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CRYSTALLINE MEASURES IN TWO DIMENSIONS

Yves Meyer

Abstract: Some crystalline measures supported by a Delone set $\Lambda \subset \mathbb{R}^2$ are constructed in this note. This gives a new proof of a remarkable theorem by Pavel Kurasov and Peter Sarnak.

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1. Introduction

An atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if the following three conditions are satisfied: (i) μ is supported by a locally finite set, (ii) μ is a tempered distribution, and (iii) the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set. This is equivalent to a generalized Poisson summation formula, as is shown in [**3**]. In the late 1950s crystalline measures were defined under other names and studied by Andrew P. Guinand [**1**] and Jean-Pierre Kahane and Szolem Mandelbrojt [**3**]. These authors were motivated by the relationship between (a) the functional equation satisfied by the Riemann zeta function and (b) the standard Poisson formula. More generally let $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ be a crystalline measure on $\mathbb{R}^n$ supported by a locally finite set Λ and let $\widehat{\mu} = \sum_{s\in S} a(s)\delta_s$. We consider the Dirichlet series $\zeta(\mu, s) = \sum_{\{\lambda\in\Lambda,\, \lambda\neq 0\}} c(\lambda)|\lambda|^{-s}$ in the complex variable s. Let $\gamma_n = \frac{\pi^{-n/2}2^{1-n}}{\Gamma(n/2)}$, where Γ denotes the Euler Gamma function. Then $\zeta(\mu, s)$ is an entire function in the complex plane if $0 \notin S$, while $\zeta(\mu, s) - \gamma_n \frac{a(0)}{s-n}$ can be extended as an entire function of $s \in \mathbb{C}$ if $0 \in S$ [**3**], [**7**]. The connection between the Poisson summation formula and the properties of the Riemann zeta function is developed in Titchmarsh's treatise and can be traced back to Riemann. When Kahane and Mandelbrojt wrote their paper it was debated whether or not a crystalline measure is necessarily a generalized Dirac comb. But this same year Guinand discovered a revolutionary crystalline measure. In his seminal work [**1**] Guinand described an explicit atomic measure μ supported by the set $\Lambda = \{\pm\sqrt{k+1/9},\, k \in \mathbb{N}\cup\{0\}\}$ and claimed that $\widehat{\mu} = \mu$. This beautiful example of a crystalline measure is rooted in Guinand's work on number theory. The fascinating problems raised by Guinand were forgotten for more than fifty years. Fortunately in 2015 Nir Lev and Alexander Olevskii gave a new life to Guinand's work and constructed a crystalline measure which is not a generalized Dirac comb [**5**]. This was important since the proof given by Guinand in [**1**] was incomplete, as was noticed by Olevskii in [**5**]. A few months later Guinand's claims were proved [**7**]. The revival of Guinand's work could be related to the discovery of quasi-crystals by Dan Shechtman (1982). The connection between crystalline measures and quasi-crystals is discussed in [**6**]. Up to now all examples of crystalline measures have been one-dimensional. Non-trivial two-dimensional crystalline measures are constructed in this note. It gives a simple proof of a remarkable theorem of Kurasov and Sarnak [**4**].

The Dirac measure located at $a \in \mathbb{R}^n$ is denoted by δ_a or $\delta_a(x)$. A *purely atomic measure* is a linear combination $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ of Dirac measures, where the coefficients $c(\lambda)$ are real or complex numbers and $\sum_{|\lambda|\le R} |c(\lambda)|$ is finite for every $R > 0$. Then Λ is a countable set of points of $\mathbb{R}^n$. A subset $\Lambda \subset \mathbb{R}^n$ is *locally finite* if $\Lambda \cap B$ is finite for every bounded set B. Equivalently, Λ can be ordered as a sequence $\{\lambda_j,\ j = 1, 2, \dots\}$ and $|\lambda_j|$ tends to infinity with j. A measure μ is a *tempered distribution* if it has a polynomial growth at infinity in the sense given by Laurent Schwartz in [**10**]. For instance, the measure $\sum_1^\infty 2^k\delta_k$ is not a tempered distribution, while $\sum_1^\infty k^3\delta_k$ and $\sum_1^\infty 2^k[\delta_{(k+2^{-k})} - \delta_k]$ are tempered distributions. The *Fourier transform* $\mathcal{F}(f) = \widehat{f}$ of an integrable function f is defined by $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-ix\cdot y) f(x)\,dx$. The *distributional Fourier transform* $\widehat{\mu}$ of a tempered measure μ is defined by $\langle \widehat{\mu}, \phi\rangle = \langle \mu, \widehat{\phi}\rangle$, $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$.

