

∞ -OPERADS AS SYMMETRIC MONOIDAL ∞ -CATEGORIES

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Abstract: We use Lurie’s symmetric monoidal envelope functor to give two new descriptions of ∞ -operads: as certain symmetric monoidal ∞ -categories whose underlying symmetric monoidal ∞ -groupoids are free, and as certain symmetric monoidal ∞ -categories equipped with a symmetric monoidal functor to finite sets (with disjoint union as tensor product). The latter leads to a third description of ∞ -operads, as a localization of a presheaf ∞ -category, and we use this to give a simple proof of the equivalence between Lurie’s and Barwick’s models for ∞ -operads.

2020 Mathematics Subject Classification: 18N70.

Key words: ∞ -operads, symmetric monoidal ∞ -categories.

1. Introduction

The close relationship between symmetric monoidal categories and (symmetric) operads goes back to the birth of operad theory in algebraic topology: both operads [23] and PROPs [21], which are a special class of symmetric monoidal categories, were introduced to describe homotopy-coherent algebraic structures on topological spaces, and it was quickly realized that operads could be viewed as a special kind of PROP (see for instance the discussion in Adams’s book [1, §2.3], or [18, §7]). The relationship has also been analysed in the context of logic and computer science, notably by Hermida [13].

In the setting of ∞ -categories, the relationship between ∞ -operads and symmetric monoidal ∞ -categories is more pronounced, since in the approach of Lurie [20] both notions are defined as certain functors to the category \mathbb{F}_* of finite pointed sets;¹ for operads this corresponds to the construction of categories of operators of May–Thomason [24]. There is then an evident forgetful functor

$$U: \text{SMCat}_\infty \longrightarrow \text{Opd}_\infty,$$

where Opd_∞ is the ∞ -category of ∞ -operads and SMCat_∞ is that of symmetric monoidal ∞ -categories. Lurie ([20, §2.2.4]) established an adjunction

$$\text{Opd}_\infty \begin{array}{c} \xrightarrow{\text{Env}} \\ \xleftarrow{U} \end{array} \text{SMCat}_\infty,$$

where the left adjoint Env is given by an explicit construction, the *symmetric monoidal envelope* of an ∞ -operad. Neither of the two functors is fully faithful, though, and so does not immediately exhibit one notion as a special case of the other.²

J. K. gratefully acknowledges support from grants MTM2016-80439-P (AEI/FEDER, UE) and PID2020-116481GB-I00 of Spain and 2017-SGR-1725 of Catalonia, and was also supported through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D grant number CEX2020-001084-M.

¹Also known as Segal’s category Γ^{op} .

²For U , this is because morphisms between symmetric monoidal ∞ -categories in Opd_∞ correspond to *lax* symmetric monoidal functors.

In the present contribution, we exploit this adjunction to establish new conceptually simple characterizations of ∞ -operads, leading to an easy proof of the equivalence between Lurie’s and Barwick’s notions of ∞ -operads.

1.1. Overview. We start out by tweaking the adjunction in two ways, so as to give two new characterizations of ∞ -operads in terms of symmetric monoidal ∞ -categories:

Theorem 1.1.1 (cf. Propositions 2.4.6 and 2.4.16). *The symmetric monoidal envelope gives an equivalence between ∞ -operads in the sense of Lurie [20] and*

- (1) *symmetric monoidal ∞ -categories \mathcal{C}^\otimes with a symmetric monoidal functor to \mathbb{F}^\amalg , the category of finite sets with disjoint union as tensor product, such that*
 - (a) *every object of \mathcal{C} is equivalent to a tensor product of objects that lie over the terminal object $\mathbf{1}$ in \mathbb{F} ,*
 - (b) *condition (*) below holds for any objects x_1, \dots, x_n and y_1, \dots, y_m that lie over $\mathbf{1}$;*
 - (2) *symmetric monoidal ∞ -categories \mathcal{C}^\otimes with a map of ∞ -groupoids $X \rightarrow \mathcal{C}^\simeq$ such that*
 - (a) *the underlying symmetric monoidal ∞ -groupoid of \mathcal{C}^\otimes is free on X , i.e. the induced morphism $\mathrm{Sym}(X) \simeq \coprod_{n=0}^\infty X_{h\Sigma_n}^{\times n} \rightarrow \mathcal{C}^\simeq$ is an equivalence,*
 - (b) *condition (*) below holds for any objects x_1, \dots, x_n and y_1, \dots, y_m in X .*
- (*) *The morphism*

$$\coprod_{\phi \in \mathrm{Map}_{\mathbb{F}}(\mathbf{n}, \mathbf{m})} \prod_{i=1}^m \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \phi^{-1}(i)} x_j, y_i \right) \longrightarrow \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j=1}^n x_j, \bigotimes_{i=1}^m y_i \right),$$

given by tensoring maps together, is an equivalence.

Remark 1.1.2. The condition (*) appearing in both characterizations can be traced back to the class of PROPs singled out by Boardman and Vogt in [6, Lemma 2.43]. More recently, it has been studied in different guises in the 1-categorical literature under the name of the *hereditary condition* (cf. [22, 7, 17, 4]; see also [25]). From that perspective, characterization (2) can be seen as the ∞ -categorical version of the equivalence of [4, 8] between the *Feynman categories* of Kaufmann and Ward [17] and coloured operads.

Remark 1.1.3. We are not aware of any direct precursor to characterization (1), but it fits well with Weber’s 2-categorical approach to operad theory [28], where operads are essentially monads cartesian over the symmetric monoidal category monad. Perhaps it should also be mentioned that in the theory of operadic categories of Batanin and Markl [5], which can be seen as a generalization of Barwick’s idea of operator categories [3], it is an essential feature that everything lives over the category of finite sets.

Using the first characterization, we proceed to give a third: viewing ∞ -categories as complete Segal spaces, we can describe symmetric monoidal ∞ -categories over \mathbb{F}^\amalg as functors $\mathcal{F} \rightarrow \mathcal{S}$ satisfying completeness and Segal conditions for a certain category \mathcal{F} , giving an equivalence

$$\mathrm{SMCat}_{\infty/\mathbb{F}^\amalg} \simeq \mathrm{CSeg}_{\mathcal{F}}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{F}, \mathcal{S}).$$

We can then identify ∞ -operads as those complete Segal \mathcal{F} -spaces that satisfy some further conditions:

Theorem 1.1.4 (Theorem 3.4.5). *There is an equivalence*

$$\mathrm{Opd}_\infty \xrightarrow{\sim} \mathrm{CSeg}'_{\mathcal{F}}(\mathcal{S}),$$

where $\mathrm{CSeg}'_{\mathcal{F}}(\mathcal{S})$ is a certain full subcategory of $\mathrm{CSeg}_{\mathcal{F}}(\mathcal{S})$.

Since $\mathrm{CSeg}'_{\mathcal{F}}(\mathcal{S})$ is by definition an accessible localization of a presheaf ∞ -category, this result implies in particular that Opd_∞ is a presentable ∞ -category, without appealing to a presentation of Opd_∞ by a model category. Our main motivation for this characterization, however, is that it is a key ingredient in the final result of the paper: via an obvious comparison functor between \mathcal{F} and Barwick’s category $\Delta_{\mathbb{F}}^{\mathrm{op}}$, we obtain with very little work an equivalence between $\mathrm{CSeg}'_{\mathcal{F}}(\mathcal{S})$ and Barwick’s definition of ∞ -operads as presheaves on $\Delta_{\mathbb{F}}$ satisfying Segal and completeness conditions. Thus we get a simple proof of the equivalence between Lurie’s and Barwick’s approaches to ∞ -operads:

Corollary 1.1.5. *There is an equivalence of ∞ -categories*

$$\mathrm{Opd}_\infty \simeq \mathrm{CSeg}_{\Delta_{\mathbb{F}}^{\mathrm{op}}}(\mathcal{S})$$

between Lurie’s and Barwick’s models for ∞ -operads.

This theorem was already proved by Barwick [3] by a rather different method (which involves studying the nerve adjunction for a functor $\Delta_{\mathbb{F}} \rightarrow \mathrm{Opd}_\infty$). Note that Barwick’s result is substantially more general than ours, giving an equivalence between two definitions of ∞ -operads over any *perfect operator category*, where the explicit description of the monoidal envelope required for our proof typically fails.³

1.2. Some basic notation. This paper is written in the language of ∞ -categories, and all terms such as (co)limits and commutative diagrams should be understood in their fully homotopy-coherent/ ∞ -categorical sense.

- \mathbb{F} is (a skeleton of) the category of finite sets, with objects $\mathbf{n} = \{1, \dots, n\}$ ($n = 0, 1, \dots$).
- \mathcal{S} is the ∞ -category of (small) ∞ -groupoids/spaces/homotopy types.
- Cat_∞ is the ∞ -category of (small) ∞ -categories.
- If \mathcal{C} is an ∞ -category, $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{L}}$ denotes the full subcategory of the overcategory $\mathrm{Cat}_{\infty/\mathcal{C}}$ spanned by the left fibrations to \mathcal{C} .
- If \mathcal{C} is an ∞ -category, $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{coc}}$ denotes the subcategory of the overcategory $\mathrm{Cat}_{\infty/\mathcal{C}}$ with objects the cocartesian fibrations to \mathcal{C} and morphisms the functors over \mathcal{C} that preserve cocartesian morphisms.
- If \mathcal{C} is an ∞ -category, we write \mathcal{C}^{\simeq} for its underlying ∞ -groupoid, i.e. the subcategory containing only equivalences.

2. From Lurie’s ∞ -operads to symmetric monoidal ∞ -categories

In this section we first review the basic notions of commutative monoids in ∞ -categories (and in particular symmetric monoidal ∞ -categories) in Subsection 2.1 and ∞ -operads (in the sense of [20]) in Subsection 2.2. Then we recall the symmetric monoidal envelope of an ∞ -operad in Subsection 2.3 before we study its image and prove Theorem 1.1.1 in Subsection 2.4.

³Our approach does also work in the particular case of non-symmetric (or planar) ∞ -operads, however.

2.1. Commutative monoids and symmetric monoidal ∞ -categories. We now recall the ∞ -categorical notion of commutative monoids, originally introduced by Segal [27]. As a special case, this also gives the definition of symmetric monoidal ∞ -categories.

