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IWASAWA THEORY OF HILBERT MODULAR FORMS FOR ANTICYCLOTOMIC EXTENSIONS WITHOUT IHARA'S LEMMA

Bingyong Xie

Abstract: Following Bertolini and Darmon's method, with "Ihara's lemma" among other conditions Longo and Wang proved one divisibility of the Iwasawa main conjecture for Hilbert modular forms of weight 2 and general low even parallel weight in the anticyclotomic setting respectively. In this paper, we remove the "Ihara's lemma" condition in their results.

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1. Introduction

Iwasawa theory studies the mysterious relationship between pure arithmetic objects and special values of complex L-functions. Its precise statement is usually called the "main conjecture" and provides an equality between a quantity measuring Selmer groups and a p-adic L-function (interpolating the special values of a complex L-function). Its proof is usually divided into two parts, one part proving one divisibility by Ribet's method, and the other proving the converse divisibility by Euler systems.

In [**1**] Bertolini and Darmon proved one divisibility of the Iwasawa main conjecture for elliptic curves over $\mathbb{Q}$ in the anticyclotomic setting. Note that Bertolini and Darmon assumed a p-isolated condition among other technical conditions. The p-isolated condition was removed by Pollack and Weston [**16**]. In [**5**] Chida and Hsieh generalized this divisibility to elliptic modular forms of even weights $< p-1$. Their results were generalized to the setting of Hilbert modular forms by Longo [**13**] for parallel weight 2, and by Wang [**18**] for even parallel weights $< p-1$. There are other generalizations obtained by Fouquet [**8**] and Nekovář [**15**].

One of the crucial ingredients for the Euler system argument in [**1**] is Ihara's lemma for Shimura curves. In the case of elliptic modular forms, the required Ihara's lemma is Theorem 12 in [**7**]. In the totally real case, [**7**, Theorem 12] is partially generalized by Jarvis [**11**]. It seems that in the unpublished paper [**4**] Ihara's lemma was proved under the conditions that the base totally real number field F is sufficiently small, i.e. $[F:\mathbb{Q}] < p$, and that the level of the Hilbert modular form in question is sufficiently large. In [**14**] Manning and Shotton proved Ihara's lemma under the hypothesis that the image of $\bar{\rho}_f$ (a modulo p representation defined in our text) contains a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$. Thus under this strong hypothesis Longo's and Wang's results are unconditional.

In this paper we remove the condition of Ihara's lemma, and thus obtain an unconditional result for all totally real number fields. We need to preserve technical conditions in [**13, 18**] other than Ihara's lemma. Instead of proving Ihara's lemma, we take an approach of avoiding it.

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Let F be a totally real number field and $\mathfrak{p}$ a place of F above p. Let K be a totally imaginary quadratic extension of F. We form the anticyclotomic $\mathbb{Z}_p^{[F_\mathfrak{p}:\mathbb{Q}_p]}$-extension K_∞ of K. Put $\Gamma = \mathrm{Gal}(K_\infty/K)$.

Let f be a new Hilbert cusp form of parallel weight $k \geq 2$. Let us write the level $\mathfrak{n}$ of f in the form $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$, where $\mathfrak{n}^+$ is only divisible by prime ideals that split in K, and $\mathfrak{n}^-$ is only divisible by prime ideals that do not split in K. We assume that $\mathfrak{n}^-$ is the product of different prime ideals whose cardinal number has the same parity as $[F:\mathbb{Q}]$. This condition ensures that f comes from a modular form on a definite quaternion algebra with discriminant $\mathfrak{n}^-$. We also assume $p \nmid \mathfrak{n}D_{K/F}$ and f is ordinary at p. Specifically, one of the two Hecke eigenvalues of f at each place of F above p is a p-adic unit.

Let $\rho_f\colon G_F \to \mathrm{GL}_2(E_f)$ be the p-adic Galois representation attached to f (see [**19, 17**] among other references), where E_f is the defining field of ρ_f. Then $\det \rho_f = \epsilon^{k-1}$, where ϵ is the p-adic cyclotomic character of $G_F = \mathrm{Gal}(\overline{F}/F)$. We consider the self-dual twist of ρ_f, namely $\rho_f^* = \rho_f \otimes \epsilon^{\frac{2-k}{2}}$. Let V_f be the underlying representation space for ρ_f^*. Fix a G_F-stable lattice T_f of V_f, and put $A_f = V_f/T_f$.

Let $\mathrm{Sel}(K_\infty, A_f)$ be the minimal Selmer group of A_f. Put $\Lambda = \mathcal{O}_f[[\Gamma]]$, where $\mathcal{O}_f$ is the ring of integers in E_f. Then $\mathrm{Sel}(K_\infty, A_f)$ and its Pontryagin dual $\mathrm{Sel}(K_\infty, A_f)^\vee$ are Λ-modules.

On the other hand, one can attach to f an anticyclotomic p-adic L-function $L_p(K_\infty, f) \in \Lambda$ that interpolates the special values $L(f/K, \chi, k/2)$ of the L-function attached to f (where χ runs over anticyclotomic characters).

Conjecture 1.1 (Iwasawa main conjecture)**.** *The* Λ*-module* $\mathrm{Sel}(K_\infty, A_f)$ *is a cofinitely generated cotorsion module, and its characteristic ideal* $\mathrm{char}_\Lambda \mathrm{Sel}(K_\infty, A_f)^\vee \in \Lambda$ *satisfies*

$$\mathrm{char}_\Lambda \mathrm{Sel}(K_\infty, A_f)^\vee = (L_p(K_\infty, f)).$$

Our main result is the following:

Theorem 1.2. *Assume that f satisfies the conditions* (CR$^+$)*,* (PO)*, and* ($\mathfrak{n}^+$-DT) *given in* [**18**]*. Then* $\mathrm{Sel}(K_\infty, A_f)$ *is a cofinitely generated cotorsion* Λ*-module, and*

$$\mathrm{char}_\Lambda \mathrm{Sel}(K_\infty, A_f)^\vee \mid (L_p(K_\infty, f)).$$

As applications of Theorem 1.2, we have the following consequences.

Corollary 1.3. *Let A be a modular elliptic curve (or more generally a modular abelian variety of* GL_2*-type) over F. Assume that $F_\mathfrak{p} = \mathbb{Q}_p$ and that the modular form attached to A satisfies the assumption in Theorem 1.2. Then $A(K_\infty)$ is finitely generated.*

In [**10**] Hung proved the vanishing of the analytic μ-invariant, generalizing the result of Chida and Hsieh [**6**]. Combining Theorem 1.2 and Hung's result, we obtain the following:

Corollary 1.4. *Keep the assumption of Theorem 1.2. Then the algebraic μ-invariant of the* Λ*-module* $\mathrm{Sel}(K_\infty, A_f)^\vee$ *is zero.*

Corollaries 1.3 and 1.4 were already obtained by Longo [**13**] and Wang [**18**] respectively, under the assumption of Ihara's lemma.

The strategy for the proof of Theorem 1.2 is to use the Euler system of Heegner points $\{\kappa_{\mathscr{D}}(\mathfrak{l})_m\}_\mathfrak{l}$ to bound the Selmer groups. In [**18**] these Heegner points were shown to satisfy two properties called the First Reciprocity Law and the Second

Reciprocity Law. The Second Reciprocity Law requires Ihara's lemma. Our input is to prove a weaker form of the Second Reciprocity Law without Ihara's lemma. Our weaker version is sufficient for us to run through Bertolini and Darmon's Euler system argument to prove Theorem 1.2. This is done in Section 5. See Proposition 4.6 and Corollary 4.14 for the precise statements of the First Reciprocity Law and the weaker version of the Second Reciprocity Law.

Both the original Second Reciprocity Law

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m) \tag{1.1}$$

(with $\mathfrak{l}_1$ and $\mathfrak{l}_2$ being different n-admissible primes) and our weaker version are based on an analysis of the specialization modulo ω (= the uniformizing element of $\mathcal{O} \supset \mathcal{O}_f$) of Heegner points to supersingular points. Starting from an (N,n)-admissible form (Δ, g) (Definition 2.4), using this specialization we obtain a map

$$\gamma\colon B''^{\times}\backslash\widehat{B}''^{\times}/Y\mathfrak{U}'' \longrightarrow \mathcal{O}_n$$

(see Subsection 4.2 for the meanings of the notations), which is expected to define a new (N,n)-admissible form. In [**18**], N is taken to coincide with n. Our (N,n)-admissible form is called n-admissible form in loc. cit.

In [**18**] Ihara's lemma is used to show that γ is nonzero modulo ω, i.e. the order of γ is zero, so that γ really defines an (N,n)-admissible form denoted by g'' in our text. Wang ([**18**]) showed that

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = \theta_m(g''). \tag{1.2}$$

With $\mathfrak{l}_1$ and $\mathfrak{l}_2$ exchanged one obtains another n-admissible form h'' such that

$$v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m) = \theta_m(h'').$$

Then the multiplicity one result $g'' = h''$ yields (1.1).

Both (1.1) and (1.2) are needed in Bertolini and Darmon's (inductive) Euler system argument. We sketch the Euler system argument as follows. The reader may consult the text for notations.

Let $\varphi\colon \mathcal{O}[[\Gamma]] \to \mathcal{O}'$ be a homomorphism. Enlarging $\mathcal{O}$ if necessary one may assume $\mathcal{O} = \mathcal{O}'$. One needs to show that the length of

$$\mathrm{Sel}_{\Delta}(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O}$$

is bounded by $2\,\mathrm{ord}\,\varphi(\theta(g))$. For this we consider the following two exact sequences:

$$\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}_1}, T_n) \oplus \widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}_2}, T_n) \xrightarrow{\eta_s} \mathrm{Sel}_{\Delta}(K_\infty, A_n)^\vee \longrightarrow S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \longrightarrow 0$$

and

$$\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_1}, T_n) \oplus \widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_2}, T_n) \xrightarrow{\eta_f} \mathrm{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty, A_n)^\vee \longrightarrow S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \longrightarrow 0.$$

Let $e_{\mathfrak{l}}$ be the (global) order of $\varphi(\kappa_{\mathscr{D}}(\mathfrak{l}))$. There exists

$$\kappa'(\mathfrak{l}) \in \mathrm{Sel}_{\Delta\mathfrak{l}}(K_\infty, T_n) \otimes_\varphi \mathcal{O}$$

such that

$$\varphi(\kappa_{\mathscr{D}}(\mathfrak{l})) = \omega^{e_{\mathfrak{l}}}\kappa'(\mathfrak{l}).$$

Furthermore, $(\partial_{\mathfrak{l}_1}\kappa'(\mathfrak{l}_1), 0)$ and $(0, \partial_{\mathfrak{l}_2}\kappa'(\mathfrak{l}_2))$ are annihilated by η_s^φ, while $(v_{\mathfrak{l}_1}\kappa'(\mathfrak{l}_2), 0)$ and $(0, v_{\mathfrak{l}_2}\kappa'(\mathfrak{l}_1))$ are annihilated by η_f^φ.

The First Reciprocity Law implies that the order of $\partial_{\mathfrak{l}}\kappa'(\mathfrak{l})$ is $\mathrm{ord}\,\varphi(\theta(g)) - e_{\mathfrak{l}}$. From this one obtains that the length of the image of η_s^φ is at most

$$2\,\mathrm{ord}\,\varphi(\theta(g)) - (e_{\mathfrak{l}_1} + e_{\mathfrak{l}_2}).$$

So by the first exact sequence it suffices to control $S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2}\otimes_\varphi\mathcal{O}$, which also lies in the second exact sequence.

To apply the second exact sequence one needs to make a good choice of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ to force $\eta_f^\varphi=0$ so that

$$\mathrm{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty,A_n)^\vee\otimes_\varphi\mathcal{O}\cong S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2}\otimes_\varphi\mathcal{O}.$$

One chooses $\mathfrak{l}_1$ such that $e_{\mathfrak{l}_1}$ is minimal. Then one chooses $\mathfrak{l}_2$ such that

$$\operatorname{ord}\varphi(v_{\mathfrak{l}_2}\kappa_{\mathscr{D}}(\mathfrak{l}_1))=e_{\mathfrak{l}_1}$$

or the same $\operatorname{ord}v_{\mathfrak{l}_2}\kappa'(\mathfrak{l}_1)=0$, which implies that $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_2},T_n)\otimes_\varphi\mathcal{O}$ is annihilated by η_f^φ. When Ihara's lemma holds, by the Second Reciprocity Law (1.1) we get

$$\operatorname{ord}\varphi(v_{\mathfrak{l}_2}\kappa_{\mathscr{D}}(\mathfrak{l}_1))=\operatorname{ord}\varphi(v_{\mathfrak{l}_1}\kappa_{\mathscr{D}}(\mathfrak{l}_2)).$$

Combining this with the trivial fact $e_{\mathfrak{l}_2}\le\operatorname{ord}\varphi(v_{\mathfrak{l}_1}\kappa_{\mathscr{D}}(\mathfrak{l}_2))$ and the minimality of $e_{\mathfrak{l}_1}$, one obtains

$$\operatorname{ord}\varphi(v_{\mathfrak{l}_1}\kappa_{\mathscr{D}}(\mathfrak{l}_2))=e_{\mathfrak{l}_2}.$$

Thus $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_1},T_n)\otimes_\varphi\mathcal{O}$ is annihilated by η_f^φ as well.

Then one uses (1.2) to finish the inductive argument.

In our approach, we deal with (1.1) and (1.2) separately.

Instead of Ihara's lemma, we use the global Tate pairing to prove a weaker version of (1.1). We show that $v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))$ and $v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2))$ coincide with each other after multiplying by $\theta(g)$. Indeed, by relations like

$$\sum_v\langle\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m,\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m\rangle_v=0$$

provided by the global Tate pairing between $H^1(K_m,T_{f,n})$ and itself (noting that $T_{f,n}\cong A_{f,n}$) we obtain

$$\theta(g)\cdot v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))=\theta(g)\cdot v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2))$$

up to multiplication by a unit in $\mathcal{O}_n[[\Gamma]]$. Note that this holds for any m-admissible $\mathfrak{l}_1\ne\mathfrak{l}_2$. For the good choice of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ made above, we have

$$\varphi(\theta(g)\cdot v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)))\ne0$$

in $\mathcal{O}_n$,[1] from which we deduce

$$\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)))=\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)))\tag{1.3}$$

up to multiplication by a unit in $\mathcal{O}_n$. So, without Ihara's lemma we again obtain $\eta_f^\varphi=0$.

The reader should note that we show (1.3) only for carefully chosen pairs $(\mathfrak{l}_1,\mathfrak{l}_2)$, rather than random pairs.

For (1.2), without Ihara's lemma, the order of γ, denoted by n_0 in our Proposition 4.15, may be nonzero. Fortunately, we can bound n_0 by $v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))$. Especially, for our good choice of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ we have $n_0<n$. Then we obtain from γ an $(N,n-n_0)$-admissible form denoted by g'' such that

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))=\omega^{n_0}\theta(g'').\tag{1.4}$$

Thus we have a weaker version of (1.2). In the (inductive) Euler system argument, (1.2) is used to show that $2\operatorname{ord}\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)))$ bounds $\mathrm{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty,A_n)^\vee\otimes_\varphi\mathcal{O}$, since this module is bounded by $2\operatorname{ord}\varphi(\theta(g''))$ by the inductive assumption. Clearly, our weaker version (1.4) is sufficient for this purpose.

[1] This requires a further technical condition which is clear in our text.

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Notations. Let $D_{K/F}$ denote the relative difference of K with respect to F. Fix a prime number $p \nmid \mathfrak{n} D_{K/F}$ and a prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ above p.

Let $\widetilde{K}_m$ be the ring class field over K of conductor $\mathfrak{p}^m$ and put $G_m = \mathrm{Gal}(\widetilde{K}_m/K)$. Set $\widetilde{K}_\infty = \bigcup_m \widetilde{K}_m$.

Let K_∞ be the unique subfield of $\widetilde{K}_\infty$ such that $\Gamma := \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^{[F_\mathfrak{p}:\mathbb{Q}_p]}$. Put $K_m = \widetilde{K}_m \cap K_\infty$ and $\Gamma_m = \mathrm{Gal}(K_m/K)$.

Let ϵ denote the p-adic cyclotomic character of $G_F = \mathrm{Gal}(\overline{F}/F)$.

We will fix an isomorphism $\jmath : \mathbb{C} \cong \mathbb{C}_p$.

2. Automorphic forms and Galois representations

2.1. Galois representation attached to f. Throughout this paper we will fix a Hilbert cusp newform f of parallel weight $k \geq 2$ and trivial central character. Let $\mathfrak{n}$ be the conductor of f, and we decompose $\mathfrak{n}$ into $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$, where $\mathfrak{n}^+$ is the product of primes split in K, and $\mathfrak{n}^-$ is the product of primes inert or ramified in K. We assume that $\mathfrak{n}$ is coprime to p.

We assume that $\mathfrak{n}^-$ satisfies the following two conditions:

(sq-fr) $\mathfrak{n}^-$ is square-free, that is, $\mathfrak{n}^-$ is the product of different primes.

(card) The cardinal number of prime factors of $\mathfrak{n}^-$ has the same parity as $[F : \mathbb{Q}]$.

By [**19, 17**] (among other references) up to isomorphisms there exists a unique p-adic Galois representation

$$\rho_f : G_F \longrightarrow \mathrm{GL}_2(\mathbb{C}_p)$$

that satisfies the following two properties.

- ρ_f is unramified outside $p\mathfrak{n}$.
- If $\mathfrak{l}$ is a prime of $\mathcal{O}_F$ not dividing $p\mathfrak{n}$, then for the geometric Frobenius $\mathrm{Frob}_\mathfrak{l}$ at $\mathfrak{l}$, the characteristic polynomial of $\rho_f(\mathrm{Frob}_\mathfrak{l})$ is $x^2 - a_\mathfrak{l}(f)x + \mathbf{N}(\mathfrak{l})^{k-1}$. Here, $a_\mathfrak{l}(f)$ is the Hecke eigenvalue of f at $\mathfrak{l}$.

Here we view $a_\mathfrak{l}(f)$ as an element of $\mathbb{C}_p$ via $\jmath$. Let E_f be the defining field of ρ_f, which contains all $a_\mathfrak{l}(f)$. Let $\mathcal{O}_f$ be the ring of integers in E_f.

A consequence of the latter property is

$$\det \rho_f = \epsilon^{k-1}.$$

The reader may consult [**19, 17**] for the construction of ρ_f and more properties of ρ_f.

