}$ is a directly indecomposable subgroup of $\mathbb{G}$, $\bigcup_i \varphi_i'(\mathbb{K}^{\perp}) = \mathbb{G}_{\mathbb{K}^{\perp}}$, and $C_{\mathbb{G}}(\mathbb{G}_{\mathbb{K}^{\perp}}) = \mathbb{G}_{\mathbb{K}}$.

(In the terminology of [**15**], the homomorphisms φ_i all factor through the principal branch of the (tribal) Makanin–Razborov diagram for G.)

Definition 7.3. In the notation above, given a limit group G over a coherent RAAG $\mathbb{G}$, we call a graph tower G_l into which G embeds a *graph tower associated to* G over $\mathbb{G}$, and the family of discriminating homomorphisms $\{\varphi_i\}$, as described above, a *principal* discriminating family of homomorphisms.

We will use the following graph of groups decomposition of graph towers, obtained in [**15**].

Lemma 7.4 (cf. Lemma 5.3 in [**15**]). *A graph tower $\mathfrak{T}_l = (G_l, \mathbb{H}_l, \pi_l)$ of height l over $\mathbb{G}$ admits one of the following decompositions, where $\mathbb{K} = \mathbb{K}_l$.*

- (i1) $G_{l-1} *_{C_{G_{l-1}}(\mathbb{K}^{\perp})} (C_{G_{l-1}}(\mathbb{K}^{\perp}) \times \langle x_1^l, \dots, x_{m_l}^l \rangle)$ *(in this case the floor is basic and $\mathbb{K}^{\perp}$ is non-abelian);*
- (i2) $G_{l-1} *_{C_{G_{l-1}}(\mathbb{K}^{\perp})} (C_{G_{l-1}}(\mathbb{K}^{\perp}) \times \langle x_1^l, \dots, x_{m_l}^l \mid [x_i^l, x_j^l] = 1,\ 1 \le i, j \le m_l,\ i \neq j \rangle)$ *(in this case the floor is basic and $\mathbb{K}^{\perp}$ is abelian);*
- (ii1) $G_{l-1} *_{C_{G_{l-1}}(u)} (C_{G_{l-1}}(u) \times \langle x_1^l, \dots, x_{m_l}^l \mid [x_i^l, x_j^l] = 1,\ 1 \le i, j \le m_l,\ i \neq j \rangle)$ *(in this case the floor is abelian and u is non-trivial irreducible root element);*
- (ii2) $G_{l-1} *_{C_{G_{l-1}}(\mathbb{K}^{\perp})} (C_{G_{l-1}}(\mathbb{K}^{\perp}) \times \langle x_1^l, \dots, x_{m_l}^l \mid [x_i^l, x_j^l] = 1,\ 1 \le i, j \le m_l,\ i \neq j \rangle)$ *(in this case the floor is abelian and $\mathbb{K}^{\perp}$ is non-abelian);*
- (iii) $G_{l-1} *_{C_{G_{l-1}}(\mathbb{K}^{\perp}) \times \langle u_{2g+1}, \dots, u_m \rangle} (\langle u_{2g+1}, \dots, u_m, x_1^l, \dots, x_{m_l}^l \mid W \rangle \times C_{G_{l-1}}(\mathbb{K}^{\perp}))$ *(in this case the floor is quadratic and $\mathbb{K}^{\perp}$ is non-abelian).*

Note that, as $\mathbb{H}_l$ is discriminated by the principal family $\{\varphi_i\}$, the elements $u, u_i \in \mathbb{H}_l$ from cases (ii1) and (iii) are root block elements.

In the remainder of the section, to distinguish between orthogonal complements of parabolic subgroups in various $\mathbb{H}_j$, given a subset Y of the canonical generating set X of $\mathbb{H}_j$ we shall write $\operatorname{lk}_j(Y) = \{x \in X : x \notin Y \text{ and } [x, Y] = 1\}$ and $\langle Y \rangle^{\perp_j} = \langle \operatorname{lk}_j(Y) \rangle \le \mathbb{H}_j$. We shall also write C_j for C_{G_j}, $\mathbf{x}^j$ for $\{x_1^j, \dots, x_{m_j}^j\}$, and, in case (iii), $\mathbf{u}$ for $\{u_{2g+1}, \dots, u_m\}$.