Notation 2.1.1. We write \mathbb{F}_* for (a skeleton of) the category of finite pointed sets. We will make use of two equivalent descriptions of this category:

- (1) The objects of \mathbb{F}_* are the pointed sets $\langle n \rangle = (\{0, 1, \dots, n\}, 0)$ ($n = 0, 1, \dots$) and the morphisms $\langle n \rangle \rightarrow \langle m \rangle$ are the functions that preserve the base point.
- (2) The objects of \mathbb{F}_* are the sets $\mathbf{n} = \{1, \dots, n\}$ ($n = 0, 1, \dots$), and morphisms from \mathbf{n} to \mathbf{m} are isomorphism classes⁴ of *spans*

$$\mathbf{n} \longleftarrow \mathbf{x} \longrightarrow \mathbf{m},$$

where the backwards map is injective. Spans are composed by taking pullbacks. To pass between these two descriptions, note that giving a pointed map $\langle n \rangle \rightarrow \langle m \rangle$ is the same thing as giving a map of sets $I \rightarrow \mathbf{m}$, where I is the subset of $\langle n \rangle$ that is not mapped to the base point. (Up to unique isomorphism, I can be replaced by an object in the chosen skeleton.)

Definition 2.1.2. A morphism $\phi: \langle n \rangle \rightarrow \langle m \rangle$ in \mathbb{F}_* is *active* if $\phi^{-1}(0) = \{0\}$ and *inert* if $\phi|_{\langle n \rangle \setminus \phi^{-1}(0)}$ is an isomorphism. The inert and active morphisms form a factorization system on \mathbb{F}_* ; in particular, every morphism factors uniquely up to isomorphism as an inert morphism followed by an active morphism.

Remark 2.1.3. In terms of the second description of \mathbb{F}_* , a span

$$\mathbf{n} \longleftarrow \mathbf{k} \longrightarrow \mathbf{m}$$

is active if the inclusion $\mathbf{n} \hookrightarrow \mathbf{k}$ is an isomorphism, and inert if the map $\mathbf{k} \rightarrow \mathbf{m}$ is an isomorphism.

Notation 2.1.4. For $\langle n \rangle \in \mathbb{F}_*$ and $i = 1, \dots, n$, we write $\rho_i: \langle n \rangle \rightarrow \langle 1 \rangle$ for the inert map given by

$$\rho_i(j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Alternatively, this is the span

$$\mathbf{n} \longleftarrow \{i\} \xrightarrow{=} \mathbf{1}.$$

Definition 2.1.5. Let \mathcal{C} be an ∞ -category with finite products. A *commutative monoid* in \mathcal{C} is a functor $M: \mathbb{F}_* \rightarrow \mathcal{C}$ such that for every $\langle n \rangle \in \mathbb{F}_*$ the natural morphism

$$M(\langle n \rangle) \longrightarrow \prod_{i=1}^n M(\langle 1 \rangle),$$

determined by the maps ρ_i , is an equivalence. We write $\text{CMon}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathbb{F}_*, \mathcal{C})$ spanned by the commutative monoids.

Definition 2.1.6. A *symmetric monoidal ∞ -category* is a commutative monoid in the ∞ -category Cat_∞ of ∞ -categories.

⁴In fact the groupoid of such spans is discrete, so from an ∞ -categorical viewpoint taking isomorphism classes does nothing.

Remark 2.1.7. Equivalently, using the straightening equivalence between functors to Cat_∞ and cocartesian fibrations, we can view a symmetric monoidal ∞ -category as a cocartesian fibration over \mathbb{F}_* .

Notation 2.1.8. We write SMCat_∞ for the ∞ -category of symmetric monoidal ∞ -categories. This can be viewed as a full subcategory of either $\text{Fun}(\mathbb{F}_*, \text{Cat}_\infty)$ or $\text{Cat}_{\infty/\mathbb{F}_*}^{\text{coc}}$.

2.2. Lurie’s ∞ -operads. Here we recall Lurie’s definition of ∞ -operads from [20, §2.1.1] and its relation to symmetric monoidal ∞ -categories.

Definition 2.2.1. An ∞ -operad is a functor $\pi: \mathcal{O} \rightarrow \mathbb{F}_*$ such that:

- (1) \mathcal{O} has π -cocartesian morphisms over inert morphisms in \mathbb{F}_* .
- (2) For every $\langle n \rangle \in \mathbb{F}_*$, the functor

$$\mathcal{O}_{\langle n \rangle} \longrightarrow \prod_{i=1}^n \mathcal{O}_{\langle 1 \rangle},$$

given by cocartesian transport along the maps $\rho_i: \langle n \rangle \rightarrow \langle 1 \rangle$, is an equivalence.

- (3) For $X \in \mathcal{O}_{\langle n \rangle}$, if $\bar{\rho}_i: X \rightarrow X_i$ is a cocartesian morphism over ρ_i ($i = 1, \dots, n$), for any $Y \in \mathcal{O}_{\langle m \rangle}$ the commutative square

$$\begin{array}{ccc} \text{Map}_{\mathcal{O}}(Y, X) & \xrightarrow{(\bar{\rho}_{i,*})} & \prod_{i=1}^n \text{Map}_{\mathcal{O}}(Y, X_i) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathbb{F}_*}(\langle m \rangle, \langle n \rangle) & \xrightarrow{(\rho_{i,*})} & \prod_{i=1}^n \text{Map}(\langle m \rangle, \langle 1 \rangle) \end{array}$$

is a pullback square.

Remark 2.2.2. It is not hard to see that a symmetric monoidal ∞ -category, viewed as a cocartesian fibration over \mathbb{F}_* , is precisely an ∞ -operad that is also a cocartesian fibration.

Definition 2.2.3. If $\pi: \mathcal{O} \rightarrow \mathbb{F}_*$ is an ∞ -operad, we say a morphism in \mathcal{O} is *inert* if it is a cocartesian morphism over an inert morphism in \mathbb{F}_* , and *active* if it lies over an active morphism in \mathbb{F}_* . By [20, Proposition 2.1.2.5], the inert and active morphisms form a factorization system on \mathcal{O} .

Definition 2.2.4. If $p: \mathcal{O} \rightarrow \mathbb{F}_*$ and $q: \mathcal{P} \rightarrow \mathbb{F}_*$ are ∞ -operads, then a *morphism of ∞ -operads* from \mathcal{O} to \mathcal{P} is a commutative triangle

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{f} & \mathcal{P} \\ p \searrow & & \swarrow q \\ & \mathbb{F}_* & \end{array}$$

such that f preserves inert morphisms. We write Opd_∞ for the subcategory of $\text{Cat}_{\infty/\mathbb{F}_*}$ whose objects are the ∞ -operads and whose morphisms are the morphisms of ∞ -operads.

Remark 2.2.5. By Remark 2.2.2, if we view symmetric monoidal ∞ -categories as cocartesian fibrations, then the subcategory of $\text{Cat}_{\infty/\mathbb{F}_*}$ corresponding to SMCat_∞ is contained in Opd_∞ , so that we have a forgetful functor $U: \text{SMCat}_\infty \rightarrow \text{Opd}_\infty$. Note

that this is not fully faithful: a morphism in SMCat_∞ is required to preserve all co-cartesian morphisms and corresponds to a symmetric monoidal functor, while a morphism in Opd_∞ is only required to preserve the cocartesian morphisms that lie over inert maps in \mathbb{F}_* . Such a morphism can be interpreted as a lax symmetric monoidal functor.

Definition 2.2.6. Suppose \mathcal{O} is an ∞ -operad. Given a full subcategory \mathcal{C} of $\mathcal{O}_{\langle 1 \rangle}$, the full subcategory of \mathcal{O} spanned by the objects that lie in $\mathcal{C}^{\times n} \subseteq \mathcal{O}_{\langle 1 \rangle}^{\times n}$ under the equivalence $\mathcal{O}_{\langle 1 \rangle}^{\times n} \simeq \mathcal{O}_{\langle n \rangle}$, for all n , is again an ∞ -operad. We refer to this as the *full suboperad* of \mathcal{O} spanned by the objects in \mathcal{C} .

2.3. Symmetric monoidal envelopes. In this subsection we recall the construction of symmetric monoidal envelopes from [20, §2.2.4].

Notation 2.3.1. Let $\text{Act}(\mathbb{F}_*)$ denote the full subcategory of the arrow category $\mathbb{F}_*^{[1]}$ whose objects are the active morphisms. We write $s, t: \text{Act}(\mathbb{F}_*) \rightarrow \mathbb{F}_*$ for the source and target projections. If $i: \mathbb{F}_* \rightarrow \text{Act}(\mathbb{F}_*)$ denotes the functor that assigns to each object its identity map, then $si = ti = \text{id}_{\mathbb{F}_*}$.

Definition 2.3.2. For \mathcal{O} an ∞ -operad, we write $\text{Env}(\mathcal{O}) \rightarrow \mathbb{F}_*$ for the fibre product $\mathcal{O} \times_{\mathbb{F}_*} \text{Act}(\mathbb{F}_*)$ along s , with the map to \mathbb{F}_* induced by t . This gives a functor $\text{Env}: \text{Opd}_\infty \rightarrow \text{Cat}_{\infty/\mathbb{F}_*}$.

Lemma 2.3.3. *There is a natural pullback square*

$$(2.1) \quad \begin{array}{ccc} \mathcal{O} & \xrightarrow{i_{\mathcal{O}}} & \text{Env}(\mathcal{O}) \\ \downarrow & & \downarrow \\ \mathbb{F}_* & \xrightarrow{i} & \text{Act}(\mathbb{F}_*) . \end{array}$$

Proof: By definition we have a commutative diagram

$$\begin{array}{ccccc} & & \text{Env}(\mathcal{O}) & \longrightarrow & \mathcal{O} \\ & & \downarrow & & \downarrow \\ \mathbb{F}_* & \xrightarrow{i} & \text{Act}(\mathbb{F}_*) & \xrightarrow{s} & \mathbb{F}_* , \\ & \searrow & \text{id} & \nearrow & \\ & & & & \end{array}$$

where the square is a pullback square. The pullback along i is therefore indeed given by $\mathcal{O} \rightarrow \mathbb{F}_*$. □

Remark 2.3.4. Since $ti = \text{id}$, we can view $i_{\mathcal{O}}$ as a natural map $\mathcal{O} \rightarrow \text{Env}(\mathcal{O})$ over \mathbb{F}_* .

