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CLASSICAL AND UNIFORM EXPONENTS OF MULTIPLICATIVE p-ADIC APPROXIMATION

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Abstract: Let p be a prime number and ξ an irrational p-adic number. Its irrationality exponent $\mu(\xi)$ is the supremum of the real numbers μ for which the system of inequalities

$$0 < \max\{|x|, |y|\} \le X, \quad |y\xi - x|_p \le X^{-\mu}$$

has a solution in integers x, y for arbitrarily large real number X. Its multiplicative irrationality exponent $\mu^{\times}(\xi)$ (resp., uniform multiplicative irrationality exponent $\widehat{\mu}^{\times}(\xi)$) is the supremum of the real numbers $\widehat{\mu}$ for which the system of inequalities

$$0 < |xy|^{1/2} \le X, \quad |y\xi - x|_p \le X^{-\widehat{\mu}}$$

has a solution in integers x, y for arbitrarily large (resp., for every sufficiently large) real number X. It is not difficult to show that $\mu(\xi) \le \mu^{\times}(\xi) \le 2\mu(\xi)$ and $\widehat{\mu}^{\times}(\xi) \le 4$. We establish that the ratio between the multiplicative irrationality exponent $\mu^{\times}$ and the irrationality exponent μ can take any given value in $[1, 2]$. Furthermore, we prove that $\widehat{\mu}^{\times}(\xi) \le (5+\sqrt{5})/2$ for every p-adic number ξ.

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1. Introduction

Let α be an irrational real number. Its irrationality exponent $\mu(\alpha)$ is the supremum of the real numbers μ for which

$$0 < |y\alpha - x| \le \max\{|x|, |y|\}^{-\mu+1} \tag{1.1}$$

or, equivalently,

$$0 < |\alpha - x/y| \le \max\{|x|, |y|\}^{-\mu} \tag{1.2}$$

has infinitely many solutions in nonzero integers x, y. Since, for all nonzero integers x, y with $|y\alpha - x| \le 1$, we have

$$\min\{|x|, |y|\} \ge \min\{|\alpha|, |\alpha|^{-1}\} \cdot \max\{|x|, |y|\} - 1,$$

the integers $|x|$ and $|y|$ in (1.1) and (1.2) have the same order of magnitude and we can replace $\max\{|x|, |y|\}$ in (1.1) and (1.2) by $|xy|^{1/2}$. The same observation does not hold for rational approximation in p-adic fields, where similar definitions give rise to two different irrationality exponents.

Throughout this paper, we let p denote a prime number and $\mathbb{Q}_p$ the field of p-adic numbers. Let ξ be an irrational p-adic number. The irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ for which

$$0 < |y\xi - x|_p \le \max\{|x|, |y|\}^{-\mu} \tag{1.3}$$

has infinitely many solutions in nonzero integers x, y. Unlike in the real case, the integers $|x|$ and $|y|$ in (1.3) do not necessarily have the same order of magnitude, and one of them can be much larger than the other one. This has recently been pointed out by de Mathan [**14**], who studied whether p-adic numbers ξ such that

$$\inf_{x,y\neq 0} |xy| \cdot |y\xi - x|_p > 0$$

actually exist; see also [**1, 8**].

Consequently, it is meaningful to also consider the multiplicative irrationality exponent $\mu^\times(\xi)$ of ξ defined as the supremum of the real numbers $\mu^\times$ for which

$$0 < |y\xi - x|_p \leq (|xy|^{1/2})^{-\mu^\times} \tag{1.4}$$

has infinitely many solutions in nonzero integers x, y. It follows from the Minkowski theorem (see [**11**] or [**12**]) and the obvious inequalities $\max\{|x|,|y|\} \leq |xy| \leq (\max\{|x|,|y|\})^2$ valid for all nonzero integers x, y that we have

$$2 \leq \mu(\xi) \leq \mu^\times(\xi) \leq 2\mu(\xi). \tag{1.5}$$

An easy covering argument given in Section 4 shows that $\mu^\times(\xi) = 2$ for almost all p-adic numbers ξ. Furthermore, the right-hand side inequality of (1.5) can be an equality: for any sufficiently large integer c, the p-adic number $\xi_c = 1 + \sum_{j\geq 1} p^{c^j}$ is well approximated by integers obtained by truncating its Hensel expansion and it satisfies $\mu^\times(\xi_c) = 2\mu(\xi_c)$; see Theorem 2.1.

Our first results, contained in Section 2, are concerned with the study of the spectra of the exponents of approximation μ and $\mu^\times$, that is, the set of values taken by these exponents. We also investigate the spectrum of their quotient $\mu^\times/\mu$ and show that it is equal to the whole interval $[1,2]$.

Besides the exponents of approximation μ and $\mu^\times$, we consider the uniform exponents $\widehat{\mu}$ and $\widehat{\mu}^\times$ defined as follows.

Definition 1.1. Let ξ be an irrational p-adic number. The uniform irrationality exponent $\widehat{\mu}(\xi)$ of ξ is the supremum of the real numbers $\widehat{\mu}$ for which the system

$$0 < \max\{|x|,|y|\} \leq X, \quad |y\xi - x|_p \leq X^{-\widehat{\mu}} \tag{1.6}$$

has a solution in integers x, y for every sufficiently large real number X. The uniform multiplicative irrationality exponent $\widehat{\mu}^\times(\xi)$ of ξ is the supremum of the real numbers $\widehat{\mu}^\times$ for which the system

$$0 < |xy|^{1/2} \leq X, \quad |y\xi - x|_p \leq X^{-\widehat{\mu}^\times} \tag{1.7}$$

has a solution in integers x, y for every sufficiently large real number X.

Let us note that, besides the classical exponent (where the points (x,y) belong to a square of area $4X^2$ centered at the origin) and the multiplicative exponent (where the points (x,y) belong to a set of area $16X^2(\log X)$ bounded by four branches of hyperbola), we can also consider weighted exponents (where the points (x,y) belong to a rectangle of area $4X^2$ centered at the origin). Although most of our results can be extended to the weighted setting, for simplicity we restrict our attention to the somehow more natural exponents $\mu^\times$ and $\widehat{\mu}^\times$ defined above.

We point out that x and y are not assumed to be coprime in (1.3), (1.4), (1.6), or in (1.7). Adding this assumption would not change the values of $\mu(\xi)$ and $\mu^\times(\xi)$, but would change the values of the uniform exponents at some p-adic numbers ξ. As in the real case, it is not difficult to show that $\widehat{\mu}(\xi) = 2$ for every irrational p-adic number ξ (this follows from [**13**, Satz 2]; see also Lemma 5.1 below). This implies that every irrational p-adic number ξ satisfies

$$2 = \widehat{\mu}(\xi) \le \widehat{\mu}^{\times}(\xi) \le 2\widehat{\mu}(\xi) = 4. \tag{1.8}$$

Furthermore, the example of the p-adic numbers ξ_c defined above shows that the exponent $\widehat{\mu}^{\times}$ takes values exceeding 2; see Theorem 2.1. Thus, unlike $\widehat{\mu}$, this exponent is not trivial and deserves to be studied more closely. Among several results, stated in Section 3, we prove that $\widehat{\mu}^{\times}$ is bounded from above by $(5+\sqrt{5})/2$, thereby improving the upper bound 4 given by (1.8). Thus, unlike (1.5), the inequalities in (1.8) are not best possible.

Throughout this paper, the numerical constants implied by $\ll$ are always positive and depend at most on the prime number p. Furthermore, the symbol $\asymp$ means that both inequalities $\ll$ and $\gg$ hold.

2. On the spectra of $\mu^{\times}$, $\widehat{\mu}^{\times}$, and $\mu^{\times}/\mu$

We begin with explicit examples of lacunary Hensel expansions, which include the p-adic numbers ξ_c defined in Section 1.

Theorem 2.1. *Let $(a_k)_{k\ge 0}$ be an increasing sequence of non-negative integers with $a_0 = 0$ and $a_{k+1} \ge 2a_k$ for every sufficiently large integer k. Define*

$$\xi = \sum_{k=0}^{\infty} p^{a_k} = 1 + p^{a_1} + p^{a_2} + \cdots .$$

Set

$$c = \liminf_{k\to\infty} \frac{a_{k+1}}{a_k}, \quad d = \limsup_{k\to\infty} \frac{a_{k+1}}{a_k},$$

where c, d are in $[2, +\infty]$. Then, we have

$$\mu(\xi) = d, \quad \mu^{\times}(\xi) = 2d, \tag{2.1}$$

and

$$3 - \frac{1}{c} \le \widehat{\mu}^{\times}(\xi) \le 3 + \frac{1}{d-1}. \tag{2.2}$$

The left-hand equality of (2.1) was established in [**5**], the best rational approximations being given by the integers $\sum_{j=0}^{J} p^{a_j}$, with $J \ge 1$, obtained by truncation of the Hensel expansion of ξ. In view of the definition of $\mu^{\times}$ and of (1.5), this implies the right-hand equality of (2.1). The left-hand inequality of (2.2) is proved in Section 5, while the right-hand inequality is derived from (3.1) below. For small values of d, Theorem 3.1 below slightly sharpens the right-hand inequality of (2.2). We believe that the left-hand inequality in (2.2) is actually an equality.

Recall that a p-adic Liouville number is, by definition, an irrational p-adic number whose irrationality exponent is infinite. The case where c and d are infinite yields the following statement.

Corollary 2.2. *The p-adic Liouville number*

$$\xi_{\infty} := \sum_{j=1}^{\infty} p^{j!}$$

satisfies $\widehat{\mu}^{\times}(\xi_{\infty}) = 3$. Consequently, the spectrum of $\widehat{\mu}^{\times}$ contains 3.

The inequalities in (1.5) motivate the study of the joint spectrum of the exponents μ and $\mu^{\times}$ and of the spectrum of their quotient $\mu^{\times}/\mu$, which, by (1.5), is included in the interval $[1, 2]$.

Theorem 2.3. *For any pair of real numbers* $(\mu, \mu^{\times})$ *satisfying*

$$\mu^{\times} > 5 + \sqrt{17}, \quad \frac{\mu^{\times}}{2} \le \mu \le \mu^{\times}, \tag{2.3}$$

there exists a p-adic number ξ *such that* $\mu^{\times}(\xi) = \mu^{\times}$ *and* $\mu(\xi) = \mu$. *Consequently, the spectrum of the quotient* $\mu^{\times}/\mu$ *is equal to the whole interval* $[1, 2]$.

The restriction $\mu^{\times} > 5 + \sqrt{17}$ in Theorem 2.3 comes from the proof and has no reason to be best possible. We believe that (2.3) can be replaced by the inequalities $\max\{2, \mu^{\times}/2\} \le \mu \le \mu^{\times}$. The proof of Theorem 2.3 is technical. First, we construct in Section 5 well-approximable p-adic numbers ζ whose best approximations are controlled in a suitable way. Then, in Section 6, we show how to modify the Hensel expansion of ζ to get a p-adic number ξ satisfying the conclusion of Theorem 2.3.