Our next goal is to give a more precise description of a graph tower when the base group is a coherent RAAG. In particular, we show that the graph tower can be built from the coherent RAAG $\mathbb{G}$ by first building floors of type (i2) with a non-abelian centraliser and of type (ii2) – obtaining a graph tower which is a coherent RAAG $\mathbb{G}'$ (discriminated by $\mathbb{G}$); and then building floors over $\mathbb{G}'$ for which the corresponding amalgamation is over a free abelian subgroup; see Theorem 7.7.

Lemma 7.5. *Let $\mathbb{G}$ be a coherent RAAG and let $\mathfrak{T} = (G_l, \mathbb{H}_l, \pi_l)$ be a graph tower associated to a limit group G over $\mathbb{G}$. Let $\mathbb{K}$ be a canonical parabolic co-irreducible subgroup of $\mathbb{H}_l$. If $\mathbb{K}^\perp$ is non-abelian, then $C_{G_l}(\mathbb{K}^\perp)$ is finitely generated torsion-free abelian.*

Proof: By Remark 7.2, the graph tower G_l is $\mathbb{G}$-discriminated by the principal family of homomorphisms $\{\varphi_i'\}$ and, if $\mathbb{K}$ is co-irreducible, then $\varphi_i'(\mathbb{K}) < \mathbb{G}_\mathbb{K}$, $\mathbb{G}_\mathbb{K}$ is co-irreducible, and $\bigcup_i \varphi_i'(\mathbb{K}^\perp) = \mathbb{G}_\mathbb{K}^\perp = \mathbb{G}_{\mathbb{K}^\perp}$, so $\mathbb{K}^\perp$ is discriminated by $\mathbb{G}_{\mathbb{K}^\perp}$. In particular, as $\mathbb{K}^\perp$ is non-abelian, it follows that $\mathbb{G}_{\mathbb{K}^\perp}$ is also non-abelian.

Then the centraliser $C_{G_l}(\mathbb{K}^\perp)$ is discriminated by $C_\mathbb{G}(\mathbb{G}_{\mathbb{K}^\perp})$. Since $\mathbb{G}_{\mathbb{K}^\perp}$ is non-abelian and $\mathbb{G}$ is coherent, we have that $C_\mathbb{G}(\mathbb{G}_{\mathbb{K}^\perp})$ is finitely generated torsion-free abelian and since $C_{G_l}(\mathbb{K}^\perp)$ is discriminated by $C_\mathbb{G}(\mathbb{G}_{\mathbb{K}^\perp})$, $C_{G_l}(\mathbb{K}^\perp)$ is also torsion-free abelian and by the structure of the tower, it is finitely generated. □

Lemma 7.6. *Let $\mathbb{G}$ be a coherent RAAG, let L be a limit group over $\mathbb{G}$, and let G_l be a graph tower of height l associated to L. Then*

(i) *G_l is a tower of height $k \le l$ over a coherent RAAG $\mathbb{G}'$, where all the floors of G_l over $\mathbb{G}'$ are of type* (i2) *with an abelian centraliser amalgamated; or of type* (ii1)*; or of type* (iii) *and*

(ii) *G_l is in $\mathcal{C}$.*

Proof: Let the tower consist of the sequence $\mathfrak{T}_0, \dots, \mathfrak{T}_l$, where $\mathfrak{T}_i = (G_i, \mathbb{H}_i, \pi_i)$, for $1 \le i \le l$.

From Lemmas 2.5 and 3.8, when $i = 0$ the following properties are satisfied.

- G_i is in $\mathcal{C}(X_i, W_i)$ and
- if Y is a subset of X_i such that $\langle Y \rangle$ is abelian, then there exists $u \in \langle Y \rangle$ such that

$$C_{G_i}(Y) = C_{G_i}(u). \tag{8}$$

Assume that all towers of height at most $l-1$ over a coherent RAAG satisfy (8) and say a floor $G_i \le G_{i+1}$ has type (i'2) if it has type (i2) with an abelian centraliser amalgamated. Note that from [**15**, Proof of Theorem 7.1], floors of type (i1) occur only at the lowest levels of the tower. That is, we may assume that $G_i \le G_{i+1}$ is of type (i1) for $0 \le$

$i \leq k$, and that no further floors of type (i1) occur in the construction. Then G_{k+1} is a coherent RAAG, and replacing $\mathbb{G}$ with G_{k+1}, we may assume there are no floors of type (i1).