Theorem 2.3.5 ([20, Propositions 2.2.4.4 and 2.2.4.9]). *The construction Env gives a functor $\text{Opd}_\infty \rightarrow \text{SMCat}_\infty$, which is left adjoint to the forgetful functor $U: \text{SMCat}_\infty \rightarrow \text{Opd}_\infty$, with unit transformation given by the natural maps $i_{(-)}$.*

Remark 2.3.6. If $\pi: \mathcal{O} \rightarrow \mathbb{F}_*$ is an ∞ -operad, an object of $\text{Env}(\mathcal{O})$ over $\langle n \rangle$ is given by an object $X \in \mathcal{O}$ together with an active morphism $\alpha: \pi(X) \rightarrow \langle n \rangle$ in \mathbb{F}_* . A morphism $(X, \alpha) \rightarrow (Y, \beta)$ in $\text{Env}(\mathcal{O})$ is given by a morphism $\phi: X \rightarrow Y$ in \mathcal{O} and a commutative square

$$\begin{array}{ccc} \pi(X) & \xrightarrow{\pi(\phi)} & \pi(Y) \\ \downarrow \alpha & & \downarrow \beta \\ \langle n \rangle & \xrightarrow{\psi} & \langle m \rangle . \end{array}$$

If the underlying map ψ is active, then the uniqueness of factorizations forces ϕ to be an active map in \mathcal{O} . In particular, since every object of \mathbb{F}_* has a unique active map to $\langle 1 \rangle$, the underlying ∞ -category $\text{Env}(\mathcal{O})_{\langle 1 \rangle}$ can be identified with the subcategory \mathcal{O}^{act} containing only the active maps in \mathcal{O} . Given a morphism $\psi: \langle n \rangle \rightarrow \langle m \rangle$, the cocartesian morphism $(X, \alpha) \rightarrow \psi_!(X, \alpha)$ can be described as follows: the inert-active factorization of $\psi \circ \alpha$ gives a commutative square

$$(2.2) \quad \begin{array}{ccc} \pi(X) & \xrightarrow{i} & \langle k \rangle \\ \downarrow \alpha & & \downarrow a \\ \langle n \rangle & \xrightarrow{\psi} & \langle m \rangle, \end{array}$$

where i is inert and a is active. Since \mathcal{O} is an ∞ -operad there is a cocartesian morphism $X \rightarrow i_!X$ in \mathcal{O} , and $\psi_!(X, \alpha)$ is given by $(i_!X, a)$ with the cocartesian morphism in \mathcal{O} together with the commutative square (2.2). In particular, if we think of objects of $\text{Env}(\mathcal{O})_{\langle 1 \rangle} \simeq \mathcal{O}^{\text{act}}$ as sequences of objects in $\mathcal{O}_{\langle 1 \rangle}$, then their tensor product is given by concatenation. Given a morphism of ∞ -operads $F: \mathcal{O} \rightarrow \mathcal{C}^\otimes$, where \mathcal{C}^\otimes is a symmetric monoidal ∞ -category, the canonical extension of F to a symmetric monoidal functor $\text{Env}(\mathcal{O}) \rightarrow \mathcal{C}^\otimes$ takes (X, α) to the codomain $\alpha_!F(X)$ of the cocartesian morphism from $F(X)$ over α .

Remark 2.3.7. For the terminal ∞ -operad \mathbb{F}_* we can describe $\text{Env}(\mathbb{F}_*)$ even more explicitly: we can identify the underlying category $\text{Env}(\mathbb{F}_*)_{\langle 1 \rangle} \simeq \mathbb{F}_*^{\text{act}}$ with the category \mathbb{F} of finite sets, and under this identification the “concatenation” symmetric monoidal structure corresponds to disjoint union. In other words, the symmetric monoidal ∞ -category $\text{Env}(\mathbb{F}_*)$ is equivalent to the coproduct symmetric monoidal structure on \mathbb{F} , that is to say,

$$\text{Env}(\mathbb{F}_*) \simeq \mathbb{F}^{\amalg}.$$

2.4. Two descriptions of ∞ -operads via envelopes. In this section we will use the symmetric monoidal envelope functor to give two descriptions of ∞ -operads in terms of symmetric monoidal ∞ -categories and thus prove Theorem 1.1.1.

For the first description we want to consider symmetric monoidal ∞ -categories equipped with a map to \mathbb{F}^{\amalg} . Since we saw in Remark 2.3.7 that Env takes the terminal ∞ -operad \mathbb{F}_* to \mathbb{F}^{\amalg} , we have a functor

$$\text{Env}' : \text{Opd}_\infty \longrightarrow \text{SMCat}_{\infty/\mathbb{F}^{\amalg}}$$

that just applies Env to the unique map to the terminal object in Opd_∞ .

Lemma 2.4.1. *The functor Env' has a right adjoint*

$$U' : \text{SMCat}_{\infty/\mathbb{F}^{\amalg}} \longrightarrow \text{Opd}_\infty,$$

given by applying the forgetful functor U and then pulling back along the unit map $\mathbb{F}_* \rightarrow \mathbb{F}^{\amalg}$.

Proof: This is a special case of [19, Proposition 5.2.5.1]. □

Remark 2.4.2. In other words, if \mathcal{C}^\otimes is a symmetric monoidal ∞ -category over \mathbb{F}_* , then we have a pullback square

$$\begin{array}{ccc} U'(\mathcal{C}^\otimes) & \longrightarrow & \mathcal{C}^\otimes \\ \downarrow & & \downarrow \\ \mathbb{F}_* & \xrightarrow{i} & \mathbb{F}^{\amalg}. \end{array}$$

Proposition 2.4.3. *The functor $\text{Env}' : \text{Opd}_\infty \rightarrow \text{SMCat}_{\infty/\mathbb{F}^\Pi}$ is fully faithful.*

Proof: It suffices to show that the unit transformation $\text{id} \rightarrow U' \text{Env}'$ is an equivalence, which follows from the pullback square (2.1) in Lemma 2.3.3. \square

Notation 2.4.4. For a symmetric monoidal functor $\mathcal{C}^\otimes \rightarrow \mathbb{F}^\Pi$, we write

$$\mathcal{C}_{(1)}^\otimes := U' \mathcal{C}^\otimes$$

for the pullback along $\mathbb{F}_* \rightarrow \mathbb{F}^\Pi$, and $\mathcal{C}_{(1)} := (\mathcal{C}_{(1)}^\otimes)_{(1)}$ for the fibre of \mathcal{C} over $\mathbf{1} \in \mathbb{F}$. Note that since $\mathbf{1}$ has no endomorphisms in \mathbb{F} , the inclusion $\mathcal{C}_{(1)} \rightarrow \mathcal{C}$ exhibits $\mathcal{C}_{(1)}$ as a full subcategory, and thus $\mathcal{C}_{(1)}^\otimes$ is the full suboperad of \mathcal{C}^\otimes spanned by objects of \mathcal{C} that lie over $\mathbf{1}$.

Corollary 2.4.5. *The category Opd_∞ is equivalent to the full subcategory of $\text{SMCat}_{\infty/\mathbb{F}^\Pi}$ consisting of symmetric monoidal ∞ -categories \mathcal{C}^\otimes over \mathbb{F}^Π such that the counit map*

$$(2.3) \quad \text{Env}(\mathcal{C}_{(1)}^\otimes) \longrightarrow \mathcal{C}^\otimes$$

is an equivalence.

We will now describe this full subcategory more explicitly:

Proposition 2.4.6. *For $\mathcal{C}^\otimes \in \text{SMCat}_{\infty/\mathbb{F}^\Pi}$, the counit map (2.3) is an equivalence if and only if the two following conditions hold:*

- (1) *Every object in \mathcal{C} is equivalent to a tensor product $x_1 \otimes \cdots \otimes x_n$ with $x_i \in \mathcal{C}_{(1)}$.*
- (2) *Given objects x_1, \dots, x_n and y_1, \dots, y_m in $\mathcal{C}_{(1)}$, the morphism*

$$\coprod_{\phi \in \text{Map}_{\mathbb{F}}(\mathbf{n}, \mathbf{m})} \prod_{i=1}^m \text{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \phi^{-1}(i)} x_j, y_i \right) \longrightarrow \text{Map}_{\mathcal{C}} \left(\bigotimes_{j=1}^n x_j, \bigotimes_{i=1}^m y_i \right),$$

given by tensoring maps together, is an equivalence.

Remark 2.4.7. Condition (2) is the so-called ‘‘hereditary condition’’ considered in [22, 7, 25, 17, 4].

Proof: A morphism in $\text{SMCat}_{\infty/\mathbb{F}^\Pi}$ is an equivalence if and only if the functor of underlying ∞ -categories is an equivalence. It therefore suffices to show that the given conditions are equivalent to the functor

$$\epsilon : (\mathcal{C}_{(1)}^\otimes)^{\text{act}} \simeq \text{Env}(\mathcal{C}_{(1)}^\otimes)_{(1)} \longrightarrow \mathcal{C}$$

being an equivalence.

An object of $(\mathcal{C}_{(1)}^\otimes)^{\text{act}}$ can be described as a list (x_1, \dots, x_n) , where each x_i is an object of $\mathcal{C}_{(1)}$, and $\epsilon(x_1, \dots, x_n)$ is the tensor product $x_1 \otimes \cdots \otimes x_n$. Condition (1) therefore corresponds precisely to ϵ being essentially surjective.