Let

$$\rho_f^* = \rho_f \otimes \epsilon^{\frac{2-k}{2}}$$

be the self-dual twist of ρ_f, and V_f the underlying representation space for ρ_f^*. The representation ρ_f^* has the following properties.

- ρ_f^* is unramified outside $p\mathfrak{n}$.
- $\rho_f^*|_{G_{F_v}} = \begin{pmatrix} \chi_v^{-1}\epsilon^{\frac{k}{2}} & * \\ 0 & \chi_v \epsilon^{\frac{2-k}{2}} \end{pmatrix}$ for each $v|p$. Here χ_v is the unramified character such that $\chi_v(\mathrm{Frob}_v) = a_v(f)$.
- $\rho_f^*|_{G_{F_\mathfrak{l}}} = \begin{pmatrix} \pm\epsilon & * \\ 0 & \pm 1 \end{pmatrix}$ for each $\mathfrak{l}$ dividing $\mathfrak{n}$ exactly once.

Fix a G_F-stable lattice T_f of V_f. We use $\bar{\rho}_f$ to denote the residual Galois representation of T_f.

We state the conditions (CR^+), (PO), and ($\mathfrak{n}^+$-DT) in Theorem 1.2.

Hypothesis ($\mathbf{CR}^+$). (1) $p > k+1$ and $(\#(\mathcal{O}_F/\mathfrak{p})^\times)^{k-1} > 5$.

(2) The restriction of $\bar{\rho}_f$ to $G_{F(\sqrt{p^*})}$ is irreducible, where $p^* = (-1)^{\frac{p-1}{2}}p$.

(3) $\bar{\rho}_f$ is ramified at $\mathfrak{l}$ if $\mathfrak{l}|\mathfrak{n}^-$ and $\mathrm{N}(\mathfrak{l})^2 \equiv 1 \pmod{p}$.

(4) If $\mathfrak{n}_{\bar{\rho}}$ denotes the Artin conductor of $\bar{\rho}_f$, then $\mathfrak{n}/\mathfrak{n}_{\bar{\rho}}$ is coprime to $\mathfrak{n}_{\bar{\rho}}$.

Hypothesis (PO). $a_v^2(f) \not\equiv 1 \pmod{p}$ for all $v|p$ if $k=2$.

Hypothesis ($\mathfrak{n}^+$-DT). If $\mathfrak{l}||\mathfrak{n}^+$ and $\mathrm{N}(\mathfrak{l}) \equiv 1 \pmod{p}$, then $\bar{\rho}_f$ is ramified at $\mathfrak{l}$.

We also need an auxiliary condition ($\mathfrak{n}^+$-min).

Hypothesis ($\mathfrak{n}^+$-min). If $\mathfrak{l}|\mathfrak{n}^+$, then $\bar{\rho}_f$ is ramified at $\mathfrak{l}$.

Throughout this paper, we fix a finite extension E of E_f, and let $\mathcal{O}$ be the ring of integers in E. So $\mathcal{O}_f \subset \mathcal{O}$. Let ω be a uniformizer of $\mathcal{O}$. For each positive integer n we put $\mathcal{O}_n = \mathcal{O}/\omega^n$. Consider E, $\mathcal{O}$, and $\mathcal{O}_n$ as coefficient rings, and let G_F act trivially on them.

Set $T_{\mathcal{O}} = T_f \otimes_{\mathcal{O}_f} \mathcal{O}$, $V_E = V_f \otimes_{E_f} E$, and $A = V_E/T_{\mathcal{O}}$. For each n we put

$$T_n = (T_{\mathcal{O}})/\omega^n = T_f \otimes_{\mathcal{O}_f} \mathcal{O}_n$$

and

$$A_n = \ker(A \xrightarrow{\omega^n} A).$$

They are all G_F-modules.

Remark 2.1. By assumption (2) in (CR^+), $\bar{\rho}_f$ is itself irreducible. So, the G_F-stable lattice T_f of V_f is unique up to isomorphisms. Hence, up to isomorphisms T_n and A_n are independent of the choice of T_f.

Lemma 2.2. *Suppose that assumption (4) in* (CR^+) *holds. If $\mathfrak{l}|\mathfrak{n}^+$ and $\bar{\rho}_f$ is ramified at $\mathfrak{l}$, then $H^0(F_{\mathfrak{l}}^{\mathrm{nr}}, A)$ is divisible.*

Proof: By [**17**], and via the local Langlands correspondence, the Frobenius-simplification of the Weil–Deligne representation attached to $\rho_{f,\mathfrak{l}}$ is the Weil–Deligne representation attached to $\pi_{f,\mathfrak{l}}$. Thus the Artin conductor of $\rho_{f,\mathfrak{l}}$ is equal to the conductor of $\pi_{f,\mathfrak{l}}$ [**9**]. As ϵ is unramified at $\mathfrak{l}$, the Artin conductor of $\rho^*_{f,\mathfrak{l}}$ is equal to that of $\rho_{f,\mathfrak{l}}$.

When $\bar{\rho}_f$ is ramified at $\mathfrak{l}$, assumption (4) in (CR^+) ensures that the conductor of $\pi_{f,\mathfrak{l}}$ is equal to the Artin conductor of $\bar{\rho}_{f,\mathfrak{l}}$. Therefore, the Artin conductor of $\rho^*_{f,\mathfrak{l}}$ is equal to that of $\bar{\rho}_{f,\mathfrak{l}}$. Our assertion follows. □

Definition 2.3 ([**18**, Definition 2.2.1])**.** A prime ideal $\mathfrak{l}$ of $\mathcal{O}_F$ is said to be *n-admissible* for f if the following conditions hold.

(a) $\mathfrak{l} \nmid p\mathfrak{n}$.

(b) $\mathfrak{l}$ is inert in K.

(c) $\mathrm{N}(\mathfrak{l})^2 - 1$ is not divisible by p.

(d) ω^n divides $\mathrm{N}(\mathfrak{l})^{\frac{k}{2}} + \mathrm{N}(\mathfrak{l})^{\frac{k-2}{2}} - \epsilon_{\mathfrak{l}} a_{\mathfrak{l}}(f)$, where $\epsilon_{\mathfrak{l}} = \pm 1$.

2.2. (N, n)-admissible form. In this subsection we recall the definition of n-admissible forms [**18**].

Let B_Δ be a quaternion algebra over F with discriminant Δ. Suppose Δ is coprime to p. For each $v \nmid \Delta$ we fix an isomorphism $(B_\Delta)_v \cong M_2(F_v)$.

Let $\mathfrak{n}^+$ be an ideal of $\mathcal{O}_F$ coprime to $p\Delta$, and let $R_{\mathfrak{n}^+} \subset B_\Delta$ be an Eichler order of level $\mathfrak{n}^+$. Then for each $v|\mathfrak{n}^+$ with $v^t||\mathfrak{n}^+$,

$$(R_{\mathfrak{n}^+})_v = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathcal{O}_{F_v}) : c \in \pi_v^t \mathcal{O}_{F_v} \right\},$$

where π_v is a uniformizing element of F_v.

For a fixed positive integer N we put

$$\mathfrak{U} = \mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N} = \left\{ x \in \widehat{R}_{\mathfrak{n}^+}^\times : x_\mathfrak{p} \equiv \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \pmod{\mathfrak{p}^N},\ a, b \in \mathcal{O}_{F_\mathfrak{p}} \right\}.$$

Let $\mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N)$ be the (commutative) Hecke algebra generated by

$$\{T_v, S_v : v \nmid \mathfrak{p}\mathfrak{n}^+\Delta\} \cup \{U_v : v|\mathfrak{p}\mathfrak{n}^+\Delta\} \cup \{\langle a\rangle : a \in \mathcal{O}_{F,\mathfrak{p}}^\times\}.$$

Here, as usual, for $v \nmid \mathfrak{p}\mathfrak{n}^+\Delta$

$$T_v = \left[\mathfrak{U}\left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathfrak{U}\right], \quad S_v = \left[\mathfrak{U}\left(\begin{smallmatrix} \pi_v & 0 \\ 0 & \pi_v \end{smallmatrix}\right)\mathfrak{U}\right];$$

for $v|\mathfrak{p}\mathfrak{n}^+$,

$$U_v = \left[\mathfrak{U}\left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathfrak{U}\right];$$

for $v|\Delta$, we choose an element π'_v of $(B_\Delta)_v$ whose norm is a uniformizing element of F_v, and put

$$U_v = [\mathfrak{U}\pi'_v\mathfrak{U}];$$

for $v = \mathfrak{p}$

$$\langle a\rangle = [\mathfrak{U}\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathfrak{U}].$$

To define n-admissible forms we need the notion of algebraic modular forms with values in p-adic rings.

Let Φ be a finite extension of $\mathbb{Q}_p$ that contains images of all embeddings $\sigma\colon F \hookrightarrow \overline{\mathbb{Q}}_p$. Let Ω be the maximal ideal of $\mathcal{O}_{\overline{\mathbb{Q}}_p}$. Then $\mathfrak{p}_\sigma := \sigma^{-1}(\Omega)$ is a maximal ideal of $\mathcal{O}_F$ lying above p. We extend σ continuously to $F_{\mathfrak{p}_\sigma}$. Let A be an $\mathcal{O}_\Phi$-algebra. Then we have a decomposition

$$A \otimes_\mathbb{Z} \mathcal{O}_F \cong \bigoplus_\sigma A, \quad a \otimes b \mapsto (a\sigma(b))_\sigma,$$

where σ runs over all embeddings $F \hookrightarrow \Phi$.

For each embedding σ let

$$L_{k,\sigma}(A) = A[X_\sigma, Y_\sigma]_{k-2}$$

be the space of homogenous polynomials of degree $k-2$ with two variables over A; we have an action of $M_2(\mathcal{O}_{F_{\mathfrak{p}_\sigma}})$ on $L_k(A)$ by

$$\widehat{\rho}_{k,\sigma}(g)P(X_\sigma, Y_\sigma) = P((X_\sigma, Y_\sigma)g).$$

We use $\rho_{k,\sigma}$ to denote the action $\det^{\frac{2-k}{2}} \cdot \widehat{\rho}_{k,\sigma}|_{\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}_\sigma}})}$ of $\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}_\sigma}})$. Put

$$L_k(A) = \bigotimes_\sigma L_{k,\sigma}(A) \cong \bigotimes_{\mathfrak{q}|p} \bigotimes_{\sigma:\mathfrak{p}_\sigma=\mathfrak{q}} L_{k,\sigma}(A).$$

Then we consider $L_k(A)$ as a $\mathrm{GL}_2(\mathcal{O}_{F_p})$-module by the action

$$\rho_k(u_p) = \bigotimes_{\mathfrak{q}|p} \bigotimes_{\sigma:\mathfrak{p}_\sigma=\mathfrak{q}} \rho_{k,\sigma}(\sigma(u_\mathfrak{q})).$$

Similarly, we consider $L_k(A)$ as a $M_2(\mathcal{O}_{F_p})$-module by the action

$$\widehat{\rho}_k(u_p) = \bigotimes_{\mathfrak{q}|p} \bigotimes_{\sigma:\mathfrak{p}_\sigma=\mathfrak{q}} \widehat{\rho}_{k,\sigma}(\sigma(u_\mathfrak{q})).$$

Note that ρ_k is self-dual; this means that there is a ρ_k-invariant pairing $\langle\cdot,\cdot\rangle_k$ on $L_k(A) \times L_k(A)$.

Now, let B_Δ be definite. One defines the space $S_k^{B_\Delta}(\mathfrak{U}, A)$ of *algebraic modular forms of level* $\mathfrak{U}$ *and weight* k by

$$S_k^{B_\Delta}(\mathfrak{U}, A) = \{f\colon B_\Delta^\times \backslash \widehat{B}_\Delta^\times \longrightarrow L_k(A) \mid f(bu) = \rho_k(u_p)^{-1} f(b) \quad \forall\, u \in \mathfrak{U}\}.$$

It is equipped with a natural $\mathbb{T}_{B_\Delta}(\mathfrak{n}^+, \mathfrak{p}^N)$-action, as follows: for any $[\mathfrak{U}x\mathfrak{U}] \in \mathbb{T}_{B_\Delta}(\mathfrak{n}^+, \mathfrak{p}^N)$, if $x_\mathfrak{p} = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $a \in \mathcal{O}_{F_\mathfrak{p}}^\times$, one defines

$$[\mathfrak{U}x\mathfrak{U}]f(b) = \sum_{u \in \mathfrak{U}/\mathfrak{U} \cap x\mathfrak{U}x^{-1}} \rho_k(u_\mathfrak{p} x_\mathfrak{p}) f(bux);$$

if $x_\mathfrak{p} = \left(\begin{smallmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, one defines

$$[\mathfrak{U}x\mathfrak{U}]f(b) = \sum_{u \in \mathfrak{U}/\mathfrak{U} \cap x\mathfrak{U}x^{-1}} \rho_k(u_\mathfrak{p}) \widehat{\rho}_k(x_\mathfrak{p}) f(bux).$$

When $k = 2$, $S_2^{B_\Delta}(\mathfrak{U}, A)$ can be naturally identified with $A[B_\Delta^\times \backslash \widehat{B}_\Delta^\times / \mathfrak{U}]$; it is compatible with Hecke actions if we define the Hecke action on the divisor group of the Shimura set $B_\Delta^\times \backslash \widehat{B}_\Delta^\times / \mathfrak{U}$ via Picard functoriality.

Set $Y = \widehat{F}^\times$. Then there is an action of Y on $S_k^{B_\Delta}(\mathfrak{U}, A)$.

Let f be the Hilbert modular form of level $\mathfrak{n}$ and weight k as in the introduction. In particular, f is ordinary at p. Put

$$\mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N)_\mathcal{O} = \mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N) \otimes \mathcal{O}.$$

One can attach to f a Hecke character

$$\lambda_{f,N}\colon \mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N)_\mathcal{O} \longrightarrow \mathcal{O}$$

as follows. As in Subsection 2.1, let $\{a_v(f)\}_v$ be the system of Hecke eigenvalues attached to f. Set

$$\alpha_v(f) = \begin{cases} \text{the unit root of } x^2 - a_v(f)x + \mathrm{N}(v)^{k-1} & \text{if } v|p, \\ a_v(f)\mathrm{N}(v)^{\frac{2-k}{2}} & \text{if } v \nmid p. \end{cases}$$

Then we define $\lambda_{f,N}$ by

$$\begin{aligned} \lambda_{f,N}(T_v) &= a_v(f), \\ \lambda_{f,N}(S_v) &= 1 && \text{for } v \nmid p\mathfrak{n}, \\ \lambda_{f,N}(U_v) &= \alpha_v(f) && \text{for } v | p\mathfrak{n}, \\ \lambda_{f,N}(\langle a \rangle) &= a^{\frac{2-k}{2}} && \text{for } a \in \mathcal{O}_{F_\mathfrak{p}}^\times. \end{aligned}$$

Definition 2.4 ([**18**, Definition 5.1.1])**.** Let N and n be two positive integers. By an (N, n)*-admissible form* we mean a pair $\mathscr{D} = (\Delta, g)$ such that

(a) Δ is a square-free product of prime ideals (in $\mathcal{O}_F$) inert or ramified in K, $\mathfrak{n}^-|\Delta$, $\Delta/\mathfrak{n}^-$ is a product of n-admissible prime ideals, and the cardinal number of prime factors of $\Delta/\mathfrak{n}^-$ is even;

(b) $g \in S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O}_n)^Y$ such that

$$g \pmod{\omega} \neq 0$$

and

$$\lambda_g \equiv \lambda_{f,N} \pmod{\omega^n}.$$

Let $\mathcal{I}_g$ be the kernel of λ_g.

When $n = N$, (N, n)-admissible forms are just n-admissible forms defined by [**18**, Definition 5.1.1].

Let $\tau_N \in \widehat{B}_\Delta^\times$ be the Atkin–Lehner element given by

$$\tau_{N,v} = \begin{pmatrix} 0 & 1 \\ \omega_v^{\mathrm{ord}_v(\mathfrak{p}^N\mathfrak{n}^+)} & 0 \end{pmatrix}.$$

Then τ_N normalizes $\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}$ and gives an involution, called the Atkin–Lehner involution, on $B_\Delta^\times \backslash \widehat{B}_\Delta^\times / \mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}$. We define a perfect pairing

$$\langle \cdot, \cdot \rangle_N \colon S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, A)^Y \times S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, A)^Y \longrightarrow A$$

by

$$\langle f, g \rangle_N = \sum_b f(b) g(b\tau_N) \sharp (B_\Delta^\times \cap b\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N} b^{-1}/F^\times)^{-1},$$

where b runs over the Shimura set $B_\Delta^\times \backslash \widehat{B}_\Delta^\times / \mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}$. We have that the action of $\mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N)_\mathcal{O}$ is self-adjoint with respect to this pairing.

For each $g \in S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O}_n)^Y$ we define the map

$$\psi_g \colon S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O})^Y \longrightarrow \mathcal{O}_n, \quad h \longmapsto \langle g, h \rangle_{\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}}.$$

Via the identity

$$S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O})^Y \cong \mathcal{O}[B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}],$$

we have

$$\psi_g(x\,\tau_N) = g(x). \tag{2.1}$$

Proposition 2.5 ([**18**, Proposition 5.1.2])**.** *Assume* (CR$^+$) *and* ($\mathfrak{n}^+$-DT)*. If $n \leq N$, and if (Δ, g) is an (N, n)-admissible form, then we have an isomorphism*

$$\psi_g \colon S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O})^Y / \mathcal{I}_g \xrightarrow{\sim} \mathcal{O}_n.$$

Proof: When $n = N$, this is [**18**, Proposition 5.1.2]. For the general case $n \leq N$, we only need to slightly adjust the proof of [**18**, Proposition 5.1.2]. Let P_k be the ideal $\{\langle a \rangle - a^{\frac{k-1}{2}} : a \in \mathcal{O}_{F_\mathfrak{p}}^\times\}$ which is clearly contained in $\mathcal{I}_g$, and let $\mathfrak{m}$ be the maximal ideal containing $\mathcal{I}_g$. In loc. cit. it is shown that

$$S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O})_\mathfrak{m}^Y / (P_k, \omega^N) \simeq S_k^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+}, \mathcal{O})_\mathfrak{m}^Y / (\omega^N).$$

Since $n \leq N$, it follows that

$$S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O})_\mathfrak{m}^Y / (P_k, \omega^n) \simeq S_k^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+}, \mathcal{O})_\mathfrak{m}^Y / (\omega^n).$$

By [**18**, Theorem 9.2.4] $S_k^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+}, \mathcal{O})_\mathfrak{m}^Y$ is a cyclic $\mathbb{T}_\Delta(\mathfrak{n}^+)_\mathcal{O}$-module. Thus $S_2^{B_\Delta}(\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}, \mathcal{O})^Y / \mathcal{I}_g$ is generated by some h as a $\mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N)_\mathcal{O}$-module. Since ψ_g is surjective, $\psi_g(h) \in \mathcal{O}_n^\times$. Now our assertion follows from the fact that $\mathbb{T}_\Delta(\mathfrak{n}^+, \mathfrak{p}^N)_\mathcal{O}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_N$. □

2.3. Gross points and Theta elements. We define Gross points and Theta elements following [**18**].

If Δ in Subsection 2.2 is a product of primes inert or ramified in K, then K can be embedded into B_Δ. We choose a basis of $B_\Delta = K \oplus KJ$ over K such that

- $J^2 = \beta \in F^\times$ is totally negative, and $Jt = \bar{t}J$ for $t \in K$;
- $\beta \in (\mathcal{O}_{F_v}^\times)^2$ for all $v|\mathfrak{p}\mathfrak{n}^+$ and $\beta \in \mathcal{O}_{F_v}^\times$ for all $v|D_{K/F}$.