If $G_{l-1} \leq G_l$ is of type (ii1), then, since u is a non-trivial block element of $\mathbb{K}^\perp$, if $\mathbb{K}^\perp$ is non-abelian, by Lemma 7.5 we have that $C_{G_{l-1}}(u)$ is abelian. Notice that if $\mathbb{K}^\perp$ is abelian, since homomorphisms φ_i of the principal family preserve disjoint commutation (see [**15**, Lemma 6.29]) it follows that $\varphi_i(\mathbb{K}^\perp)$ is cyclic. Hence, it follows that for all $x \in \mathbb{K}^\perp$ one has $C_{G_{l-1}}(x) = C_{G_{l-1}}(\mathbb{K}^\perp)$. Therefore if $\mathbb{K}^\perp$ is abelian, without loss of generality, we can assume that $G_{l-1} \leq G_l$ is of type a$'$2). Assuming then that $C_{G_{l-1}}(u)$ is abelian, from Theorem 4.2, G_l is in $\mathcal{C}$.

If $G_{l-1} \leq G_l$ is of type a$'$2) with an abelian centraliser $C_{G_{l-1}}(\mathbb{K}^\perp)$, then (8) implies there is $u \in \mathbb{K}^\perp$ such that $C_{G_{l-1}}(\mathbb{K}^\perp) = C_{G_{l-1}}(u)$, so G_l is in $\mathcal{C}$, using Theorem 4.2 again.

If $G_{l-1} \leq G_l$ is of type (iii), then the surface S corresponding to the right hand side of the graph of groups decomposition described in Lemma 7.4(iii) has m boundary components and genus g. We form a surface S' from S by attaching an $m+1$-punctured sphere to S, identifying m of the punctures to the m boundary components of S; so S' has genus $g+m-1$ and one boundary component. Now G_l may be obtained from G_{l-1} by attaching the boundary component of S' along a path representing the word $u_{2g+1} \cdots u_m$ in G_{l-1}.

More formally, set $t = u_{2g+1} \cdots u_m \in G_{l-1} < G_l$. Observe that

$$G_l \simeq G_{l-1} *_{\langle z\rangle \times C_{G_{l-1}}(\mathbb{K}^\perp)} (\langle x_1^l, \dots, x_{m_l}^l, v \mid V(x_1^l, \dots, x_{m_l}^l) \cdot v\rangle \times C_{\Gamma_{l-1}}(\mathbb{K}^\perp)),$$

where $\langle z\rangle \simeq \mathbb{Z}$, $z \mapsto t$ in G_{l-1}, $z \mapsto v$ in the second vertex group, and V is a quadratic word in the x_i^l. From the definition we have $C_{G_{l-1}}(t) = C_B(v) = \langle v\rangle \times C_{\Gamma_{l-1}}(\mathbb{K}^\perp)$, where B is the right hand vertex group, and since $G_{l-1} \in \mathcal{C}$, we have $Z_{G_{l-2}}(t) = \langle t\rangle$ and $O_{G_{l-1}}(t)$. Therefore we may write

$$G_l = G_{l-1} *_{C_{G_{l-1}}(t)} (\Sigma \times O_{G_{l-1}}(t)), \tag{9}$$

where $\Sigma = \langle x_1^l, \dots, x_{m_l}^l, v \mid V(x_1^l, \dots, x_{m_l}^l) \cdot v\rangle$. Therefore, from Corollary 4.4, G_l is in $\mathcal{C}$.

If $G_{l-1} \leq G_l$ is of type (ii2), then $G_l = G_{l-1} *_C (C \times A)$, where $C = C_{G_{l-1}}(\mathbb{K}^\perp)$, $\mathbb{K}^\perp$ is non-abelian, and A is the free abelian group with basis $\mathbf{x}^l$. From Lemma 7.5, C is finitely generated torsion-free abelian. Let Y be a basis for C. Then, for all $y \in Y$, $C_{G_{l-1}}(y)$ contains $\mathbb{K}^\perp$, so is non-abelian and thus canonical, and $y \in K_{G_{l-1}}(y)$, which is contained in the centre of $C_{G_{l-1}}(y)$, so every generator of $K_{G_{l-1}}(y)$ commutes with $\mathbb{K}^\perp$. It follows that Y is a subset of the generators of G_{l-1}. We have $G_{l-2} \le G_{l-1}$ of type a$'$2), (ii1), or (iii) whence, from the above, $G_{l-1} = G_{l-2} *_D B$, where $D = C_{G_{l-2}}(u)$, for some $u \in G_{l-2}$, and $B = \langle D, \mathbf{x}^{l-1}\rangle$. If $y \in \mathbf{x}^{l-1}$, then $C_{G_{l-1}}(y)$ is abelian in each case, a contradiction. Hence $y \in G_{l-2}$. Therefore $Y \subseteq G_{l-2}$ and $C = C_{G_{l-1}}(\mathbb{K}^\perp) = \langle Y\rangle = C_{G_{l-2}}(\mathbb{K}^\perp)$. If $u \in C_{G_{l-2}}(\mathbb{K}^\perp)$, then $\mathbb{K}^\perp \le C_{G_{l-2}}(u)$, which is abelian. As $\mathbb{K}^\perp$ is non-abelian it follows that $C_{G'_{l-1}}(u) = C_{G_{G_{l-2}}}(u)$, from which in turn we obtain $G_l = G'_{l-1} *_D B$. Then G'_{l-1} satisfies the inductive hypotheses and $G'_{l-1} \le G_l$ is of type a$'$2), (ii1), or (iii), so G_l is in $\mathcal{C}$.