If μ is a crystalline measure, its distributional Fourier transform is also a crystalline measure. A product $P\mu$ between a crystalline measure and a finite trigonometric sum P is still a crystalline measure. If μ is a crystalline measure and A is an affine transformation of $\mathbb{R}^n$ into itself, the image measure μ_A of μ by A is still a crystalline measure. Let $\Gamma \subset \mathbb{R}^n$ be a lattice. The distributional Fourier transform of the Dirac comb $\mu = \operatorname{vol}(\Gamma) \sum_{\gamma\in\Gamma} \delta_\gamma$ is the Dirac comb $\widehat{\mu} = (2\pi)^n \sum_{y\in\Gamma^*} \delta_y$ on the dual lattice Γ^*.

Definition 1.1. Let σ_j, $1 \le j \le N$, be N Dirac combs, each σ_j being supported by a coset $x_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^n$, $1 \le j \le N$. Let $P_j(x) = \sum_{y\in F_j} c_j(y) \exp(2\pi i y\cdot x)$ be a finite trigonometric sum. Let $\mu_j = P_j\,\sigma_j$. Then $\mu = \mu_1 + \cdots + \mu_N$ is called a generalized Dirac comb.

A generalized Dirac comb is a trivial example of a crystalline measure. A Delone set $\Lambda \subset \mathbb{R}^n$ is defined by the following property: there exist two positive numbers $R > r > 0$ such that any ball of radius r, whatever its center, contains at most a point $\lambda \in \Lambda$ and any ball of radius R, whatever its center, contains at least a point $\lambda \in \Lambda$. Pavel Kurasov and Peter Sarnak constructed a one-dimensional crystalline measure whose support is a Delone set [**4**] and which is not a generalized Dirac comb. This extends to any dimension since the tensor product $\mu_1 \otimes \mu_2$ between two crystalline measures μ_1 and μ_2 on $\mathbb{R}$ is a crystalline measure on $\mathbb{R}^2$. We construct a crystalline measure σ on $\mathbb{R}^2$ which is not a tensor product between two crystalline measures on $\mathbb{R}$. More precisely, $\sigma = \sum_{\lambda \in \Lambda} \delta_\lambda$, where Λ is a Delone set.

2. Crystalline measures on $\mathbb{R}^2$

Let $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ be the torus. Let $r \in [0,1]$ and let θ_r be the 1-periodic continuous function defined on $\mathbf{T}$ by $\cos(2\pi\theta_r(x)) = r\cos(2\pi x)$ and $0 \le \theta_r(x) \le 1/2$. We have $\theta_0(x) = 1/4$ identically and $\theta_1(x) = |x|$ if $-1/2 \le x \le 1/2$. The function θ_r is analytic if $0 \le r < 1$. The derivative of $\theta_r(x)$ is

$$\frac{d\,\theta_r}{dx} = \frac{r\sin(2\pi x)}{\sqrt{1 - r^2\cos^2(2\pi x)}}.$$

We have $\theta_r(1/2 - x) + \theta_r(x) = 1/2$ and $\|\theta_r - 1/4\|_\infty < 1/4$ if $0 \le r < 1$.

Lemma 2.1. *For any $m \in \mathbb{N}$ the 1-periodic function $\cos(2\pi m\theta_r(x))$ is a trigonometric polynomial. More precisely we have*

$$\cos(2\pi m\theta_r(x)) = \sum_0^m \alpha_r(k,m)\cos(2\pi kx). \tag{1}$$

The Chebyshev polynomial of degree m is denoted by P_m and we have

$$\begin{aligned}\cos(2\pi m\theta_r(x)) &= P_m[\cos(2\pi\theta_r(x))] \\ &= P_m[r\cos(2\pi x)] = \sum_0^m \alpha_r(k,m)\cos(2\pi kx),\end{aligned}$$

which ends the proof. We have $\alpha_r(0,0) = 1$ and $\alpha_r(k,0) = 0$ if $k \ne 0$, $\alpha(1,1) = r$ and $\alpha(k,1) = 0$ if $k \ne 1$.

Lemma 2.2. *There exist an integer $m_0(r)$ and a constant $c(r) > 0$ such that if $m \ge m_0(r)$ and $0 \le k \le \frac{r}{10}m$, we have*

$$|\alpha_r(k,m)| \ge c(r)m^{-1/2}. \tag{2}$$

Indeed we have

$$\alpha_r(k,m) = 2\int_0^1 \cos(2\pi m\theta_r(x))\cos(2\pi kx)\,dx. \tag{3}$$

To prove (2) it suffices to estimate the right-hand side of (3) by van der Corput's lemma.