A morphism in $(\mathcal{C}_{(1)}^\otimes)^{\text{act}}$ from (x_1, \dots, x_n) to (y_1, \dots, y_m) is given by a map $\phi : \mathbf{n} \rightarrow \mathbf{m}$ in \mathbb{F} together with a morphism $f_i : \bigotimes_{j \in \phi^{-1}(i)} x_j \rightarrow y_i$ lying over the unique map $\phi^{-1}(i) \rightarrow \mathbf{1}$, for every $i = 1, \dots, m$. The functor ϵ takes this to the morphism

$$\bigotimes_{i=1}^m f_i : \bigotimes_{j=1}^n x_j \xrightarrow{\sim} \bigotimes_{i=1}^m \left(\bigotimes_{j \in \phi^{-1}(i)} x_j \right) \longrightarrow \bigotimes_{i=1}^m y_i$$

in \mathcal{C} . (The unnamed equivalence is explicit: it is given by permutation of tensor factors according to the bijection $\sigma_\phi: \mathbf{n} \xrightarrow{\sim} \mathbf{n}$ obtained by factoring $\phi = \lambda_\phi \circ \sigma_\phi$, where λ_ϕ is monotone and σ_ϕ is a bijection monotone on fibres. For the present purposes, these permutations do not play any significant role.) In other words, there is an equivalence

$$\mathrm{Map}_{(\mathcal{C}_{(1)}^\otimes)_{\mathrm{act}}}((x_1, \dots, x_n), (y_1, \dots, y_m)) \simeq \prod_{\phi \in \mathrm{Map}_{\mathbb{F}}(\mathbf{n}, \mathbf{m})} \prod_{i=1}^m \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \phi^{-1}(i)} x_j, y_i \right),$$

and the map to $\mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j=1}^n x_j, \bigotimes_{i=1}^m y_i \right)$ is given by tensoring maps together (after appropriately permuting tensor factors). Condition (2) therefore corresponds precisely to ϵ being fully faithful. \square

Remark 2.4.8. Using the functor $\mathcal{C} \rightarrow \mathbb{F}$, the morphism in (2) fits into a commutative triangle

$$\begin{array}{ccc} \prod_{\phi \in \mathrm{Map}_{\mathbb{F}}(\mathbf{n}, \mathbf{m})} \prod_{i=1}^m \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \phi^{-1}(i)} x_j, y_i \right) & \longrightarrow & \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j=1}^n x_j, \bigotimes_{i=1}^m y_i \right) \\ & \searrow & \swarrow \\ & \mathrm{Map}_{\mathbb{F}}(\mathbf{n}, \mathbf{m}), & \end{array}$$

so that passing to fibres we can equivalently phrase (2) as:

(2') For every morphism $\phi: \mathbf{n} \rightarrow \mathbf{m}$ in \mathbb{F} , the map

$$\prod_{i=1}^m \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \phi^{-1}(i)} x_j, y_i \right) \longrightarrow \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{j=1}^n x_j, \bigotimes_{i=1}^m y_i \right)_{\phi},$$

given by tensoring morphisms, is an equivalence.

In particular, taking ϕ to be $\mathrm{id}_{\mathbf{n}}$ we have equivalences

$$\prod_{i=1}^n \mathrm{Map}_{\mathcal{C}_{(1)}}(x_i, y_i) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}} \left(\bigotimes_{i=1}^n x_i, \bigotimes_{i=1}^n y_i \right)_{\mathrm{id}_{\mathbf{n}}} \simeq \mathrm{Map}_{\mathcal{C}_{(n)}} \left(\bigotimes_{i=1}^n x_i, \bigotimes_{i=1}^n y_i \right),$$

where $\mathcal{C}_{(n)}$ is the fibre of $\mathcal{C} \rightarrow \mathbb{F}$ at \mathbf{n} . This says precisely that the functor $\mathcal{C}_{(1)}^{\times n} \rightarrow \mathcal{C}_{(n)}$ is fully faithful. On the other hand, condition (1) amounts to requiring the same functors to be essentially surjective. In the presence of condition (2) (or equivalently (2')) we can therefore replace (1) by

(1') For every n , the functor $\mathcal{C}_{(1)}^{\times n} \rightarrow \mathcal{C}_{(n)}$, induced by the tensor product, is an equivalence.

Alternatively, since the full faithfulness of this functor is also part of (2), we can replace (1) with

(1'') For every n , the map of spaces $(\mathcal{C}_{(1)}^{\simeq})^{\times n} \rightarrow \mathcal{C}_{(n)}^{\simeq}$, induced by the tensor product, is an equivalence.

Finally, note that we can reformulate (2') for all objects at once as:

(2'') For every morphism $\phi: \mathbf{n} \rightarrow \mathbf{m}$ in \mathbb{F} , the map

$$\prod_{i=1}^m \mathrm{Map}(\Delta^1, \mathcal{C})_{\mathbf{n}_i \rightarrow \mathbf{1}} \longrightarrow \mathrm{Map}(\Delta^1, \mathcal{C})_{\phi},$$

given by tensoring morphisms, is an equivalence.

This is equivalent to (2') since we have a commutative square

$$\begin{array}{ccc} \prod_{i=1}^m \text{Map}(\Delta^1, \mathcal{C})_{\mathbf{n}_i \rightarrow \mathbf{1}} & \longrightarrow & \text{Map}(\Delta^1, \mathcal{C})_{\phi} \\ \downarrow & & \downarrow \\ \prod_{i=1}^m \mathcal{C}_{(n_i)}^{\simeq} \times \mathcal{C}_{(1)}^{\simeq} & \xrightarrow{\sim} & \mathcal{C}_{(n)}^{\simeq} \times \mathcal{C}_{(m)}^{\simeq}, \end{array}$$

where the bottom horizontal map is an equivalence by (1''), and the maps on fibres are those in (2').

Now we turn to the second description:

Definition 2.4.9. The forgetful functor $\text{CMon}(\mathcal{S}) \rightarrow \mathcal{S}$ has a left adjoint $\text{Sym}: \mathcal{S} \rightarrow \text{CMon}(\mathcal{S})$. Since the underlying ∞ -groupoid functor $(-)^{\simeq}: \text{Cat}_{\infty} \rightarrow \mathcal{S}$ preserves products, it induces a functor $\text{CMon}(\text{Cat}_{\infty}) \rightarrow \text{CMon}(\mathcal{S})$, and we define PROP_{∞} as the pullback

$$\begin{array}{ccc} \text{PROP}_{\infty} & \longrightarrow & \text{CMon}(\text{Cat}_{\infty}) \\ \downarrow & & \downarrow (-)^{\simeq} \\ \mathcal{S} & \xrightarrow{\text{Sym}} & \text{CMon}(\mathcal{S}). \end{array}$$

An object of PROP_{∞} is thus a symmetric monoidal ∞ -category \mathcal{C} together with an ∞ -groupoid X and an equivalence of symmetric monoidal ∞ -groupoids $\text{Sym } X \simeq \mathcal{C}^{\simeq}$.

Remark 2.4.10. As the name suggests, we think of the objects of PROP_{∞} as a good ∞ -categorical analogue of the classical notion of PROPs, but we will not try to justify this here. Note, however, that PROPs are usually defined to be symmetric monoidal categories whose underlying *set* of objects is a free commutative monoid, while our definition corresponds for ordinary categories to having a free underlying symmetric monoidal *groupoid*. This condition does have the advantage of being invariant under equivalence, whereas with the more traditional definition *every* symmetric monoidal category is equivalent to a PROP (since every commutative monoid in sets admits a surjective map from a free one). On the other hand, this probably means that our ∞ -category PROP_{∞} does not correspond to the Quillen model structure on simplicial PROPs of Hackney and Robertson [11].

Proposition 2.4.11. *For any ∞ -operad \mathcal{O} , the functor $i_{\mathcal{O}}: \mathcal{O} \rightarrow \text{Env}(\mathcal{O})$ restricts to a morphism of ∞ -groupoids $\mathcal{O}_{(1)}^{\simeq} \rightarrow \text{Env}(\mathcal{O})_{(1)}^{\simeq}$ that is adjoint to an equivalence of commutative monoids*

$$\text{Sym}(\mathcal{O}_{(1)}^{\simeq}) \xrightarrow{\simeq} \text{Env}(\mathcal{O})_{(1)}^{\simeq}.$$

Proof: Consider the subcategory \mathcal{O}^{int} of \mathcal{O} containing only the inert morphisms. This is also an ∞ -operad, and for any ∞ -operad \mathcal{P} we have equivalences

$$\text{Map}_{\text{Opd}_{\infty}}(\mathcal{O}^{\text{int}}, \mathcal{P}) \simeq \text{Map}_{\text{Cat}_{\infty/\mathbb{F}_*^{\text{int}}}}(\mathcal{O}^{\text{int}}, \mathcal{P} \times_{\mathbb{F}_*} \mathbb{F}_*^{\text{int}}) \simeq \text{Map}_{\text{Cat}_{\infty}}(\mathcal{O}_{(1)}^{\simeq}, \mathcal{P}_{(1)}),$$

where the first equivalence is obtained by pulling back along $\mathbb{F}_*^{\text{int}} \rightarrow \mathbb{F}_*$ and the second holds because \mathcal{O}^{int} and $\mathcal{P} \times_{\mathbb{F}_*} \mathbb{F}_*^{\text{int}}$ are the cocartesian fibrations over $\mathbb{F}_*^{\text{int}}$ for the right Kan extensions of $\mathcal{O}_{(1)}^{\simeq}$ and $\mathcal{P}_{(1)}$ along the inclusion $\{1\} \hookrightarrow \mathbb{F}_*$, respectively.

It follows that for a symmetric monoidal ∞ -category \mathcal{C}^\otimes we have a natural equivalence

$$\mathrm{Map}_{\mathrm{SMCat}_\infty}(\mathrm{Env}(\mathcal{O}^{\mathrm{int}}), \mathcal{C}^\otimes) \simeq \mathrm{Map}(\mathcal{O}_{\langle 1 \rangle}^\simeq, \mathcal{C}).$$

Thus $\mathrm{Env}(\mathcal{O}^{\mathrm{int}})$ has the universal property of the free symmetric monoidal ∞ -category on $\mathcal{O}_{\langle 1 \rangle}^\simeq$ (which is also the free symmetric monoidal ∞ -groupoid).

On the other hand, from the construction of Env we see that the symmetric monoidal functor $\mathrm{Env}(\mathcal{O}^{\mathrm{int}}) \rightarrow \mathrm{Env}(\mathcal{O})$ induced by the inclusion of $\mathcal{O}^{\mathrm{int}}$ is an equivalence on underlying symmetric monoidal ∞ -groupoids. This shows that the inclusion of $\mathcal{O}_{\langle 1 \rangle}^\simeq$ exhibits the underlying symmetric monoidal ∞ -groupoid of $\mathrm{Env}(\mathcal{O})$ as free, which is what we wanted to prove. \square

Corollary 2.4.12. *The functor $\mathrm{Env}: \mathrm{Opd}_\infty \rightarrow \mathrm{SMCat}_\infty \simeq \mathrm{CMon}(\mathrm{Cat}_\infty)$ fits into a commutative square*

$$\begin{array}{ccc} \mathrm{Opd}_\infty & \xrightarrow{\mathrm{Env}} & \mathrm{CMon}(\mathrm{Cat}_\infty) \\ (-)_{\langle 1 \rangle}^\simeq \downarrow & & \downarrow (-)^\simeq \\ \mathcal{S} & \xrightarrow{\mathrm{Sym}} & \mathrm{CMon}(\mathcal{S}), \end{array}$$

and so the functor Env factors uniquely through a functor $\mathrm{Env}'': \mathrm{Opd}_\infty \rightarrow \mathrm{PROP}_\infty$ over \mathcal{S} .