Let dim denote the Hausdorff dimension. The p-adic analogue of the theorem of Jarník and Besicovitch ([**9, 10**]) asserts that, for every real number $\mu \ge 2$, we have

$$\dim(\{\xi \in \mathbb{Q}_p : \mu(\xi) \ge \mu\}) = \dim(\{\xi \in \mathbb{Q}_p : \mu(\xi) = \mu\}) = \frac{2}{\mu};$$

see [**3**] for a more general p-adic result. Combining this result with (1.5) and an easy covering argument given in Section 4, we deduce that

$$\dim(\{\xi \in \mathbb{Q}_p : \mu^{\times}(\xi) \ge \mu^{\times}\}) = \dim(\{\xi \in \mathbb{Q}_p : \mu^{\times}(\xi) = \mu^{\times}\}) = \frac{2}{\mu^{\times}}$$

holds for every real number $\mu^{\times} \ge 2$. Consequently, the spectrum of $\mu^{\times}$ is equal to the whole interval $[2, +\infty]$. It would be interesting to construct explicitly, for any real number $\mu^{\times} \ge 2$, a p-adic number $\xi_{\mu}^{\times}$ satisfying $\mu^{\times}(\xi_{\mu}^{\times}) = \mu^{\times}$. For $\mu^{\times} \ge 4$, such examples are given in Theorem 2.1.

Problem 2.4. For any real number $\mu^{\times}$ with $2 \le \mu^{\times} < 4$, construct explicitly a p-adic number $\xi_{\mu}^{\times}$ such that $\mu^{\times}(\xi_{\mu}^{\times}) = \mu^{\times}$.

The construction presented in Section 5 below may be helpful for answering Problem 2.4. However, if we impose the additional natural condition $\mu^{\times}(\xi_{\mu}^{\times}) > \mu(\xi_{\mu}^{\times})$, new difficulties occur.

In a subsequent work, we will study more closely the classical and uniform multiplicative exponents of p-adic numbers whose Hensel expansion is given by a classical combinatorial sequence, like the Thue–Morse sequence or a Sturmian sequence. Let us just note that the p-adic Thue–Morse number

$$\xi_{TM} = 1 + p^3 + p^5 + p^6 + p^9 + p^{10} + \cdots,$$

whose Hensel expansion is given by the Thue–Morse word over $\{0, 1\}$, satisfies $\mu(\xi_{TM}) = 2$ (see [**7**]) and $\mu^{\times}(\xi_{TM}) \ge 3$, where presumably this inequality is in fact an equality.

3. Upper bounds for the uniform exponent $\widehat{\mu}^{\times}$

In the main result of this section, we improve the trivial upper bound 4 given in (1.8) for the exponent of uniform approximation $\widehat{\mu}^{\times}$.

Theorem 3.1. *Any irrational p-adic number ξ satisfies*

$$\widehat{\mu}^{\times}(\xi) \le 3 + \frac{2}{\mu^{\times}(\xi) - 2}, \tag{3.1}$$

$$\mu^{\times}(\xi) \ge \widehat{\mu}^{\times}(\xi)^2 - 3\widehat{\mu}^{\times}(\xi) + 3, \tag{3.2}$$

and

$$\widehat{\mu}^{\times}(\xi) \le \frac{5+\sqrt{5}}{2} = 3.6180\ldots \tag{3.3}$$

The first assertion of Theorem 3.1 is stronger than the third one only when $\mu^{\times}(\xi)$ exceeds $3 + \sqrt{5} = 5.23\ldots$

If $\widehat{\mu}^{\times}(\xi) \ge 3$, then the combination of (3.1) and (3.2) gives

$$\widehat{\mu}^{\times}(\xi) \le 3 + \frac{2}{\widehat{\mu}^{\times}(\xi)^2 - 3\widehat{\mu}^{\times}(\xi) + 1},$$

thus

$$(\widehat{\mu}^{\times}(\xi) - 1)(\widehat{\mu}^{\times}(\xi)^2 - 5\widehat{\mu}^{\times}(\xi) + 5) \le 0,$$

and we obtain (3.3). Therefore, to establish Theorem 3.1, it is sufficient to prove (3.1) and (3.2).

Note that (3.2) is of interest only for putative ξ with $\widehat{\mu}^{\times}(\xi) > 3$. Combined with (1.5) it implies that

$$\mu(\xi) \ge \frac{\widehat{\mu}^{\times}(\xi)^2 - 3\widehat{\mu}^{\times}(\xi) + 3}{2}.$$

In particular, if $\widehat{\mu}^{\times}(\xi) > (3+\sqrt{13})/2 = 3.3027\ldots$, then $\mu(\xi) > 2$, thus, ξ is very well approximable. In other words, if $\mu(\xi) = 2$, then $\widehat{\mu}^{\times}(\xi) \le (3+\sqrt{13})/2$.

We display an immediate consequence of (3.1).

Corollary 3.2. *Any p-adic Liouville number ξ satisfies $\widehat{\mu}^{\times}(\xi) \le 3$.*

In view of Corollary 2.2, the upper bound 3 for Liouville numbers obtained in Corollary 3.2 is best possible. We cannot exclude that $\widehat{\mu}^{\times}$ is always bounded by 3.

For the proof of Theorem 3.1, we introduce the sequence $(x_k^{\times}, y_k^{\times})_{k\ge1}$ of multiplicative best approximations to ξ, defined in Section 7. We are able to get the stronger conclusion $\widehat{\mu}^{\times}(\xi) \le 3$ under certain conditions.

Theorem 3.3. *Assume that at least one of the following two claims holds:*

(i) *There exist $c > 0$ and arbitrarily large k such that $|x_k^{\times}| \ge c|y_k^{\times}|$ and $|x_{k+1}^{\times}| \ge c|y_{k+1}^{\times}|$.*

(ii) *There exist $c > 0$ and arbitrarily large k such that $|x_k^{\times}| \le c|y_k^{\times}|$ and $|x_{k+1}^{\times}| \le c|y_{k+1}^{\times}|$.*

Then we have $\widehat{\mu}^{\times}(\xi) \le 3$.

The upper bound $\frac{5+\sqrt{5}}{2}$ in Theorem 3.1 is obtained when, simultaneously, $|x_{2k}^{\times}|$ is very small compared to $|y_{2k}^{\times}|$ and $|x_{2k+1}^{\times}|$ is very large compared to $|y_{2k+1}^{\times}|$, or vice versa, for every sufficiently large integer k. We cannot exclude the existence of a p-adic number whose sequence of multiplicative best approximations has this property.

The main difference with the classical setting occurs when we estimate the p-adic value of the difference between distinct rational numbers. Let x, y, x', y' be nonzero integers, not divisible by p and such that $xy' \neq x'y$. Then, $|x/y - x'/y'|_p^{-1} = |xy' - x'y|_p^{-1}$ is at most equal to $|xy'| + |x'y|$, which can be much larger than the product $|xy|^{1/2}$ times $|x'y'|^{1/2}$, in particular when simultaneously $|x|$ is much larger than $|y|$ and $|y'|$ is much larger than $|x'|$. Thus, we cannot avoid using the trivial estimate $|xy' - x'y|_p^{-1} \leq 2\max\{|x|, |y|\}\max\{|x'|, |y'|\}$, which involves the sup norm.

It follows from (3.2) that

$$\dim(\{\xi \in \mathbb{Q}_p : \widehat{\mu}^\times(\xi) \geq \mu^\times\}) \leq \frac{2}{(\mu^\times)^2 - 3\mu^\times + 3}, \quad \mu^\times \in \left[3, \frac{5+\sqrt{5}}{2}\right].$$

Our results motivate the following question.

Problem 3.4. Determine the Hausdorff dimension of the sets

$$\{\xi \in \mathbb{Q}_p : \widehat{\mu}^\times(\xi) \geq \mu^\times\}, \quad \{\xi \in \mathbb{Q}_p : \widehat{\mu}^\times(\xi) = \mu^\times\}, \quad \mu^\times \in \left[2, \frac{5+\sqrt{5}}{2}\right].$$

We end this section with a remark. It follows from Theorem 3.1 that any p-adic number ξ with $\widehat{\mu}^\times(\xi) = \frac{5+\sqrt{5}}{2}$ also satisfies $\mu^\times(\xi) = 3 + \sqrt{5}$. A similar situation occurs with the extremal numbers defined by Roy [**16**]. These are transcendental real numbers α whose uniform exponent of quadratic approximation takes the maximal possible value, that is, for which we have $\widehat{w}_2(\alpha) = (3+\sqrt{5})/2$. Roy ([**16**]) proved that they satisfy

$$1 + w_2^*(\alpha) = 3 + \sqrt{5}, \quad 1 + \widehat{w}_2(\alpha) = \frac{5+\sqrt{5}}{2}, \tag{3.4}$$

where w_2^* and $\widehat{w}_2$ denote classical and uniform exponents of quadratic approximation. Subsequently, Moshchevitin ([**15**]) established that every irrational, non-quadratic real number α satisfies

$$w_2^*(\alpha) \geq \widehat{w}_2(\alpha)(\widehat{w}_2(\alpha) - 1),$$

that is,

$$1 + w_2^*(\alpha) \geq (1 + \widehat{w}_2(\alpha))^2 - 3(1 + \widehat{w}_2(\alpha)) + 3, \tag{3.5}$$

with equality when α satisfies (3.4). Furthermore, by [**6**, Inequality (2.5)], we also have

$$1 + \widehat{w}_2(\alpha) \leq 3 + \frac{2}{(1 + w_2^*(\alpha)) - 2}, \tag{3.6}$$

with equality when α satisfies (3.4). Since (3.5) and (3.6) are analogous to (3.2) and (3.1), respectively, this may suggest that the bounds of Theorem 3.1 are best possible.

4. Proofs of Theorem 2.1 and of the metrical statements

We begin with a brief proof of the metrical statements given in Sections 1 and 2. We direct the reader to [**2**, Chapter 6] for a concise introduction to the metric theory of p-adic numbers. Recall that for a p-adic number ξ and a positive integer k, the Haar measure of the closed disc $D(\xi, p^k)$ centered at ξ and of radius p^k,

$$D(\xi, p^k) = \{\zeta \in \mathbb{Q}_p : |\zeta - \xi|_p \leq p^{-k}\},$$

is equal to p^{-k}.

Let $\varepsilon > 0$ be given. Let ξ be a p-adic number in $\mathbb{Z}_p$ for which there are infinitely many integer pairs (x, y) with $xy \neq 0$ and $|y\xi - x|_p < (|xy|)^{-1-\varepsilon}$. Then, for every $X \geq 2$, there are infinitely many pairs (x, y) of coprime integers with $|xy| \geq X$ and $|\xi - x/y|_p < (|xy|)^{-1-\varepsilon}$. Thus, ξ is an element of the set

$$\bigcup_{|xy| \geq X} D(x/y, |xy|^{-1-\varepsilon}),$$

whose Haar measure is

$$\ll \sum_{1 \leq x \leq X} \sum_{y \geq X/x} (xy)^{-1-\varepsilon} + \sum_{1 \leq y \leq X} \sum_{x \geq X/y} (xy)^{-1-\varepsilon} + \sum_{x,y \geq X} (xy)^{-1-\varepsilon} \ll X^{-\varepsilon}(\log X).$$

Since the latter quantity tends to 0 as X tends to infinity, we deduce that the Haar measure of the set of p-adic numbers ξ satisfying $\mu^\times(\xi) > 2 + 2\varepsilon$ is zero. As ε is arbitrary, we conclude that almost all ξ satisfy $\mu^\times(\xi) \leq 2$.