The first ingredient of our construction is given by an arbitrary one-dimensional crystalline measure μ. We have

$$\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda,$$

where $\Lambda \subset \mathbb{R}$ is a locally finite set of real numbers. To avoid cumbersome discussions Λ is assumed to be a Delone set and $c(\lambda)$ is assumed to be a bounded sequence. Then all the series that form part of the proof of Theorem 2.5 converge in the distributional sense. The distributional Fourier transform of μ is

$$\widehat{\mu} = \sum_{s\in S} \gamma(s)\delta_s,$$

where $S \subset \mathbb{R}$ is a locally finite set. Before constructing the two-dimensional crystalline measure σ_r let us define its support $M_r = M_r(\Lambda)$. If $r \in (0,1)$, this support is the disjoint union of two pieces. We have $M_r = M_r^+ \cup M_r^-$, where the two locally finite sets $M_r^\pm \subset \mathbb{R}^2$ are defined by

$$M_r^\pm = \{(k \pm \theta_r(\lambda), \lambda);\ k\in\mathbb{Z},\ \lambda\in\Lambda\}.$$

Lemma 2.3. *We have*

$$M_r = \{(x_1,x_2)\in\mathbb{R}^2;\ \cos(2\pi x_1) = r\cos(2\pi x_2),\ x_2\in\Lambda\}.$$

Moreover, $M_0 = \{k\pm 1/4,\ \lambda,\ k\in\mathbb{Z},\ \lambda\in\Lambda\}$ and $M_1 = \{k\pm\theta_1(\lambda),\ \lambda,\ k\in\mathbb{Z},\ \lambda\in\Lambda\}$. If $r\in(0,1)$, then M_r^+ and M_r^- are two disjoint sets.

Definition 2.4. With the preceding notations the atomic measure σ_r is defined by

$$\sigma_r = \sum_{k\in\mathbb{Z}}\sum_{\lambda\in\Lambda} c(\lambda)\delta_{(k+\theta_r(\lambda),\lambda)} + \sum_{k\in\mathbb{Z}}\sum_{\lambda\in\Lambda} c(\lambda)\delta_{(k-\theta_r(\lambda),\lambda)}.$$

This measure σ_r is 1-periodic with respect to the first variable. It can also be written as

$$\sigma_r = \sum_{y\in M_r} \kappa(y)\delta_y,$$

where $\kappa(y) = c(\lambda)$ if $y = k\pm\theta_r(\lambda)$, $k\in\mathbb{Z}$, $\lambda\in\Lambda$.

Theorem 2.5. *Let $r \in [0,1]$. Then the atomic measure σ_r is a crystalline measure supported by the Delone set M_r.*

If $r = 0$, this is a trivial fact, and σ_0 is a generalized Dirac comb if μ is a Dirac comb. If $r = 1$, Theorem 2.5 is still a trivial fact. Indeed for any $\lambda \in \Lambda$ we have $\sum_{k\in\mathbb{Z}}(\delta_{(k+\theta_1(\lambda),\lambda)} + \delta_{(k-\theta_1(\lambda),\lambda)}) = \sum_{k\in\mathbb{Z}}(\delta_{(k+\lambda,\lambda)} + \delta_{(k-\lambda,\lambda)})$. Therefore σ_1 is a trivial crystalline measure. Indeed σ_1 is the sum of the two images of $\tau = \sum_{k\in\mathbb{Z}}\sum_{\lambda\in\Lambda} c(\lambda)\delta_{(k,\lambda)}$ by the mappings $(x_1,x_2) \mapsto (x_1 \pm x_2, x_2)$. Before proving Theorem 2.5 let us consider a third example. Let $\alpha > 0$ be irrational and let $\mu_\alpha = \sum_{k\in\mathbb{Z}} \delta_{\alpha k}$. If $r \in (0,1)$, then

$$\sigma_r = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \delta_{(k+\theta_r(\alpha l),\alpha l)} + \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \delta_{(k-\theta_r(\alpha l),\alpha l)}.$$

This atomic measure is a genuine sum of Dirac measures carried by the Delone set M_r. Theorem 2.5 gives a new proof of the theorem by Kurasov and Sarnak [**4**]. The vector space of all crystalline measures supported by M_r will be described in this note (Theorem 3.3) when μ is a Dirac comb supported on $\alpha\mathbb{Z}$, where $\alpha > 1$, $\alpha \notin \mathbb{Q}$.