Lemma 2.4.13. *The functor $\mathrm{Env}'': \mathrm{Opd}_\infty \rightarrow \mathrm{PROP}_\infty$ has a right adjoint U'' , which takes $(\mathcal{C}^\otimes, \mathrm{Sym}(X) \simeq \mathcal{C}^\simeq)$ to the full suboperad of \mathcal{C}^\otimes on the objects in the subspace $X \subseteq \mathrm{Sym}(X) \simeq \mathcal{C}^\simeq$.*

Proof: Given $\mathcal{O} \in \mathrm{Opd}_\infty$ and $(\mathcal{C}^\otimes, \alpha: \mathrm{Sym}(X) \simeq \mathcal{C}^\simeq) \in \mathrm{PROP}_\infty$ we have a natural pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{PROP}_\infty}(\mathrm{Env}''(\mathcal{O}), (\mathcal{C}^\otimes, \alpha)) & \longrightarrow & \mathrm{Map}_{\mathrm{SMCat}_\infty}(\mathrm{Env}(\mathcal{O}), \mathcal{C}^\otimes) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}}(\mathcal{O}_{\langle 1 \rangle}^\simeq, X) & \longrightarrow & \mathrm{Map}_{\mathrm{CMon}(\mathcal{S})}(\mathrm{Env}(\mathcal{O})_{\langle 1 \rangle}^\simeq, \mathcal{C}^\simeq). \end{array}$$

We can rewrite the right-hand part of this square using the adjunction $\mathrm{Env} \dashv U$ as well as the free-forgetful adjunction for commutative monoids as

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{PROP}_\infty}(\mathrm{Env}''(\mathcal{O}), (\mathcal{C}^\otimes, \alpha)) & \longrightarrow & \mathrm{Map}_{\mathrm{Opd}_\infty}(\mathcal{O}, U\mathcal{C}^\otimes) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}}(\mathcal{O}_{\langle 1 \rangle}^\simeq, X) & \longleftarrow & \mathrm{Map}_{\mathcal{S}}(\mathcal{O}_{\langle 1 \rangle}^\simeq, \mathcal{C}^\simeq), \end{array}$$

where the bottom horizontal map is now the inclusion of those maps that factor through $X \hookrightarrow \mathrm{Sym}(X) \simeq \mathcal{C}^\simeq$. The pullback is then precisely the space of ∞ -operad maps $\mathcal{O} \rightarrow U\mathcal{C}^\otimes$ that factor through the full suboperad $U''\mathcal{C}^\otimes$ on the objects in X , so that we have a natural equivalence

$$\mathrm{Map}_{\mathrm{PROP}_\infty}(\mathrm{Env}''(\mathcal{O}), (\mathcal{C}^\otimes, \alpha)) \simeq \mathrm{Map}_{\mathrm{Opd}_\infty}(\mathcal{O}, U''(\mathcal{C}^\otimes, \alpha)),$$

as required. \square

Proposition 2.4.14. *The functor $\mathrm{Env}'': \mathrm{Opd}_\infty \rightarrow \mathrm{PROP}_\infty$ is fully faithful.*

Proof: It suffices to show that the unit transformation $\text{id} \rightarrow U'' \text{Env}''$ is an equivalence, which follows from the pullback square (2.1) in Lemma 2.3.3, since this exhibits \mathcal{O} as the full suboperad of $\text{Env}''(\mathcal{O})$ spanned by objects that lie over $\mathbf{1}$ in \mathbb{F} , which are precisely the objects that lie in $\mathcal{O}_{\langle \mathbf{1} \rangle}^{\simeq}$ under the equivalence $\text{Sym } \mathcal{O}_{\langle \mathbf{1} \rangle}^{\simeq} \simeq \text{Env}''(\mathcal{O})_{\langle \mathbf{1} \rangle}^{\simeq}$. \square

Corollary 2.4.15. *The category Opd_{∞} is equivalent to the full subcategory of PROP_{∞} consisting of pairs $(\mathcal{C}^{\otimes}, \alpha : \text{Sym}(X) \simeq \mathcal{C}^{\simeq})$ such that the counit map*

$$(2.4) \quad \text{Env}(U''(\mathcal{C}^{\otimes}, \alpha)) \longrightarrow \mathcal{C}^{\otimes}$$

is an equivalence.

We can also give an explicit description of this subcategory:

Proposition 2.4.16. *For $(\mathcal{C}^{\otimes}, \alpha : \text{Sym}(X) \simeq \mathcal{C}^{\simeq})$ in PROP_{∞} , the counit map (2.4) is an equivalence if and only if the following “hereditary” condition holds:*

(*) *Given objects x_1, \dots, x_n and y_1, \dots, y_m in $X \subseteq \mathcal{C}^{\simeq}$, the morphism*

$$\coprod_{\phi \in \text{Map}_{\mathbb{F}}(\mathbf{n}, \mathbf{m})} \prod_{i=1}^m \text{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \phi^{-1}(i)} x_j, y_i \right) \longrightarrow \text{Map}_{\mathcal{C}} \left(\bigotimes_{j=1}^n x_j, \bigotimes_{i=1}^m y_i \right),$$

given by tensoring maps together, is an equivalence.

Remark 2.4.17. The PROP structure together with the hereditary condition can be seen as an ∞ -categorical version of what Kaufmann and Ward call *Feynman categories* ([17, Definition 1.1]; see also [4, 3.2] for a version closer to ours). Corollary 2.4.15 and Proposition 2.4.16 together are then an ∞ -categorical version of the equivalence established in [4, 5.16] between Feynman categories and operads.

Proof of Proposition 2.4.16: A morphism in PROP_{∞} is an equivalence if and only if it projects to an equivalence in both \mathcal{S} and SMCat_{∞} . By construction the counit maps to an equivalence in \mathcal{S} , and the forgetful functor from SMCat_{∞} to Cat_{∞} is conservative, so it suffices to show that the given condition is equivalent to the functor

$$\epsilon : \text{Env}(U''\mathcal{C})_{\langle \mathbf{1} \rangle} \longrightarrow \mathcal{C}$$

being an equivalence.

From Proposition 2.4.11 we see that the underlying map of symmetric monoidal ∞ -groupoids of ϵ is an equivalence, so that ϵ is in particular essentially surjective. This means we only need to show that the given condition is equivalent to ϵ being fully faithful. That in turn follows from identifying the mapping spaces in $\text{Env}(U''\mathcal{C})_{\langle \mathbf{1} \rangle} \simeq (U''\mathcal{C})^{\text{act}}$ as in the proof of Proposition 2.4.6. \square

3. From symmetric monoidal ∞ -categories to presheaves

Our goal in this section is to use the description of ∞ -operads as a full subcategory of $\text{SMCat}_{\infty/\mathbb{F}\Pi}$ from Proposition 2.4.6 to give a presentation of ∞ -operads as a localization of a presheaf ∞ -category. We first recall the description of ∞ -categories as complete Segal spaces, and the more general notion of Segal \mathcal{O} -spaces over an algebraic pattern \mathcal{O} , in Subsection 3.1. Then we prove in Subsection 3.2 that overcategories in Segal \mathcal{O} -spaces can be described as Segal spaces for another algebraic pattern. We apply this to symmetric monoidal ∞ -categories in Subsection 3.3, which in particular gives a presentation of $\text{SMCat}_{\infty/\mathbb{F}\Pi}$, and then finally apply this to describe ∞ -operads in Subsection 3.4.

3.1. Segal spaces. Here we briefly recall Rezk’s definition of ∞-categories as complete Segal spaces. As we will consider several similar structures, it is convenient to do so using some terminology from [9]:

Definition 3.1.1. An *algebraic pattern* is an ∞-category \mathcal{O} equipped with a factorization system (whereby every morphism factors as an *inert* morphism followed by an *active* morphism) and a collection of *elementary objects*. We write \mathcal{O}^{int} and \mathcal{O}^{act} for the subcategories containing only the inert and active maps, respectively, and $\mathcal{O}^{\text{el}} \subseteq \mathcal{O}^{\text{int}}$ for the full subcategory of elementary objects and inert maps between them.

The purpose of algebraic patterns is to be an abstract general setting for Segal conditions, as we proceed to explain. A basic example is the category Δ^{op} (as explained in Example 3.1.8 below).

Notation 3.1.2. If \mathcal{O} is an algebraic pattern, then for $X \in \mathcal{O}$ we write

$$\mathcal{O}_{X/}^{\text{el}} := \mathcal{O}^{\text{el}} \times_{\mathcal{O}^{\text{int}}} \mathcal{O}_{X/}^{\text{int}}$$

for the ∞-category of inert maps from X to elementary objects.

Definition 3.1.3. Let \mathcal{O} be an algebraic pattern and \mathcal{C} an ∞-category with limits of shape $\mathcal{O}_{X/}^{\text{el}}$ for all $X \in \mathcal{O}$. Then a *Segal \mathcal{O} -object* in \mathcal{C} is a functor $F: \mathcal{O} \rightarrow \mathcal{C}$ such that for all $X \in \mathcal{O}$ the natural map

$$F(X) \longrightarrow \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} F(E)$$

is an equivalence. We call a Segal \mathcal{O} -object in the ∞-category \mathcal{S} a *Segal \mathcal{O} -space*.

Remark 3.1.4. Equivalently, a Segal \mathcal{O} -object is a functor $F: \mathcal{O} \rightarrow \mathcal{C}$ such that the restriction $F|_{\mathcal{O}^{\text{int}}}$ is a right Kan extension of $F|_{\mathcal{O}^{\text{el}}}$.

Notation 3.1.5. If \mathcal{O} is an algebraic pattern, we write $\text{Seg}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the Segal \mathcal{O} -objects.

Example 3.1.6. We consider the category \mathbb{F}_* as an algebraic pattern using the factorization system of Definition 2.1.2 and with $\langle 1 \rangle$ as the unique elementary object. Then a Segal \mathbb{F}_* -object in an ∞-category \mathcal{C} is precisely a commutative monoid in the sense of Definition 2.1.5.