To define Gross points we need to choose a precise isomorphism

$$\prod_{v\nmid\Delta} i_v\colon \widehat{B}_\Delta^{(\Delta)} \longrightarrow M_2(\widehat{F}^{(\Delta)}).$$

For this we fix a CM type Σ of K. Choose an element ϑ such that

- $\mathrm{Im}(\sigma(\vartheta)) > 0$ for all $\sigma \in \Sigma$;
- $\{1, \vartheta_v\}$ is a basis of $\mathcal{O}_{K_v}$ over $\mathcal{O}_{F_v}$ for all $v|D_{K/F}\mathfrak{p}\mathfrak{n}$;
- ϑ is a local uniformizer at each prime v that is ramified in K.

Then we require that for each $v|\mathfrak{p}\mathfrak{n}^+$, i_v is given by

$$i_v(\vartheta) = \begin{pmatrix} T(\vartheta) & -N(\vartheta) \\ 1 & 0 \end{pmatrix}, \quad i_v(J) = \sqrt{\beta}\begin{pmatrix} -1 & T(\vartheta) \\ 0 & 1 \end{pmatrix},$$

where $T(\vartheta) = \vartheta + \bar{\vartheta}$ and $N(\vartheta) = \vartheta\bar{\vartheta}$; for $v \nmid \mathfrak{p}\mathfrak{n}^+\Delta$, $i_v(\mathcal{O}_{K_v}) \subset M_2(\mathcal{O}_{F_v})$.

Now we define Gross points. For $v|\mathfrak{n}^+$ we put $\zeta_v = (\vartheta - \bar{\vartheta})^{-1}\begin{pmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(K_w) = \mathrm{GL}_2(F_v)$ if $v = w\bar{w}$ in K. If m is a positive integer, we put

$$\zeta_{\mathfrak{p}}^{(m)} = \begin{cases} \begin{pmatrix} \vartheta & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varpi_{\mathfrak{p}}^m & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K_{\mathfrak{P}}) = \mathrm{GL}_2(F_{\mathfrak{p}}) & \text{if } \mathfrak{p} = \mathfrak{P}\overline{\mathfrak{P}}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \varpi_{\mathfrak{p}}^m & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mathfrak{p} \text{ is inert.} \end{cases}$$

Set $\zeta^{(m)} = \zeta_{\mathfrak{p}}^{(m)} \prod_{v|\mathfrak{n}^+} \zeta_v \in \widehat{B}_\Delta^\times$.

Let R_m be the order $\mathcal{O}_F + \mathfrak{p}^m\mathcal{O}_K$ of K. If $m \geq N$, then $(\zeta^{(m)})^{-1}\widehat{R}_m^\times\zeta^{(m)} \subset \mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N}$. Thus we have a map

$$\begin{aligned} x_m\colon K^\times\backslash\widehat{K}^\times/Y\widehat{R}_m^\times &\longrightarrow B_\Delta^\times\backslash\widehat{B}_\Delta^\times/Y\mathfrak{U}_{\mathfrak{n}^+,\mathfrak{p}^N} \\ a &\longmapsto [a\zeta^{(m)}]. \end{aligned}$$

If $\mathscr{D} = (\Delta, g)$ is an (N, n)-admissible form ($n \leq N$), for each $m \geq N$ we define

$$\Theta_m(g) = \frac{1}{\alpha_{\mathfrak{p}}^m} \sum_{[a]_m \in G_m} g(x_m(a))[a]_m \in \mathcal{O}_n[G_m],$$

where $a \mapsto [a]_m$ is the map induced by the normalized geometrical reciprocity law. These elements $\Theta_m(g)$ are compatible in the sense that $\pi_{m+1,m}(\Theta_{m+1}(g)) = \Theta_m(g)$. Here, $\pi_{m+1,m}$ is the quotient map $\mathcal{O}_n[G_{m+1}] \to \mathcal{O}_n[G_m]$.

Let $\pi_m\colon G_m \to \Gamma_m$ be the natural map, and put

$$\theta_m(g) = \pi_m(\Theta_m(g)) \in \mathcal{O}_n[\Gamma_m].$$

Then $\theta_m(g)$ ($m \geq 1$) are compatible and thus define an element $\theta(g)$ of $\mathcal{O}_n[[\Gamma]]$.

Now we restrict to the case $\Delta = \mathfrak{n}^-$, and put $B = B_{\mathfrak{n}^-}$. Let $\widehat{R}_{\mathfrak{n}^+}$ be an Eichler order in B of level $\mathfrak{n}^+$.

By the Jacquet–Langlands correspondence we find a $\mathbb{C}_p$-automorphic representation π' for the group $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$ corresponding to f (more precisely $\jmath(f)$) and an eigenform $f_B \in S_k^B(\widehat{R}_{\mathfrak{n}^+}^\times, \mathbb{C}_p)$ with the property $T_v f_B = a_v(f) f_B$ for $v \nmid \mathfrak{n}$ and $U_v f_B = \alpha_v(f) f_B$ for $v|\mathfrak{n}$. Put

$$\varphi_B(x) = \langle \rho_{k,\infty}(x_\infty)\mathbf{v}_0, f_B(x^\infty)\rangle_k,$$

where $\mathbf{v}_0 = X^{\frac{k-2}{2}}Y^{\frac{k-2}{2}}$. Then φ_B is in the π'-part of the space of $\mathbb{C}_p$-automorphic forms for G. We normalize f_B such that φ_B takes values in $\mathcal{O}$ (enlarging E if necessary) and is nonzero modulo ω.

Define the $\mathfrak{p}$-stabilization $\varphi_B^\dagger$ of φ_B as

$$\varphi_B^\dagger = \varphi_B - \frac{1}{\alpha_\mathfrak{p}}\pi'\left(\left(\begin{smallmatrix}1 & 0\\ 0 & \omega_\mathfrak{p}\end{smallmatrix}\right)\right)\varphi_B.$$

Then we define

$$\Theta_m(f) = \frac{1}{\alpha_\mathfrak{p}^m}\sum_{[a]_m\in G_m}\varphi_B^\dagger(x_m(a))[a]_m \in \mathcal{O}[G_m].$$

These elements $\Theta_m(f)$ are compatible, meaning $\pi_{m+1,m}(\Theta_{m+1}(f)) = \Theta_m(f)$. Then we define $\theta_m(f)$ and $\theta(f)$ as above.

Finally we define the p-adic L-adic function $L_p(K_\infty, f)$ by $L_p(K_\infty, f) = \theta(f)^2$. Hung ([**10**]) proved an interpolation formula for $L_p(K_\infty, f)$. We do not state it here, since we will not use it.

Proposition 2.6 ([**10**, Theorem 6.9])**.** *We have that the analytic μ-invariant of $L_p(K_\infty, f)$ is zero, i.e. $L_p(K_\infty, f) \not\equiv 0 \pmod{\omega}$. In particular, $L_p(K_\infty, f) \neq 0$.*

Proposition 2.7 ([**18**, Proposition 7.4.2])**.** *If $\Delta = \mathfrak{n}^-$, there exists an (N,N)-admissible form $\mathscr{D}_N = (\mathfrak{n}^-, f_N^\dagger)$ such that*

$$\theta_m(\mathscr{D}_N) \equiv \theta_m(f) \pmod{\omega^N}$$

for each $m \geq N$. In particular

$$\theta(\mathscr{D}_N) \equiv \theta(f) \pmod{\omega^N}.$$

3. Selmer groups

For the convenience of readers, we recall the definition of Selmer groups. See [**1, 5, 13, 18**] for more details.

3.1. Basic properties of Selmer groups. Let L be a finite extension of F. For each place $\mathfrak{l}$ of F and each discrete G_F-module M, we put

$$H^1(L_\mathfrak{l}, M) = \bigoplus_{\lambda|\mathfrak{l}} H^1(L_\lambda, M), \quad H^1(I_{L_\mathfrak{l}}, M) = \bigoplus_{\lambda|\mathfrak{l}} H^1(I_{L_\lambda}, M),$$

where λ runs through all places of L above $\mathfrak{l}$. Denote by

$$\mathrm{res}_\mathfrak{l}\colon H^1(L, M) \longrightarrow H^1(L_\mathfrak{l}, M)$$

the restriction map at $\mathfrak{l}$.

We define the *finite part* $H^1(L_\mathfrak{l}, M)$ as

$$H^1_{\mathrm{fin}}(L_\mathfrak{l}, M) = \ker(H^1(L_\mathfrak{l}, M) \longrightarrow H^1(I_{L_\mathfrak{l}}, M))$$

and the *singular quotient* as

$$H^1_{\mathrm{sing}}(L_\mathfrak{l}, M) = H^1(L_\mathfrak{l}, M)/H^1_{\mathrm{fin}}(L_\mathfrak{l}, M).$$

One has the following exact sequence:

$$\textstyle\bigoplus_{\lambda|\mathfrak{l}} H^1(G_{L_\lambda}/I_{L_\lambda}, M^{I_{L_\lambda}}) \longrightarrow H^1(L_\mathfrak{l}, M) \xrightarrow{\partial_\mathfrak{l}} \bigoplus_{\lambda|\mathfrak{l}} H^1(I_{L_\lambda}, M)^{G_{L_\lambda}/I_{L_\lambda}}.$$

Then $H^1_{\mathrm{fin}}(L_\mathfrak{l}, M)$ coincides with the image of the map

$$\bigoplus_{\lambda|\mathfrak{l}} H^1(G_{L_\lambda}/I_{L_\lambda}, M^{I_{L_\lambda}}) \longrightarrow H^1(L_\mathfrak{l}, M),$$

and $H^1_{\mathrm{sing}}(L_{\mathfrak{l}}, M)$ is naturally isomorphic to the image of $\partial_{\mathfrak{l}}$. By abuse of notation, the composition map $\partial_{\mathfrak{l}} \circ \mathrm{res}_{\mathfrak{l}}$ is also denoted by $\partial_{\mathfrak{l}}$. If an element $s \in H^1(G_L, M)$ satisfies $\partial_{\mathfrak{l}}(s) = 0$, then $\mathrm{res}_{\mathfrak{l}}(s)$ is in $H^1_{\mathrm{fin}}(L_{\mathfrak{l}}, M)$ and we will denote it as $v_{\mathfrak{l}}(s)$.

If $\mathfrak{l}|\mathfrak{n}^-$, if $\mathfrak{l}$ is n-admissible, or if $\mathfrak{l}|p$, then the restriction $\rho_f^*|_{G_{F_{\mathfrak{l}}}}$ of A_n to $G_{F_{\mathfrak{l}}}$ sits in a $G_{F_{\mathfrak{l}}}$-equivariant short exact sequence of free $\mathcal{O}_{f,n}$-modules

$$0 \longrightarrow F_{\mathfrak{l}}^+ A_n \longrightarrow A_n \longrightarrow F_{\mathfrak{l}}^- A_n \longrightarrow 0,$$

where $G_{F_{\mathfrak{l}}}$ acts on $F_{\mathfrak{l}}^+ A_n$ by $\pm\epsilon$ (resp. $\chi^{-1}\epsilon^{k/2}$) if $\mathfrak{l}|\mathfrak{n}^-$ or $\mathfrak{l}$ is n-admissible (resp. $\mathfrak{l}|p$). Here, when $\mathfrak{l}|p$, χ is the unramified character of $G_{F_{\mathfrak{l}}}$ such that $\chi(\mathrm{Frob}) = \alpha_{\mathfrak{l}}$, where $\alpha_{\mathfrak{l}}$ is the unit root of the Hecke polynomial $x^2 - a_{\mathfrak{l}}(f)x + N(\mathfrak{l})^{k-1}$. Then we define the *ordinary part* of $H^1_{\mathrm{ord}}(L_{\mathfrak{l}}, A_n)$ to be the image of

$$H^1(G_{L_{\mathfrak{l}}}, F_{\mathfrak{l}}^+ A_n) \longrightarrow H^1(G_{L_{\mathfrak{l}}}, A_n).$$

We define $H^1_{\mathrm{ord}}(L_{\mathfrak{l}}, T_n)$ similarly.

Let Δ ($\mathfrak{n}^-|\Delta$) be a square-free product of prime ideals in $\mathcal{O}_F$ such that $\Delta/\mathfrak{n}^-$ is a product of n-admissible prime ideals. Let S be a finite (maybe empty) set of places of F that are coprime to $p\Delta\mathfrak{n}$.

Definition 3.1. We define the Selmer group $\mathrm{Sel}^S_{\Delta}(G_L, M)$, where $M = A_n$ or T_n, to be the group of elements $s \in H^1(G_L, M)$ such that

(a) $\mathrm{res}_{\mathfrak{l}}(s) \in H^1_{\mathrm{fin}}(L_{\mathfrak{l}}, M)$ if $\mathfrak{l} \nmid p\Delta$ and $\mathfrak{l} \notin S$;

(b) $\mathrm{res}_{\mathfrak{l}}(s) \in H^1_{\mathrm{ord}}(L_{\mathfrak{l}}, M)$ for all $\mathfrak{l}|p\Delta$;

(c) $\mathrm{res}_{\mathfrak{l}}(s)$ is arbitrary if $\mathfrak{l} \in S$.

The group $\mathrm{Gal}(K_m/F)$ acts on $H^1(K_m, T_n)$ and $H^1(K_m, A_n)$.

Lemma 3.2. $\mathrm{Gal}(K_m/F)$ *preserves* $\mathrm{Sel}^S_{\Delta}(G_{K_m}, T_n)$ *and* $\mathrm{Sel}^S_{\Delta}(G_{K_m}, A_n)$.

Proof: If $\mathfrak{l} \nmid p\Delta$ and if $\mathfrak{l} \notin S$, then for each place λ of K_m above $\mathfrak{l}$, the largest unramified extension of $K_{m,\lambda}$ is Galois over $F_{\mathfrak{l}}$. Thus $\mathrm{Gal}(K_m/F)$ acts on

$$\bigoplus_{\lambda|\mathfrak{l}} H^1(G_{K_{m,\lambda}}/I_{K_{m,\lambda}}, T_n^{I_{K_{m,\lambda}}})$$

and thus preserves $H^1_{\mathrm{fin}}(K_{m,\mathfrak{l}}, T_n)$.

If $\mathfrak{l}|p\Delta$, then $\mathrm{Gal}(K_m/F)$ preserves $H^1_{\mathrm{ord}}(K_{m,\mathfrak{l}}, T_n)$. This follows from the fact that $G_{F_{\mathfrak{l}}}$ preserves the subspace $F_{\mathfrak{l}}^+ T_n$ of T_n used to define the ordinary part. □

Proposition 3.3 ([**13**, Proposition 7.5], [**18**, Theorem 7.1.2])**.** *Assume* (CR$^+$) *holds. Let $t \le n$ be positive integers. Let κ be a nonzero element in $H^1(K, T_t)$. Then there exist infinitely many n-admissible primes $\mathfrak{l}$ such that $\partial_{\mathfrak{l}}(\kappa) = 0$ and the map*

$$v_{\mathfrak{l}} \colon \langle\kappa\rangle \longrightarrow H^1_{\mathrm{fin}}(K_{\mathfrak{l}}, T_t)$$

is injective, where $\langle\kappa\rangle$ denotes the $\mathcal{O}$-submodule of $H^1(K, T_t)$ generated by κ.

We put

$$H^1(K_\infty, A_n) = \varinjlim_r H^1(K_r, A_n), \quad \widehat{H}^1(K_\infty, T_n) = \varprojlim_m H^1(K_m, T_n),$$

$$H^1(K_{\infty,\mathfrak{l}}, A_n) = \varinjlim_m H^1(K_{m,\mathfrak{l}}, A_n), \quad \text{and} \quad \widehat{H}^1(K_{\infty,\mathfrak{l}}, T_n) = \varprojlim_m H^1(K_{m,\mathfrak{l}}, T_n).$$

The finite parts and the singular quotients $H^1_?(K_{\infty,\mathfrak{l}}, A_n)$ and $\widehat{H}^1_?(K_{\infty,\mathfrak{l}}, T_n)$ for $? \in \{\mathrm{fin}, \mathrm{sing}\}$ are defined similarly.

For each $\mathfrak{l}$ we have the local Tate pairing

$$\langle \cdot, \cdot \rangle_{\mathfrak{l}} \colon \widehat{H}^1(K_{\infty,\mathfrak{l}}, T_n) \times H^1(K_{\infty,\mathfrak{l}}, A_n) \longrightarrow E/\mathcal{O}.$$

Proposition 3.4. (a) *If $\mathfrak{l}$ splits in K, then* $H^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}},A_n)=0$ *and* $\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}},T_n)=0$.

(b) *If $\mathfrak{l}$ is inert in K, then* $\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}},T_n)\cong H^1_{\mathrm{sing}}(K_{\mathfrak{l}},T_n)\otimes_{\mathcal{O}}\mathcal{O}[[\Gamma]]$.

(c) *If $\mathfrak{l}\nmid p$, then* $H^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}},A_n)$ *and* $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}},T_n)$ *are orthogonal to each other under the pairing* $\langle\cdot,\cdot\rangle_{\mathfrak{l}}$.

(d) *If $\mathfrak{l}$ is n-admissible, then* $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}},T_n)$, $\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}},T_n)$, *and* $\widehat{H}^1_{\mathrm{ord}}(K_{\infty,\mathfrak{l}},T_n)$ *are free of rank* 1 *over* $\mathcal{O}[[\Gamma]]/(\omega^n)$.