Similarly, for a given real number $\mu^\times \geq 2$, the sum

$$\sum_{x,y \geq 1,\, xy \geq X} (xy)^{-s\mu^\times/2}$$

converges for every $s > 2/\mu^\times$. This eventually implies that

$$\dim(\{\xi \in \mathbb{Q}_p : \mu^\times(\xi) \geq \mu^\times\}) \leq \frac{2}{\mu^\times}.$$

We omit the details.

Proof of Theorem 2.1: Let ξ be as in the theorem and define the rational integers

$$Q_k = \sum_{j=0}^{k} p^{a_j}, \quad k \geq 1.$$

Then with $c_k = a_{k+1}/a_k$ for $k \geq 1$ we get

$$|\xi - Q_k|_p = \left| \sum_{j=k+1}^{\infty} p^{a_j} \right|_p \asymp p^{-a_{k+1}} \asymp Q_{k+1}^{-1} \asymp Q_k^{-c_k}, \quad k \geq 1.$$

This in particular shows that $\mu(\xi) \geq d$ and $\mu^\times(\xi) \geq 2d$, since there are arbitrarily large k such that c_k is arbitrarily close to d. The equality $\mu(\xi) = d$ was established in [**5**]. By (1.5), this gives $\mu^\times(\xi) \leq 2d$ and proves (2.1). Set $Q_k^\times = \sqrt{Q_k}$ for $k \geq 1$. For a given integer X, let k be the index defined by $Q_k^\times \leq X < Q_{k+1}^\times$. We then have

$$Q_k^\times = \sqrt{Q_k \cdot 1} \leq X, \quad |\xi - Q_k|_p \asymp (Q_k^\times)^{-2c_k}.$$

Let M be the largest integral power of p smaller than $X/Q_k^\times$. Then

$$\sqrt{(M \cdot 1) \cdot (M \cdot Q_k)} \leq X$$

and

$$|M\xi - MQ_k|_p \ll M^{-1}(Q_k^\times)^{-2c_k} \ll \frac{Q_k^\times}{X}(Q_k^\times)^{-2c_k} \ll X^{\frac{1-2c_k}{c_k} - 1} = X^{-3+\frac{1}{c_k}}.$$

By the definition of c, we get the lower bound $\widehat{\mu}^\times(\xi) \geq 3 - 1/c$ in (2.2). The upper bound follows from (3.1). $\square$

5. Auxiliary results

The following easy lemma will be used several times in the sequel of the paper.

Lemma 5.1. *Let ξ be in $\mathbb{Q}_p$. If (x_1, y_1) and (x_2, y_2) are two linearly independent pairs of integers, then, setting*

$$X_i = \max\{|x_i|, |y_i|\}, \quad L_i = |y_i\xi - x_i|_p, \quad i = 1, 2,$$

we have

$$1 \le 2X_1X_2 \max\{L_1, L_2\}.$$

Lemma 5.1 easily implies that every irrational p-adic number ξ satisfies $\widehat{\mu}(\xi) \le 2$, thus $\widehat{\mu}(\xi) = 2$, a fact already stated in Section 1.

Proof: It follows from the identity

$$x_1y_2 - x_2y_1 = y_1(y_2\xi - x_2) - y_2(y_1\xi - x_1)$$

that

$$|x_1y_2 - x_2y_1|_p \le \max\{|y_2\xi - x_2|_p, |y_1\xi - x_1|_p\}.$$

Since $x_1y_2 \ne x_2y_1$, we get

$$|x_1y_2 - x_2y_1|_p \ge \frac{1}{|x_1y_2 - x_2y_1|} \ge \frac{1}{|x_1y_2| + |x_2y_1|} \ge \frac{1}{2X_1X_2}.$$

The combination of these inequalities proves the lemma. □

The proof of Theorem 2.3 is semi-constructive and uses

Theorem 5.2. *For any $\tilde{\mu} > 2$ and any $\epsilon > 0$, there exists ξ in $\mathbb{Z}_p$ with the following properties. There exists a sequence $((x_{j,0}, x_{j,1}))_{j\ge1}$ of pairs of coprime integers not divisible by p, whose moduli tend to infinity, satisfying the following properties:*

(i) *We have*

$$|x_{j,0}| \asymp |x_{j,1}|, \quad |x_{j,1}\xi - x_{j,0}|_p \asymp |x_{j,0}|^{-\tilde{\mu}} \asymp |x_{j,1}|^{-\tilde{\mu}}, \quad j \ge 1.$$

(ii) *We have*

$$\lim_{j\to\infty} \frac{\log|x_{j+1,0}|}{\log|x_{j,0}|} = \infty.$$

(iii) *For every integer pair (z_0, z_1) linearly independent of any pair $(x_{j,0}, x_{j,1})$ with $j \ge 1$, we have*

$$|z_1\xi - z_0|_p \gg \max\{|z_0|, |z_1|\}^{-2-\epsilon}. \tag{5.1}$$

The first property implies $\mu(\xi) \ge \tilde{\mu}$. The second one states that there are large gaps between consecutive very good approximations. The third one asserts that at most finitely many of the other approximations are very good, thus (when $\epsilon < \mu - 2$) we have $\mu(\xi) = \tilde{\mu}$. We note that (5.1) may be sharpened; indeed, using refined estimates the proof actually yields the lower bound $\gg_\epsilon Z^{-2}(\log Z)^{-1-\epsilon}$, where $Z = \max\{|z_0|, |z_1|\}$.

Preparation for the proof of Theorem 5.2: Fix $\epsilon > 0$. We first construct a p-adic number ξ in such a way that we control the quality of its best rational approximations, apart possibly from some good approximations (x, y), for which $|y\xi - x| \gg \max\{|x|, |y|\}^{-2-\epsilon}$.

More precisely, for a given sequence $(\mu_n)_{n\geq 1}$ with $\mu_n \geq 2+\epsilon$ for $n \geq 1$, we find ξ as the p-adic limit of a sequence of rationals p_n/q_n with $p_n \asymp q_n$ and upon writing

$$L_i = |p_i q_{i+1} - p_{i+1} q_i|_p, \quad H_i = \max\{|p_i|, |q_i|\}, \quad i \geq 0,$$

we have

$$H_n^{-\mu_n} \leq L_n \leq p H_n^{-\mu_n}, \quad n \geq 1, \tag{5.2}$$

and (5.1) holds for any (z_0, z_1) linearly independent of all pairs (p_n, q_n).

We use Schneider's continued fraction algorithm (see e.g. [**4**]) to construct a sequence p_n/q_n that converges at some given rate with respect to the p-adic metric to some p-adic number ξ. Start with

$$p_{-1} = 1,\, q_{-1} = 0, \quad p_0 = 0,\, q_0 = 1.$$

Then $|p_{-1}q_0 - q_{-1}p_0| = 1$. Then recursively, for $n \geq 0$, let

$$p_{n+1} = p_n + b_{n+1} p_{n-1}, \quad q_{n+1} = q_n + b_{n+1} q_{n-1}, \tag{5.3}$$

where each $b_n = p^{g_n}$ is an integer power of p to be suitably chosen. For all n, since $|b_n|_p = b_n^{-1}$ we calculate

$$\begin{aligned} |p_n q_{n+1} - p_{n+1} q_n|_p &= |p_n(q_n + b_{n+1} q_{n-1}) - q_n(p_n + b_{n+1} p_{n-1})|_p \\ &= |b_{n+1}(p_n q_{n-1} - p_{n-1} q_n)|_p \\ &= \frac{1}{b_{n+1}} \cdot |p_{n-1} q_n - p_n q_{n-1}|_p. \end{aligned}$$

Setting $L_n = |p_n q_{n+1} - p_{n+1} q_n|_p$ and $H_n = \max\{p_n, q_n\}$ for $n \geq 0$, we see that

$$L_n = \frac{1}{b_{n+1}} \cdot L_{n-1}, \quad n \geq 1. \tag{5.4}$$

Set $b_1 = 1$. Then, $p_1 = q_1 = H_1 = 1$ and it is easy to see that none of the p_n, q_n with $n \geq 1$ is divisible by p, hence

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|_p = |p_n q_{n+1} - p_{n+1} q_n|_p = \frac{1}{b_{n+1}} \cdot |p_{n-1} q_n - p_n q_{n-1}|_p.$$

Set $b_2 = p^2$. Then, $p_2 = 1$ and $q_2 = H_2 = p^2 + 1$. Since $p_1 = q_1 = 1$, it follows from (5.3) that $H_n = q_n$ for $n \geq 0$. Since $b_n \geq p$ for $n \geq 2$, the rational numbers p_n/q_n form a Cauchy sequence and thus converge with respect to the p-adic metric to some p-adic number ξ. Observe that

$$|q_n \xi - p_n|_p = \left| \xi - \frac{p_n}{q_n} \right|_p = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|_p = |p_{n+1} q_n - p_n q_{n+1}|_p = L_n, \quad n \geq 2,$$

where the second equality holds because

$$\left| \xi - \frac{p_{n+1}}{q_{n+1}} \right|_p < \left| \xi - \frac{p_n}{q_n} \right|_p.$$

Set $b_3 = p$. Since $H_3 L_2 H_2 = (H_2 + p) p^{-3} H_2 > p$ and

$$H_{n+1} L_n H_n \geq b_{n+1} H_{n-1} \cdot \frac{L_{n-1}}{b_{n+1}} \cdot H_n = H_n L_{n-1} H_{n-1}, \quad n \geq 3,$$

we get that

$$H_n^2 L_{n-1} > H_n L_{n-1} H_{n-1} > p, \quad n \geq 3. \tag{5.5}$$

It follows from (5.3) and the choice $b_1 = 1$, $b_2 = p^2$ that

$$q_{n+1} = H_{n+1} = H_n + b_{n+1} H_{n-1} \leq 2 \max\{H_n, b_{n+1} H_{n-1}\}. \tag{5.6}$$

Assume that for some fixed integer $N \geq 2$ we have constructed $p_1/q_1, \ldots, p_N/q_N$ with the desired approximation properties. We describe how to choose b_{N+1} (or g_{N+1}) to get the next p_{N+1}/q_{N+1}. Set $\gamma_N = L_{N-1}H_N H_{N-1}$ and observe that the inequality

$$L_{N-1} \leq \gamma_N \cdot H_N^{-1} H_{N-1}^{-1} \tag{5.7}$$

holds. Now, define recursively

$$g_{n+1} = \left\lfloor \frac{\log H_n^{\mu_n} L_{n-1}}{\log p} \right\rfloor, \quad n \geq N,$$

which is the largest integer such that $b_{n+1} = p^{g_{n+1}} \leq H_n^{\mu_n} L_{n-1}$. We readily conclude from (5.4) that then indeed (5.2) holds for all $n \geq 1$. Furthermore, it follows from (5.5) and $\mu_n \geq 2$ that $g_{n+1} \geq 1$ for $n \geq 3$.