Notation 3.1.7. We write Δ for the simplex category, i.e. the category of ordered sets $[n] := \{0 < 1 < \dots < n\}$ ($n = 0, 1, \dots$) and order-preserving maps between them. A morphism $\phi: [n] \rightarrow [m]$ in Δ is called *inert* if it is a subinterval inclusion, i.e. $\phi(i) = \phi(0) + i$ for all i , and *active* if it preserves the end points, i.e. $\phi(0) = 0$ and $\phi(n) = m$. The active and inert morphisms form a factorization system on Δ . For $0 \leq i \leq j \leq n$, we write $\iota_{ij}: [j-i] \hookrightarrow [n]$ for the inert map in Δ given by $\iota_{ij}(t) = i+t$, i.e. the inclusion of $\{i, i+1, \dots, j\}$.

Example 3.1.8. We view Δ^{op} as an algebraic pattern using this inert-active factorization system, with $[0]$ and $[1]$ as the elementary objects. A Segal Δ^{op} -object in an ∞-category \mathcal{C} is then a simplicial object $F: \Delta^{\text{op}} \rightarrow \mathcal{C}$ such that the natural map

$$F([n]) \longrightarrow F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1])$$

determined by the inert maps $[0], [1] \hookrightarrow [n]$ in Δ (i.e. the maps ι_{ii} and $\iota_{i(i+1)}$) is an equivalence. In particular, a Segal Δ^{op} -space is precisely a *Segal space* in the sense of Rezk [26].

Notation 3.1.9. Let $E^1 \in \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})$ denote the nerve of the generic equivalence, i.e. the category with two objects and a unique morphism between any pair of objects. (Equivalently, this is the simplicial set with n -simplices $E_n^1 = \{0, 1\}^n$.)

Definition 3.1.10. For $X \in \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})$, an *equivalence* in X is a morphism $E^1 \rightarrow X$. We write $X^{\text{eq}} := \text{Map}_{\text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})}(E^1, X)$ for the space of equivalences in X . The Segal space X is *complete* if the map $X_0 \rightarrow X^{\text{eq}}$ given by composition with $E^1 \rightarrow \Delta^0$ is an equivalence. (In other words, X is complete if it is local with respect to this morphism.) We write $\text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S}) \subseteq \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})$ for the full subcategory spanned by the complete Segal spaces.

Theorem 3.1.11 (Joyal–Tierney [16]). *The restricted Yoneda embedding*

$$\text{Cat}_{\infty} \longrightarrow \text{Fun}(\Delta^{\text{op}}, \mathcal{S})$$

along the functor $\Delta \rightarrow \text{Cat}_{\infty}$ given by viewing the partially ordered sets $[n]$ as (∞) -categories induces an equivalence

$$\text{Cat}_{\infty} \xrightarrow{\sim} \text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S}).$$

3.2. Slices via Segal conditions. In this subsection we prove that if B is a Segal \mathcal{O} -space, then we can describe the overcategory $\text{Seg}_{\mathcal{O}}(\mathcal{S})/B$ as the ∞ -category of Segal \mathcal{B} -spaces, where $\mathcal{B} \rightarrow \mathcal{O}$ is the left fibration corresponding to B .

The starting point is the following observation:

Proposition 3.2.1 ([10, Corollary 9.8]). *Let \mathcal{B} be an ∞ -category and let $\pi: \mathcal{E} \rightarrow \mathcal{B}$ be a left fibration. Then the functor*

$$\pi_!: \text{Fun}(\mathcal{E}, \mathcal{S}) \longrightarrow \text{Fun}(\mathcal{B}, \mathcal{S}),$$

given by left Kan extension along π , induces an equivalence

$$(3.1) \quad \text{Fun}(\mathcal{E}, \mathcal{S}) \xrightarrow{\sim} \text{Fun}(\mathcal{B}, \mathcal{S})/E,$$

where the value of $\pi_!$ at the terminal object is the functor $E: \mathcal{B} \rightarrow \mathcal{S}$ corresponding to the left fibration π .

Remark 3.2.2. Under the straightening equivalence between $\text{Fun}(\mathcal{B}, \mathcal{S})$ and $\text{Cat}_{\infty}^{\text{L}}/\mathcal{B}$, the functor $\pi_!$ is given by composition with π , and the equivalence (3.1) boils down to the observation that if we have a commutative triangle

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{E} \\ & \searrow & \swarrow \pi \\ & \mathcal{B} & \end{array}$$

then f is a left fibration if and only if πf is a left fibration.

Definition 3.2.3. Let \mathcal{O} be an algebraic pattern, and suppose $\pi: \mathcal{B} \rightarrow \mathcal{O}$ is a left fibration. Then \mathcal{B} inherits a factorization system where the inert and active morphisms are simply those that lie over inert and active morphisms in \mathcal{O} . If the functor $B: \mathcal{O} \rightarrow \mathcal{S}$ corresponding to π is a Segal \mathcal{O} -space, we view \mathcal{B} as an algebraic pattern via this factorization system and with all objects that lie over elementary objects in \mathcal{O} as its elementary objects.

Remark 3.2.4. Suppose \mathcal{O} is an algebraic pattern and $\mathcal{B} \rightarrow \mathcal{O}$ is the left fibration corresponding to a Segal \mathcal{O} -space. Then for every object \bar{X} in \mathcal{B} lying over X in \mathcal{O} the functor π induces an equivalence

$$\mathcal{B}_{\bar{X}/}^{\text{el}} \xrightarrow{\sim} \mathcal{O}_{X/}^{\text{el}},$$

since there is a unique (cocartesian) morphism over every (inert) morphism $X \rightarrow E$ in \mathcal{O} . This means that a functor $F: \mathcal{B} \rightarrow \mathcal{S}$ is a Segal \mathcal{B} -space if and only if for every $\bar{X} \in \mathcal{B}$ lying over $X \in \mathcal{O}$ the natural map

$$F(\bar{X}) \longrightarrow \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} F(\bar{E})$$

is an equivalence, where $\bar{X} \rightarrow \bar{E}$ is the cocartesian morphism lying over $X \rightarrow E$.

Proposition 3.2.5. *Let \mathcal{O} be an algebraic pattern, and suppose $\pi: \mathcal{B} \rightarrow \mathcal{O}$ is a left fibration corresponding to a Segal \mathcal{O} -space B . Then the equivalence (3.1) from Proposition 3.2.1 restricts to an equivalence*

$$\pi_! : \text{Seg}_{\mathcal{B}}(\mathcal{S}) \xrightarrow{\sim} \text{Seg}_{\mathcal{O}}(\mathcal{S})_{/B}.$$

Proof: Since we know from Proposition 3.2.1 that $\pi_!$ gives an equivalence

$$\text{Fun}(\mathcal{B}, \mathcal{S}) \xrightarrow{\sim} \text{Fun}(\mathcal{O}, \mathcal{S})_{/B},$$

it suffices to show that the full subcategories of Segal objects are identified under this equivalence. For $F: \mathcal{B} \rightarrow \mathcal{S}$ and $X \in \mathcal{O}$, we have a commutative square

$$\begin{array}{ccc} \pi_! F(X) & \longrightarrow & \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} \pi_! F(E) \\ \downarrow & & \downarrow \\ B(X) & \xrightarrow{\sim} & \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} B(E). \end{array}$$

The functor $\pi_! F$ is a Segal \mathcal{O} -space if and only if the top horizontal morphism is an equivalence in every such square. Since B is a Segal \mathcal{O} -space, we know that the bottom horizontal morphism is an equivalence, and hence this condition is equivalent to all these squares being pullbacks. This in turn is equivalent to the map on fibres over every point of $B(X)$ being an equivalence for every $X \in \mathcal{O}$.

Since limits commute, we can identify the map on fibres over $p \in B(X)$ as

$$(3.2) \quad \pi_! F(X)_p \longrightarrow \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} \pi_! F(E)_{p_E},$$

where p_E is the image of p in $B(E)$ under the map corresponding to $X \rightarrow E$ in \mathcal{O} .

Since the functor π is a left fibration, the left Kan extension $\pi_!$ is computed fibre-wise, i.e.

$$\pi_! F(X) \simeq \text{colim}_{p \in B(X)} F(p).$$

For a space T , the straightening equivalence $\text{Fun}(T, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}_{/T}$ is given by taking colimits, with inverse given by taking fibres. Hence we have a natural identification of $\pi_! F(X)_p$ with $F(p)$, under which the map (3.2) corresponds to the Segal map

$$F(p) \longrightarrow \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} F(p_E).$$

As we saw in Remark 3.2.4, asking for this to be an equivalence for all $X \in \mathcal{O}$ and $p \in B(X)$ is precisely asking for F to be a Segal \mathcal{B} -space. \square

In the special case where \mathcal{O} is Δ^{op} , we can use Proposition 3.2.5 to get a description of the overcategory $\text{Cat}_{\infty/\mathcal{C}}$ in terms of complete Segal conditions; this description can also be found in [2] and [15].

Remark 3.2.6. Let $\pi: \mathcal{B} \rightarrow \Delta^{\text{op}}$ be a left fibration corresponding to a Segal space B . An object $b \in \mathcal{B}$ over $[n] \in \Delta^{\text{op}}$ corresponds to a morphism between left fibrations $i_b: \Delta^{\text{op}}_{/[n]} \rightarrow \mathcal{B}$ over Δ^{op} . For $b \in \mathcal{B}_0$, composition with i_b then restricts to a functor

$$i_b^*: \text{Seg}_{\mathcal{B}}(\mathcal{S}) \longrightarrow \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S}),$$

since through the equivalence of Proposition 3.2.5 the functor i_b^* corresponds to base change along i_b , which preserves the Segal condition since limits commute.

Definition 3.2.7. Let $\pi: \mathcal{B} \rightarrow \Delta^{\text{op}}$ be a left fibration corresponding to a Segal space B . We say a Segal \mathcal{B} -space F is *complete* if the Segal spaces i_b^*F are complete for all $b \in \mathcal{B}_0$, or equivalently the fibres $(\pi_! F)_b$ are all complete. We write $\text{CSeg}_{\mathcal{B}}(\mathcal{S}) \subseteq \text{Seg}_{\mathcal{B}}(\mathcal{S})$ for the full subcategory of complete Segal \mathcal{B} -spaces.