If $G_{l-1} \le G_l$ is of type (i2) with a non-abelian centraliser, we shall show that we can replace the sequence of extensions $G_{l-2} < G_{l-1} < G_l$ by a sequence $G_{l-2} < G'_{l-1} < G_l$, where $G'_{l-1} < G_l$ is of type a$'$2), (ii1), or (iii) and G'_{l-1} is in $\mathcal{C}$. To simplify notation write $R = G_{l-2}$, $S = G_{l-1}$, and $T = G_l$, so R and S are in $\mathcal{C}$, say R is in $\mathcal{C}(X_R, W_R)$ and S is in $\mathcal{C}(X_S, W_S)$, $R \le S$ is of type a$'$2), (ii1), or (iii) and $S \le T$ is of type (i2) with a non-abelian centraliser. Then (from the discussion above) there are elements $r \in R$ and $s \in S$ such that either

Case (i) $S = R *_{C_R(r)} (C_R(r) \times A)$ or

Case (ii) $S = R *_{C_R(r)} (O_R(r) \times \Sigma)$,

where A is free abelian with basis $\mathbf{x}_A$ and Σ is the appropriate analogue of (9): generated by $\mathbf{x}_A$ and r, with V a quadratic word in the generators $\mathbf{x}_A$. From Lemma 7.4, $T = S *_{C_S(\mathbb{K}^\perp)} (C_S(\mathbb{K}^\perp) \times B)$, for some co-irreducible subgroup $\mathbb{K}$ of H_{l-1} such that $\mathbb{K}^\perp$ is abelian, $C_S(\mathbb{K}^\perp)$ is non-abelian, and B is free abelian with basis $\mathbf{x}_B$. From property (8), there exists $s \in S$ such that $C_S(s) = C_S(\mathbb{K}^\perp)$, so

$$T = S *_{C_S(s)} (C_S(s) \times B),$$

and, without loss of generality, we may assume $s \in W_S$.

Next define

$$S' = R *_{C_R(s)} (C_R(s) \times B)$$

and either

(i) $T' = S' *_{C_{S'}(r)} (C_{S'}(r) \times A)$, if S is given by Case (i) above, or

(ii) $T' = S' *_{C_{S'}(r)} (O_{S'}(r) \times \Sigma)$, if S is given by Case (ii).

The inductive hypothesis implies that $S' \in \mathcal{C}$ and satisfies condition (8) and so, from Theorem 4.2 or Corollary 4.4, $T' \in \mathcal{C}$.

As $C_S(s)$ is non-abelian, it follows from Lemma 4.3 or the proof of Corollary 4.4 that $s \in R$ and either $s \notin C_R(r)$ and $C_S(s) = C_R(s)$ or $s \in C_R(r)$ and $C_S(s) = \mathrm{Z}_R(s) \times \langle O_R(s), B\rangle$. If $s \notin C_R(r)$, then $r \notin C_R(s)$ so $C_{S'}(r) = C_R(r)$ and it follows, in the case when S is given by Case (i), that $T = T'$. In Case (ii), we have $C_R(r) = Z_R(r) \times O_R(r)$, where $Z_R(r)$ is cyclic and non-canonical. If $r \notin C_R(s)$, then $C_{S'}(r) = C_R(r)$ and so $C_{S'}(r) = Z'_S(r) \times O_{S'}(r)$ with $O_{S'}(r) = O_R(r)$. Again this implies $T = T'$.