The proof of Theorem 2.5 is straightforward. The series used in this proof converge in the distributional sense since Λ is a Delone set and $c(\lambda)$ is a bounded sequence. The distributional Fourier transform $\widehat{\sigma}_r(x)$ of σ_r is $\widehat{\sigma}_r(x) = S_r = S_r^+(x) + S_r^-(x)$, where

$$S_r^\pm(x) = \sum_{y\in M_r^\pm} \kappa(y)\exp(-ix\cdot y).$$

We first consider

$$S_r^+(x) = \sum_{k\in\mathbb{Z}}\sum_{\lambda\in\Lambda} c(\lambda)\exp(-ix_1(k+\theta_r(\lambda)))\exp(-ix_2\lambda).$$

We first sum on $k \in \mathbb{Z}$ and apply the standard Poisson formula. We obtain

$$\begin{aligned} S_r^+(x) &= 2\pi \sum_{\lambda\in\Lambda}\left[\exp(-ix_1\theta_r(\lambda))\sum_{m\in\mathbb{Z}}\delta_{2\pi m}(x_1)\right]c(\lambda)\exp(-ix_2\lambda) \\ &= 2\pi \sum_{\lambda\in\Lambda}\left[\sum_{m\in\mathbb{Z}}\exp(-2\pi i m\theta_r(\lambda))\delta_{2\pi m}(x_1)\right]c(\lambda)\exp(-ix_2\lambda) \\ &= 2\pi \sum_{m\in\mathbb{Z}}\delta_{2\pi m}(x_1)\left[\sum_{\lambda\in\Lambda}\exp(-2\pi i m\theta_r(\lambda))c(\lambda)\exp(-ix_2\lambda)\right]. \end{aligned}$$

For any $m \in \mathbb{Z}$ we consider the distribution

$$g_m^+(x_2) = \sum_{\lambda\in\Lambda} c(\lambda) \exp(-2\pi i m\theta_r(\lambda)) \exp(-ix_2\lambda).$$

Then

$$S_r^+ = 2\pi \sum_{m\in\mathbb{Z}} \delta_{2\pi m} \otimes g_m^+.$$

Once this is reached we consider the corresponding series

$$S_r^-(x) = \sum_{k\in\mathbb{Z}} \sum_{\lambda\in\Lambda} c(\lambda) \exp(-ix_1(k-\theta_r(\lambda))) \exp(-ix_2\lambda)$$

and we end with $S_r^- = 2\pi \sum_{m\in\mathbb{Z}} \delta_{2\pi m} \otimes g_m^-$, where for any $m \in \mathbb{Z}$ we define a distribution

$$g_m^-(x_2) = \sum_{\lambda\in\Lambda} c(\lambda) \exp(2\pi i m\theta_r(\lambda)) \exp(-ix_2\lambda).$$

We do not want to compute $g_m^\pm$. Finally,

$$G_m(x_2) = g_m^+(x_2) + g_m^-(x_2) = 2 \sum_{\lambda\in\Lambda} c(\lambda) \cos(2\pi m\theta_r(\lambda)) \exp(-ix_2\lambda).$$

We obviously have $G_{-m} = G_m$.

Lemma 2.6. *We have*

$$S_r = 2\pi \sum_{m\in\mathbb{Z}} \delta_{2\pi m} \otimes G_m.$$

If $m = 0$, $G_0(x_2) = 2\sum_{\lambda\in\Lambda} c(\lambda)\exp(-ix_2\lambda) = 2\sum_{s\in S} \gamma(s)\delta_s$. If $m = \pm 1$, the definition of θ_r implies $\cos(2\pi\theta_r(\lambda)) = r\cos(2\pi\lambda)$ and we have

$$\begin{aligned} G_{\pm 1}(x_2) &= 2r \sum_{\lambda\in\Lambda} c(\lambda)\cos(2\pi\lambda)\exp(-ix_2\lambda) \\ &= r \sum_{s\in S} \gamma(s)(\delta_{s+2\pi} + \delta_{s-2\pi}). \end{aligned}$$

If $m \geq 2$, Lemma 2.1 yields

$$G_m(x_2) = 2 \sum_{s\in S} \sum_{\{k\in\mathbb{Z};\, |k|\leq |m|\}} \alpha_r(k,|m|)\gamma(s)\delta_{s+2\pi k}.$$

Therefore $S_r = S_r^+ + S_r^- = 4\pi\sum_{m\in\mathbb{Z}} \delta_{2\pi m} \otimes G_m$ is a sum of weighted Dirac masses on the locally finite set $U = \{(2\pi m, s+2\pi k),\ s \in S,\ |k| \leq |m|,\ m \in \mathbb{Z}\}$. More precisely we have

$$\widehat{\sigma}_r = S_r = 2\pi \sum_{m\in\mathbb{Z}} \sum_{s\in S} \sum_{\{k\in\mathbb{Z};\, |k|\leq|m|\}} \alpha_r(|k|,|m|)\gamma(s)\, \delta_{2\pi m} \otimes \delta_{s+2\pi k}. \tag{4}$$

The definition of $\alpha_r(k,m)$ is given in (1). This ends the proof of Theorem 2.5.