By an easy induction, it follows from (5.3) that $p_k \asymp q_k$ for $k \geq 1$. Moreover, it is clear from the recursion (5.3) that p_n and q_n are coprime for all n.

Let us estimate the growth of the height sequence $(H_n)_{n\geq 1}$. For the initial value $n = N$, in the event that the maximum in (5.6) is $b_{n+1}H_{n-1} = b_{N+1}H_{N-1}$, by (5.4) and (5.2) we have

$$\begin{aligned} H_{N+1} &\leq 2b_{N+1}H_{N-1} = 2\frac{L_{N-1}}{L_N}H_{N-1} \leq 2L_{N-1}H_N^{\mu_N}H_{N-1} \\ &= 2H_N^{\mu_N}(L_{N-1}H_{N-1}) \leq 2\gamma_N H_N^{\mu_N-1}, \end{aligned} \tag{5.8}$$

where we have used our induction assumption (5.7). Then by (5.8) in view of (5.2) for n (which we have verified above) we have

$$L_N \leq pH_N^{-\mu_N} \leq 2p\gamma_N \cdot H_{N+1}^{-1}H_N^{-1}.$$

In other words, in the next step, similarly to (5.7), we have

$$L_N \leq \gamma_{N+1}H_{N+1}^{-1}H_N^{-1}, \quad \gamma_{N+1} = 2p\gamma_N.$$

Thus, similarly as in (5.8) above, we infer

$$H_{N+2} \leq 2\gamma_{N+1}H_{N+1}^{\mu_{N+1}-1}.$$

Iterating this process we see that there exists a real number $c > 1$ such that, for all $n \geq N$, we have

$$L_n \leq \gamma_N(2p)^n \cdot H_{n+1}^{-1}H_n^{-1} \ll c^n \cdot H_{n+1}^{-1}H_n^{-1} \tag{5.9}$$

and

$$H_n \leq \gamma_N(4p)^n \cdot H_{n-1}^{\mu_{n-1}-1} \ll c^n \cdot H_{n-1}^{\mu_{n-1}-1}. \tag{5.10}$$

Otherwise, if the maximum in (5.6) is H_{N+1}, then we directly get

$$H_{N+1} \leq 2H_N \leq 2H_N^{\mu_N-1},$$

since $\mu_n \geq 2 + \epsilon > 2$, which is even stronger than the estimates derived in the first case and we infer the same result.

It follows from (5.5) and (5.2) that

$$p < H_{n+1}H_nL_n \leq H_{n+1}H_n(pH_n^{-\mu_n}), \quad n \geq 3,$$

thus $H_{n+1} \geq H_n^{1+\epsilon}$ for large n. This shows that the sequence $(\log H_n)_{n\geq 2}$ grows exponentially fast, so in particular

$$c^n = H_n^{o(1)}, \quad n \to \infty.$$

It then follows from (5.9) and (5.10) that, for every $\eta > 0$, we have

$$L_n \ll_\eta H_{n+1}^{-1+\eta}H_n^{-1+\eta}, \quad H_n \ll_\eta H_{n-1}^{\mu_{n-1}-1+\eta}, \quad n \geq 2, \tag{5.11}$$

where the implicit positive constants depend only on η.

Now take an integer pair (z_0, z_1) which satisfies

$$|z_1\xi - z_0|_p \leq Z^{-2-\epsilon/2}, \tag{5.12}$$

where $Z = \max\{|z_0|, |z_1|\}$. We assume that (z_0, z_1) is linearly independent of all (p_n, q_n) and that z_0 and z_1 are coprime. Let k be the index such that $L_k \leq |z_1\xi - z_0|_p < L_{k-1}$. It then follows from Lemma 5.1 that

$$1 \leq 2H_{k-1}ZL_{k-1}.$$

Combined with (5.2), this gives

$$Z \geq \frac{H_{k-1}^{\mu_{k-1}-1}}{2p},$$

thus, by (5.11), we obtain

$$\frac{\log H_k}{\log Z} \leq 1 + \eta,$$

for arbitrarily small $\eta > 0$ and large enough k. On the other hand, again by Lemma 5.1 and (5.12), we get

$$1 \leq 2ZH_k|z_1\xi - z_0|_p \leq 2Z^{-1-\epsilon/2}H_k,$$

thus

$$H_k \geq \frac{Z^{1+\epsilon/2}}{2}.$$

By choosing $\eta = \epsilon/3$, we end up with a contradiction for large k. Thus, (5.12) cannot hold if Z is large enough.

Completion of the proof of Theorem 5.2: We choose for $(\mu_n)_{n\geq 1}$ the sequence

$$2+\epsilon, \tilde{\mu}, 2+\epsilon, 2+\epsilon, \ldots, 2+\epsilon, \tilde{\mu}, 2+\epsilon, 2+\epsilon, \ldots,$$

with very long blocks of $2+\epsilon$ separating two occurrences of $\tilde{\mu}$. We identify $x_{j,1} = q_{\sigma(j)}$ and $x_{j,0} = p_{\sigma(j)}$ for all $j \geq 1$, where the injective map $\sigma\colon \mathbb{N} \to \mathbb{N}$ is defined so that $\sigma(j)$ is the j-th index, where $\mu_n = \tilde{\mu}$. The property $\gcd(p, x_{j,0}x_{j,1}) = 1$ holds since we have noticed that $\gcd(p, p_nq_n) = 1$ for all n. Moreover, the large gaps guarantee the second claim of the theorem. It then follows from the observations above that $\gcd(x_{j,0}, x_{j,1}) = 1$ and $x_{j,0} \asymp x_{j,1}$ for all j, and the estimate $|x_{j,1}\xi - x_{j,0}|_p \asymp x_{j,0}^{-\tilde{\mu}}$ is immediate from (5.2). For the remaining p_n/q_n with n not in the image of σ, we have $\mu_n = 2+\epsilon$, and the estimate (5.1) is implied by (5.2) again. For all other pairs (z_0, z_1) we have already shown that (5.12) does not hold if $\max\{|z_0|, |z_1|\}$ is large enough. This completes the proof. □

6. Proof of Theorem 2.3

We prove Theorem 2.3 by using a similar strategy as in [**17**, Theorem 3.7]. The idea is to start with a p-adic number ζ given by Theorem 5.2 and to change its Hensel expansion by replacing its digits by 0 in certain large intervals J_i in order to obtain a p-adic number ξ with the requested properties. This will induce good *integer* approximations to ξ and thereby imply that $\mu^\times(\xi)$ is rather large, say $\mu^\times(\xi) \geq s\mu(\zeta)$ for some real number $s \geq 1$. We will see that the good approximations $x_{j,0}/x_{j,1}$ to ζ give rise to equally good rational approximations $y_{j,0}/y_{j,1}$ to ξ, thereby showing $\mu(\xi) \geq \mu(\zeta)$ as well. The most technical part is to show that there are no better rational approximations, that is, to verify the upper bounds $\mu(\xi) \leq \mu(\zeta)$ and $\mu^\times(\xi) \leq s\mu(\zeta)$. Here we essentially use the method developed in [**17**, Theorem 3.7] to show that putative good approximations to ξ would induce good approximations to ζ which are not among the $x_{j,0}/x_{j,1}$, in contradiction with Theorem 5.2.

Proof of Theorem 2.3: Fix t in $[1,2]$ and $\mu > 2$. Let ζ be in $\mathbb{Z}_p$, which satisfies the hypotheses of Theorem 5.2 for a small enough ϵ with $0 < \epsilon < 1/2$ depending on μ (this will be made more precise later) and with

$$\mu(\zeta) = \tilde{\mu} = t\mu.$$

Let $(x_{j,0}, x_{j,1})_{j\geq 1}$ denote the sequence of integer pairs given by Theorem 5.2. Without loss of generality, we assume $x_{j,0} > 0$ for $j \geq 1$ and that $x_{1,0}$ and $|x_{1,1}|$ are large. Let the Hensel expansion of ζ be

$$\zeta = \sum_{i=0}^{\infty} a_i p^i, \quad a_i \in \{0, 1, \dots, p-1\}.$$

For $j \geq 1$, set $\sigma_j = \lfloor \log x_{j,0} / \log p \rfloor$ so that

$$x_{j,0} \asymp |x_{j,1}| \asymp p^{\sigma_j}.$$

Then the second claim of Theorem 5.2 implies that σ_{j+1}/σ_j tends to infinity with j. Partition the integers greater than or equal to σ_1 into the intervals $I_j := [\sigma_j, \sigma_{j+1}) \cap \mathbb{Z}$.

We construct ξ with the desired properties by manipulating the Hensel expansion of ζ. First, we derive from the sequence $(\sigma_j)_{j\geq 1}$ two other positive integer sequences $(\tau_j)_{j\geq 1}$, $(\nu_j)_{j\geq 1}$ defined by

$$\nu_j = \lfloor t\mu\sigma_j \rfloor + C, \quad \tau_j = \lfloor \mu\nu_j \rfloor, \tag{6.1}$$

for some large positive integer constant C. For $x_{1,0}$ and $|x_{1,1}|$ sufficiently large, we have

$$\sigma_1 < \nu_1 < \tau_1 < \sigma_2 < \nu_2 < \tau_2 < \cdots,$$

and since the quotient σ_{j+1}/σ_j tends to infinity with j we also have

$$\lim_{j\to\infty} \frac{\sigma_{j+1}}{\tau_j} = \infty. \tag{6.2}$$

Let

$$J_j = \{\nu_j, \nu_j + 1, \dots, \tau_j\} = [\nu_j, \mu\nu_j] \cap \mathbb{Z}, \quad j \geq 1,$$

so that $J_j \subseteq I_j$, for j sufficiently large. Consider the p-adic number

$$\xi = \sum_{i=0}^{\infty} b_i p^i, \quad b_i \in \{0, 1, \dots, p-1\},$$

derived from ζ by setting

$$b_i = \begin{cases} 0, & i \in \cup_j (J_j \setminus \{\nu_j, \tau_j\}), \\ 1, & i \in \cup_j \{\nu_j, \tau_j\}, \\ a_i, & i \notin \cup_j J_j. \end{cases}$$

In other words the Hensel expansions of ξ and ζ coincide outside the intervals J_j, whereas the digits of ξ are all zero inside J_j, except at the first and last position of any J_j, where for technical reasons we put the digit 1. We will show that

$$\mu^{\times}(\xi) = 2\mu, \quad \mu(\xi) = \tilde{\mu} = t\mu, \tag{6.3}$$

if μ is sufficiently large. This will prove the theorem as t is arbitrary in $[1,2]$.