(e) *Assume* (CR$^+$) *and* (PO) *hold. If $\mathfrak{l}$ is n-admissible or if $\mathfrak{l}|p\mathfrak{n}^-$, then* $H^1_{\mathrm{ord}}(K_{\infty,\mathfrak{l}},A_n)$ *and* $\widehat{H}^1_{\mathrm{ord}}(K_{\infty,\mathfrak{l}},T_n)$ *are orthogonal to each other under the pairing* $\langle\cdot,\cdot\rangle_{\mathfrak{l}}$.

Proof: This is [**18**, Proposition 2.4.1, Lemma 2.4.2, Proposition 2.4.4]. □

We define

$$\mathrm{Sel}^S_\Delta(K_\infty,A_n)=\varinjlim_m \mathrm{Sel}^S_\Delta(K_m,A_n),\quad \widehat{\mathrm{Sel}}^S_\Delta(K_\infty,T_n)=\varprojlim_m \mathrm{Sel}^S_\Delta(K_m,T_n).$$

If S is empty, we drop S from the above notations. When $S=\emptyset$ and $\Delta=\mathfrak{n}^-$, we drop both S and Δ from the notations; the Selmer group in Theorem 1.2 is in this case.

3.2. Control theorems.

Lemma 3.5. *Assume* (CR$^+$) *holds. Let L/K be a finite extension contained in K_∞.*

(a) *The restriction maps*

$$H^1(K,A_n)\longrightarrow H^1(L,A_n)^{\mathrm{Gal}(L/K)}$$

and

$$\mathrm{Sel}^S_\Delta(K,A_n)\longrightarrow \mathrm{Sel}^S_\Delta(L,A_n)^{\mathrm{Gal}(L/K)}$$

are isomorphisms.

(b) *If S contains all prime $\mathfrak{q}|\mathfrak{n}^+$ with $\bar\rho_{f,\mathfrak{q}}$ unramified, then*

$$\mathrm{Sel}^S_{\mathfrak{n}^-}(L,A_n)=\mathrm{Sel}^S_{\mathfrak{n}^-}(L,A)[\omega^n]. \tag{3.1}$$

In particular, if further ($\mathfrak{n}^+$-min) *holds, then for any set S of primes,* (3.1) *holds.*

(c) *If S contains all prime $\mathfrak{q}|\mathfrak{n}^+$ with $\bar\rho_{f,\mathfrak{q}}$ unramified, then for any $m\le n$*

$$\mathrm{Sel}^S_\Delta(L,A_m)=\mathrm{Sel}^S_\Delta(L,A_n)[\omega^m]. \tag{3.2}$$

In particular, if further ($\mathfrak{n}^+$-min) *holds, then for any set S of primes,* (3.2) *holds.*

Proof: Assertion (a) is [**18**, Proposition 2.5.1(1)]. Next we prove (b).

Since L/K is abelian, by (CR$^+$) we have $A_1^{G_L}=0$. Then $A_m^{G_L}=0$ for every m, and thus $A^{G_L}=0$. So from the exact sequence

$$0\longrightarrow A_n\longrightarrow A\xrightarrow{\omega^n}A\longrightarrow 0$$

we obtain the isomorphism $H^1(G_L,A_n)\cong H^1(G_L,A)[\omega^n]$ and the injectivity of $\mathrm{Sel}^S_\Delta(L,A_n)\hookrightarrow\mathrm{Sel}^S_\Delta(L,A)[\omega^n]$. To prove the surjectivity of $\mathrm{Sel}^S_\Delta(L,A_n)\hookrightarrow\mathrm{Sel}^S_\Delta(L,A)[\omega^n]$, it suffices to prove

(i) $H^1(L^{\mathrm{ur}}_{\mathfrak{l}},A_n)\to H^1(L^{\mathrm{ur}}_{\mathfrak{l}},A)$ is injective for $\mathfrak{l}\nmid p\Delta$ and $\mathfrak{l}\notin S$.

(ii) $H^1(L_{\mathfrak{l}},A_n/F^+_{\mathfrak{l}}A_n)\to H^1(L_{\mathfrak{l}},A/F^+_{\mathfrak{l}}A)$ is injective for $\mathfrak{l}|p\mathfrak{n}^-$.

For (i) if $\mathfrak{l} \nmid \mathfrak{n}^+$, the action of $I_{L,\mathfrak{l}}$ is trivial and the claim follows immediately. If $\mathfrak{l}|\mathfrak{n}^+$, then by Lemma 2.2, $H^0(F_{\mathfrak{l}}^{\mathrm{nr}}, A)$ is divisible. The claim again follows.

For (ii) if $\mathfrak{l}|\mathfrak{n}^-$, the actions of $G_{L_\mathfrak{l}}$ on $A_n/F_{\mathfrak{l}}^+A_n$ and $A/F_{\mathfrak{l}}^+A$ are trivial, and the claim is clear. If $\mathfrak{l}|p$, then $G_{L_\mathfrak{l}}$ acts on $A_m/F_{\mathfrak{l}}^+A_m$ by $\chi_\mathfrak{l}\epsilon^{1-\frac{k}{2}}$, where $\chi_\mathfrak{l}$ is an unramified character. Thus $H^0(L_\mathfrak{l}, A_m/F_{\mathfrak{l}}^+A_m) = 0$ for each m. Then $H^0(L_\mathfrak{l}, A/F_{\mathfrak{l}}^+A) = 0$. The claim follows.

The proof of (c) is similar to that of (b). One only needs to note that, for each $\mathfrak{l}|\Delta$, the action of $G_{L_\mathfrak{l}}$ on $A_n/F_{\mathfrak{l}}^+A_n$ is trivial. □

Theorem 3.6 ([**18**, Proposition 7.2.3])**.** *Assume the conditions* (CR$^+$), (PO)*, and* ($\mathfrak{n}^+$-min) *hold. For each positive integer n there exists a finite set S of n-admissible prime ideals such that $\widehat{\mathrm{Sel}}_{\Delta}^{S}(K_\infty, T_n)$ is free over $\mathcal{O}_n[[\Gamma]]$.*

Theorem 3.7. *If $\mathrm{Sel}_{\mathfrak{n}^-}(K_\infty, A)$ is $\mathcal{O}[[\Gamma]]$-cotorsion and the algebraic μ-invariant of $\mathrm{Sel}_{\mathfrak{n}^-}(K_\infty, A)^\vee$ vanishes, then for any finite set S of n-admissible primes that do not divide $p\mathfrak{n}\Delta$, $\widehat{\mathrm{Sel}}_{\Delta}^{S}(K_\infty, T_n)$ is free over $\mathcal{O}_n[[\Gamma]]$.*

Proof: This was essentially proved by Wang [**18**, Chapter 10] following Kim, Pollack, and Weston's idea [**12**]. However, the assertion in the above form is not clearly stated in loc. cit., so we give a sketch of the proof.

Let

$$\Phi_n \colon \{\text{cofinitely generated } \mathcal{O}[[\Gamma]]\text{-modules}\} \longrightarrow \{\text{finitely generated } \Lambda/\omega^n\text{-modules}\}$$

be the functor defined by $\Phi(M) = \varprojlim_m M[\omega^n]^{\Gamma_m}$. It follows from Lemma 3.5(a) that

$$\begin{aligned}\Phi_n(\mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+S}(K_\infty, A)) &\cong \varprojlim_m \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+S}(K_\infty, A)[\omega^n]^{\Gamma_m} \\ &= \varprojlim_m \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+S}(K_m, A_n) \cong \widehat{\mathrm{Sel}}_{\mathfrak{n}^-}^{\mathfrak{n}^+S}(K_\infty, T_n).\end{aligned}$$

The functor Φ_n satisfies the following properties.

- If A and B are pseudo-isomorphic cofinitely generated $\mathcal{O}[[\Gamma]]$-modules, then $\Phi_n(A) = \Phi_n(B)$.
- If Y is a finitely cotorsion $\mathcal{O}[[\Gamma]]$-module with vanishing (algebraic) μ-invariant, then $\Phi_n(Y) = 0$.
- If $Y = \mathcal{O}[[\Gamma]]/\omega^t$ with $t \geq n$, then $\Phi_n(Y^\vee) = \mathcal{O}[[\Gamma]]/\omega^n$.

Wang ([**18**, Lemma 10.1.2]) showed that for any finite set S away from $p\mathfrak{n}\Delta$, $\mathrm{Sel}_{\mathfrak{n}^-}^{S\mathfrak{n}^+}(K_\infty, A)$ sits in the exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A) \longrightarrow \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+S}(K_\infty, A) \longrightarrow \textstyle\prod_{v\in S} \mathcal{H}_v \longrightarrow 0,$$

where $\mathcal{H}_v = \varinjlim_m \prod_{w|v} H^1(K_{m,w}, A)$. When $v \in S$ is n-admissible, $\mathcal{H}_v \cong (\mathcal{O}[[\Gamma]]/\omega^{t_v})^\vee$ for some $t_v \geq n$ [**18**, Lemma 10.1.3]. Thus $\mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+S}(K_\infty, A)^\vee$ is pseudo-isomorphic to

$$\left(\bigoplus_{v\in S} \mathcal{O}[[\Gamma]]/\omega^{t_v}\right) \times Y,$$

where Y is a torsion $\mathcal{O}[[\Gamma]]$-module with $\mu(Y) = 0$. Hence, by the above properties of Φ_n, $\widehat{\mathrm{Sel}}_{\mathfrak{n}^-}^{S\mathfrak{n}^+}(K_\infty, T_n)$ is free of rank $\sharp S$ over $\mathcal{O}[[\Gamma]]/\omega^n$.

For $\mathfrak{l} \in \Delta$ we have the following exact sequence:

$$0 \longrightarrow \widehat{\mathrm{Sel}}_{\mathfrak{l}\mathfrak{n}^-}^{\mathfrak{n}^+ S}(K_\infty, T_n) \longrightarrow \widehat{\mathrm{Sel}}_{\mathfrak{n}^-}^{\mathfrak{l}\mathfrak{n}^+ S}(K_\infty, T_n) \longrightarrow \widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}}, T_n) \longrightarrow 0.$$

So, the freeness of $\widehat{\mathrm{Sel}}_{\mathfrak{n}^-}^{\mathfrak{l}\mathfrak{n}^+ S}(K_\infty, T_n)$ and $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}}, T_n)$ implies the freeness of $\widehat{\mathrm{Sel}}_{\mathfrak{l}\mathfrak{n}^-}^{\mathfrak{n}^+ S}(K_\infty, T_n)$. Repeating this several times we obtain the freeness of $\widehat{\mathrm{Sel}}_{\Delta}^{\mathfrak{n}^+ S}(K_\infty, T_n)$. By Proposition 3.4(a), we have $\widehat{\mathrm{Sel}}_{\Delta}^{S}(K_\infty, T_n) = \widehat{\mathrm{Sel}}_{\Delta}^{\mathfrak{n}^+ S}(K_\infty, T_n)$. □

4. Euler system of Heegner points

Fix $N \geq n \geq 1$. Let $\mathscr{D} = (\Delta, g)$, $\mathfrak{n}^- | \Delta$, be an (N, n)-admissible form.

4.1. Shimura curves. In this subsection we collect necessary results on Shimura curves [**13, 18**].

Let $\mathfrak{l} \nmid \Delta$ be an n-admissible prime ideal of f with $\epsilon_\mathfrak{l} \alpha_\mathfrak{l} = \mathrm{N}(\mathfrak{l}) + 1 \pmod{\omega^n}$. One defines the character of Hecke algebra

$$\lambda_g^{[\mathfrak{l}]} : \mathbb{T}_\Delta(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N)_\mathcal{O} \longrightarrow \mathcal{O}_n$$

by $\lambda_g^{[\mathfrak{l}]}(U_\mathfrak{l}) = \epsilon_\mathfrak{l}$, and let $\mathcal{I}_g^{[\mathfrak{l}]}$ be the kernel of $\lambda_g^{[\mathfrak{l}]}$.

Let B' be the quaternion algebra with discriminant $\Delta\mathfrak{l}$ that splits at exactly one real place. Then we have an isomorphism $\phi : \widehat{B}_\Delta^{(\mathfrak{l})} \cong \widehat{B}'^{(\mathfrak{l})}$. Let $\mathcal{O}_{B'_\mathfrak{l}}$ be the maximal order of $B'_\mathfrak{l}$. Put

$$\mathfrak{U}' = \mathfrak{U}'_{\mathfrak{n}^+, \mathfrak{p}^N} = \phi((\mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N})^{(\mathfrak{l})}) \mathcal{O}_{B'_\mathfrak{l}}^\times.$$

With $\mathfrak{U}'$ instead of $\mathfrak{U} = \mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N}$ we have a Hecke algebra $\mathbb{T}_{\Delta\mathfrak{l}}(\mathfrak{n}^+, \mathfrak{p}^N)$.

Associated to $(B', Y\mathfrak{U}')$ there is a Shimura curve $M_N^{[\mathfrak{l}]}$ with complex points

$$M_N^{[\mathfrak{l}]}(\mathbb{C}) = B'^\times \backslash (\mathbf{P}^1(\mathbb{C}) - \mathbf{P}^1(\mathbb{R})) \times \widehat{B}'^\times / Y\mathfrak{U}';$$

$M_N^{[\mathfrak{l}]}$ is smooth and projective over F. We write $[z, b']_N$ for the point in $M_N^{[\mathfrak{l}]}$ corresponding to $z \in \mathbf{P}^1(\mathbb{C}) - \mathbf{P}^1(\mathbb{R})$ and $b' \in \widehat{B}'^\times$.

Let $F_{\mathfrak{l}^2}$ be the unramified extension of $F_\mathfrak{l}$ of degree 2. The Shimura curve $M_N^{[\mathfrak{l}]}$ admits a regular semistable model over $\mathcal{O}_{F_{\mathfrak{l}^2}}$ such that all irreducible components of its special fiber are smooth. One associates a graph $\mathcal{G}$ to the special fiber as follows.

The set of vertices in $\mathcal{G}$ which correspond to irreducible components of $M_N^{[\mathfrak{l}]}$ is identified with

$$\mathcal{V}(\mathcal{G}) = B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N} \times \mathbb{Z}/2\mathbb{Z}.$$

The set of oriented edges which correspond to ordered singular points on the special fiber is identified with

$$\vec{\mathcal{E}}(\mathcal{G}) = B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N} \times \mathbb{Z}/2\mathbb{Z}.$$

We choose an orientation of $\vec{\mathcal{E}}(\mathcal{G})$ such that the source and target maps $s, t \colon \mathcal{E}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ are given by

$$\begin{aligned} s \colon \mathcal{E}(\mathcal{G}) = B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N} &\longrightarrow \mathcal{V}(\mathcal{G}) = B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N} \times \mathbb{Z}/2\mathbb{Z} \\ B_\Delta^\times b Y\mathfrak{U}_{\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N} &\longmapsto (B_\Delta^\times b Y\mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N}, 0) \end{aligned}$$

and

$$\begin{aligned} t \colon \mathcal{E}(\mathcal{G}) = B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N} &\longrightarrow \mathcal{V}(\mathcal{G}) = B_\Delta^\times \backslash \widehat{B}_\Delta^\times / Y\mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N} \times \mathbb{Z}/2\mathbb{Z} \\ B_\Delta^\times b Y\mathfrak{U}_{\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N} &\longmapsto (B_\Delta^\times b Y\mathfrak{U}_{\mathfrak{n}^+, \mathfrak{p}^N}, 1). \end{aligned}$$

Let $J_N^{[\mathfrak{l}]}$ be the Jacobian of $M_N^{[\mathfrak{l}]}$, and let $\Phi^{[\mathfrak{l}]}$ be the component group of the Néron model of $J_N^{[\mathfrak{l}]}$ over $F_{\mathfrak{l}^2}$. Let $r_{\mathfrak{l}}\colon J_N^{[\mathfrak{l}]} \to \Phi^{[\mathfrak{l}]}$ be the reduction map.

There is a natural action of $\mathbb{T}_{\Delta\mathfrak{l}}(\mathfrak{n}^+, \mathfrak{p}^N)$ on $J_N^{[\mathfrak{l}]}$ via Picard functoriality. Note that

$$\mathbb{T}_{\Delta}^{(\mathfrak{l})}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N) \simeq \mathbb{T}_{\Delta\mathfrak{l}}^{(\mathfrak{l})}(\mathfrak{n}^+, \mathfrak{p}^N).$$

We extend it to a homomorphism

$$\varphi_*\colon \mathbb{T}_{\Delta}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N) \longrightarrow \mathbb{T}_{\Delta\mathfrak{l}}(\mathfrak{n}^+, \mathfrak{p}^N)$$

which sends $U_{\mathfrak{l}} = \left[\mathfrak{U}\left(\begin{smallmatrix}\pi_{\mathfrak{l}} & 0\\ 0 & 1\end{smallmatrix}\right)\mathfrak{U}\right]$ to $U_{\mathfrak{l}} = [\mathfrak{U}'\pi_{\mathfrak{l}}'\mathfrak{U}']$. Via φ_* we obtain an action of $\mathbb{T}_{\Delta}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N)$ on $J_N^{[\mathfrak{l}]}$. It induces an action of $\mathbb{T}_{\Delta}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N)$ on $\Phi^{[\mathfrak{l}]}$.

We need the relation between $\Phi^{[\mathfrak{l}]}$ and $\mathcal{G}$.

Let

$$d_* = t_* - s_*\colon \mathbb{Z}[\mathcal{E}(\mathcal{G})] \longrightarrow \mathbb{Z}[\mathcal{V}(\mathcal{G})]$$

be the boundary map, and

$$d^*\colon t^* - s^*\colon \mathbb{Z}[\mathcal{V}(\mathcal{G})] \longrightarrow \mathbb{Z}[\mathcal{E}(\mathcal{G})]$$

its dual. Put $\mathbb{Z}[\mathcal{V}(\mathcal{G})]_0 = \mathrm{im}(d_*)$. By [**2**, Section 9.6, Theorem 1] there exists a natural identification

$$\Phi^{[\mathfrak{l}]} \simeq \mathbb{Z}[\mathcal{V}(\mathcal{G})]_0/d_*d^*.$$

One can identify $\mathbb{Z}[\mathcal{V}(\mathcal{G})]$ with $(S_2^{B_\Delta}(\mathfrak{U}, \mathbb{Z})^Y)^{\oplus 2}$, and identify $\mathbb{Z}[\mathcal{V}(\mathcal{G})]_0$ with a submodule $(S_2^{B_\Delta}(\mathfrak{U}, \mathbb{Z})^Y)_0^{\oplus 2}$ of $(S_2^{B_\Delta}(\mathfrak{U}, \mathbb{Z})^Y)^{\oplus 2}$. Define an action of $\mathbb{T}_{\Delta}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N)$ on $(S_2^{B_\Delta}(\mathfrak{U}, \mathbb{Z})^Y)^{\oplus 2}$ by

$$t(x,y) = (t(x), t(y)), \quad t \in \mathbb{T}_{\Delta}^{(\mathfrak{l})}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N),$$

and

$$\tilde{U}_{\mathfrak{l}}(x,y) = (-\mathrm{N}(\mathfrak{l})y, x + T_{\mathfrak{l}}(y)).$$

Here, in the event of confusion with the diagonal action we use the notation $\tilde{U}_{\mathfrak{l}}$ instead of $U_{\mathfrak{l}}$.