The right-hand side of (4) can be written $S = \rho * \widetilde{\mu}$, where $\widetilde{\mu} = \delta_0 \otimes \widehat{\mu}$ and

$$\rho = \sum_{m \in \mathbb{Z}} \sum_{\{k \in \mathbb{Z};\, |k| \le |m|\}} \alpha_r(|k|, |m|)\, \delta_{2\pi m} \otimes \delta_{2\pi k} \tag{5}$$

is an atomic measure supported by $(2\pi\mathbb{Z})^2$.

This remark paves the way to a second proof of Theorem 2.5. We start with a $\mathbb{Z}^2$-periodic measure W which is given a simple geometric definition. Then we prove directly that σ_r coincides with the pointwise product between the measure W and the measure $dx_1\, d\mu(x_2)$. This pointwise product between two measures is defined in Lemma 2.10. The Fourier transform of W is computed and coincides with ρ. The Fourier transform of $dx_1\, d\mu(x_2)$ is obviously $\widetilde{\mu}$. Finally, the Fourier transform of the pointwise product between W and $dx_1\, d\mu(x_2)$ is the convolution product between ρ and $\widetilde{\mu}$. The support of $\rho * \widetilde{\mu}$ is locally finite, as is checked immediately. The distributional Fourier transform of σ_r is the atomic measure $\rho * \widetilde{\mu}$ which is supported by a locally finite set. This ends the proof of Theorem 2.5.

Here are the details. Let us define W. We first consider the curve $\mathcal{C}_+$ on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ defined by $x_1 = \theta_r(x_2)$. Similarly $\mathcal{C}_-$ is defined by $x_1 = -\theta_r(x_2)$. Let w_+ be the measure on $\mathcal{C}_+$ which is the image of the measure dx_2 on $\mathbb{T}$ by the mapping $x_2 \mapsto (\theta_r(x_2), x_2)$. Then we have

Lemma 2.7. *The Fourier coefficients of w_+ are*

$$\widehat{w}_+(k_1, k_2) = \int_0^1 \exp(-2\pi i k_1 \theta_r(x_2)) \exp(-2\pi i k_2 x_2)\, dx_2.$$

This is obvious by the definition of the direct image of a measure. In a similar way we consider the measure w_- on $\mathcal{C}_-$ which is the image of the normalized measure dx_2 on $\mathbb{T}$ by the mapping $x_2 \mapsto (-\theta_r(x_2), x_2)$. Then the Fourier coefficients of w_- are

$$\widehat{w}_-(k_1, k_2) = \int_0^1 \exp(2\pi i k_1 \theta_r(x_2)) \exp(-2\pi i k_2 x_2)\, dx_2.$$

Finally, we consider $w = w_+ + w_-$. Its Fourier coefficients are

$$\widehat{w}(k_1, k_2) = 2 \int_0^1 \cos(2\pi k_1 \theta_r(x_2)) \exp(-2\pi i k_2 x_2)\, dx_2. \tag{6}$$

But $\cos(2\pi k \theta_r(x)) = \sum_0^k \alpha_r(k, m) \cos(2\pi k x)$. This yields

$$\widehat{w}(k_1, k_2) = \alpha_r(|k_2|, |k_1|).$$

This together with (6) implies

$$w(x_1,x_2)=\sum\sum\alpha_r(|k_2|,|k_1|)\exp(2\pi i(k_1x_1+k_2x_2)). \tag{7}$$

The right-hand side of (7) is $\widehat{\rho}(x)$ when ρ is defined by (5).

Viewed as two $\mathbb{Z}^2$-periodic measures on $\mathbb{R}^2$ the measures $w_\pm$ and w are denoted by $W_\pm$ and W. The support of W_+ is the union of the pairwise disjoint curves $\mathcal{C}_l^+$, $l\in\mathbb{Z}$, defined by the equations $y_1=\theta_r(y_2)+l$. Similarly the support of W_- is the union of the pairwise disjoint curves $\mathcal{C}_l^-$, $l\in\mathbb{Z}$, defined by the equations $y_1=-\theta_r(y_2)+l$.