We start with the easiest of the four inequalities, namely

$$\mu^{\times}(\xi) \geq 2\mu. \tag{6.4}$$

Define the integers

$$N_j = \sum_{i=0}^{\nu_j} b_i p^i, \quad j \geq 1. \tag{6.5}$$

Clearly $N_j \ll p^{\nu_j}$. Moreover, as ξ has digits 0 at places ranging from ν_j+1 to $\tau_j-1 \approx \mu\nu_j$, the integers N_j approximate ξ at the order roughly μ, hence

$$|\xi - N_j|_p \ll p^{-\tau_j} \ll p^{-\nu_j\mu} \ll N_j^{-\mu} = (\sqrt{1\cdot N_j})^{-2\mu}. \tag{6.6}$$

We directly deduce (6.4) from (6.6). We note that $|\xi - N_j|_p \geq p^{-\tau_j}$ since $b_{\tau_j}=1$, and furthermore, since $b_{\nu_j}=1$, in fact we get $N_j \asymp p^{\nu_j}$. So we can refine (6.6) as

$$|\xi - N_j|_p \asymp p^{-\nu_j\mu} \asymp N_j^{-\mu}. \tag{6.7}$$

Next we show that

$$\mu(\xi) \geq t\mu. \tag{6.8}$$

By (6.7) the pairs $(x_0,x_1)=(N_j,1)$ induce approximations of quality exactly μ. By manipulating the pairs $(x_{j,0},x_{j,1})$ associated to ζ, we construct better approximating sequences $(y_{j,0})_{j\geq 1}$, $(y_{j,1})_{j\geq 1}$ such that, for any given $\varepsilon_1>0$ and sufficiently large j, we have

$$|y_{j,1}\xi - y_{j,0}|_p \ll \max\{|y_{j,0}|,|y_{j,1}|\}^{-t\mu+\varepsilon_1}. \tag{6.9}$$

This obviously implies (6.8). In the sequel, $\varepsilon_2,\varepsilon_3,\ldots$ denote positive real numbers that can be taken arbitrarily small as the index j tends to infinity.

To construct suitable $y_{j,0}$, $y_{j,1}$, recall that $|x_{j,0}| \asymp |x_{j,1}| \asymp p^{\sigma_j}$ and $\sigma_j < \nu_j < \tau_j < \sigma_{j+1}$ for $j\geq 1$, with

$$\lim_{j\to\infty}\frac{\nu_j}{\sigma_j}=t\mu, \quad \lim_{j\to\infty}\frac{\sigma_{j+1}}{\tau_j}=+\infty, \quad \lim_{j\to\infty}\frac{\tau_j}{\nu_j}=\mu.$$

For $i\geq 1$ define

$$u_i=\sum_{j\in J_i} a_j p^j - p^{\nu_i} - p^{\tau_i}, \quad u^{(i)}=u_1+u_2+\cdots+u_i.$$

Notice that by construction $\zeta-\xi$ is the infinite sum $u_1+u_2+\cdots$.

Moreover, assuming that $\epsilon<(\mu-2)/2$, we note that

$$p^{\tau_i(\frac{1}{2}-\epsilon)} \ll |u^{(i)}| \ll p^{\tau_i}, \quad i\geq 1. \tag{6.10}$$

The right-hand estimate is obvious. If the left-hand one is not satisfied, then $a_j=0$ for $\lfloor\eta_i\rfloor \leq j \leq \tau_i-1$, where $\eta_i := \left(\frac{1}{2}-\epsilon\right)\tau_i$ and $a_{\tau_i}=1$. But then the integer $M_i = \sum_{j\leq\lfloor\eta_i\rfloor} a_j p^j$ satisfies

$$|\zeta - M_i|_p \ll p^{-\tau_i} \ll M_i^{-\tau_i/\eta_i} \ll M_i^{-1/(\frac{1}{2}-\epsilon)} = M_i^{-2-\epsilon-\frac{3\epsilon+2\epsilon^2}{1-2\epsilon}},$$

a contradiction with Theorem 5.2 for large i.

We claim that if we set

$$y_{j,0}=x_{j,0}-u^{(j-1)}x_{j,1}, \quad y_{j,1}=x_{j,1}, \tag{6.11}$$

then indeed (6.9) holds. We rearrange

$$\begin{aligned} |y_{j,1}\xi - y_{j,0}|_p &= |x_{j,1}(\xi+u^{(j-1)}) - x_{j,0}|_p \\ &= |x_{j,1}(\xi+u^{(j-1)}-\zeta) + (x_{j,1}\zeta - x_{j,0})|_p \\ &\leq \max\{|x_{j,1}(\xi+u^{(j-1)}-\zeta)|_p, |x_{j,1}\zeta - x_{j,0}|_p\}. \end{aligned} \tag{6.12}$$

By assumption the latter term satisfies

$$|x_{j,1}\zeta - x_{j,0}|_p \ll x_{j,0}^{-t\mu}. \tag{6.13}$$

To estimate the former expression, note that by construction the Hensel expansions of $\xi + u^{(j-1)}$ and ζ coincide up to the $(\nu_j - 1)$-th digit (the last digit before the interval J_j starts). Thus, we have

$$|x_{j,1}(\xi + u^{(j-1)} - \zeta)|_p \le |\xi + u^{(j-1)} - \zeta|_p \ll p^{-\nu_j} \ll x_{j,0}^{-\nu_j/\sigma_j} \ll x_{j,0}^{-t\mu}, \tag{6.14}$$

where the last estimate follows from (6.1). By combining (6.12), (6.13), and (6.14), we derive

$$|y_{j,1}\xi - y_{j,0}|_p \ll x_{j,0}^{-t\mu} \ll |x_{j,1}|^{-t\mu} = |y_{j,1}|^{-t\mu}. \tag{6.15}$$

Now for given $\varepsilon_2 > 0$ and j large enough, we get from (6.2) the estimate

$$|u^{(j-1)}| \ll p^{\tau_{j-1}} < p^{\varepsilon_2 \sigma_j} \ll x_{j,0}^{\varepsilon_2}.$$

Combined with (6.15) and $x_{j,0} \asymp |x_{j,1}|$, this gives

$$|y_{j,0}| = |x_{j,0} - u^{(j-1)}x_{j,1}| \ll x_{j,0} + |u^{(j-1)}| \cdot |x_{j,1}| \ll |x_{j,1}|^{1+\varepsilon_2} = |y_{j,1}|^{1+\varepsilon_2}, \tag{6.16}$$

hence we derive (6.9) from (6.15), and consequently (6.8) follows.

At this point we notice that the reverse inequality $|y_{j,0}| \gg |y_{j,1}|$ follows similarly via

$$|y_{j,0}| = |x_{j,0} - u^{(j-1)}x_{j,1}| \gg |x_{j,1}| \cdot |u^{(j-1)}| \ge |x_{j,1}| = |y_{j,1}|, \tag{6.17}$$

where we use that $x_{j,0} \asymp |x_{j,1}|$ and the fact that $|u^{(j)}|$ tends to infinity with j by (6.10). So we keep in mind for the sequel that all the integers $x_{j,0}$, $|x_{j,1}|$, $|y_{j,0}|$, $|y_{j,1}|$ are of comparable size, in the sense that, for every $\eta > 0$ and for every sufficiently large j, we have

$$\max\{x_{j,0}, |x_{j,1}|, |y_{j,0}|, |y_{j,1}|\} \le (\min\{x_{j,0}, |x_{j,1}|, |y_{j,0}|, |y_{j,1}|\})^{1+\eta}.$$

Next we show the reverse estimate

$$\mu(\xi) \le t\mu. \tag{6.18}$$

Assume otherwise that there are integers x, y with $\max\{|x|, |y|\}$ arbitrarily large and $\theta > t\mu$ such that

$$|y\xi - x|_p \le \max\{|x|, |y|\}^{-\theta}. \tag{6.19}$$

We may assume that x and y are coprime. We distinguish two cases.

Case 1: The pair (x, y) is among the pairs $\pm(y_{j,0}, y_{j,1})$ defined in (6.11) above. We show the reverse estimate to (6.9), that is,

$$|y_{j,1}\xi - y_{j,0}|_p \gg \max\{|y_{j,0}|, |y_{j,1}|\}^{-t\mu}, \quad j \ge 1. \tag{6.20}$$

This clearly contradicts (6.19) for these pairs. By assumption the reverse estimate to (6.13) holds as well, i.e.

$$|x_{j,1}\zeta - x_{j,0}|_p \gg x_{j,0}^{-t\mu}.$$

Recall that for a, b in $\mathbb{Q}_p$ with $|a|_p \ne |b|_p$ we have $|a + b|_p = \max\{|a|_p, |b|_p\}$. Now by taking C large enough in (6.1), we can guarantee that

$$|x_{j,1}(\xi + u^{(j-1)} - \zeta)|_p < |x_{j,1}\zeta - x_{j,0}|_p,$$

thus we get

$$|x_{j,1}\zeta - x_{j,0}|_p = |x_{j,1}(\xi + u^{(j-1)}) - x_{j,0}|_p = |y_{j,1}\xi - y_{j,0}|_p,$$

by (6.11). We conclude that

$$|y_{j,1}\xi - y_{j,0}|_p \gg x_{j,0}^{-t\mu} \gg |x_{j,1}|^{-t\mu} = |y_{j,1}|^{-t\mu} \ge \max\{|y_{j,0}|, |y_{j,1}|\}^{-t\mu},$$

which is our desired lower bound (6.20).

Case 2: The pair (x,y) is not among the pairs $\pm(y_{j,0},y_{j,1})$. Write

$$H=\max\{|x|,|y|\}.$$

In fact we show that then

$$|y\xi-x|_p \gg H^{-\mu-\varepsilon_3}. \tag{6.21}$$

Since $\mu\le t\mu$ this clearly implies (6.18). Note that the bound is optimal as by (6.7) it is attained with $\varepsilon_3=0$ by $(x,y)=(N_j,1)$. However, by the same argument, we can exclude these pairs and the pairs $(-N_j,-1)$ in our investigation. For other pairs, we verify (6.21) indirectly by showing that any pair (x,y) that violates the inequality induces a reasonably good rational approximation to ζ which is not among the $x_{j,0}/x_{j,1}$, contradicting the third claim of Theorem 5.2. So, assume that for some (x,y) as above we have

$$|y\xi-x|_p\le H^{-\mu-\varepsilon_3}. \tag{6.22}$$

Below (6.17) we noticed that $|y_{j,1}|=|x_{j,1}|$ and $|y_{j,0}|$ are of comparable size, all being roughly equal to $x_{j,0}\asymp|x_{j,1}|$. In particular, the sequences $(|y_{j,0}|)_{j\ge1}$ and $(|y_{j,1}|)_{j\ge1}$ are increasing. For a pair (x,y) satisfying (6.22), let h be the index with

$$\max\{|y_{h,0}|,|y_{h,1}|\}<H\le\max\{|y_{h+1,0}|,|y_{h+1,1}|\}.$$

By (6.22),

$$\max\{|y_{h,0}|,|y_{h,1}|\}H|y\xi-x|_p\le H^{2-\mu-\varepsilon_3}<\frac{1}{2},$$

thus Lemma 5.1 and (6.9) imply that

$$1\le 2\max\{|y_{h,0}|,|y_{h,1}|\}H\max\{|y_{h,0}|,|y_{h,1}|\}^{-t\mu+\varepsilon_1}.$$

Likewise, again by (6.9),

$$\max\{|y_{h+1,0}|,|y_{h+1,1}|\}H|y_{h+1,1}\xi-y_{h+1,0}|_p<\frac{1}{2},$$

thus Lemma 5.1 and (6.22) imply that

$$1\le 2\max\{|y_{h+1,0}|,|y_{h+1,1}|\}H|y\xi-x|_p\le 2\max\{|y_{h+1,0}|,|y_{h+1,1}|\}H^{1-\mu-\varepsilon_3}.$$