Assume then that $s \in C_R(r)$ and $C_S(s) = \mathrm{Z}_R(s) \times \langle O_R(s), B\rangle$. As $S' \in \mathcal{C}$, we have $C_{S'}(r) = Z_{S'}(r) \times O_{S'}(r)$. As $r \in C_R(s)$, we have $\mathbf{x}_B \subseteq C_{S'}(r)$. Hence, if $C_{S'}(r)$ is abelian, then it is contained in $C_R(s) \times B$, so $C_{S'}(r) = C_R(r) \times B$. Therefore, from ($\mathcal{C}5$), $r \in Z_{S'}(r)$ and $O_R(r) \subseteq O_{S'}(r)$. As both $O_R(r)$ and $O_{S'}(r)$ are canonical and $X_{S'} = X_R \cup \mathbf{x}_B$, we have $O_{S'}(r) = \langle O_R(r), \mathbf{x}_B\rangle$. Moreover, from Remark 3.5(1), $Z_R(r) = Z_{S'}(r)$, so $C_{S'}(r) = Z_R(r) \times \langle O_R(r), \mathbf{x}_B\rangle$.

Now we have relative presentations $S = \langle R, A \mid [C_R(r), \mathbf{x}_A]\rangle$, in Case (i), and $S = \langle R, A \mid [O_R(r), \mathbf{x}_A], V \cdot r\rangle$, in Case (ii), whence

Case (i) $T = \langle R, A, B \mid [C_R(r), \mathbf{x}_A], [\mathrm{Z}_R(s), \mathbf{x}_B], [O_R(s), \mathbf{x}_B], [\mathbf{x}_A, \mathbf{x}_B]\rangle$ and

Case (ii) $T = \langle R, A, B \mid [O_R(r), \mathbf{x}_A], V \cdot r, [\mathrm{Z}_R(s), \mathbf{x}_B], [O_R(s), \mathbf{x}_B], [\mathbf{x}_A, \mathbf{x}_B]\rangle$.

Also, S' has relative presentation $S' = \langle R, B \mid [\mathrm{Z}_R(s), \mathbf{x}_B], [O_R(s), \mathbf{x}_B]\rangle$, and using the description of $C_{S'}(r)$ above, in each case we see that T' and T have the same relative presentation. Thus $T = T'$ and T is in $\mathcal{C}$.

Therefore, in all cases G_l is in $\mathcal{C}$ and it remains to show that G_l satisfies condition (8). To simplify notation we use the notation above with $S = G_{l-1}$ and $T = G_l$, where S is generated by X_S and T is generated by $X_T = X_S \cup \mathbf{x}_B$. From the above we may now assume that the floor $S < T$ is of type a$'$2), (ii1), or (iii). In all cases we have $T = S *_C D$, where $C = C_S(s)$, for some $s \in S$, and D is either of the form $D = C \times B$ or $D = O_s(s) \times \Sigma$, where B is free abelian with basis $\mathbf{x}_B$ and Σ is a surface group generated by $\mathbf{x}_B$ and s. Assume that $Y \subseteq X_T$ is such that $\langle Y\rangle$ is abelian. If $Y \nsubseteq X_S$, then Y contains an element z of $\mathbf{x}_B$, so $C_T(Y) \le D$. If $S < T$ is not of type (iii), then, as $\langle Y\rangle$ and C are abelian, $C_T(Y) = C \times B = C_T(z)$, as required. If $S < T$ is of type (iii), then, $Y \cap \mathbf{x}_B = \{z\}$ (as $\langle Y\rangle$ is abelian) and $C_T(Y) = \langle z\rangle \times O_S(s) = C_T(z)$ again. Hence we may assume that $Y \subseteq X_S$. If $Y \nsubseteq C$, then $C_T(Y) = C_S(Y)$, and we have $u \in \langle Y\rangle$ such that $C_S(Y) = C_S(u)$, from (8) in S. If $u \in C$, then $C_T(u)$ contains elements outside S, a contradiction, so $u \notin C$ and $C_T(u) = C_S(u) = C_T(Y)$. This leaves the case $Y \subseteq C_S(s)$. In this case there exists $u \in \langle Y\rangle$ such that $C_S(Y) = C_s(u)$ and then $C_T(Y) \le C_T(u)$. As in the proof of Lemma 4.3 and Corollary 4.4, if $C_S(u)$ is abelian, then $C_T(u) = C_T(s)$ is abelian, in which case $C_T(s) \le C_T(Y)$, so $C_T(Y) = C_T(u)$. Otherwise $C_T(u)$ is non-abelian and $C_T(u) = \mathrm{Z}_S(u) \times \langle O_S(u), \mathbf{x}_B\rangle$. In this case if $g \in C_S(u)$, then $g \in C_S(Y) \leq C_T(Y)$, by definition of u, and $\mathbf{x}_B \subseteq C_T(Y)$, so $C_T(u) = C_T(Y)$ again. Therefore, in all cases (8) holds. $\square$