Lemma 2.8. *The distributional Fourier transform of W is ρ.*

This is immediate from (5) and (7). Our second proof of Theorem 2.5 is an immediate consequence of the following lemma:

Lemma 2.9. *The measure σ_r is the pointwise product between W and $dx_1\mu(x_2)$.*

This pointwise product between two measures makes sense as the following lemma indicates:

Lemma 2.10. *Let f be a real-valued continuous function of the real variable t and let $F\colon\mathbb{R}\mapsto\mathbb{R}^2$ be the mapping defined by $F(t)=(t,f(t))$. Let l be the Lebesgue measure on $\mathbb{R}$. Let $\xi=F_*(l)$ be the Radon measure on $\mathbb{R}^2$ which is the pushforward of l by F. For any Radon measure μ on $\mathbb{R}$ the pushforward $F_*(\mu)$ is the pointwise product between $\mu\otimes l$ and ξ.*

The measure μ is the weak limit of a sequence μ_j of continuous functions. Then for any continuous compactly supported function ϕ on $\mathbb{R}^2$ we have

$$\begin{aligned}\langle F_*(\mu),\phi\rangle&=\int\phi(t,f(t))\,d\mu(t)=\lim_{j\to\infty}\int\phi(t,f(t))\mu_j(t)\,dt\\&=\lim_{j\to\infty}\langle(\xi)(\mu_j\otimes l),\phi\rangle=\langle(\xi)(\mu\otimes l),\phi\rangle.\end{aligned}$$

This last limit is the definition of the product between ξ and $\mu\otimes l$. It ends the proof of Lemma 2.10. This obvious lemma is applied to each Dirac measure δ_λ, $\lambda\in\Gamma$. The product between W and $dx_1\otimes\delta_\lambda(x_2)$ is $\delta_{(\theta_r(\lambda),\lambda)}+\delta_{(-\theta_r(\lambda),\lambda)}$. This implies Lemma 2.9. Lemma 2.9 implies that the Fourier transform of σ_r is given by the convolution product between $\widehat{W}$ and $\widehat{dx_1\otimes\mu}$. We have $\widehat{W}=\rho$ and $\widehat{dx_1\otimes\mu}=\widetilde{\mu}$. Our second proof of Theorem 2.5 is complete.

3. Mean-periodic measures

In this section it is assumed that the atomic measure μ used in the construction of σ_r is the Dirac comb on $\alpha\mathbb{Z}$, where $\alpha > 1$, $\alpha \notin \mathbb{Q}$. Then the distributional Fourier transform ν of the crystalline measure σ_r belongs to a larger class $\mathcal{W}$ of crystalline measures which are studied in this section. This distributional Fourier transform is a mean-periodic measure which can be calculated explicitly by simple geometric rules. This observation is proved now and paves the way to the definition of $\mathcal{W}$.

Here are the proofs of these claims. Let $r \in (0,1)$ and let us consider the two atomic measures τ_1 and τ_2 on $\mathbb{R}^2$ defined by $\tau_1 = r\delta_{(0,2\pi)} + r\delta_{(0,-2\pi)} - \delta_{(2\pi,0)} - \delta_{(-2\pi,0)}$ and $\tau_2 = \delta_{(0,2\pi/\alpha)} - \delta_{(0,0)}$. We have $P_1(x) = \widehat{\tau}_1 = 2r\cos(2\pi x_2) – 2\cos(2\pi x_1)$. Similarly $P_2(x) = \widehat{\tau}_2 = \exp(-2\pi i x_2/\alpha) - 1$. Then $M_r \subset \mathbb{R}^2$ can be defined by $P_1(x) = P_2(x) = 0$. Indeed $P_2(x) = 0$ implies $x_2 = k\alpha$, $k \in \mathbb{Z}$, and $P_1(x) = 0$ yields $\cos(2\pi x_1) = r\cos(2k\pi\alpha)$. This implies $P_1\sigma_r = P_2\sigma_r = 0$ since σ_r is supported by M_r. Therefore $\nu = \widehat{\sigma}_r$ satisfies

$$\nu * \tau_1 = \nu * \tau_2 = 0. \tag{8}$$

Any solution of (8) is a mean-periodic measure. We forget σ_r now and focus on this convolution equation.