Consequently, we have established that

$$\max\{|y_{h+1,0}|,|y_{h+1,1}|\}^{\frac{1}{\mu-1}+\varepsilon_4}\gg H\gg\max\{|y_{h,0}|,|y_{h,1}|\}^{t\mu-1-\varepsilon_4}. \tag{6.23}$$

However, the right-hand estimate is not sufficient. We show the stronger lower bound

$$H\gg\max\{|y_{h,0}|,|y_{h,1}|\}^{t\mu(\mu-1)-\varepsilon_5}, \tag{6.24}$$

again by application of Lemma 5.1. For simplicity write

$$s=\frac{\log H}{\log x_{h,0}}.$$

Recalling that all $x_{j,0}$, $x_{j,1}$, $y_{j,0}$, $y_{j,1}$ are in absolute value roughly of the same size, we have to show $s\ge t\mu(\mu-1)-\varepsilon_6$ for arbitrarily small $\varepsilon_6>0$. According to (6.23), upon increasing ε_4 to take into account the implied constants if necessary, we can assume $s\ge t\mu-1-\varepsilon_7$ for arbitrarily small $\varepsilon_7>0$. On the one hand, with N_j as in (6.5) we deduce from (6.6) that

$$\max\{|y\xi-x|_p,|\xi-N_h|_p\}\ll\max\{H^{-\mu-\varepsilon_3},p^{-\nu_h\mu}\}$$

and, since

$$H=x_{h,0}^s,\quad p^{\nu_h}\asymp x_{h,0}^{\nu_h/\sigma_h}\asymp x_{h,0}^{t\mu},$$

we get

$$\max\{|y\xi-x|_p,|\xi-N_h|_p\}\ll x_{h,0}^{-\mu\min\{t\mu,s\}}.$$

On the other hand, as $\gcd(x,y) = \gcd(N_j,1) = 1$ and we have assumed $(x,y) \neq \pm(N_j,1)$, these pairs are linearly independent. Hence, from Lemma 5.1 and $N_h \ll p^{\nu_h} \ll x_{h,0}^{t\mu}$, we get

$$\max\{|y\xi - x|_p, |\xi - N_h|_p\} \gg H^{-1}N_h^{-1} \gg x_{h,0}^{-s-t\mu}.$$

This gives the lower bound

$$s + t\mu + \varepsilon_8 \geq \mu \min\{s, t\mu\}. \tag{6.25}$$

If $s \leq t\mu$, then we get $s \leq t\mu/(\mu-1) + \varepsilon_8$. However, in view of $s \geq t\mu - 1 - \varepsilon_7$ noticed above, as ε_7 and ε_8 can be arbitrarily small this gives a contradiction as soon as

$$\mu > 1 + \frac{t\mu}{t\mu - 1} = 2 + \frac{1}{t\mu - 1}.$$

Since $t\mu \geq \mu > 2$ a sufficient condition is

$$\mu > 3. \tag{6.26}$$

If $s > t\mu$, then we derive from (6.25) that $s + t\mu + \varepsilon_8 \geq t\mu^2$ or equivalently $s \geq t\mu(\mu-1) - \varepsilon_8$. Thus, as ε_8 can be arbitrarily small, we have shown (6.24).

Next observe that the triangle inequality gives

$$|x + yu^{(h)} - y\zeta|_p = |(x - y\xi) + y(\xi + u^{(h)} - \zeta)|_p \leq \max\{|x - y\xi|_p, |y(\xi + u^{(h)} - \zeta)|_p\},$$

so combined with (6.14) applied for $j = h+1$ and with (6.22) we conclude that

$$|x + yu^{(h)} - y\zeta|_p \ll \max\{x_{h+1,0}^{-t\mu}, H^{-\mu-\varepsilon_3}\}. \tag{6.27}$$

From (6.23) and since $|y_{h,0}| \ll |y_{h,1}|^{1+\varepsilon_2} = |x_{h,1}|^{1+\varepsilon_2}$ by (6.16), we use that $\mu > 2$ and $t \geq 1$ to check that

$$H \ll x_{h+1,0}^{\frac{1}{\mu-1}+\varepsilon_9} \ll x_{h+1,0}^{t},$$

so the right-hand expression in the maximum in (6.27) is larger than the left-hand one. With (6.23) and (6.24), and since $|y_{j,1}|$, $|y_{j,0}|$, and $x_{j,0}$ are of comparable size and τ_h/σ_h tends to $(t\mu)\mu = t\mu^2$ with h by (6.1), we estimate

$$\begin{aligned}\max\{|y|, |x + yu^{(h)}|\} &\leq |x| + |y| \cdot |u^{(h)}| \leq H + H \cdot p^{\tau_h+1} \ll H \cdot x_{h,0}^{\tau_h/\sigma_h} \\ &\ll H \cdot (H^{\frac{1}{(\mu-1)t\mu}+\varepsilon_{10}})^{\tau_h/\sigma_h} \ll H^{2+\frac{1}{\mu-1}+\varepsilon_{11}}.\end{aligned}$$

To sum up, we have shown

$$\begin{aligned}\max\{|y|, |x + yu^{(h)}|\} &\ll H^{2+\frac{1}{\mu-1}+\varepsilon_{11}}, \\ |x + yu^{(h)} - y\zeta|_p &\leq H^{-\mu-\varepsilon_3}.\end{aligned} \tag{6.28}$$

From (6.28), we get

$$\begin{aligned}-\frac{\log|x + yu^{(h)} - y\zeta|_p}{\log\max\{|y|, |x + yu^{(h)}|\}} &\geq (\mu + \varepsilon_3) \cdot \frac{\log H}{\log\max\{|y|, |x + yu^{(h)}|\}} \\ &\geq (\mu + \varepsilon_3) \cdot \left(2 + \frac{1}{\mu - 1} + \varepsilon_{11}\right)^{-1} > \frac{\mu^2 - \mu}{2\mu - 1} - \varepsilon_{12}.\end{aligned}$$

Thus we have found integers z_0, z_1 with

$$-\frac{\log|z_1\zeta - z_0|_p}{\log\max\{|z_0|, |z_1|\}} \geq \frac{\mu^2 - \mu}{2\mu - 1} - \varepsilon_{12}.$$

Thereby, as ε_{10} can be arbitrarily small, if

$$\mu > \frac{5 + \sqrt{17}}{2}, \tag{6.29}$$

then $\frac{\mu^2-\mu}{2\mu-1} > 2$ and we have constructed an approximation of order greater than 2 to ζ. If $\epsilon > 0$ from Theorem 5.2 for our ζ has been chosen small enough (depending on the given μ), concretely for

$$\epsilon = \frac{1}{2}\left(\frac{\mu^2-\mu}{2\mu-1} - 2\right),$$

by the assumptions of the theorem and since $\gcd(x,y) = \gcd(x_{j,0}, x_{j,1}) = 1$, this implies $(x + yu^{(h)}, y) = (x_{j,0}, x_{j,1})$ for some j. We assume this is the case and will derive a contradiction.

We first show that j cannot exceed h. Note that the case $j = h+1$ has already been treated in Case 1. Since $x_{j,0} \asymp |x_{j,1}|$ and $|u^{(j)}|$ tends to infinity with j by (6.10), we must have

$$|x| \asymp |y| \cdot |u^{(h)}| \asymp x_{j,0} \cdot p^{\tau_h} \asymp x_{j,0} \cdot x_{h,0}^{\tau_h/\sigma_h} \asymp x_{j,0} \cdot x_{h,0}^{t\mu^2}.$$

If $j \geq h+2$, then this clearly contradicts $|x| \leq H \ll x_{h+1,0}$ from (6.23). Now, we assume that $j \leq h$. It follows from (6.14) that

$$\begin{aligned}
|\xi y - x|_p &= |\xi x_{j,1} - x_{j,0} + x_{j,1}u^{(h)}|_p \\
&= |(u^{(h)} + \xi - \zeta)x_{j,1} + (\zeta x_{j,1} - x_{j,0})|_p \\
&\geq |\zeta x_{j,1} - x_{j,0}|_p - |(u^{(h)} + \xi - \zeta)x_{j,1}|_p \\
&\gg x_{j,0}^{-t\mu} - x_{h+1,0}^{-t\mu}.
\end{aligned}$$

Now the crude estimate $x_{h+1,0} \gg x_{h,0}^{\sigma_{h+1}/\sigma_h} \gg x_{h,0}^{\tau_h/\sigma_h} \gg x_{h,0}^{t\mu^2} \gg x_{h,0}^4 \gg x_{j,0}^4$ suffices to derive

$$|\xi y - x|_p \gg x_{j,0}^{-t\mu}.$$

But on the other hand by assumption (6.22) and (6.24) we get

$$|\xi y - x|_p \ll H^{-\mu} \ll x_{j,0}^{-t\mu^2(\mu-1)+\varepsilon_{13}}.$$

The combination of the latter inequalities gives the desired contradiction and ends the proof of (6.18). Thus, by (6.8), we have $\mu(\xi) = t\mu$.

In view of (6.4), it only remains for us to show

$$\mu^\times(\xi) \leq 2\mu. \tag{6.30}$$

We again distinguish between rationals $y_{j,0}/y_{j,1}$ and other rationals. Concerning the first family, again because $|y_{j,0}|$ and $|y_{j,1}|$ are of comparable size and by (6.20), we indeed derive that

$$|y_{j,1}\xi - y_{j,0}|_p \gg \left(\sqrt{|y_{j,0} \cdot y_{j,1}|}\right)^{-t\mu-\varepsilon_{14}}.$$

Thus the exponent restricted to this family satisfies $\mu^\times(\xi) \leq t\mu + \varepsilon_{14} \leq 2\mu + \varepsilon_{14}$. Since ε_{14} can be taken arbitrarily small the claim follows. Finally for the latter family, where $(x,y) \neq \pm(y_{j,0}, y_{j,1})$ and $xy \neq 0$, we conclude with (6.21) via

$$|y\xi - x|_p \gg \max\{|x|,|y|\}^{-\mu-\varepsilon_3} \geq (\sqrt{|xy|})^{-2\mu-2\varepsilon_3},$$

again giving $\mu^\times(\xi) \leq 2\mu + 2\varepsilon_3$, and the claim (6.30) follows as ε_3 can be taken arbitrarily small. The proof of (6.3) is complete. Thus, we have established that $\mu^\times(\xi) = 2\mu$.