Theorem 7.7. *Let $\mathbb{G}$ be a coherent RAAG and let $\mathfrak{T} = (G, \mathbb{H}, \pi)$ be a graph tower associated to a limit group L over $\mathbb{G}$. Then G has a graph of groups decomposition (in the same generating set as G) where:*

(i) *the graph of the decomposition is a tree;*
(ii) *edge groups are finitely generated free abelian;*
(iii) *vertex groups are either graph towers of lower height, or a finitely generated free abelian group or the direct product of a finitely generated free abelian group and a non-exceptional surface group.*

Proof: Note that by Lemma 7.6, without loss of generality, we can assume that all the floors of G are of type (i2) with an abelian centraliser amalgamated; or of type (ii1); or of type (iii). We prove the statement by induction on the height l of the graph tower. If the height is 0, then $G = \mathbb{G}$. In this case, G is a coherent RAAG and admits a graph of groups decomposition (in the same set of generators) where all the vertex groups and edge groups are finitely generated torsion-free abelian groups; see [**20**].

Assume that $l > 0$ and let $\mathfrak{T} = (G_l, \mathbb{H}_l, \pi_l)$. If $G_{l-1} < G_l$ is of type (iii), then G_l has a splitting as in (9), and this decomposition is of the required type (on the original generating set).

If $G_{l-1} < G_l$ is of type (ii1), we may assume, as in the proof of Lemma 7.6, that $C_{G_{l-1}}(u)$ is abelian, and in this case, as in the case when $G_{l-1} < G_l$ is of type (i2) with an abelian centraliser, Lemma 7.4 exhibits an appropriate decomposition. $\square$

Corollary 7.8. *Let $\mathbb{G}$ be a coherent RAAG and let $\mathfrak{T} = (G, \mathbb{H}, \pi)$ be a graph tower associated to a limit group L over $\mathbb{G}$. Then G is coherent.*

Proof: We use induction on the height l of the graph tower. If the height is 0, then $G = \mathbb{G}$ and hence is coherent.

Let $l > 0$ and $G_l = G$. By Theorem 7.7, the group G_l admits a decomposition as an amalgamated product with finitely generated abelian edge group. By the induction hypothesis, the graph tower G_{l-1} is coherent. Furthermore, in all the cases, the other vertex group is a direct product of a coherent group and an abelian group and hence, by Lemma 2.9, vertex groups of the decomposition of G_l as an amalgamated product are coherent.

Therefore, since an amalgamated product of coherent groups over a finitely generated abelian subgroup is coherent (see [**50**, Lemma 4.8]) it follows that G_l is coherent. $\square$

Corollary 7.9. *Limit groups over coherent RAAGs are coherent. In particular, they are finitely presented.*

Proof: If L is a limit group over a coherent RAAG $\mathbb{G}$, then L is a subgroup of a graph tower G over $\mathbb{G}$ by Remark 7.2, and G is coherent by Corollary 7.8, and hence the result follows from the fact that limit groups are finitely generated by definition. $\square$

8. Characterisation of limit groups over coherent RAAGs

We are now in a position to show that limit groups over coherent RAAGs are exactly the finitely generated subgroups of their $\mathbb{Z}[t]$-completions.

Theorem 8.1. *Let $\mathbb{G}$ be a coherent RAAG. Then a finitely generated group G is a limit group over $\mathbb{G}$ if and only if G is a subgroup of the $\mathbb{Z}[t]$-completion $\mathbb{G}^{\mathbb{Z}[t]}$ of $\mathbb{G}$.*

Proof: It follows from Theorem 6.3, that $\mathbb{G}^{\mathbb{Z}[t]}$ is discriminated by $\mathbb{G}$. Since discrimination passes to subgroups, it follows that every finitely generated subgroup of $\mathbb{G}^{\mathbb{Z}[t]}$ is a limit group over $\mathbb{G}$.

Let us prove the converse. By [**15**, Theorem 8.1], it follows that any limit group over $\mathbb{G}$ is a subgroup of a graph tower $\mathfrak{T}$ over $\mathbb{G}$ of height l. Hence, we are left to show that graph towers over $\mathbb{G}$ embed into $\mathbb{G}^{\mathbb{Z}[t]}$. We proceed by induction on the height l of the tower $\mathfrak{T} = (G_l, \mathbb{H}_l, \pi_l)$. Assume, by induction, that graph towers over $\mathbb{G}$ of height $l-1$ embed into $\mathbb{G}^{\mathbb{Z}[t]}$. In the light of Lemma 7.6, without loss of generality we may assume that all the floors of G are of type (i2) with an abelian centraliser; or of type (ii1); or of type (iii).