Let $\mathcal{W}$ be the vector space consisting of all atomic measures $\nu(x_1, x_2)$ on $\mathbb{R}^2$ which are supported by a locally finite set and such that:

$$\nu(x_1, x_2 + 2\pi/\alpha) = \nu(x_1, x_2) \tag{9}$$

and

$$\nu(x_1+2\pi, x_2)+\nu(x_1 - 2\pi, x_2) = r\nu(x_1, x_2+2\pi) + r\nu(x_1, x_2 - 2\pi). \tag{10}$$

Lemma 3.1. *Any measure $\nu \in \mathcal{W}$ is a tempered distribution.*

Lemma 3.1 is proved in the next section. If this result is accepted, the distributional Fourier transform σ of ν makes sense. Then (10) implies $\widehat{\tau}_1\widehat{\nu} = 0$. The zero set of $\widehat{\tau}_1$ in $\mathbb{R}^2$ is a collection of pairwise disjoint curves $\mathcal{C}_l^{\pm}$, $l \in \mathbb{Z}$, defined by the equations $y_1 = \pm\theta_r(y_2) + l$. Each $\mathcal{C}_l^{\pm}$, $l \in \mathbb{Z}$, is contained in a vertical strip. Similarly (9) implies $\widehat{\tau}_2\widehat{\nu} = 0$. The zero set of $\widehat{\tau}_2$ is a collection of horizontal lines defined by $y_2 = \alpha q$, $q \in \mathbb{Z}$. Each of these lines is transverse to each $\mathcal{C}_l^{\pm}$. Therefore the distribution $\sigma = \widehat{\nu}$ is a sum of weighted Dirac measures on M_r. If ν itself is a sum of weighted Dirac masses on a locally finite set, then ν and σ are necessarily two crystalline measures.

The characterization of $\mathcal{W}$ is given in Theorem 3.2 and completed in Theorem 3.3. We consider the vertical strip $\mathcal{S} = [-2\pi, 2\pi] \times \mathbb{R}$ and the rectangle $R = [-2\pi, 2\pi] \times [-2\pi/\alpha, 2\pi/\alpha]$. The restriction of an atomic measure ν to a line or to a point is denoted the same way as if ν were a continuous function. Then we say that (10) holds for $x_1 = 0$ if

$$\nu(2\pi, x_2) + \nu(-2\pi, x_2) = r\nu(0, x_2 + 2\pi) + r\nu(0, x_2 - 2\pi). \tag{11}$$

Theorem 3.2. *Let ν_0 be an atomic measure which satisfies* (9) *and* (11) *and whose support is a locally finite set $F \subset \mathcal{S}$. Then there exists a unique crystalline measure $\nu \in \mathcal{W}$ whose restriction to $\mathcal{S}$ is ν_0.*

The proof is almost obvious. We rewrite (10) as an evolution equation and treat x_1 as a time variable. We obtain

$$\nu(x_1, x_2)+\nu(x_1-4\pi, x_2)=r\nu(x_1-2\pi, x_2+2\pi)+r\nu(x_1-2\pi, x_2-2\pi). \tag{12}$$

The initial condition is ν_0. Then (12) is used to move from the vertical strip $\mathcal{S} = [-2\pi, 2\pi] \times \mathbb{R}$ to the vertical strip $\mathcal{S}_1 = [0, 4\pi] \times \mathbb{R}$. Next we iterate to define ν on the vertical strip $\mathcal{S}_m = [2(m-1)\pi, 2m\pi] \times \mathbb{R}$, $m \geq 2$. The treatment of the left vertical strips is identical. The extended atomic measure ν still satisfies (9) and (10). This ends the proof. As was stated earlier, the inverse Fourier transform σ of an atomic measure ν defined by Theorem 3.2 is a crystalline measure supported by M_r.

Here is a slight improvement on Theorem 3.2. We start from an arbitrary atomic measure ν_R carried be a finite subset of $R = [-2\pi, 2\pi] \times [-2\pi/\alpha, 2\pi/\alpha]$. Does there exist a unique crystalline measure $\nu \in \mathcal{W}$ whose restriction to R is ν_R? Here is the answer. We first construct an atomic measure ν_0 enjoying the following three properties: (a) ν_0 is carried by $\mathcal{S} = [-2\pi, 2\pi] \times \mathbb{R}$, (b) ν_0 satisfies (9) on $\mathcal{S}$ and (11), and (c) the restriction of ν_0 to R is ν_R. Once ν_0 is constructed the crystalline measure $\nu \in \mathcal{W}$ is defined as above. The only obstruction to the existence of $\nu \in \mathcal{W}$ is the construction of ν_0. There are two issues. The first obstruction comes from the periodicity of ν_0 given by (9). It forces the restriction $\nu_R(x_1, \pi/\alpha)$ of ν_R to the upper horizontal side of R to be identical to the restriction $\nu_R(x_1, -\pi/\alpha)$ of ν_R to the lower horizontal side of R. Then ν_R is the restriction to R of a unique atomic measure ν_0 satisfying (9). It remains to check (11). This provides the second obstruction. Indeed let us consider the restrictions of ν_R to the two vertical sides of R and to $\{0\} \times [-2\pi/\alpha, 2\pi/\alpha]$. These three atomic measures are extended by the periodicity defined by (9) and the corresponding atomic measures are denoted by θ, η, and κ. These three measures shall satisfy (11). This reads