We see that $\mu > (5+\sqrt{17})/2$ is the most restrictive one among the conditions (6.26), (6.29) we have collected on the way, which imposes $\mu^\times = 2\mu > 5 + \sqrt{17}$, that is, the restriction made in the theorem. □

7. Proofs of Theorems 3.1 and 3.3

Proof of (3.1) *in Theorem 3.1:* Assume $\mu^\times(\xi) > 2$ since (3.1) is trivial otherwise. Assume that for some $\mu > 2$ there exist nonzero coprime integers x, y with $|xy|$ arbitrarily large and such that

$$|y\xi - x|_p = (Q^\times)^{-\mu}, \quad Q^\times = \sqrt{|xy|}. \tag{7.1}$$

Set $Q = \max\{|x|, |y|\}$. Define A in $[1, 2]$ and τ by

$$Q = (Q^\times)^A, \quad \tau = \frac{\mu - A}{2}.$$

In the sequel, $\varepsilon_1, \varepsilon_2, \ldots$ denote positive real numbers that can be taken arbitrarily small as $Q^\times$ tends to infinity. Set

$$X = (Q^\times)^{\tau - \varepsilon_1}.$$

By (7.1), for any positive integer M with $\sqrt{|Mx \cdot My|} = MQ^\times \le X$, we have

$$\begin{aligned} |My\xi - Mx|_p \ge M^{-1}(Q^\times)^{-\mu} &\ge \frac{Q^\times}{X}(Q^\times)^{-\mu} \\ &\ge X^{-\frac{\mu-1}{\tau} - 1 - \varepsilon_2} = X^{-\frac{2(\mu-1)}{\mu-A} - 1 - \varepsilon_2}. \end{aligned} \tag{7.2}$$

Consider an integer pair $(\tilde{x}, \tilde{y})$ linearly independent of (x, y) and with $\sqrt{|\tilde{x}\tilde{y}|} \le X$. Set $\tilde{X} := \max\{|\tilde{x}|, |\tilde{y}|\}$ and observe that $\tilde{X} \le X^2$. By construction, we have

$$|y\xi - x|_p = (Q^\times)^{-\mu} = Q^{-\frac{\mu}{A}} \le Q^{-1}X^{-2} \le Q^{-1}\tilde{X}^{-1},$$

where we have used that

$$X^2 = (Q^\times)^{\mu - A - 2\varepsilon_1} = Q^{\frac{\mu - A}{A} - \varepsilon_3}.$$

It then follows from Lemma 5.1 that

$$\begin{aligned} |\tilde{y}\xi - \tilde{x}|_p \gg Q^{-1}\tilde{X}^{-1} \gg Q^{-1}X^{-2} &= X^{-2-\frac{A}{\tau - \varepsilon_1}} \\ &= X^{-2-\frac{A}{\tau} - \varepsilon_4} = X^{-\frac{2\mu}{\mu - A} - \varepsilon_4}. \end{aligned} \tag{7.3}$$

Since μ can be chosen arbitrarily close to $\mu^\times(\xi)$, we deduce from (7.2) and (7.3) that

$$\widehat{\mu}^\times(\xi) \le \sup_{A \in [1,2]} \max\left\{ \frac{3\mu^\times(\xi) - 2 - A}{\mu^\times(\xi) - A}, \frac{2\mu^\times(\xi)}{\mu^\times(\xi) - A} \right\}. \tag{7.4}$$

For $\mu^\times(\xi) \ge 4$ it is readily checked that, for any A in $[1, 2]$, the quantity on the left is greater than or equal to the quantity on the right. Hence, for $\mu^\times(\xi) \ge 4$, we have proved that

$$\widehat{\mu}^\times(\xi) \le \sup_{A \in [1,2]} \frac{3\mu^\times(\xi) - 2 - A}{\mu^\times(\xi) - A} = 3 + \frac{2}{\mu^\times(\xi) - 2}. \tag{7.5}$$

Since (7.5) clearly holds if $\mu^\times(\xi) < 4$ (recall that $\widehat{\mu}^\times(\xi) \le 4$ is always true), this proves the first claim (3.1) of the theorem. □

Actually, in the preceding proof, we have shown a slightly stronger result than (7.4), which we state in the following corollary for later use.

Corollary 7.1. *Let ξ be an irrational p-adic number. Assume that there exist A in $[1,2]$ and an infinite sequence $\mathcal{S}$ of pairs of nonzero integers (x,y) such that*

$$\limsup_{(x,y)\in\mathcal{S},\, \max\{|x|,|y|\}\to\infty} \frac{-\log|y\xi - x|_p}{\log\sqrt{|xy|}} = \mu^\times(\xi)$$

and

$$\lim_{(x,y)\in\mathcal{S},\, \max\{|x|,|y|\}\to\infty} \frac{\log\max\{|x|,|y|\}}{\log\sqrt{|xy|}} = A.$$

Then, we have

$$\widehat{\mu}^\times(\xi) \le \max\left\{ \frac{3\mu^\times(\xi) - 2 - A}{\mu^\times(\xi) - A}, \frac{2\mu^\times(\xi)}{\mu^\times(\xi) - A} \right\}. \tag{7.6}$$

In particular, if we have $A = 1$, that is, if $|x|$ and $|y|$ are of comparable size for every pair (x,y) in $\mathcal{S}$, then we obtain $\widehat{\mu}^\times(\xi) \le 3$.

Proof: The estimate (7.6) comes directly from the proof of Theorem 3.1 above. For the last assertion, observe that when $A = 1$ the left-hand side of the maximum in (7.6) is equal to 3, while the right-hand side is at most equal to 3 when $\mu^\times(\xi) \ge 3$. □

For a given p-adic number ξ with

$$\inf_{x,y\ne 0} |xy| \cdot |y\xi - x|_p = 0 \tag{7.7}$$

(this can be assumed, since otherwise $\mu^\times(\xi) = \widehat{\mu}^\times(\xi) = 2$), we define the sequence of integer pairs $((\tilde{x}_k^\times, \tilde{y}_k^\times))_{k\ge 1}$ by taking a pair of *coprime* integers (x,y) minimizing $|y\xi - x|_p$ among all the integer pairs with $0 < \sqrt{|xy|} \le Q$, and letting the positive real number Q grow to infinity. Write $\tilde{Q}_k^\times = \sqrt{|\tilde{x}_k^\times \tilde{y}_k^\times|}$ for $k \ge 1$. By construction, we have

$$\tilde{Q}_1^\times < \tilde{Q}_2^\times < \cdots, \quad |\tilde{y}_1^\times \xi - \tilde{x}_1^\times|_p > |\tilde{y}_2^\times \xi - \tilde{x}_2^\times|_p > \cdots.$$

However, we cannot guarantee that $\tilde{Q}_k^\times |\tilde{y}_k^\times \xi - \tilde{x}_k^\times|_p > \tilde{Q}_{k+1}^\times |\tilde{y}_{k+1}^\times \xi - \tilde{x}_{k+1}^\times|_p$ for every $k \ge 1$. Therefore, we extract a subsequence $((\tilde{x}_{i_k}^\times, \tilde{y}_{i_k}^\times))_{k\ge 1}$ from $((\tilde{x}_k^\times, \tilde{y}_k^\times))_{k\ge 1}$, where $i_1 = 1$ and, for $k \ge 1$, the index i_{k+1} is the smallest index $j > i_k$ such that $\tilde{Q}_j^\times |\tilde{y}_j^\times \xi - \tilde{x}_j^\times|_p < \tilde{Q}_{i_k}^\times |\tilde{y}_{i_k}^\times \xi - \tilde{x}_{i_k}^\times|_p$. This gives an infinite subsequence since ξ satisfies (7.7).

To simplify the notation, put $x_k^\times = \tilde{x}_{i_k}^\times$, $y_k^\times = \tilde{y}_{i_k}^\times$, and $Q_k^\times = \tilde{Q}_{i_k}^\times$, for $k \ge 1$.

For $k \ge 1$, by construction, $|y_{k+1}^\times \xi - x_{k+1}^\times|_p$ is smaller than $|My_k^\times \xi - Mx_k^\times|_p$ as soon as the positive integer M satisfies $M \le Q_{k+1}^\times / Q_k^\times$. Observe that, by the remark on the coprimality of x and y following Definition 1.1, we have

$$\mu^\times(\xi) = \limsup_{k\to\infty} \frac{-\log|y_k^\times \xi - y_k^\times|_p}{\log Q_k^\times}$$

and

$$\widehat{\mu}^\times(\xi) = 1 + \liminf_{k\to\infty} \frac{-\log|y_k^\times \xi - y_k^\times|_p - \log Q_k^\times}{\log Q_{k+1}^\times}.$$

We begin with an auxiliary result relating the sequence $(Q_k^\times)_{k\ge 1}$ with $\mu^\times(\xi)$ and $\widehat{\mu}^\times(\xi)$.

Lemma 7.2. *With the above notation, we have*

$$\limsup_{k\to\infty} \frac{\log Q_{k+1}^\times}{\log Q_k^\times} \le \frac{\mu^\times(\xi) - 1}{\widehat{\mu}^\times(\xi) - 1}.$$

Proof: By the definitions of the limsup and of the liminf, we get that, for every $\varepsilon > 0$ and every large k, we have

$$(\mu^\times(\xi) + \varepsilon) \log Q_k^\times \geq -\log |y_k^\times \xi - x_k^\times|_p$$

and

$$(\widehat{\mu}^\times(\xi) - 1 - \varepsilon) \log Q_{k+1}^\times \leq -\log |y_k^\times \xi - x_k^\times|_p - \log Q_k^\times.$$

This gives

$$(\widehat{\mu}^\times(\xi) - 1 - \varepsilon) \log Q_{k+1}^\times \leq (\mu^\times(\xi) - 1 + \varepsilon) \log Q_k^\times,$$

and the lemma follows. □

Proof of Theorem 3.3: We establish (i) and observe that (ii) can be proved analogously.

Assume that we have $|x_k^\times| \gg |y_k^\times|$ and $|x_{k+1}^\times| \gg |y_{k+1}^\times|$. Recall that $Q_k^\times = \sqrt{|x_k^\times y_k^\times|}$. Define α_k and β_k by

$$(Q_k^\times)^{\alpha_k} = |x_k^\times|, \quad (Q_k^\times)^{\beta_k} = |y_k^\times|,$$

and note that $\alpha_k + \beta_k = 2$. Define μ_k (it would be more appropriate to write $\mu_k^\times$, but for the sake of readability we choose to drop the $^\times$) by

$$|y_{k+1}^\times \xi - x_{k+1}^\times|_p < |y_k^\times \xi - x_k^\times|_p = (Q_k^\times)^{-\mu_k}.$$

Now, as in Lemma 5.1, we get

$$|x_{k+1}^\times y_k^\times - x_k^\times y_{k+1}^\times|_p = |y_{k+1}^\times (y_k^\times \xi - x_k^\times) - y_k^\times (y_{k+1}^\times \xi - x_{k+1}^\times)|_p \leq (Q_k^\times)^{-\mu_k}.$$

Thus,

$$|x_{k+1}^\times y_k^\times - x_k^\times y_{k+1}^\times| \geq (Q_k^\times)^{\mu_k},$$

which implies that

$$\max\{|x_{k+1}^\times y_k^\times|, |x_k^\times y_{k+1}^\times|\} \gg (Q_k^\times)^{\mu_k}.$$