Suppose first that the floor $G_{l-1} < G_l$ is of type (i2), where $C_{G_{l-1}}(\mathbb{K}^\perp)$ is abelian. In this case, the statement is obvious, since $G_l = G_{l-1} *_A B$, where A and B are finitely generated free abelian groups. Similarly, if the floor $G_{l-1} < G_l$ is of type (ii1), then G_l is an extension of an abelian centraliser, hence the statement follows in this case.

We are left to consider the case when the floor $G_{l-1} < G_l$ is of type (iii). In this case, as in the proof of Theorem 7.7, G_l has the decomposition (9) and, as in Corollary 4.4, G_l embeds in the group $T^* = \langle G, y \mid [C, y]\rangle = G *_C (C \times \langle y\rangle)$ of (5), with $G = G_{l-1}$ and $C = C_{G_{l-1}}(u)$.

As $G_{l-1} < T^*$ is of type (i2) with an abelian centraliser, the statement follows. $\square$

9. Limit groups over coherent RAAGs are CAT(0)

A group that acts properly discontinuously and co-compactly by isometries on a proper CAT(0) space is called a CAT(0) *group.*

In this section we prove that limit groups over coherent RAAGs are CAT(0) and deduce that the conjugacy problem is decidable for these groups. We recall the following two definitions; see [**1**].

Definition 9.1. Let X be a connected locally CAT(0) space. A subspace C of X is a *core* of X if it is compact and locally CAT(0) (with respect to the induced path metric) and the inclusion $C \to X$ is a homotopy equivalence.

Definition 9.2. A connected locally CAT(0) space Y is geometrically coherent if for every covering space $X \to Y$ with X connected and $\pi_1(X)$ finitely generated and every compact subset $K \subset X$ it follows that X contains a core $C \supset K$.

If G is the fundamental group of a geometrically coherent space Y and H is a finitely generated subgroup of G, then, taking a covering space $X \to Y$ with $\pi_1(X) = H$ and a core C of X, by definition $\pi_1(C) = H$ and C is locally CAT(0), so H is finitely presented and the Cartan–Hadamard theorem (see [**11**] for example) implies that H is CAT(0).

Example 9.3 (see Example 2.4 in [**1**])**.** A flat torus $T = \mathbb{R}^n/\Lambda$ is geometrically coherent. Indeed, any connected covering space admits a metric splitting $T' \times E$ where T' is a flat torus or a point and E is Euclidean space $\mathbb{R}^k$ or a point. Thus $T' \times \{\text{point}\}$ is a (convex) core.

Theorem 9.4 ([**1**, Theorem 2.6])**.** *Let M and N be geometrically coherent locally* CAT(0) *spaces. Let Y be the space obtained from the disjoint union of M, N, and a finite collection of n-tori T_i*

$$M \sqcup \bigsqcup_i T_i \times [0,1] \sqcup N$$

by gluing $T_i \times \{0\}$ to a convex space in M by a local isometry and gluing $T_i \times \{1\}$ to a convex space in N by a local isometry.

Then Y is geometrically coherent.

Proof: Proof follows from [**1**, Theorem 2.6 and Remark 2.8]. □

For a description of the Salvetti complex of a RAAG we refer to Charney's survey [**18**].

Lemma 9.5. *The Salvetti complexes of coherent RAAGs are geometrically coherent.*

Proof: Let Γ be such that $\mathbb{G}(\Gamma)$ is coherent and let $C(\Gamma)$ be the corresponding Salvetti complex.

The graph Γ contains a clique which induces a decomposition of $\mathbb{G}(\Gamma)$ as an amalgamated product, where vertex groups are RAAGs corresponding to proper full subgraphs of Γ and the amalgamation is over a free abelian group corresponding to the clique; see [**20**]. Hence, $\mathbb{G}(\Gamma)$ is the fundamental group of a tree S of free abelian groups; see [**20**]. We use induction on the number of vertices in S.