Proposition 4.1 ([**18**, Proposition 4.4.1])**.** *We have the following* $\mathbb{T}_{\Delta}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N)$*-module isomorphism:*

$$\Phi^{[\mathfrak{l}]} \simeq (S_2^{B_\Delta}(\mathfrak{U}, \mathbb{Z})^Y)_0^{\oplus 2}/(\tilde{U}_{\mathfrak{l}}^2 - 1).$$

Write

$$\Phi_{\mathcal{O}}^{[\mathfrak{l}]} = \Phi^{[\mathfrak{l}]} \otimes_{\mathbb{Z}} \mathcal{O}.$$

Corollary 4.2. *We have an isomorphism*

$$\Phi_{\mathcal{O}}^{[\mathfrak{l}]}/\mathcal{I}_g^{[\mathfrak{l}]} \simeq S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y/\mathcal{I}_g \xrightarrow{\psi_g} \mathcal{O}_n.$$

When $n = N$, this is [**18**, Theorem 5.1.3].

Proof: Let $\mathfrak{m}^{[\mathfrak{l}]}$ be the maximal ideal of $\mathbb{T}_{\Delta}(\mathfrak{l}\mathfrak{n}^+, \mathfrak{p}^N)_{\mathcal{O}}$ containing $\mathcal{I}_g^{[\mathfrak{l}]}$.

Note that $(S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y)^{\oplus 2}/(S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y)_0^{\oplus 2}$ is Eisenstein, while $\mathfrak{m}^{[\mathfrak{l}]}$ is not Eisenstein. Thus

$$(S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y)_{0\,\mathfrak{m}^{[\mathfrak{l}]}}^{\oplus 2} = (S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y)_{\mathfrak{m}^{[\mathfrak{l}]}}^{\oplus 2}.$$

By Proposition 4.1 we obtain

$$(\Phi_{\mathcal{O}}^{[\mathfrak{l}]})_{\mathfrak{m}^{[\mathfrak{l}]}} \simeq (S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y)_{\mathfrak{m}^{[\mathfrak{l}]}}^{\oplus 2}/(\tilde{U}_{\mathfrak{l}}^2 - 1) \simeq (S_2^{B_\Delta}(\mathfrak{U}, \mathcal{O})^Y)_{\mathfrak{m}^{[\mathfrak{l}]}}^{\oplus 2}/(\tilde{U}_{\mathfrak{l}} - \epsilon_{\mathfrak{l}}).$$

Hence,

$$\begin{aligned}\Phi_{\mathcal{O}}^{[\mathfrak{l}]}/\mathcal{I}_g^{[\mathfrak{l}]} &\simeq ((S_2^{B_\Delta}(\mathfrak{U},\mathcal{O})^Y)^{\oplus 2}/(\tilde{U}_\mathfrak{l}-\epsilon_\mathfrak{l}))\otimes \mathbb{T}_\Delta(\mathfrak{l}\mathfrak{n}^+,\mathfrak{p}^N)/\mathcal{I}_g^{[\mathfrak{l}]}\\ &\simeq (S_2^{B_\Delta}(\mathfrak{U},\mathcal{O})^Y/(\epsilon_\mathfrak{l}T_\mathfrak{l}-\mathbf{N}(\mathfrak{l})-1))\otimes \mathbb{T}_\Delta(\mathfrak{n}^+,\mathfrak{p}^N)/\mathcal{I}_g\\ &\simeq S_2^{B_\Delta}(\mathfrak{U},\mathcal{O})^Y/\mathcal{I}_g.\end{aligned}$$

By Proposition 2.5, ψ_g is an isomorphism. □

Let $T_p(J_N^{[\mathfrak{l}]})$ be the p-adic Tate module of $J_N^{[\mathfrak{l}]}$. Then $T_p(J_N^{[\mathfrak{l}]})$ is a $\mathbb{T}_\Delta(\mathfrak{l}\mathfrak{n}^+,\mathfrak{p}^N)$-module.

Proposition 4.3. *We have an isomorphism of G_F-modules*

$$T_p(J_N^{[\mathfrak{l}]})_\mathcal{O}/\mathcal{I}_g^{[\mathfrak{l}]} \simeq T_n.$$

When $n=N$, this is [**18**, Theorem 5.1.4].

Proof: Let $\mathfrak{m}^{[\mathfrak{l}]}$ be the maximal ideal of $\mathbb{T}_\Delta(\mathfrak{l}\mathfrak{n}^+,\mathfrak{p}^N)_\mathcal{O}$ containing $\mathcal{I}_g^{[\mathfrak{l}]}$. In the proof of [**18**, Theorem 5.1.4] it is shown that $T_p(J_N^{[\mathfrak{l}]})_\mathcal{O}/\mathfrak{m}^{[\mathfrak{l}]} \simeq T_1$. By irreducibility of T_1, to finish the proof one only needs to show that the exponent of $T_p(J_N^{[\mathfrak{l}]})_\mathcal{O}/\mathcal{I}_g^{[\mathfrak{l}]}$ is ω^n. On one hand, its exponent is at most ω^n. On the other hand, when n' is sufficiently large, $J_N^{[\mathfrak{l}]}[p^{n'}]_\mathcal{O}/\mathcal{I}_g^{[\mathfrak{l}]}$ maps onto $\Phi_\mathcal{O}^{[\mathfrak{l}]}/\mathcal{I}_g^{[\mathfrak{l}]}$, together with Corollary 4.2, which implies that the exponent is at least ω^n. □

Let

$$\mathrm{Kum}\colon J_N^{[\mathfrak{l}]}(K_m)_\mathcal{O} \longrightarrow H^1(K_m, T_p(J_N^{[\mathfrak{l}]})_\mathcal{O})$$

be the Kummer map.

Proposition 4.4 ([**18**, Theorem 5.2.2])**.** *We have the following commutative diagram:*

$$\begin{array}{ccc} J_N^{[\mathfrak{l}]}(K_m)_\mathcal{O}/\mathcal{I}_g^{[\mathfrak{l}]} & \xrightarrow{\mathrm{Kum}} & H^1(K_m,T_n)\\ \downarrow r_\mathfrak{l} & & \downarrow \partial_\mathfrak{l}\\ \Phi_\mathcal{O}^{[\mathfrak{l}]} & \xrightarrow[\simeq]{\psi_g} & H^1_{\mathrm{sing}}(K_{m,\mathfrak{l}},T_n)\ . \end{array}$$

4.2. First and Second Reciprocity Laws. We choose an auxiliary prime $\mathfrak{q}_0 \nmid \Delta\mathfrak{n}^+$ such that $1+\mathrm{N}(\mathfrak{q}_0)-\alpha_{\mathfrak{q}_0}(f)\in\mathcal{O}^\times$.

The inclusion $t'(K^\times)\subset B'^\times\subset \mathrm{GL}_2(\mathbb{R})$ gives an action of $K^\times$ on $\mathbf{P}^1(\mathbb{C})-\mathbf{P}^1(\mathbb{R})$. This action has two fixed points; we choose one of them and denote it by z'.

For $m\geq N$ and $a\in\widehat{K}^\times$ we define the Heegner point

$$P_m(a)=[z',\phi(a^{(\mathfrak{l})}\zeta^{(m)}\tau_N)]_N\in M_N^{[\mathfrak{l}]}(\mathbb{C}).$$

See Subsections 2.2 and 2.3 for the notations $\zeta^{(m)}$ and τ_N. By the theory of complex multiplication $P_m(a)$ is defined over the ring class field $\widetilde{K}_m$.

We define a map

$$\begin{aligned}\xi_{\mathfrak{q}_0}\colon\ \mathrm{Div}(M_N^{[\mathfrak{l}]}(\widetilde{K}_m)) &\longrightarrow J_N^{[\mathfrak{l}]}(\widetilde{K}_m)_\mathcal{O}\\ P &\longmapsto \frac{1}{1+\mathrm{N}(\mathfrak{q}_0)-\alpha_{\mathfrak{q}_0}(f)}\,\mathrm{cl}((1+\mathrm{N}(\mathfrak{q}_0)-T_{\mathfrak{q}_0})P).\end{aligned}$$

Put

$$D_m=\sum_{\sigma\in\mathrm{Gal}(\widetilde{K}_m/K_m)}\xi_{\mathfrak{q}_0}(P_m(1)^\sigma)=\sum_{[a]_m\in\mathrm{Gal}(\widetilde{K}_m/K_m)}\xi_{\mathfrak{q}_0}(P_m(a))\in J_N^{[\mathfrak{l}]}(K_m)_\mathcal{O}.$$

We define the cohomology class $\kappa_{\mathscr{D}}(\mathfrak{l})_m$ by

$$\kappa_{\mathscr{D}}(\mathfrak{l})_m := \frac{1}{\alpha_{\mathfrak{p}}^m}\,\mathrm{Kum}(D_m)\,(\mathrm{mod}\ \mathcal{I}_g^{[\mathfrak{l}]}) \in H^1(K_m, T_p(J_N^{[\mathfrak{l}]})_{\mathcal{O}}/\mathcal{I}_g^{[\mathfrak{l}]}) = H^1(K_m, T_n).$$

When m varies, these $\kappa_{\mathscr{D}}(\mathfrak{l})_m$ are compatible for the corestriction maps [**18**, Lemma 5.4.1], and thus give rise to an element $\kappa_{\mathscr{D}}(\mathfrak{l})$ of $\widehat{H}^1(K_\infty, T_n)$.

Proposition 4.5 ([**13**, Lemma 7.16], [**18**, Proposition 5.4.2])**.** *$\kappa_{\mathscr{D}}(\mathfrak{l})$ belongs to $\widehat{\mathrm{Sel}}_{\Delta\mathfrak{l}}(K_\infty, T_n)$.*

By Proposition 3.4(d), $\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}}, T_n)$ is free of rank 1 over $\mathcal{O}[[\Gamma]]/(\omega^n)$. Choosing a base of $\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}}, T_n)$ we may identify $\widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}}, T_n)$ with $\mathcal{O}[[\Gamma]]/(\omega^n)$.

Proposition 4.6 (First Reciprocity Law [**18**, Theorem 6.1.2])**.** *Let $m \geq N \geq n$. For each (N,n)-admissible form $\mathscr{D} = (\Delta, g)$ and each n-admissible prime $\mathfrak{l} \nmid \mathfrak{q}_0\Delta$, we have*

$$\partial_{\mathfrak{l}}(\kappa_{\mathscr{D}}(\mathfrak{l})_m) = \theta_m(g) \in \mathcal{O}_n[\Gamma_m]$$

up to multiplication by a unit of $\mathcal{O}_n[\Gamma_m]$.

Proof: By Proposition 4.4 one has

$$\partial_{\mathfrak{l}}(\kappa_{\mathscr{D}}(\mathfrak{l})) = \sum_{\sigma\in\Gamma_m} \psi_g(r_{\mathfrak{l}}(D_m^\sigma))\sigma.$$

But

$$\psi_g(r_{\mathfrak{l}}(D_m^\sigma)) = \sum_{[b]_m\in\mathrm{Gal}(\tilde{K}_m/K_m)} \langle g, x_m(ab)\tau_N\rangle = \sum_{[b]_m\in\mathrm{Gal}(\tilde{K}_m/K_m)} g(x_m(ab)),$$

where $a \in \widehat{K}^\times$ satisfies $\pi_m([a]_m) = \sigma$. Thus

$$\partial_{\mathfrak{l}}(\kappa_{\mathscr{D}}(\mathfrak{l})_m) = \sum_{[a]_m\in G_m} g(x_m(a))\pi_m([a]_m),$$

as desired. □

We fix two different n-admissible prime ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ ($\mathfrak{l}_1, \mathfrak{l}_2 \nmid \mathfrak{q}_0\Delta$). Then $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are inert in K. We fix a place $\mathfrak{l}_2'$ of K_m above $\mathfrak{l}_2$, and a place $\tilde{\mathfrak{l}}_2'$ of $\widetilde{K}_m$ above $\mathfrak{l}_2'$.

We have already seen that the image of the map

$$J_N^{[\mathfrak{l}_1]}(K_{\mathfrak{l}_2})_{\mathcal{O}}/\mathcal{I}_g^{[\mathfrak{l}_1]} \longrightarrow H^1(K_{\mathfrak{l}_2}, T_n)$$

is contained in $H^1_{\mathrm{fin}}(K_{\mathfrak{l}_2}, T_n) \cong \mathcal{O}_n$, and that the reduction map

$$J_N^{[\mathfrak{l}_1]}(K_{\mathfrak{l}_2})_{\mathcal{O}}/\mathcal{I}_g^{[\mathfrak{l}_1]} \longrightarrow J^{[\mathfrak{l}_1]}(k_{\mathfrak{l}_2})/\mathcal{I}_g^{[\mathfrak{l}_1]}$$

is an isomorphism, where $k_{\mathfrak{l}_2}$ is the residue field of $K_{\mathfrak{l}_2}$.

Let B'' be the definite quaternion algebra with discriminant $\Delta\mathfrak{l}_1\mathfrak{l}_2$. Then there is an isomorphism

$$\psi : \widehat{B}''^{(\mathfrak{l}_2)} \cong \widehat{B}'^{(\mathfrak{l}_2)}.$$

Let $\mathcal{O}_{B''_{\mathfrak{l}_1}}$ and $\mathcal{O}_{B''_{\mathfrak{l}_2}}$ be the maximal orders of $B''_{\mathfrak{l}_1}$ and $B''_{\mathfrak{l}_2}$ respectively. Put

$$\mathfrak{U}'' = \psi((\mathfrak{U}'_{\mathfrak{n}^+,\mathfrak{p}^N})^{(\mathfrak{l}_2)})\mathcal{O}^\times_{B''_{\mathfrak{l}_1}}\mathcal{O}^\times_{B''_{\mathfrak{l}_2}}.$$

By [**20**, Section 5.4] we have an isomorphism

$$\iota : B''^\times\backslash\widehat{B}''^\times/Y\mathfrak{U}'' \cong \mathcal{S}_{\mathfrak{l}_2},$$

where $\mathcal{S}_{\mathfrak{l}_2}$ is the set of supersingular points in $J_N^{[\mathfrak{l}_1]}(k_{\mathfrak{l}_2})$. Let $\mathbb{T}_\Delta(\mathfrak{l}_1\mathfrak{n}^+, \mathfrak{p}^N)$ act on $\mathrm{Div}(\mathcal{S}_{\mathfrak{l}_2})$ via Picard functoriality.

The reduction $\mathrm{red}_{\tilde{\mathfrak{l}}'_2}(P_m(a))$ of the CM point $P_m(a)$ modulo $\tilde{\mathfrak{l}}'_2$ is in $\mathcal{S}_{\mathfrak{l}_2}$. We choose ι such that

$$\mathrm{red}_{\tilde{\mathfrak{l}}'_2}([z', b']) = \iota(\psi^{-1}(b'^{(\mathfrak{l}_2)})).$$

In particular we have

$$\mathrm{red}_{\tilde{\mathfrak{l}}'_2}(P_m(a)) = \iota(x_m(a)\tau_N).$$

So, restricting the isomorphism

$$J_N^{[\mathfrak{l}_1]}(k_{\mathfrak{l}_2})_{\mathcal{O}}/\mathcal{I}_g^{[\mathfrak{l}_1]} \longrightarrow \mathcal{O}_n$$

to $\mathcal{S}_{\mathfrak{l}_2}$ we obtain a map

$$\gamma\colon \mathrm{Div}(\mathcal{S}_{\mathfrak{l}_2}) \longrightarrow \mathcal{O}_n.$$

Write $\overline{T}$ for the image of $T \in \mathbb{T}_\Delta(\mathfrak{l}_1\mathfrak{n}^+, \mathfrak{p}^n)_{\mathcal{O}}$ in $\mathbb{T}_\Delta(\mathfrak{l}_1\mathfrak{n}^+, \mathfrak{p}^n)_{\mathcal{O}}/\mathcal{I}_g^{[\mathfrak{l}_1]}$.

Proposition 4.7 ([**13**, Lemma 7.17])**.** *For $x \in \mathrm{Div}(\mathcal{S}_{\mathfrak{l}_2})$ the following relations hold:*

(a) *For $\mathfrak{q} \nmid \Delta\mathfrak{n}^+\mathfrak{l}_1$, one has $\gamma(T_{\mathfrak{q}}x) = \overline{T}_{\mathfrak{q}}\gamma(x)$.*
(b) *For $\mathfrak{q}|\Delta\mathfrak{n}^+\mathfrak{l}_1$, one has $\gamma(U_{\mathfrak{q}}x) = \overline{U}_{\mathfrak{q}}\gamma(x)$.*
(c) $\gamma(T_{\mathfrak{l}_2}x) = \overline{T}_{\mathfrak{l}_2}\gamma(x)$.
(d) $\gamma(\mathrm{Frob}_{\mathfrak{l}_2}(x)) = \epsilon_{\mathfrak{l}_2}\gamma(x)$*, where* $\mathrm{Frob}_{\mathfrak{l}_2}$ *is the Frobenius of F at $\mathfrak{l}_2$.*

The relation between γ and the system $\{\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m : m \geq N\}$ is given by the following.

Proposition 4.8. *If (Δ, g) is an (N, n)-admissible form, and if $m \geq N$, then*

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = \frac{1}{\alpha_{\mathfrak{p}}^m} \sum_{[a]_m \in G_m} \gamma \circ \iota(x_m(a)\tau_N)\pi_m([a]_m)$$

in $\mathcal{O}_n[\Gamma_m]$.

Proposition 4.8 is more or less contained in [**13, 18**], but it is not stated in the above form.