Set $\delta_k = \min\{\alpha_k, \beta_k\}$. Since $|x_k^\times| \gg |y_k^\times|$, we may assume that $\delta_k = \beta_k$ (if necessary, we absorb the numerical constant in $\ll$). We have either

$$|x_{k+1}^\times| \gg \frac{(Q_k^\times)^{\mu_k}}{|y_k^\times|} = (Q_k^\times)^{\mu_k - \beta_k},$$

which gives

$$Q_{k+1}^\times = \sqrt{|x_{k+1}^\times y_{k+1}^\times|} \geq \sqrt{|x_{k+1}^\times|} \gg (Q_k^\times)^{\frac{\mu_k - \delta_k}{2}},$$

or

$$|x_{k+1}^\times| \gg |y_{k+1}^\times| \gg \frac{(Q_k^\times)^{\mu_k}}{|x_k^\times|} = (Q_k^\times)^{\mu_k - \alpha_k} \gg (Q_k^\times)^{\mu_k - 2 + \delta_k},$$

which gives

$$Q_{k+1}^\times = \sqrt{|x_{k+1}^\times y_{k+1}^\times|} \geq (Q_k^\times)^{\mu_k - 2 + \delta_k}.$$

To sum up, we have proved that

$$\frac{\log Q_{k+1}^\times}{\log Q_k^\times} \geq \min\left\{\frac{\mu_k - \delta_k}{2}, \mu_k - 2 + \delta_k\right\}.$$

Let $\varepsilon > 0$ be a given real number. For k large enough, it then follows from the proof of Lemma 7.2 that

$$\frac{\log Q_{k+1}^\times}{\log Q_k^\times} \leq \frac{\mu_k - 1}{\widehat{\mu}^\times(\xi) - 1} + \varepsilon. \tag{7.8}$$

We deduce that

$$\begin{aligned}\widehat{\mu}^\times(\xi) &\le 1+\max\Big\{\frac{2\mu_k-2}{\mu_k-\delta_k},\frac{\mu_k-1}{\mu_k-2+\delta_k}\Big\}+\tilde{\varepsilon}\\ &\le 1+\max\Big\{\frac{2\mu_k-2}{\mu_k-1},\frac{\mu_k-1}{\mu_k-2}\Big\}+\tilde{\varepsilon}=1+\max\Big\{2,\frac{\mu_k-1}{\mu_k-2}\Big\}+\tilde{\varepsilon},\end{aligned}$$

where $\tilde{\varepsilon}$ tends to 0 with ε. If $\mu_k \le 3$ for arbitrarily large k as above, then the upper bound $\widehat{\mu}^\times(\xi)\le 3$ follows from (7.8). Otherwise, we have $\mu_k > 3$ for every sufficiently large k and, since ε can be taken arbitrarily small, we conclude that $\widehat{\mu}^\times(\xi)\le 3$, as asserted. □

Proof of (3.2) *in Theorem 3.1:* First, note that (3.2) clearly holds when $\widehat{\mu}^\times(\xi)\le 3$, since we then have

$$\widehat{\mu}^\times(\xi)^2-3\widehat{\mu}^\times(\xi)+3\le\widehat{\mu}^\times(\xi)\le\mu^\times(\xi).$$

Consequently, we assume throughout this proof that $\widehat{\mu}^\times(\xi)>3$. By (3.1), we then have

$$3<\mu^\times(\xi)<+\infty.$$

Observe also that (3.2) can be rewritten as

$$\widehat{\mu}^\times(\xi)\le\frac{3+\sqrt{4\mu^\times(\xi)-3}}{2}. \tag{7.9}$$

Define μ_k by

$$|x_k^\times\xi-y_k^\times|_p=\Big(\sqrt{|x_k^\times y_k^\times|}\Big)^{-\mu_k}=(Q_k^\times)^{-\mu_k},$$

and α_k, β_k, δ_k by

$$(Q_k^\times)^{\alpha_k}=|x_k^\times|,\quad (Q_k^\times)^{\beta_k}=|y_k^\times|,\quad \delta_k=\min\{\alpha_k,\beta_k\}.$$

We can assume that

$$\mu^\times(\xi)=\limsup_{\ell\to\infty}\mu_{2\ell},$$

and, in view of Theorem 3.3, that for all large even integers k we have

$$|x_k^\times|>|y_k^\times|,\quad |x_{k+1}^\times|<|y_{k+1}^\times|.$$

Then $\delta_k=\beta_k$. Below, k denotes a sufficiently large even integer.

Let $\varepsilon>0$ be a given real number. Proceeding as in the preceding proof, but with the pairs $(x_k^\times,y_k^\times)$ and $(x_{k+2}^\times,y_{k+2}^\times)$, which satisfy the inequalities $|x_{k+2}^\times|>|y_{k+2}^\times|$ and $|x_k^\times|>|y_k^\times|$, we get

$$\frac{\log Q_{k+2}^\times}{\log Q_k^\times}\ge\min\Big\{\frac{\mu_k-\delta_k}{2},\mu_k-2+\delta_k\Big\}+\varepsilon,$$

for k large. On the other hand, from the proof of Lemma 7.2 we get

$$\frac{\log Q_{k+2}^\times}{\log Q_k^\times}=\frac{\log Q_{k+2}^\times}{\log Q_{k+1}^\times}\cdot\frac{\log Q_{k+1}^\times}{\log Q_k^\times}\le\Big(\frac{\mu_k-1}{\widehat{\mu}^\times(\xi)-1}\Big)^2+\varepsilon.$$

The combination of the latter inequalities gives

$$\widehat{\mu}^\times(\xi)\le 1+\sqrt{\max\Big\{\frac{2(\mu_k-1)^2}{\mu_k-\delta_k},\frac{(\mu_k-1)^2}{\mu_k-2+\delta_k}\Big\}}+\tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ tends to 0 with ε. Take an increasing sequence $(k_j)_{j\ge1}$ of even integers such that

$$\mu^\times(\xi)=\lim_{j\to\infty}\mu_{k_j}.$$

By extracting a subsequence if needed, we may assume that the sequence $(\delta_{k_j})_{j\geq 1}$ converges and we put

$$\overline{\delta} = \lim_{j\to\infty} \delta_{k_j}.$$

We deduce that

$$\widehat{\mu}^{\times}(\xi) \leq 1 + \max\left\{ \sqrt{\frac{2}{\mu^{\times}(\xi) - \overline{\delta}}}(\mu^{\times}(\xi) - 1), \frac{\mu^{\times}(\xi) - 1}{\sqrt{\mu^{\times}(\xi) - 2 + \overline{\delta}}} \right\}.$$

Observe that in the maximum the right-hand term is larger than the left-hand term if and only if $\mu^{\times}(\xi) \leq 4 - 3\overline{\delta}$. Consequently, if $\mu^{\times}(\xi) \leq 4 - 3\overline{\delta} \leq 4$, then we get

$$\widehat{\mu}^{\times}(\xi) \leq 1 + \frac{\mu^{\times}(\xi) - 1}{\sqrt{\mu^{\times}(\xi) - 2}}.$$

Taking into account that $\mu^{\times}(\xi) > 3$, a rapid calculation shows that this inequality implies (7.9), as wanted.

So we may assume $\mu^{\times}(\xi) > 4 - 3\overline{\delta}$ and thus

$$\widehat{\mu}^{\times}(\xi) \leq 1 + \sqrt{\frac{2}{\mu^{\times}(\xi) - \overline{\delta}}}(\mu^{\times}(\xi) - 1).$$

Observe that Corollary 7.1 applied with $A = 2 - \overline{\delta}$ gives

$$\widehat{\mu}^{\times}(\xi) \leq \max\left\{ \frac{3\mu^{\times}(\xi) - 2 - A}{\mu^{\times}(\xi) - A}, \frac{2\mu^{\times}(\xi)}{\mu^{\times}(\xi) - A} \right\}, \tag{7.10}$$

where the maximum is given by the left-hand term if and only if we have $A \leq \mu^{\times}(\xi) - 2$. We distinguish two cases.

Case 1: Assume that $A \leq \mu^{\times}(\xi) - 2$, that is, $\mu^{\times}(\xi) \geq 4 - \overline{\delta}$. Then

$$\widehat{\mu}^{\times}(\xi) \leq \min\left\{ \frac{3\mu^{\times}(\xi) - 4 + \overline{\delta}}{\mu^{\times}(\xi) - 2 + \overline{\delta}}, \sqrt{\frac{2}{\mu^{\times}(\xi) - \overline{\delta}}}(\mu^{\times}(\xi) - 1) + 1 \right\} \tag{7.11}$$

holds. In view of (7.9), we can assume that

$$\frac{3\mu^{\times}(\xi) - 4 + \overline{\delta}}{\mu^{\times}(\xi) - 2 + \overline{\delta}} > \frac{3 + \sqrt{4\mu^{\times}(\xi) - 3}}{2},$$

that is,

$$\overline{\delta} < \frac{3\mu^{\times}(\xi) - 2 - (\mu^{\times}(\xi) - 2)\sqrt{4\mu^{\times}(\xi) - 3}}{1 + \sqrt{4\mu^{\times}(\xi) - 3}}.$$

Using this bound for $\overline{\delta}$, we derive from (7.11) that

$$\widehat{\mu}^{\times}(\xi) < 1 + \sqrt{\frac{(\mu^{\times}(\xi) - 1)(1 + \sqrt{4\mu^{\times}(\xi) - 3})}{\sqrt{4\mu^{\times}(\xi) - 3} - 1}} = 1 + \frac{1 + \sqrt{4\mu^{\times}(\xi) - 3}}{2},$$

which gives the bound (7.9).

Case 2: We assume that $A > \mu^\times(\xi) - 2$, that is, $\mu^\times(\xi) < 4 - \overline{\delta} \le 4$. Then (7.10) gives

$$\widehat{\mu}^\times(\xi) \le 2 + \frac{2A}{\mu^\times(\xi) - A} = \frac{2\mu^\times(\xi)}{\mu^\times(\xi) - 2 + \overline{\delta}}$$

and we get

$$\widehat{\mu}^\times(\xi) \le \min\left\{ \frac{2\mu^\times(\xi)}{\mu^\times(\xi) - 2 + \overline{\delta}}, \sqrt{\frac{2}{\mu^\times(\xi) - \overline{\delta}}}(\mu^\times(\xi) - 1) + 1 \right\}. \tag{7.12}$$

In view of (7.9), we can assume that

$$\frac{2\mu^\times(\xi)}{\mu^\times(\xi) - 2 + \overline{\delta}} > \frac{3 + \sqrt{4\mu^\times(\xi) - 3}}{2},$$

that is,

$$\overline{\delta} < \frac{\mu^\times(\xi) + 6 - (\mu^\times(\xi) - 2)\sqrt{4\mu^\times(\xi) - 3}}{3 + \sqrt{4\mu^\times(\xi) - 3}}.$$

Using this bound for $\overline{\delta}$, we derive from (7.12) that

$$\widehat{\mu}^\times(\xi) < 1 + \sqrt{\frac{(\mu^\times(\xi) - 1)^2(3 + \sqrt{4\mu^\times(\xi) - 3})}{(\mu^\times(\xi) - 3) + (\mu^\times(\xi) - 1)\sqrt{4\mu^\times(\xi) - 3}}}.$$

A careful computation shows that, since $\mu^\times(\xi) \ge 3$, we get (7.9). □

As noticed above Corollary 3.2, the upper bound (3.3) followsfrom (3.1) and (3.2). The proof of Theorem 3.1 is complete.

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