Proposition 3.2.8. *Let $\pi: \mathcal{B} \rightarrow \Delta^{\text{op}}$ be a left fibration corresponding to a simplicial space B . If B is a complete Segal space, then the functor $\pi_!$ restricts to an equivalence*

$$\text{CSeg}_{\mathcal{B}}(\mathcal{S}) \xrightarrow{\sim} \text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S})_{/B}.$$

Proof: Suppose X is a Segal space over B . We then have a commutative square

$$\begin{array}{ccc} X_0 & \longrightarrow & X^{\text{eq}} \\ \downarrow & & \downarrow \\ B_0 & \xrightarrow{\sim} & B^{\text{eq}}, \end{array}$$

where the bottom horizontal morphism is an equivalence since B is complete. The Segal space X is therefore complete if and only if this square of spaces is a pullback, which is equivalent to the map on fibres over each $b \in B_0$ being an equivalence. Thus X is complete if and only if for each $b \in B_0$ the map on fibres $X_{b,0} \rightarrow (X^{\text{eq}})_b$ is an equivalence. Since $\text{Map}(E^1, -)$ preserves limits, we can also identify the fibre $(X^{\text{eq}})_b$ with $(X_b)^{\text{eq}}$, so this condition says precisely that the Segal spaces X_b are complete for all $b \in B_0$. \square

Combining this observation with Theorem 3.1.11, we get:

Corollary 3.2.9. *For \mathcal{C} an ∞ -category, let*

$$\Delta_{/\mathcal{C}} := \Delta \times_{\text{Cat}_{\infty}} \text{Cat}_{\infty/\mathcal{C}} \longrightarrow \Delta$$

be the right fibration corresponding to \mathcal{C} viewed as a complete Segal space. Then the restricted Yoneda embedding along $\Delta_{/\mathcal{C}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$ induces an equivalence

$$\text{Cat}_{\infty/\mathcal{C}} \xrightarrow{\sim} \text{CSeg}_{\Delta_{/\mathcal{C}}}(\mathcal{S}).$$

3.3. Slices of symmetric monoidal ∞-categories. We now specialize the results of the previous section to describe the overcategories of SMCat_∞ in terms of Segal and completeness conditions. We first observe that SMCat_∞ itself admits such a description:

Definition 3.3.1. We view the product $\mathbb{F}_* \times \Delta^{\text{op}}$ as an algebraic pattern with the inert and active morphisms given by those that are inert and active in each coordinate, and with $(\langle 1 \rangle, [0])$ and $(\langle 1 \rangle, [1])$ as the elementary objects. We say that a Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -space F is *complete* if the Segal space $F(\langle 1 \rangle, -)$ is complete. We write $\text{CSeg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S}) \subseteq \text{Seg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S})$ for the full subcategory of complete Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -spaces.

Proposition 3.3.2. *The restricted Yoneda embedding along $\Delta \rightarrow \text{Cat}_\infty$ induces an equivalence*

$$\text{SMCat}_\infty \xrightarrow{\sim} \text{CSeg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S}).$$

Proof: It follows immediately from the definitions that the equivalence

$$\text{Fun}(\mathbb{F}_*, \text{Fun}(\Delta^{\text{op}}, \mathcal{S})) \simeq \text{Fun}(\mathbb{F}_* \times \Delta^{\text{op}}, \mathcal{S})$$

restricts to an equivalence

$$\text{Seg}_{\mathbb{F}_*}(\text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})) \simeq \text{Seg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S}).$$

Moreover, since complete Segal spaces are closed under limits, this restricts further to an equivalence

$$\text{Seg}_{\mathbb{F}_*}(\text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S})) \simeq \text{CSeg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S}).$$

In other words, complete Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -spaces are commutative monoids in $\text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S})$. Combining this with the equivalence of Theorem 3.1.11 now gives the result. \square

Remark 3.3.3. Let $\mathcal{M} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ be a left fibration corresponding to a Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -space M . Then a functor $F: \mathcal{M} \rightarrow \mathcal{S}$ is a Segal \mathcal{M} -space if and only if for $X \in \mathcal{M}$ lying over $(\langle k \rangle, [n])$ in $\mathbb{F}_* \times \Delta^{\text{op}}$ the natural map

$$F(X) \longrightarrow \prod_{i=1}^k F(X_{i,01}) \times_{F(X_{i,1})} \cdots \times_{F(X_{i,n-1})} F(X_{i,(n-1)n})$$

induced by the (cocartesian) maps $X \rightarrow X_{i,j}$ over $\rho_i \times \iota_{jj}$ and $X \rightarrow X_{i,(j-1)j}$ over $\rho_i \times \iota_{(j-1)j}$ is an equivalence. This condition can conveniently be split into three parts:

- (1) $F(X) \xrightarrow{\sim} F(X_{01}) \times_{F(X_1)} \cdots \times_{F(X_{n-1})} F(X_{(n-1)n})$, where X lies over $[n] \in \Delta^{\text{op}}$ and the maps $X \rightarrow X_j$ and $X \rightarrow X_{(j-1)j}$ are cocartesian over ι_{jj} and $\iota_{(j-1)j}$, respectively.
- (2) $F(X) \xrightarrow{\sim} \prod_{i=1}^k F(X_i)$, where X lies over $(\langle k \rangle, [1])$ and $X \rightarrow X_i$ is cocartesian over ρ_i .
- (3) $F(X) \xrightarrow{\sim} \prod_{i=1}^k F(X_i)$, where X lies over $(\langle k \rangle, [0])$ and $X \rightarrow X_i$ is cocartesian over ρ_i .

As a special case of Proposition 3.2.5 we have:

Corollary 3.3.4. *Let $\pi: \mathcal{M} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ be a left fibration corresponding to a Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -space M . Then the functor $\pi_!$ given by left Kan extension along π restricts to an equivalence*

$$\text{Seg}_{\mathcal{M}}(\mathcal{S}) \xrightarrow{\sim} \text{Seg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S})/M.$$

We now want to incorporate completeness into this description:

Definition 3.3.5. Let $\pi: \mathcal{M} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ be a left fibration corresponding to a Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -space M , and let $\mathcal{M}_{\langle 1 \rangle} \rightarrow \Delta^{\text{op}}$ be the fibre at $\langle 1 \rangle \in \mathbb{F}_*$, corresponding to the underlying Segal space $M(\langle 1 \rangle, -)$. Let $u_M: \mathcal{M}_{\langle 1 \rangle} \rightarrow \mathcal{M}$ denote the inclusion of this fibre; composition with u_M restricts to a functor $\text{Seg}_{\mathcal{M}}(\mathcal{S}) \rightarrow \text{Seg}_{\mathcal{M}_{\langle 1 \rangle}}(\mathcal{S})$. We say a Segal \mathcal{M} -space F is *complete* if $u_M^* F$ is complete in the sense of Definition 3.2.7, and write $\text{CSeg}_{\mathcal{M}}(\mathcal{S}) \subseteq \text{Seg}_{\mathcal{M}}(\mathcal{S})$ for the full subcategory spanned by the complete objects.

Proposition 3.3.6. *Let $\pi: \mathcal{M} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ be a left fibration corresponding to a Segal $(\mathbb{F}_* \times \Delta^{\text{op}})$ -space M . If M is complete, then the functor $\pi_!$ restricts to an equivalence*

$$\text{CSeg}_{\mathcal{M}}(\mathcal{S}) \xrightarrow{\sim} \text{CSeg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S})/M.$$

Proof: Since left Kan extensions along the left fibration π are given by taking colimits fibrewise, we have a commutative square

$$\begin{array}{ccc} \text{Seg}_{\mathcal{M}}(\mathcal{S}) & \xrightarrow{u_M^*} & \text{Seg}_{\mathcal{M}_{\langle 1 \rangle}}(\mathcal{S}) \\ \downarrow \pi_! & & \downarrow \pi_{\langle 1 \rangle, !} \\ \text{Seg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S})/M & \longrightarrow & \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})/M(\langle 1 \rangle, -), \end{array}$$

where the vertical maps are equivalences. Combining this observation with Proposition 3.2.8 now completes the proof, since $\text{CSeg}_{\mathcal{M}}(\mathcal{S})$ and $\text{CSeg}_{\mathbb{F}_* \times \Delta^{\text{op}}}(\mathcal{S})/M$ are defined as the preimages in this diagram of $\text{CSeg}_{\mathcal{M}_{\langle 1 \rangle}}(\mathcal{S})$ and $\text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S})/M(\langle 1 \rangle, -)$, respectively. \square

Corollary 3.3.7. *Suppose \mathcal{C}^{\otimes} is a symmetric monoidal ∞ -category, and let $\mathcal{M} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ be the left fibration corresponding to \mathcal{C}^{\otimes} viewed as a commutative monoid in complete Segal spaces. Then there is an equivalence of ∞ -categories*

$$\text{SMCat}_{\infty/\mathcal{C}^{\otimes}} \simeq \text{CSeg}_{\mathcal{M}}(\mathcal{S}).$$

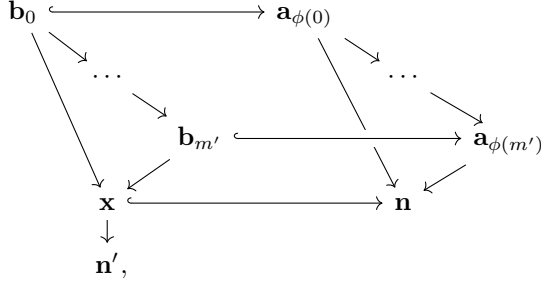
Proof: Combine Proposition 3.3.6 with Proposition 3.3.2. \square

3.4. Application to ∞ -operads. Our next goal is to combine the results of the previous subsection with those of Subsection 2.4 to obtain a new description of Opd_{∞} in terms of Segal and completeness conditions.

Definition 3.4.1. Let $\pi: \mathcal{F} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ be the left fibration corresponding to the symmetric monoidal category \mathbb{F}^{II} viewed as a commutative monoid in Segal spaces. Unwinding the definitions, the category \mathcal{F} has the following explicit description (where it is convenient to use the description of \mathbb{F}_* in terms of spans of finite sets): the objects of \mathcal{F} are sequences of maps in \mathbb{F}

$$\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow \cdots \longrightarrow & \mathbf{a}_m \\ & \searrow & \swarrow \\ & \mathbf{n}, & \end{array}$$

where this object lives over $(\mathbf{n}, [m])$ in $\mathbb{F}_* \times \Delta^{\text{op}}$. A (necessarily cocartesian) morphism over $(\mathbf{n} \leftarrow \mathbf{x} \rightarrow \mathbf{n}', [m'] \xrightarrow{\phi} [m])$ with this object as source is given by a commutative diagram



where the squares

$$\begin{array}{ccc}
 \mathbf{b}_i & \xleftarrow{\quad} & \mathbf{a}_{\phi(i)} \\
 \downarrow & & \downarrow \\
 \mathbf{x} & \xleftarrow{\quad} & \mathbf{n}
 \end{array}$$

are all pullback squares. (In other words, we restrict along ϕ , pull back along $\mathbf{x} \leftarrow \mathbf{n}$, and compose with the map $\mathbf{x} \rightarrow \mathbf{n}'$.)