Proof: All primes of K_m above $\mathfrak{l}_2$ are $\{\sigma\mathfrak{l}'_2 : \sigma \in \Gamma_m\}$. So

$$\begin{aligned} v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) &= \sum_{\sigma \in \Gamma_m} v_{\sigma\mathfrak{l}'_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) \\ &= \sum_{\sigma \in \Gamma_m} v_{\mathfrak{l}'_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m^{\sigma^{-1}})\sigma \\ &= \frac{1}{\alpha_{\mathfrak{p}}^m} \sum_{[a]_m \in G_m} v_{\tilde{\mathfrak{l}}'_2}(P_m(a))\pi_m([a]_m). \end{aligned}$$

Note that the reduction of $P_m(a)$ modulo $\tilde{\mathfrak{l}}'_2$ lies in $\mathcal{S}_{\mathfrak{l}_2}$. Thus

$$v_{\tilde{\mathfrak{l}}'_2}(P_m(a)) = \gamma(\mathrm{red}_{\tilde{\mathfrak{l}}'_2}(P_m(a))) = \gamma \circ \iota(x_m(a)\tau_N),$$

as wanted. □

Corollary 4.9. *If there exists m such that $v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) \neq 0$, then $\gamma \neq 0$.*

Proposition 4.10 ([**13**, Lemma 7.20])**.** *If Ihara's lemma holds, then γ is surjective.*

By Proposition 3.4(d), $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_1}, T_n)$ and $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_2}, T_n)$ are free of rank 1 over $\mathcal{O}[[\Gamma]]/(\omega^r)$. We may identify both $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_1}, T_n)$ and $\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_2}, T_n)$ with $\mathcal{O}[[\Gamma]]/(\omega^n)$.

Proposition 4.11 (Second Reciprocity Law [**18**, Theorem 6.6])**.** *If Ihara's lemma holds, then*

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m)$$

up to multiplication by a unit of $\mathcal{O}_n[\Gamma_m]$.

4.3. A weaker version of the Second Reciprocity Law. One expects to show that

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m)$$

without using Ihara's lemma. But we can only prove a weaker result. We will deduce from the First Reciprocity Law and Tate duality that they coincide with each other after multiplying by $\theta_m(g)$.

Let τ be a complex conjugation which depends on a choice of embedding of the algebraic closure of $\overline{F}$ in $\mathbb{C}$. For each $\sigma \in \Gamma_m$ we have $\tau\sigma = \sigma^{-1}\tau$. The homomorphism $\sigma \mapsto \sigma^{-1}$ of Γ_m induces an involution ι on $\mathcal{O}_n[\Gamma_m]$. Then τ acts on $\mathcal{O}_n[\Gamma_m]$ as ι.

For each $i \in \{1, 2\}$, as $\mathfrak{l}_i$ splits completely in K_m [**18**, Lemma 2.4.2], the number of places of K_m above $\mathfrak{l}_i$ are $[K_m : K]$. Fix a place $\mathfrak{l}'_i$ of K_m above $\mathfrak{l}_i$. Then all places of K_m above $\mathfrak{l}_i$ are $\{\sigma\mathfrak{l}'_i : \sigma \in \mathrm{Gal}(K_m/K)\}$. Note that τ permutes $\{\sigma\mathfrak{l}'_i : \sigma \in \mathrm{Gal}(K_m/K)\}$.

Note that

$$H^1_{\mathrm{fin}}(K_{m,\mathfrak{l}_i}, T_n) \cong H^1_{\mathrm{fin}}(K_{m,\tau\mathfrak{l}'_i}, T_n) \otimes_{\mathcal{O}} \mathcal{O}[\Gamma_m]$$

and

$$H^1_{\mathrm{sing}}(K_{m,\mathfrak{l}_i}, T_n) \cong H^1_{\mathrm{sing}}(K_{m,\mathfrak{l}'_i}, T_n) \otimes_{\mathcal{O}} \mathcal{O}[\Gamma_m].$$

Both $H^1_{\mathrm{fin}}(K_{m,\tau\mathfrak{l}'_i}, T_n)$ and $H^1_{\mathrm{sing}}(K_{m,\mathfrak{l}'_i}, T_n)$ are isomorphic to $\mathcal{O}_n$. We choose generators $c_{\tau\mathfrak{l}'_i}$ and $d_{\mathfrak{l}'_i}$ of $H^1_{\mathrm{fin}}(K_{m,\tau\mathfrak{l}'_i}, T_n)$ and $H^1_{\mathrm{sing}}(K_{m,\mathfrak{l}'_i}, T_n)$ such that $\langle c_{\tau\mathfrak{l}'_1}, \tau d_{\mathfrak{l}'_1}\rangle_{\tau\mathfrak{l}'_1} = 1$ and $\langle \tau c_{\tau\mathfrak{l}'_2}, d_{\mathfrak{l}'_2}\rangle_{\mathfrak{l}'_2} = 1$.

Lemma 4.12. *For each* $\sigma \in \Gamma_m$ *we have*

$$\langle \sigma c_{\tau\mathfrak{l}'_1}, \sigma\tau d_{\mathfrak{l}'_1}\rangle_{\sigma\tau\mathfrak{l}'_1} = \langle \sigma\tau c_{\tau\mathfrak{l}'_2}, \sigma d_{\mathfrak{l}'_2}\rangle_{\sigma\mathfrak{l}'_2} = 1.$$

Proof: Let $\mathrm{Res}\colon H^1(K_{\mathfrak{l}_1}, T_n) \to H^1(K_{m,\mathfrak{l}_1}, T_n)$ and $\mathrm{Cores}\colon H^1(K_{m,\mathfrak{l}_1}, T_n) \to H^1(K_{\mathfrak{l}_1}, T_n)$ be the restriction map and the corestriction map respectively.

As $\sum_{\gamma\in\Gamma_m} \gamma\tau d_{\mathfrak{l}'_1}$ is fixed by Γ_m, we have $\sum_{\gamma\in\Gamma_m} \gamma\tau d_{\mathfrak{l}'_1} = \mathrm{Res}(x)$ for some $x \in H^1(K_{\mathfrak{l}_1}, T_n)$. Then

$$\langle \sigma c_{\tau\mathfrak{l}'_1}, \sigma\tau d_{\mathfrak{l}'_1}\rangle_{\sigma\tau\mathfrak{l}'_1} = \left\langle \sigma c_{\tau\mathfrak{l}'_1}, \sum_{\gamma\in\Gamma_m} \gamma\sigma\tau d_{\mathfrak{l}'_1} \right\rangle_{\mathfrak{l}_1} = \langle \sigma c_{\tau\mathfrak{l}'_1}, \mathrm{Res}(x)\rangle_{\mathfrak{l}_1} = \langle \mathrm{Cores}(\sigma c_{\tau\mathfrak{l}'_1}), x\rangle_{\mathfrak{l}_1}.$$

As $\mathrm{Cores}(\sigma c_{\tau\mathfrak{l}'_1}) = \mathrm{Cores}(c_{\tau\mathfrak{l}'_1})$, we obtain

$$\langle \sigma c_{\tau\mathfrak{l}'_1}, \sigma\tau d_{\mathfrak{l}'_1}\rangle_{\sigma\tau\mathfrak{l}'_1} = \langle c_{\tau\mathfrak{l}'_1}, \tau d_{\mathfrak{l}'_1}\rangle_{\tau\mathfrak{l}'_1} = 1.$$

The proof of

$$\langle \sigma\tau c_{\tau\mathfrak{l}'_2}, \sigma d_{\mathfrak{l}'_2}\rangle_{\sigma\mathfrak{l}'_2} = 1$$

is similar. □

Proposition 4.6 says that there exist two units u_1 and u_2 in $\mathcal{O}_n[\Gamma_m]$ such that

$$\partial_{\mathfrak{l}_i}(\kappa_{\mathscr{D}}(\mathfrak{l}_i)_m) = u_i\theta_m(g) \cdot d_{\mathfrak{l}'_i}.$$

Let θ_1 and θ_2 be the elements in $\mathcal{O}_n[\Gamma_m]$ such that

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = \theta_1 c_{\tau\mathfrak{l}'_2} \quad \text{and} \quad v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m) = \theta_2 c_{\tau\mathfrak{l}'_1}.$$

Theorem 4.13. *We have*

$$\theta_m(g)(u_2\theta_1 + u_1\theta_2) = 0 \tag{4.1}$$

in $\mathcal{O}_{f,n}[\Gamma_m]$.

Proof: Note that T_n is self-dual, so we can form the local Tate pairing $\langle\cdot,\cdot\rangle_v$ on $H^1(K_{m,v}, T_n)$ for each place v of K_m.

For any $c_1, c_2 \in H^1(K_m, T_n)$ and each place v of K_m we write $\langle c_1, c_2\rangle_v = \langle \mathrm{res}_v(c_1), \mathrm{res}_v(c_2)\rangle_v$. Then $\sum_v \langle c_1, c_2\rangle_v = 0$. We apply this to $c_1 = \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m$ and $c_2 = \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m$ with $\gamma \in \Gamma_m$.

By Lemma 3.2, $c_1 \in \mathrm{Sel}_{\Delta\mathfrak{l}_1}(K_m, T_n)$, and $c_2 \in \mathrm{Sel}_{\Delta\mathfrak{l}_2}(K_m, T_n)$. So, when v is not above $\mathfrak{l}_1$ or $\mathfrak{l}_2$ we have

$$\langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_v = 0.$$

Hence,

$$\sum_{\sigma\in\Gamma_m} (\langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_{\sigma\mathfrak{l}_1'} + \langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_{\sigma\mathfrak{l}_2'}) = 0.$$

We write

$$u_i\theta_m(g) = \sum_{\sigma\in\Gamma_m} a_{i,\sigma}\sigma, \quad a_{i,\sigma} \in \mathcal{O}_n,$$

and

$$\theta_i = \sum_{\sigma\in\Gamma_m} b_{i,\sigma}\sigma, \quad b_{i,\sigma} \in \mathcal{O}_n.$$

Then

$$\partial_{\mathfrak{l}_1}(\tau\kappa_{\mathscr{D}}(\mathfrak{l}_1)) = \iota(u_1\theta_m(g))\tau d_{\mathfrak{l}_1'} = \sum_{\sigma\in\Gamma_m} a_{1,\sigma^{-1}}\sigma\tau d_{\mathfrak{l}_1'}$$

and

$$v_{\mathfrak{l}_1}(\gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)) = \gamma\theta_2 c_{\tau\mathfrak{l}_1'} = \sum_{\sigma\in\Gamma_m} b_{2,\sigma\gamma^{-1}}\sigma c_{\tau\mathfrak{l}_1'}.$$

By Lemma 4.12 we have

$$\begin{aligned}\langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_{\sigma\tau\mathfrak{l}_1'} &= \langle \partial_{\mathfrak{l}_1}(\tau\kappa_{\mathscr{D}}(\mathfrak{l}_1)), v_{\mathfrak{l}_1}(\gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2))\rangle_{\sigma\tau\mathfrak{l}_1'} \\ &= \langle a_{1,\sigma^{-1}}\sigma\tau d_{\mathfrak{l}_1'}, b_{2,\sigma\gamma^{-1}}\sigma c_{\tau\mathfrak{l}_1'}\rangle_{\sigma\tau\mathfrak{l}_1'} = a_{1,\sigma^{-1}}b_{2,\sigma\gamma^{-1}}.\end{aligned}$$

Hence,

$$\sum_{\sigma\in\Gamma_m} \langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_{\sigma\mathfrak{l}_1'} = \sum_{\sigma\in\Gamma_m} \langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_{\sigma\tau\mathfrak{l}_1'} = \sum_{\sigma\in\Gamma_m} a_{1,\sigma^{-1}}b_{2,\sigma\gamma^{-1}}.$$

Similarly,

$$\sum_{\sigma\in\Gamma_m} \langle \tau\kappa_{\mathscr{D}}(\mathfrak{l}_1), \gamma\kappa_{\mathscr{D}}(\mathfrak{l}_2)\rangle_{\sigma\mathfrak{l}_2'} = \sum_{\sigma\in\Gamma_m} b_{1,\sigma^{-1}}a_{2,\sigma\gamma^{-1}}.$$

Therefore,

$$\sum_{\sigma\in\Gamma_m} (a_{1,\sigma^{-1}}b_{2,\sigma\gamma^{-1}} + b_{1,\sigma^{-1}}a_{2,\sigma\gamma^{-1}}) = 0.$$

This sum is just the coefficient of γ^{-1} on the left-hand side of (4.1). This proves (4.1). □

Each element a of $\mathcal{O}_n$ can be written as $a = u\omega^s$ with u a unit in $\mathcal{O}_n$, and $s \in \{0, 1, \ldots, n\}$; we put $\mathrm{ord}(a) = s$.

Let $\varphi\colon \mathcal{O}_n[\Gamma_m] \to \mathcal{O}_n$ be a homomorphism. For each $\theta \in \mathcal{O}_n[\Gamma_m]$ we put $\mathrm{ord}_\varphi(\theta) := \mathrm{ord}(\varphi(\theta))$.

For each element x of $H^1(K_\infty, T_n)$ we write $\varphi(x)$ for its image in

$$H^1(K_\infty, T_n) \otimes_\varphi \mathcal{O}_n \cong \mathcal{O}_n,$$

and put

$$\operatorname{ord}_\varphi(x) = \operatorname{ord}(\varphi(x)).$$

Then we have

$$\operatorname{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) = \operatorname{ord}_\varphi(\theta_1)$$

and

$$\operatorname{ord}_\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m)) = \operatorname{ord}_\varphi(\theta_2).$$

Corollary 4.14. *If $\varphi\colon \mathcal{O}_n[\Gamma_m] \to \mathcal{O}_n$ is a homomorphism such that*

$$\operatorname{ord}_\varphi(\partial_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) + \operatorname{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) < n,$$

then

$$\operatorname{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) = \operatorname{ord}_\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m)).$$

Proof: By Theorem 4.13 we have

$$\varphi(\theta_m(g))(\varphi(u_2)\varphi(\theta_1) + \varphi(u_1)\varphi(\theta_2)) = 0.$$

Note that $\varphi(u_1)$ and $\varphi(u_2)$ are units of $\mathcal{O}_n$.

We write

$$\varphi(\theta_m(g)) = v\omega^r, \quad \varphi(\theta_1) = v_1\omega^{s_1}, \quad \text{and} \quad \varphi(\theta_2) = v_2\omega^{s_2},$$

where v, v_1, and v_2 are units of $\mathcal{O}_n$, and $r, s_1, s_2 \in \{0, 1, \dots, n\}$. By our assumption, $r + s_1 < n$. What we need to show is $s_1 = s_2$.

If $s_1 > s_2$, then

$$\varphi(u_2)vv_1\omega^{s_1-s_2} + \varphi(u_1)vv_2 = \varphi(u_1)vv_2(1 + (\varphi(u_1)vv_2)^{-1} \cdot \varphi(u_2)vv_1 \cdot \omega^{s_1-s_2})$$

is a unit. Indeed,

$$(\varphi(u_2)vv_1\omega^{s_1-s_2} + \varphi(u_1)vv_2) \cdot \left((\varphi(u_1)vv_2)^{-1} \cdot \sum_{i=0}^{n-1} ((\varphi(u_1)vv_2)^{-1} \cdot \varphi(u_2)vv_1 \cdot \omega^{s_1-s_2})^i \right) = 1.$$

It follows that

$$\varphi(\theta_m(g))(\varphi(u_2)\varphi(\theta_1) + \varphi(u_1)\varphi(\theta_2)) = \omega^{r+s_2}(\varphi(u_2)vv_1\omega^{s_1-s_2} + \varphi(u_1)vv_2) \neq 0$$

since $r + s_2 < r + s_1 < n$, and $\varphi(u_2)vv_1\omega^{s_1-s_2} + \varphi(u_1)vv_2$ is a unit.

If $s_1 < s_2$, we again have

$$\varphi(\theta_m(g))(\varphi(u_2)\varphi(\theta_1) + \varphi(u_1)\varphi(\theta_2)) = \omega^{r+s_1}(\varphi(u_2)vv_1 + \varphi(u_1)vv_2\omega^{s_2-s_1}) \neq 0,$$

since $r + s_1 < n$, and $\varphi(u_2)vv_1 + \varphi(u_1)vv_2\omega^{s_2-s_1}$ is a unit.

Thus we must have $s_1 = s_2$. □

4.4. Admissible form.

Proposition 4.15. *Let (Δ, g) be an (N, n)-admissible form. If $\mathfrak{l}_1$ and $\mathfrak{l}_2$ ($\mathfrak{l}_1, \mathfrak{l}_2 \nmid \mathfrak{q}_0\Delta$) are two different n-admissible prime ideals, and if $m \geq N$ is an integer such that $v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) \neq 0$, then there exists a nonnegative integer $n_0 < n$ and an $(N, n - n_0)$-admissible form $(\Delta\mathfrak{l}_1\mathfrak{l}_2, g'')$ satisfying the following.*

(a) *For any homomorphism $\varphi\colon \mathcal{O}_n[\Gamma_m] \to \mathcal{O}_n$ we have*

$$n_0 \leq \mathrm{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)).$$

(b) *We have*

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = \omega^{n_0}\theta_m(g'') \in \mathcal{O}_n[\Gamma_m]$$

up to multiplication by a unit of $\mathcal{O}_n[\Gamma_m]$.

Here, $\theta_m(g'')$ is in $\mathcal{O}_{n-n_0}[\Gamma_m]$. The homomorphism

$$\begin{aligned} \mathcal{O}_n[\Gamma_m] &\xrightarrow{\times\omega^{n_0}} \mathcal{O}_n[\Gamma_m] \\ \sum_{\sigma\in\Gamma_m} a_\sigma\sigma &\longmapsto \sum_{\sigma\in\Gamma_m} \omega^{n_0}a_\sigma\sigma \end{aligned}$$

annihilates $\omega^{n-n_0}\mathcal{O}_n[\Gamma_m]$, and thus induces a homomorphism

$$\mathcal{O}_{n-n_0}[\Gamma_m] \xrightarrow{\times\omega^{n_0}} \mathcal{O}_n[\Gamma_m].$$

Proof: Let n_0 be the largest integer such that $\mathrm{Im}(\gamma) \in \omega^{n_0}\mathcal{O}_n$. By Proposition 4.8 we have

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) \in \omega^{n_0}\mathcal{O}_n[\Gamma_m].$$

Thus for any homomorphism $\varphi\colon \mathcal{O}_n[\Gamma] \to \mathcal{O}_n$ we have

$$\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) \in \omega^{n_0}\mathcal{O}_n$$

yielding

$$\mathrm{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) \geq n_0.$$

Let $\widetilde{\gamma}$ be a map

$$\widetilde{\gamma}\colon \mathrm{Div}(\mathcal{S}_{\mathfrak{l}_2}) \longrightarrow \mathcal{O}_n$$

such that $\gamma = \omega^{n_0}\widetilde{\gamma}$. Let γ' be the composition

$$\mathrm{Div}(\mathcal{S}_{\mathfrak{l}_2}) \xrightarrow{\widetilde{\gamma}} \mathcal{O}_n \longrightarrow \mathcal{O}_{n-n_0},$$

where $\mathcal{O}_n \to \mathcal{O}_{n-n_0}$ is the natural quotient map.