If S has one vertex, then $C(\Gamma)$ is a flat torus and the statement follows from Example 9.3. Suppose that the statement of the lemma is true for any Γ' such that the corresponding tree S' has fewer than n vertices. Let Γ be such that S has precisely n vertices. Let S' be a subtree of S obtained by removing a leaf v. We have $\mathbb{G}(\Gamma) = \pi_1(S') *_{\mathbb{Z}^k} \mathbb{Z}^l$, for some $k < l \in \mathbb{N}$, and $\pi_1(S') = \mathbb{G}(\Gamma')$ is a RAAG such that the corresponding tree S' has fewer than n vertices. Notice that the generators of $\mathbb{Z}^k$ are generators of $\pi_1(S')$ and of $\mathbb{Z}^l$. Then $C(\Gamma)$ decomposes as $C(S') \sqcup T^k \times [0,1] \sqcup T^l / \sim$, where T^k and T^l are k- and l-tori, $T^k \times \{1\}$ is identified with a k-torus spanned by coordinate circles in T^l, and $T^k \times \{0\}$ is identified with a k-torus spanned by coordinate loops in $C(S')$. In particular, we glue the $T^k \times \{0\}$ and $T^k \times \{1\}$ k-tori to convex subspaces of T^l and $C(S')$, respectively, each by a local isometry. Thus, the statement follows by induction and Theorem 9.4. ☐

Abusing the terminology, we call a group geometrically coherent if it is the fundamental group of a geometrically coherent space.

Proposition 9.6. *Finite iterated centraliser extensions of coherent RAAGs are geometrically coherent.*

Proof: Let $\mathbb{G}$ be a coherent RAAG. We use induction on the number of iterated centraliser extensions. To the base of induction $\mathbb{G}$, we associate the Salvetti complex $\chi(\mathbb{G})$ of $\mathbb{G}$, which is geometrically coherent by Lemma 9.5.

Now let $\mathbb{G} = G_0 < G_1 < \cdots < G_{m-1} < G_m$ be an ICE. Arguing as in the proof of Lemma 7.6, we may assume that the centraliser $C_{G_{m-1}}(u)$ being extended in step m is abelian. Indeed, all non-abelian centralisers correspond to canonical generators of $\mathbb{G}$ and extension of such a centraliser results in a new coherent RAAG, as in the proof of Lemma 7.6.

Hence, without loss of generality, we may assume that $G_m = G_{m-1} *_C \mathbb{Z}^l$, where $C = C_{G_{m-1}}(u) = \langle u \rangle \times \mathbb{Z}^{k-1} \cong \mathbb{Z}^k$, with $k < l$, is the (abelian) centraliser of a block element u in G_{m-1} and G_{m-1} is an ICE satisfying the statement of the proposition. By induction, the corresponding core $\chi(G_{m-1})$, such that $G_{m-1} = \pi_1(\chi(G_{m-1}))$, is already defined.

We now define the core $\chi(G_m)$ as $\chi(G_{m-1}) \sqcup T^k \times [0,1] \sqcup T^l/\sim$, where T^k and T^l are k- and l-tori, $T^k \times \{1\}$ is identified with a k-torus spanned by coordinate circles in T^l, and the coordinate circles of $T^k \times \{0\}$ are identified with non-trivial loops in $\chi(G_{m-1})$ that generate the abelian subgroup C in $\pi_1(\chi(G_{m-1})) = G_{m-1}$. The induction step now follows by Theorem 9.4. □

Since every limit group over a coherent RAAG $\mathbb{G}$ is a subgroup of an iterated centraliser extension of $\mathbb{G}$, we obtain the following

Corollary 9.7. *Limit groups over coherent RAAGs are* CAT(0)*. In particular, the conjugacy problem is decidable in limit groups over coherent RAAGs.*

Proof: If L is a limit group over a coherent RAAG $\mathbb{G}$, then L is a subgroup of $\mathbb{G}^{\mathbb{Z}[t]}$ by Theorem 8.1 and $\mathbb{G}^{\mathbb{Z}[t]}$ is an ICE of $\mathbb{G}$ by Corollary 6.12 (and the subsequent comment). Moreover, it follows from the proof of Theorem 8.1 that L is a subgroup of G', where G' is obtained from $\mathbb{G}$ by a finite chain of centraliser extensions. Therefore, G' is CAT(0) and geometrically coherent by Proposition 9.6. Hence, L is isomorphic to a finitely generated subgroup of a geometrically coherent group so, from the comments following Definition 9.2, L is CAT(0). Finally, by [**11**, Theorem 1.12, III.Γ.1], the conjugacy problem in L is decidable. □

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Received on September 25, 2020.
Accepted on January 17, 2022.