$$\theta(x_2) + \eta(-2\pi, x_2) = r\kappa(x_2 + 2\pi) + r\kappa(x_2 - 2\pi).$$

We can conclude

Theorem 3.3. *Let ν_R be an atomic measure supported by a finite subset of the rectangle $R = [-2\pi, 2\pi] \times [-2\pi/\alpha, 2\pi/\alpha]$. Let us assume that ν_R satisfies* (11) *and that $\nu_R(x_1, \pi/\alpha) = \nu_R(x_1, -\pi/\alpha)$. Then ν_R is the restriction to R of a unique crystalline measure $\nu \in \mathcal{W}$. The inverse Fourier transform of ν is a crystalline measure σ supported by M_r. Conversely, any crystalline measure σ supported by M_r is obtained by this construction.*

4. Proof of Lemma 3.1

Let ν be a Radon measure satisfying (9) and (10). If g is a compactly supported continuous function, then the convolution product $f = \nu * g$ is a continuous function satisfying (9) and (10). It suffices to prove that f is a tempered distribution to conclude. This is implied by the following lemma:

Lemma 4.1. *There exists a constant C such that for every $y \in \mathbb{R}^2$ and for every continuous solution f of* (9) *and* (10) *we have*

$$\int_{R+y} |f(x)|^2\, dx \le C \int_R |f(x)|^2\, dx. \tag{13}$$

The Fourier series of the $2\pi/\alpha$-periodic continuous function $x_2 \mapsto f(x_1, x_2)$ is denoted by $\sum_k a_k(x_1) \exp(i\alpha k x_2)$. Then for any x_1 Parseval's theorem implies $\sum_k |a_k(x_1)|^2 = \alpha/2\pi \int_{-\pi/\alpha}^{\pi/\alpha} |f(x_1, x_2)|^2\, dx_2$. Next Fubini's theorem yields

$$\sum_k \int_{-2\pi}^{2\pi} |a_k(x_1)|^2\, dx_1 = \alpha/2\pi \int_R |f(x)|^2\, dx. \tag{14}$$

We now plug (10) into the Fourier expansion

$$f(x_1, x_2) = \sum_k a_k(x_1) \exp(i\alpha k x_2).$$

Then for any $k \in \mathbb{Z}$ we obtain

$$2r \cos(2\pi k\alpha)\, a_k(x_1) = a_k(x_1 + 2\pi) + a_k(x_1 - 2\pi).$$

Therefore $a_k(x_1)$ is a mean-periodic function of x_1 and can be expanded as a generalized Fourier series [**2**]. The possibly complex frequencies which appear in this expansion are the roots $u \in \mathbb{C}$ of $\cos(2\pi u) - r\cos(2\pi k\alpha) = 0$. Since $0 < r < 1$ these roots are the real numbers $\lambda_m \in T$ defined by $\lambda_m = m \pm \theta_r(k\alpha)$, $m \in \mathbb{Z}$. This implies that $a_k(x_1) = \exp(i\theta_r(k\alpha)x_1)b_k(x_1) + \exp(-i\theta_r(k\alpha)x_1)c_k(x_1)$ where b_k and c_k are two 2π-periodic continuous functions. Since $\|\theta_r - 1/4\|_\infty < 1/2$ the set T

is uniformly discrete. Therefore there exists a constant C such that for any $y_1 \in \mathbb{R}$ we have

$$\int_{y_1-2\pi}^{y_1+2\pi} |a_k(x_1)|^2\, dx_1 \le C \int_{-2\pi}^{2\pi} |a_k(x_1)|^2\, dx_1, \tag{15}$$

where C does not depend on k or y_1. Finally, (15) and (14) imply (13) when $y = (y_1, 0)$. This yields (13) in full generality since f is periodic in the second variable.

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Centre Borelli, ENS-Paris-Saclay, CNRS, Université Paris-Saclay, France
E-mail address: yves.meyer305@orange.fr

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