If $\mathfrak{q} \nmid \Delta\mathfrak{l}_1\mathfrak{l}_2\mathfrak{n}^+$, from $\gamma(T_\mathfrak{q}x) - \overline{T}_\mathfrak{q}\gamma(x) = 0$, we get

$$\widetilde{\gamma}(T_\mathfrak{q}x) - \overline{T}_\mathfrak{q}\widetilde{\gamma}(x) \in \omega^{n-n_0}\mathcal{O}_n.$$

It follows that

$$\gamma'(T_\mathfrak{q}x) - \overline{T}_\mathfrak{q}\gamma'(x) = 0.$$

The same argument shows that, if $\mathfrak{q}|\Delta\mathfrak{n}^+\mathfrak{l}_1$, then

$$\gamma'(U_\mathfrak{q}x) = \overline{U}_\mathfrak{q}\gamma'(x).$$

In particular, $\gamma'(U_{\mathfrak{l}_1}x) = \epsilon_{\mathfrak{l}_1}\gamma'(x)$. Similarly, $\gamma'(\mathrm{Frob}_{\mathfrak{l}_2}(x)) = \epsilon_{\mathfrak{l}_2}\gamma'(x)$. By [**3**, Section 9] we have $U_{\mathfrak{l}_2} = \mathrm{Frob}_{\mathfrak{l}_2}$ on $\mathrm{Div}(\mathcal{S}_{\mathfrak{l}_2})$. Hence,

$$\gamma'(U_{\mathfrak{l}_2}x) = \gamma'(\mathrm{Frob}_{\mathfrak{l}_2} x) = \epsilon_{\mathfrak{l}_2}\gamma'(x).$$

Let

$$g'' \in S_2^{B''}(\mathfrak{U}'', \mathcal{O}_{n-n_0})^Y$$

be the function such that $\psi_{g''} = \gamma'$. Since γ' is Hecke equivariant, $(\Delta\mathfrak{l}_1\mathfrak{l}_2, g'')$ is an $(N, n-n_0)$-admissible form. By (2.1) we have

$$g''(x_m(a)) = \gamma'(x_m(a)\tau_N).$$

By Proposition 4.8 we have

$$\begin{aligned} v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) &= \frac{1}{\alpha_{\mathfrak{p}}^m} \sum_{[a]_m \in G_m} \gamma(x_m(a)\tau_N)\pi_m([a]_m) \\ &= \frac{1}{\alpha_{\mathfrak{p}}^m} \sum_{[a]_m \in G_m} \omega^{n_0}\gamma'(x_m(a)\tau_N)\pi_m([a]_m) \\ &= \frac{\omega^{n_0}}{\alpha_{\mathfrak{p}}^m} \sum_{[a]_m \in G_m} g''(x_m(a))\pi_m([a]_m) = \omega^{n_0}\theta_m(g''), \end{aligned}$$

as desired. □

Remark 4.16. Proposition 4.10 says that, if Ihara's lemma holds, then $n_0 = 0$.

We can strengthen the statement of Corollary 4.14. Though it will not be used in the next section, we give it below for its own interest.

Theorem 4.17. *Assume* (CR^+) *and* $(\mathfrak{n}^+\text{-DT})$ *hold. If there exists a homomorphism*

$$\varphi\colon \mathcal{O}_n[\Gamma_m] \longrightarrow \mathcal{O}_n$$

such that

$$\mathrm{ord}_\varphi(\partial_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) + \mathrm{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m)) < n, \tag{4.2}$$

then

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m)$$

up to multiplication by a unit of $\mathcal{O}_n[\Gamma_m]$.

Proof: By Corollary 4.14 it follows from (4.2) that

$$\mathrm{ord}_\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2))) = \mathrm{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))) < n.$$

Let n_0 and g'' be as in Proposition 4.15. Then

$$n_0 \leq \mathrm{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)))$$

and

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)) = \omega^{n_0}\theta(g'') \in \mathcal{O}_n[[\Gamma]]$$

up to multiplication by a unit of $\mathcal{O}_n[[\Gamma]]$. Exchanging $\mathfrak{l}_1$ and $\mathfrak{l}_2$, by Proposition 4.15 there exists a nonnegative integer

$$n_0' \leq \mathrm{ord}_\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)))$$

and an $(N, n-n_0')$-admissible form $(\Delta\mathfrak{l}_1\mathfrak{l}_2, h'')$ such that

$$v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)) = \omega^{n_0'}\theta(h'') \in \mathcal{O}_n[[\Gamma]]$$

up to multiplication by a unit of $\mathcal{O}_n[[\Gamma]]$.

Without loss of generality we may assume that $n_0 \le n_0'$. When (CR$^+$) and ($\mathfrak{n}^+$-DT) hold, the multiplicity one theorem holds [**18**, Theorem 9.1.1], from which we obtain

$$h'' \equiv g'' \pmod{\omega^{n-n_0'}}.$$

So

$$\omega^{n_0'}\theta_m(h'') = \omega^{n_0'}\theta_m(g'')$$

in $\mathcal{O}_n[\Gamma_m]$. It follows that

$$\begin{aligned}\operatorname{ord}_\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2))) &= n_0' + \operatorname{ord}_\varphi(\theta_m(h''))\\ &= (n_0' - n_0) + (n_0 + \operatorname{ord}_\varphi \theta_m(g''))\\ &= (n_0' - n_0) + \operatorname{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))).\end{aligned}$$

Since

$$\operatorname{ord}_\varphi(v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2))) = \operatorname{ord}_\varphi(v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1))) < n,$$

we obtain $n_0' = n_0$, yielding our conclusion. □

5. Proof of Theorem 1.2

Let $\varphi\colon \mathcal{O}[[\Gamma]] \to \mathcal{O}$ be a homomorphism. For each positive integer r, let φ_r be the composition

$$\mathcal{O}[[\Gamma]] \xrightarrow{\varphi} \mathcal{O} \longrightarrow \mathcal{O}_r = \mathcal{O}/(\omega^r).$$

We write ord for the valuation of $\mathcal{O}$ whose value on ω is 1.

Theorem 5.1. *Let $N \ge r$ be two positive integers, and $\mathscr{D} = (\Delta, g)$ be an (N, r)-admissible form. Assume that $\varphi_r(\theta(g)) \ne 0$. If $t_{\varphi,g} := \operatorname{ord}(\varphi_r(\theta(g)))$ satisfies $2t_{\varphi,g} \le r$, then for each positive integer $n \le r - t_{\varphi,g}$ we have*

$$\operatorname{length}_{\mathcal{O}}(\operatorname{Sel}_\Delta(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O}) \le 2t_{\varphi,g}. \tag{5.1}$$

We fix an integer $m \ge N$ such that φ_N factors through $\mathcal{O}_N[\Gamma_m]$. Then φ_r factors through $\mathcal{O}_r[\Gamma_m]$. So $\varphi_r(\theta(g)) = \varphi_r(\theta_m(g))$ and $t_{\varphi,g} = \operatorname{ord}(\varphi_r(\theta_m(g)))$.

We prove (5.1) by induction on $t_{\varphi,g}$.

First we assume (CR$^+$), (PO), and ($\mathfrak{n}^+$-min) hold. By Theorem 3.6 there exists a finite set S of r-admissible prime ideals such that $\widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}$ is free over $\mathcal{O}_r$. We fix such a set S. Let

$$s_1, \dots, s_d \quad (d = \operatorname{rank}_{\mathcal{O}_r} \widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O})$$

be a basis of $\widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}$ over $\mathcal{O}_r$. For every element $\sum_i a_i s_i$ in $\widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}$ we define

$$\operatorname{ord}\left(\sum_i a_i s_i\right) := \min\{\operatorname{ord}(a_i) : i = 1, \dots, d\} \in \{0, 1, \dots, r\}.$$

Note that this does not depend on the choice of the basis $\{s_i : i = 1, \dots, d\}$.

For each r-admissible prime ideal $\mathfrak{l} \notin S$, considering $\kappa_\varphi(\mathfrak{l}) = \varphi(\kappa_{\mathscr{D}}(\mathfrak{l}))$ as an element of $\widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}$, we put $e_{\mathfrak{l}} = \operatorname{ord} \kappa_\varphi(\mathfrak{l})$. By Proposition 4.6, we have $e_{\mathfrak{l}} \le t_{\varphi,g}$.

Then there exists

$$\tilde{\kappa}'(\mathfrak{l}) \in \widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}$$

such that $\omega^{e_{\mathfrak{l}}}\tilde{\kappa}'(\mathfrak{l}) = \kappa_\varphi(\mathfrak{l})$.

The quotient map $T_r \to T_n$ induces a homomorphism

$$\widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O} \longrightarrow \widehat{H}^1(K_\infty, T_n) \otimes_\varphi \mathcal{O}.$$

Lemma 5.2. *Let $\kappa'(\mathfrak{l})$ be the image of $\tilde{\kappa}'(\mathfrak{l})$ in $\widehat{H}^1(K_\infty, T_n) \otimes_\varphi \mathcal{O}$.*

(a) $\operatorname{ord} \kappa'(\mathfrak{l}) = 0$.

(b) $\operatorname{ord} \partial_{\mathfrak{l}} \kappa'(\mathfrak{l}) = t_{\varphi,g} - e_{\mathfrak{l}}$.

(c) $\partial_{\mathfrak{q}} \kappa'(\mathfrak{l}) = 0$ *for* $\mathfrak{q} \nmid \Delta\mathfrak{l}\mathfrak{p}$.

(d) $\operatorname{res}_{\mathfrak{q}} \kappa'(\mathfrak{l}) \in \widehat{H}^1_{\mathrm{ord}}(K_{\infty,\mathfrak{q}}, T_n) \otimes_\varphi \mathcal{O}$ *for* $\mathfrak{q} | \Delta\mathfrak{l}\mathfrak{p}$.

Proof: Assertions (a) and (b) follow from the definition of $\kappa'(\mathfrak{l})$ and the First Reciprocity Law. The latter two assertions for $\mathfrak{q} \notin S$ follow from the fact $\tilde{\kappa}'(\mathfrak{l}) \in \widehat{\mathrm{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}$.

We assume that $\mathfrak{q} \in S$ and $\mathfrak{q} \nmid \Delta\mathfrak{l}\mathfrak{p}$. As $\mathfrak{q}$ is r-admissible, by Proposition 3.4(d) we have that $\widehat{H}^1(K_{\infty,\mathfrak{q}}, T_r) \otimes_\varphi \mathcal{O}$ is free over $\mathcal{O}[\Gamma] \otimes_\varphi \mathcal{O}$. Thus there exists $s \in \widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{q}}, T_r)$ such that $\omega^{e_{\mathfrak{l}}} s = \operatorname{res}_{\mathfrak{q}} \kappa_\varphi(\mathfrak{l})$. This means $\omega^{e_{\mathfrak{l}}}(s - \operatorname{res}_{\mathfrak{q}} \tilde{\kappa}'(\mathfrak{l})) = 0$. As $e_{\mathfrak{l}} \le t_{\varphi,g} \le r - n$, from the freeness of $\widehat{H}^1(K_{\infty,\mathfrak{q}}, T_r) \otimes_\varphi \mathcal{O}$ we obtain $s - \operatorname{res}_{\mathfrak{q}} \tilde{\kappa}'(\mathfrak{l}) \in \omega^n \widehat{H}^1(K_{\infty,\mathfrak{q}}, T_r) \otimes_\varphi \mathcal{O}$. Hence the images of s and $\operatorname{res}_{\mathfrak{q}} \kappa'(\mathfrak{l})$ in $\widehat{H}^1(K_{\infty,\mathfrak{q}}, T_n) \otimes_\varphi \mathcal{O}$ coincide with each other, which shows (c) for $\mathfrak{q} \in S$.

By the same argument we can prove (d) for $\mathfrak{q} \in S$. □

Lemma 5.3 ([**18**, Lemma 7.3.4])**.** *Let*

$$\eta_{\mathfrak{l}} \colon \widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}}, T_n) \otimes_\varphi \mathcal{O} \longrightarrow \mathrm{Sel}_\Delta(K_\infty, T_n)^\vee \otimes_\varphi \mathcal{O}$$

be the map defined by

$$\eta_{\mathfrak{l}}(c)(x) = \langle c, \operatorname{res}_{\mathfrak{l}}(x) \rangle_{\mathfrak{l}}$$

for $x \in \mathrm{Sel}_\Delta(K_\infty, A_n)[\ker(\varphi)]$ *and* $c \in \widehat{H}^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}}, T_n)$*. Then* $\eta_{\mathfrak{l}}(\partial_{\mathfrak{l}}(\kappa'(\mathfrak{l}))) = 0$.

Proof: By the global class field theory we have $\sum_{\mathfrak{q}} \langle \operatorname{res}_{\mathfrak{q}} \kappa'(\mathfrak{l}), \operatorname{res}_{\mathfrak{q}} x \rangle_{\mathfrak{q}} = 0$. When $\mathfrak{q} \ne \mathfrak{l}$, both $\operatorname{res}_{\mathfrak{q}} \kappa'(\mathfrak{l})$ and $\operatorname{res}_{\mathfrak{q}} x$ lie in the finite part or the ordinary part. Thus by Proposition 3.4(c) and (e), $\langle \operatorname{res}_{\mathfrak{q}} \kappa'(\mathfrak{l}), \operatorname{res}_{\mathfrak{q}} x \rangle_{\mathfrak{q}} = 0$ for $\mathfrak{q} \ne \mathfrak{l}$. So $\langle \partial_{\mathfrak{l}} \kappa'(\mathfrak{l}), \operatorname{res}_{\mathfrak{l}} x \rangle_{\mathfrak{l}} = \langle \operatorname{res}_{\mathfrak{l}} \kappa'(\mathfrak{l}), \operatorname{res}_{\mathfrak{l}} x \rangle_{\mathfrak{l}} = 0$. □

Choose an r-admissible prime ideal $\mathfrak{l}_1 \notin S$ such that

$$e_{\mathfrak{l}_1} = \min_{\substack{\mathfrak{l} \notin S \cup \{\mathfrak{q}_0\}: \\ r\text{-admissible}}} e_{\mathfrak{l}},$$

where $\mathfrak{q}_0$ is the prime chosen in Subsection 4.2.

Lemma 5.4 ([**18**, Lemmas 7.3.5 and 7.3.6])**.**

(a) *If* $t_{\varphi,g} = 0$*, then* $\mathrm{Sel}_\Delta(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O}$ *is trivial.*

(b) *If* $t_{\varphi,g} > 0$*, then* $e_{\mathfrak{l}_1} < t_{\varphi,g}$.

Proof: Assume that $\mathrm{Sel}_\Delta(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O} \ne 0$. Then by Nakayama's lemma

$$(\mathrm{Sel}_\Delta(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O})/(\omega) = (\mathrm{Sel}_\Delta(K_\infty, A_n)[\mathfrak{m}])^\vee \otimes_\varphi \mathcal{O}$$

is nonzero. Here, $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}[[\Gamma]]$. Let x be a nonzero element in $\mathrm{Sel}_\Delta(K_\infty, A_n)[\mathfrak{m}]$. By Lemma 3.5(a) and (c) we have

$$\mathrm{Sel}_\Delta(K_\infty, A_n)[\mathfrak{m}] = \mathrm{Sel}_\Delta(K, A_1).$$

So, by Proposition 3.3 there exists an r-admissible prime $\mathfrak{l} \notin S$ such that $v_{\mathfrak{l}}(x) \ne 0$.

We show that $e_{\mathfrak{l}} < t_{\varphi,g}$ for this $\mathfrak{l}$. Indeed, if $e_{\mathfrak{l}} = t_{\varphi,g}$, then by Lemma 5.2(b), $\partial_{\mathfrak{l}} \kappa'(\mathfrak{l})$ is indivisible, hence a generator of $H^1_{\mathrm{sing}}(K_{\infty,\mathfrak{l}}, T_n) \otimes_\varphi \mathcal{O}$. By Proposition 3.4(b) and (d) the image of $\partial_{\mathfrak{l}} \kappa'(\mathfrak{l})$ in $H^1_{\mathrm{sing}}(K_{\mathfrak{l}}, T_1)$ is a generator. As $\langle \cdot, \cdot \rangle_{\mathfrak{l}}$ induces a perfect pairing between $\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{l}}, T_1)$ and $H^1_{\mathrm{fin}}(K_{\mathfrak{l}}, A_1)$, we have $\langle \operatorname{Res}_{\mathfrak{l}} \kappa'(\mathfrak{l}), \operatorname{Res}_{\mathfrak{l}} x \rangle_{\mathfrak{l}} \ne 0$. But this implies $\eta_{\mathfrak{l}}(\kappa'(\mathfrak{l})) \ne 0$, which contradicts Lemma 5.3. □

By Lemma 5.4(a), (5.1) holds when $t_{\varphi,g} = 0$. So we assume that $t_{\varphi,g} > 0$.

Lemma 5.5. *We have*

$$e_{\mathfrak{l}_1} = \min_{\mathfrak{l}'} \operatorname{ord}(v_{\mathfrak{l}'} \kappa_\varphi(\mathfrak{l}_1)),$$

where $\mathfrak{l}'$ runs over all r-admissible prime ideals that do not divide $\mathfrak{l}_1\mathfrak{q}_0\Delta$ and are not in S.

Proof: What we need to prove is

$$\min_{\mathfrak{l}'} \operatorname{ord}(v_{\mathfrak{l}'} \tilde{\kappa}'(\mathfrak{l}_1)) = 0.$$

Let κ_1 denote the image of $\tilde{\kappa}'(\mathfrak{l}_1)$ in

$$\widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r) \otimes_\varphi \mathcal{O}_1 = \widehat{\operatorname{Sel}}^S_\Delta(K_\infty, T_r)/\mathfrak{m} \otimes_\varphi \mathcal{O} \hookrightarrow H^1(K, T_1).$$

By Lemma 5.2, $\kappa_1 \neq 0$. If $\operatorname{ord}(v_{\mathfrak{l}'}\tilde{\kappa}'(\mathfrak{l}_1)) > 0$ for each r-admissible prime ideal $\mathfrak{l}' \notin S$ that does not divide $\mathfrak{l}_1\mathfrak{q}_0\Delta$, then $v_{\mathfrak{l}'}(\kappa_1) = 0$ for each $\mathfrak{l}'$, as above. This contradicts Proposition 3.3. ☐

Let $\mathfrak{l}_2$ ($\mathfrak{l}_2 \nmid \mathfrak{l}_1\mathfrak{q}_0\Delta$ and $\mathfrak{l}_2 \notin S$) be an r-admissible prime ideal such that $\operatorname{ord} v_{\mathfrak{l}_2}(\kappa_\varphi(\mathfrak{l}_1)) = e_{\mathfrak{l}_1}$. In particular, $v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) \neq 0$. By the choice of $\mathfrak{l}_2$ and the minimality of $e_{\mathfrak{l}_1}$ we have

$$\operatorname{ord} v_{\mathfrak{l}_2}(\kappa_\varphi(\mathfrak{l}_1)) = e_{\mathfrak{l}_1} \leq e_{\mathfrak{l}_2} \leq \operatorname{ord} v_{\mathfrak{l}_1}(\kappa_\varphi(\mathfrak{l}_2)). \tag{5.2}$$

As

$$\begin{aligned}\operatorname{ord}_\varphi v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) + \operatorname{ord}_\varphi \partial_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) &= \operatorname{ord}_\varphi v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)) + \operatorname{ord}_\varphi \partial_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)) \\ &= e_{\mathfrak{l}_1} + t_{\varphi,g} < 2t_{\varphi,g} \leq r,\end{aligned}$$

by Corollary 4.14 we have

$$\operatorname{ord} v_{\mathfrak{l}_2}(\kappa_\varphi(\mathfrak{l}_1)) = \operatorname{ord}_\varphi v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = \operatorname{ord}_\varphi v_{\mathfrak{l}_1}(\kappa_{\mathscr{D}}(\mathfrak{l}_2)_m) = \operatorname{ord} v_{\mathfrak{l}_1}(\kappa_\varphi(\mathfrak{l}_2)). \tag{5.3}$$

Combining (5.2) and (5.3) we obtain

$$\operatorname{ord} v_{\mathfrak{l}_2}(\kappa_\varphi(\mathfrak{l}_1)) = e_{\mathfrak{l}_1} = e_{\mathfrak{l}_2} = \operatorname{ord} v_{\mathfrak{l}_1}(\kappa_\varphi(\mathfrak{l}_2)).$$

It follows that

$$\operatorname{ord} v_{\mathfrak{l}_1}(\kappa'(\mathfrak{l}_2)) = \operatorname{ord} v_{\mathfrak{l}_2}(\kappa'(\mathfrak{l}_1)) = 0. \tag{5.4}$$

Since $v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) \neq 0$, by Proposition 4.15 there exists an integer $r_0 < r$ and an $(N, r - r_0)$-admissible form $(\Delta\mathfrak{l}_1\mathfrak{l}_2, g'')$ such that

$$r_0 \leq \operatorname{ord}_\varphi v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = e_{\mathfrak{l}_1}$$

and

$$v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = \omega^{r_0}\theta_m(g'') \in \mathcal{O}_r[\Gamma_m]$$

up to multiplication by a unit of $\mathcal{O}_r[\Gamma_m]$. It follows that

$$r_0 + t_{\varphi,g''} = \operatorname{ord}_\varphi v_{\mathfrak{l}_2}(\kappa_{\mathscr{D}}(\mathfrak{l}_1)_m) = e_{\mathfrak{l}_1}.$$

Let $S_{\mathfrak{l}_1,\mathfrak{l}_2}$ be the subgroup of $\operatorname{Sel}_\Delta(K_\infty, A_n)$ consisting of elements that are locally trivial at the prime ideals dividing $\mathfrak{l}_1$ or $\mathfrak{l}_2$. By the definition of Selmer groups, we have the following two exact sequences:

$$\widehat{H}^1_{\text{sing}}(K_{\infty,\mathfrak{l}_1}, T_n) \oplus \widehat{H}^1_{\text{sing}}(K_{\infty,\mathfrak{l}_2}, T_n) \xrightarrow{\eta_s} \operatorname{Sel}_\Delta(K_\infty, A_n)^\vee \longrightarrow S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \longrightarrow 0$$

and

$$\widehat{H}^1_{\text{fin}}(K_{\infty,\mathfrak{l}_1}, T_n) \oplus \widehat{H}^1_{\text{fin}}(K_{\infty,\mathfrak{l}_2}, T_n) \xrightarrow{\eta_f} \operatorname{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty, A_n)^\vee \longrightarrow S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \longrightarrow 0,$$

where η_s and η_f are induced by the local Tate pairing $\langle\cdot,\cdot\rangle_{\mathfrak{l}_1} \oplus \langle\cdot,\cdot\rangle_{\mathfrak{l}_2}$.

Lemma 5.6. *We have $\eta_f^\varphi = 0$.*

Proof: From (5.4) we see

$$\big(\widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_1}, T_n) \oplus \widehat{H}^1_{\mathrm{fin}}(K_{\infty,\mathfrak{l}_2}, T_n)\big) \otimes_\varphi \mathcal{O}$$

is generated by $(v_{\mathfrak{l}_1}(\kappa'(\mathfrak{l}_2)), 0)$ and $(0, v_{\mathfrak{l}_2}(\kappa'(\mathfrak{l}_1)))$.

Let s be in $\mathrm{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty, A_n)$. By Lemma 5.2(c), for each $\mathfrak{q} \nmid \Delta\mathfrak{l}_1\mathfrak{l}_2\mathfrak{p}$,

$$\langle \kappa'(\mathfrak{l}_1), s\rangle_{\mathfrak{q}} = \langle \partial_{\mathfrak{q}}\kappa'(\mathfrak{l}_1), s\rangle_{\mathfrak{q}} = 0.$$

By Lemma 5.2(d), for each $\mathfrak{q}|\Delta\mathfrak{l}_1\mathfrak{p}$,

$$\langle \kappa'(\mathfrak{l}_1), s\rangle_{\mathfrak{q}} = 0.$$

Thus by the global Tate pairing we have $\langle v_{\mathfrak{l}_2}(\kappa'(\mathfrak{l}_1)), s\rangle_{\mathfrak{l}_2} = \langle \kappa'(\mathfrak{l}_1), s\rangle_{\mathfrak{l}_2} = 0$. The same argument shows that $\langle v_{\mathfrak{l}_1}(\kappa'(\mathfrak{l}_2)), s\rangle_{\mathfrak{l}_1} = 0$. ☐

By Lemma 5.6 we obtain

$$S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \otimes_\varphi \mathcal{O} \cong \mathrm{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O}.$$

As

$$r_0 + 2t_{\varphi,g''} \leq 2r_0 + 2t_{\varphi,g''} = 2e_{\mathfrak{l}_1} < 2t_{\varphi,g} \leq r,$$

we have

$$2t_{\varphi,g''} \leq r - r_0.$$

We also have

$$n \leq r - t_{\varphi,g} < r - e_{\mathfrak{l}_1} = (r - r_0) - t_{\varphi,g''}.$$

Hence, $(\Delta\mathfrak{l}_1\mathfrak{l}_2, g'')$ is an $(N, r - r_0)$-admissible form, $2t_{\varphi,g''} \leq r - r_0$, and $n \leq (r - r_0) - t_{\varphi,g''}$. As $t_{\varphi,g''} < t_{\varphi,g}$, by the inductive assumption we have

$$\mathrm{length}_{\mathcal{O}}\, S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \otimes_\varphi \mathcal{O} = \mathrm{length}_{\mathcal{O}}\, \mathrm{Sel}_{\Delta\mathfrak{l}_1\mathfrak{l}_2}(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O} \leq 2t_{\varphi,g''}.$$

By Lemma 5.3, η_s^φ factors through the quotient

$$\mathcal{O}/((\partial_{\mathfrak{l}_1}\kappa'(\mathfrak{l}_1)) \oplus \mathcal{O}/(\partial_{\mathfrak{l}_2}\kappa'(\mathfrak{l}_2)).$$

Thus

$$\begin{aligned}\mathrm{length}_{\mathcal{O}}\, \mathrm{Sel}_\Delta(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O} &\leq \mathrm{ord}\,\partial_{\mathfrak{l}_1}(\kappa'(\mathfrak{l}_1)) + \mathrm{ord}\,\partial_{\mathfrak{l}_2}(\kappa'(\mathfrak{l}_2)) + \mathrm{length}_{\mathcal{O}}\, S^\vee_{\mathfrak{l}_1,\mathfrak{l}_2} \otimes_\varphi \mathcal{O}\\ &\leq (t_{\varphi,g} - e_{\mathfrak{l}_1}) + (t_{\varphi,g} - e_{\mathfrak{l}_2}) + 2t_{\varphi,g''}\\ &= 2t_{\varphi,g} - 2r_0 \leq 2t_{\varphi,g}.\end{aligned}$$

This finishes the inductive argument of the proof of Theorem 5.1 in the case of ($\mathfrak{n}^+$-min). ☐

Proof of Theorem 1.2 in the case of ($\mathfrak{n}^+$-min). Let $\varphi\colon \mathcal{O}[[\Gamma]] \to \mathcal{O}'$ be a homomorphism from $\mathcal{O}[[\Gamma]]$ to the ring of integers in a finite extension of E. Enlarging E if necessary we may assume that $\mathcal{O} = \mathcal{O}'$.

If $\varphi(L_p(K_\infty, f)) = 0$, then obviously

$$\varphi(L_p(K_\infty, f)) \in \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{\mathfrak{n}^-}(K_\infty, A)^\vee \otimes_\varphi \mathcal{O}).$$

So, we may assume that $\varphi(L_p(K_\infty, f)) \neq 0$. Choose t^* larger than $\mathrm{ord}\,\varphi(L_p(K_\infty, f))$.

Let n be a nonnegative integer. We consider the $(n+t^*, n+t^*)$-admissible form $\mathscr{D}_{n+t^*} = (\mathfrak{n}^-, f^\dagger_{n+t^*})$ provided by Proposition 2.7. That is, we take $N = r = n + t^*$ and $g = f^\dagger_{n+t^*}$ in Theorem 5.1.

Since $\operatorname{ord} \varphi(L_p(K_\infty, f)) < r$,

$$\varphi_r(\theta(f^\dagger_{n+t^*}))^2 = \varphi_r(L_p(K_\infty, f)) = \varphi(L_p(K_\infty, f)) \pmod{\omega^r}$$

is nonzero in $\mathcal{O}_r$, and we have

$$2t_{\varphi, f^\dagger_{n+t^*}} = 2 \operatorname{ord} \varphi_r(\theta(f^\dagger_{n+t^*})) = \operatorname{ord} \varphi_r(L_p(K_\infty, f)) = \operatorname{ord} \varphi(L_p(K_\infty, f)) < t^* \leq r.$$

On the other hand,

$$n = r - t^* < r - \operatorname{ord} \varphi(L_p(K_\infty, f)) \leq r - \operatorname{ord} \varphi_r(\theta(f^\dagger_{n+t^*})) = r - t_{\varphi, f^\dagger_{n+t^*}}.$$

Thus by Theorem 5.1 we have

$$\operatorname{length}_{\mathcal{O}}(\operatorname{Sel}_{\mathfrak{n}^-}(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O}) \leq 2t_{\varphi, f^\dagger_{n+t^*}} = \operatorname{ord} \varphi(L_p(K_\infty, f)).$$

So, $\varphi(L_p(K_\infty, f))$ belongs to $\operatorname{Fitt}_{\mathcal{O}}(\operatorname{Sel}_{\mathfrak{n}^-}(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O})$.

Hence, $\varphi(L_p(K_\infty, f))$ belongs to

$$\operatorname{Fitt}_{\mathcal{O}}(\operatorname{Sel}_{\mathfrak{n}^-}(K_\infty, A)^\vee \otimes_\varphi \mathcal{O}) = \bigcap_n \operatorname{Fitt}_{\mathcal{O}}(\operatorname{Sel}_{\mathfrak{n}^-}(K_\infty, A_n)^\vee \otimes_\varphi \mathcal{O}).$$

Now, by [**5**, Lemma 6.11] we have

$$L_p(K_\infty, f) \in \operatorname{Fitt}_{\mathcal{O}}(\operatorname{Sel}_{\mathfrak{n}^-}(K_\infty, A)^\vee).$$

As $L_p(K_\infty, f) \neq 0$ by Proposition 2.6, $\operatorname{Sel}_{\mathfrak{n}^-}(K_\infty, A)$ is $\mathcal{O}[[\Gamma]]$-cotorsion. Taking $\mathcal{O} = \mathcal{O}_f$ we obtain the precise statement in Theorem 1.2. □

Next, we relax the condition ($\mathfrak{n}^+$-min) to ($\mathfrak{n}^+$-DT).

Lemma 5.7. *There exists a Hilbert modular form f' congruence to f modulo ω that satisfies* (CR$^+$)*,* (PO)*, and* ($\mathfrak{n}^+$-min)*.*

Proof: We need to show that, if $\mathfrak{l} | \frac{\mathfrak{n}}{\mathfrak{n}_{\bar\rho}}$, then there exists a Hilbert modular form f' of level dividing $\frac{\mathfrak{n}}{\mathfrak{l}}$ congruence to f. In the case where $\pi_\mathfrak{l}$ is special or supercuspidal, this follows directly from Jarvis's level lowering result [**11**, Theorem 0.1]. Note that our condition ($\mathfrak{n}^+$-DT) ensures that f satisfies conditions of [**11**, Theorem 0.1].

Now, let $\pi_\mathfrak{l} = \operatorname{Ind}_B^{\mathrm{GL}_2(F_\mathfrak{l})}(\chi \otimes \chi^{-1})$ be a principal series representation, where χ is a character of $F_\mathfrak{l}^\times$, and B is the Borel subgroup of $\mathrm{GL}_2(F_\mathfrak{l})$ consisting of upper-triangular invertible matrices. When the conductor $\mathfrak{n}_\chi$ of χ is $\mathfrak{l}$, f again satisfies the condition of [**11**, Theorem 0.1], and so we can apply Jarvis's result. It remains to show that, either if $\mathfrak{n}_\chi$ is $\mathcal{O}_{F_\mathfrak{l}}$, or if $\mathfrak{n}_\chi$ is divisible by $\mathfrak{l}^2$, then $\mathfrak{l} \nmid \frac{\mathfrak{n}}{\mathfrak{n}_{\bar\rho}}$. In the former case there is nothing to prove. In the latter case, observe that the conductor of $\overline{\chi} = \chi \pmod{\omega}$ is equal to that of χ. It follows that the conductor of $\bar\rho_{f,\mathfrak{l}}$ is equal to that of $\rho_{f,\mathfrak{l}}$, since $\rho_{f,\mathfrak{l}} \cong \chi \oplus \chi^{-1}$ when it is restricted to the inertia subgroup of $G_{F_\mathfrak{l}}$ [**17**]. □

Proposition 5.8. *Assume that* (CR$^+$)*,* (PO)*, and* ($\mathfrak{n}^+$-DT) *hold. Then* $\operatorname{Sel}^{\mathfrak{n}^+}_{\mathfrak{n}^-}(K_\infty, A)$ *is* $\mathcal{O}[[\Gamma]]$*-cotorsion and* $\operatorname{Sel}^{\mathfrak{n}^+}_{\mathfrak{n}^-}(K_\infty, A)^\vee$ *has vanishing μ-invariant.*

Proof: Let f' be as in Lemma 5.7. We write A' and A'_i for A and A_i attached to f'. We have already shown that Theorem 1.2 holds for f'. Combining this with Proposition 2.6 we obtain that $\operatorname{Sel}^{\mathfrak{n}^+}_{\mathfrak{n}^-}(K_\infty, A')^\vee$ has vanishing μ-invariant. In other words, $\operatorname{Sel}^{\mathfrak{n}^+}_{\mathfrak{n}^-}(K_\infty, A')[\omega]$ is finite.

By Lemma 3.5(b), taking $S = \mathfrak{n}^+$, we get

$$\mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A)[\omega] = \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A_1) = \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A_1') = \mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A')[\omega].$$

Thus $\mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A)[\omega]$ is finite, and $\mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A)^\vee$ has vanishing μ-invariant. □

Since $\mathrm{Sel}_{\mathfrak{n}^-}^{\mathfrak{n}^+}(K_\infty, A)^\vee$ has vanishing μ-invariant, by Theorem 3.7, $\widehat{\mathrm{Sel}}_\Delta(K_\infty, T_N)$ is free over $\mathcal{O}_N[[\Gamma]]$. Now repeating the argument for the case of ($\mathfrak{n}^+$-min) we finish the proof of Theorem 1.2. The only place we need to revise the argument is the proof of Lemma 5.4. Assume that $\mathrm{Sel}_\Delta(K_\infty, A_n)$ is nonzero. In general, we may not have $\mathrm{Sel}_\Delta(K_\infty, A_n)[\mathfrak{m}] = \mathrm{Sel}_\Delta(K, A_1)$ now. But by Lemma 3.5(a) and (c) we have

$$\mathrm{Sel}_\Delta(K_\infty, A_n)[\mathfrak{m}] \subseteq \mathrm{Sel}_\Delta^{\mathfrak{n}^+}(K_\infty, A_n)[\mathfrak{m}] = \mathrm{Sel}_\Delta^{\mathfrak{n}^+}(K, A_1).$$

Consider the nonzero element x in $\mathrm{Sel}_\Delta(K_\infty, A_n)[\mathfrak{m}]$ as an element in $\mathrm{Sel}_\Delta^{\mathfrak{n}^+}(K, A_1)$. If $e_\mathfrak{l} = t_{\varphi,g}$, we again obtain $\langle \mathrm{Res}_\mathfrak{l}\, \kappa'(\mathfrak{l}), \mathrm{Res}_\mathfrak{l}\, x\rangle_\mathfrak{l} \neq 0$ and $\eta_\mathfrak{l}(\kappa'(\mathfrak{l})) \neq 0$, contradicting Lemma 5.3. □

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School of Mathematical Sciences, Ministry of Education Key Laboratory of Mathematics and Engineering Applications and Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China
E-mail address: byxie@math.ecnu.edu.cn

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