Remark 3.4.2. With this description of \mathcal{F} , the requirements for a functor $\Phi: \mathcal{F} \rightarrow \mathcal{S}$ to be a Segal \mathcal{F} -space from Remark 3.3.3 amount to the following maps being equivalences:

$$\begin{aligned}
 (3.3) \quad \Phi \left(\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow & \mathbf{a}_m \\ & \searrow & \swarrow \\ & \mathbf{n} & \end{array} \right) &\longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow & \mathbf{a}_1 \\ & \searrow & \swarrow \\ & \mathbf{n} & \end{array} \right) \times_{\Phi \left(\begin{array}{c} \mathbf{a}_1 \\ \downarrow \\ \mathbf{n} \end{array} \right)} \cdots \\
 &\times_{\Phi \left(\begin{array}{c} \mathbf{a}_{m-1} \\ \downarrow \\ \mathbf{n} \end{array} \right)} \Phi \left(\begin{array}{ccc} \mathbf{a}_{m-1} & \longrightarrow & \mathbf{a}_m \\ & \searrow & \swarrow \\ & \mathbf{n} & \end{array} \right),
 \end{aligned}$$

$$(3.4) \quad \Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \swarrow \\ & \mathbf{n} & \end{array} \right) \longrightarrow \prod_{i=1}^n \Phi \left(\begin{array}{ccc} \mathbf{a}_i & \longrightarrow & \mathbf{b}_i \\ & \searrow & \swarrow \\ & \mathbf{1} & \end{array} \right),$$

$$(3.5) \quad \Phi \left(\begin{array}{c} \mathbf{a} \\ \downarrow \\ \mathbf{n} \end{array} \right) \longrightarrow \prod_{i=1}^n \Phi \left(\begin{array}{c} \mathbf{a}_i \\ \downarrow \\ \mathbf{1} \end{array} \right).$$

From Corollary 3.3.7 we then get the following:

Corollary 3.4.3. *There is an equivalence*

$$\text{SMCat}_{\infty/\mathbb{F}\amalg} \xrightarrow{\sim} \text{CSeg}_{\mathcal{F}}(\mathcal{S}),$$

where the right-hand side is the full subcategory of $\text{Fun}(\mathcal{F}, \mathcal{S})$ spanned by functors Φ satisfying conditions (3.3) to (3.5) and for which the Segal space $\Phi_{(1), \mathbf{a}}$ is complete for every $\mathbf{a} \in \mathbb{F}$.

Definition 3.4.4. We write $\text{Fun}'(\mathcal{F}, \mathcal{S})$ for the full subcategory of $\text{Fun}(\mathcal{F}, \mathcal{S})$ spanned by functors Φ such that for every object

$$\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow & \cdots & \longrightarrow & \mathbf{a}_m \\ & \searrow & & \swarrow & \\ & & \mathbf{n} & & \end{array}$$

in \mathcal{F} and every map $\mathbf{n} \rightarrow \mathbf{n}'$ in \mathbb{F} , the map

$$(3.6) \quad \Phi \left(\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow & \cdots & \longrightarrow & \mathbf{a}_m \\ & \searrow & & \swarrow & \\ & & \mathbf{n} & & \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow & \cdots & \longrightarrow & \mathbf{a}_m \\ & \searrow & & \swarrow & \\ & & \mathbf{n}' & & \end{array} \right)$$

lying over $\mathbf{n} \xleftarrow{=} \mathbf{n} \rightarrow \mathbf{n}'$ is an equivalence. We then write $\text{Seg}'_{\mathcal{F}}(\mathcal{S})$ and $\text{CSeg}'_{\mathcal{F}}(\mathcal{S})$ for the intersections of $\text{Fun}'(\mathcal{F}, \mathcal{S})$ with the full subcategories $\text{Seg}_{\mathcal{F}}(\mathcal{S})$ and $\text{CSeg}_{\mathcal{F}}(\mathcal{S})$ in $\text{Fun}(\mathcal{F}, \mathcal{S})$, respectively.

Our goal is now to prove the following:

Theorem 3.4.5. *The equivalence of Corollary 3.4.3 restricts along the fully faithful inclusion $\text{Env}' : \text{Opd}_{\infty} \hookrightarrow \text{SMCat}_{\infty/\mathbb{F}\mathbb{U}}$ from Proposition 2.4.3 to an equivalence*

$$\text{Opd}_{\infty} \xrightarrow{\sim} \text{CSeg}'_{\mathcal{F}}(\mathcal{S}).$$

We begin by simplifying the definition of $\text{Seg}'_{\mathcal{F}}(\mathcal{S})$ a bit:

Lemma 3.4.6. *Suppose Φ is in $\text{Seg}_{\mathcal{F}}(\mathcal{S})$. Then Φ lies in $\text{Seg}'_{\mathcal{F}}(\mathcal{S})$ if and only if the two following conditions hold:*

(1) *For every object \mathbf{n} in \mathbb{F} , the map*

$$\Phi \left(\begin{array}{c} \mathbf{n} \\ \parallel \\ \mathbf{n} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{c} \mathbf{n} \\ \downarrow \\ \mathbf{1} \end{array} \right)$$

over $\mathbf{n} \xleftarrow{=} \mathbf{n} \rightarrow \mathbf{1}$ is an equivalence.

(2) *For every morphism $\mathbf{n} \rightarrow \mathbf{m}$ in \mathbb{F} , the map*

$$\Phi \left(\begin{array}{ccc} \mathbf{n} & \longrightarrow & \mathbf{m} \\ & \searrow & \swarrow \\ & & \mathbf{m} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{n} & \longrightarrow & \mathbf{m} \\ & \searrow & \swarrow \\ & & \mathbf{1} \end{array} \right)$$

over $\mathbf{m} \xleftarrow{=} \mathbf{m} \rightarrow \mathbf{1}$ is an equivalence.

Proof: Clearly (1) and (2) are special cases of (3.6), so we need to prove that these special cases suffice. We first observe that condition (3.3) for $\text{Seg}_{\mathcal{F}}(\mathcal{S})$ implies that Φ lies in $\text{Seg}'_{\mathcal{F}}(\mathcal{S})$ if and only if condition (3.6) holds for $m = 0$ and $m = 1$. Now we claim that (1) and (2) are equivalent to these two cases, respectively; we will prove the case where $m = 1$, the proof for $m = 0$ being similar.

For any morphism $\mathbf{n} \rightarrow \mathbf{n}'$ in \mathbb{F} , consider the maps

$$\Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \swarrow \\ & & \mathbf{n} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \swarrow \\ & & \mathbf{n}' \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \swarrow \\ & & \mathbf{1} \end{array} \right),$$

where the first map lies over $\mathbf{n} \xleftarrow{=} \mathbf{n} \rightarrow \mathbf{n}'$ and the second lies over $\mathbf{n}' \xleftarrow{=} \mathbf{n}' \rightarrow \mathbf{1}$. Then the composite lies over $\mathbf{n} \xleftarrow{=} \mathbf{n} \rightarrow \mathbf{1}$, so that by the 2-of-3 property of equivalences, condition (3.6) holds for all maps if and only if it holds for maps of the form $\mathbf{n} \rightarrow \mathbf{1}$.

Next, consider for any map $\mathbf{b} \rightarrow \mathbf{n}$ in \mathbb{F} the maps

$$\Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \parallel \\ & & \mathbf{b} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \swarrow \\ & & \mathbf{n} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{a} & \longrightarrow & \mathbf{b} \\ & \searrow & \swarrow \\ & & \mathbf{1} \end{array} \right).$$

Here assumption (2) implies that the composite map is an equivalence, so that the second map is an equivalence if and only if the first one is. But the first map decomposes using the Segal conditions as

$$\prod_{i=1}^n \Phi \left(\begin{array}{ccc} \mathbf{a}_i & \longrightarrow & \mathbf{b}_i \\ & \searrow & \parallel \\ & & \mathbf{b}_i \end{array} \right) \longrightarrow \prod_{i=1}^n \Phi \left(\begin{array}{ccc} \mathbf{a}_i & \longrightarrow & \mathbf{b}_i \\ & \searrow & \swarrow \\ & & \mathbf{1} \end{array} \right),$$

which is also an equivalence under assumption (2). Thus the Segal conditions and (2) imply that (3.6) holds in the case $m = 1$, as required. \square

Proof of Theorem 3.4.5: We must show that under the equivalence of Corollary 3.4.3 the two conditions from Proposition 2.4.6 correspond precisely to (3.6). Equivalently, we can check that the alternative conditions (1'') and (2'') from Remark 2.4.8 correspond to those from Lemma 3.4.6. To this end, let $\mathcal{C}^\otimes \rightarrow \mathbb{F}^\Pi$ be an object of $\text{SMCat}_{\infty/\mathbb{F}^\Pi}$ and Φ the corresponding object in $\text{CSeg}_{\mathcal{F}}(\mathcal{S})$. Unwinding the definitions, we have equivalences

$$\Phi \left(\begin{array}{c} \mathbf{n} \\ \downarrow \\ \mathbf{1} \end{array} \right) \simeq \mathcal{C}_{(n)}^\simeq, \quad \Phi \left(\begin{array}{ccc} \mathbf{n} & \xrightarrow{\phi} & \mathbf{m} \\ & \searrow & \swarrow \\ & & \mathbf{1} \end{array} \right) \simeq \text{Map}(\Delta^1, \mathcal{C})_\phi,$$

under which the tensoring maps

$$(\mathcal{C}_{(1)}^\simeq)^{\times n} \longrightarrow \mathcal{C}_{(n)}^\simeq, \quad \prod_{i=1}^m \text{Map}(\Delta^1, \mathcal{C})_{\mathbf{n}_i \rightarrow \mathbf{1}} \longrightarrow \text{Map}(\Delta^1, \mathcal{C})_\phi$$

from conditions (1'') and (2'') correspond to

$$\prod_{i=1}^n \Phi \left(\begin{array}{c} \mathbf{1} \\ \parallel \\ \mathbf{1} \end{array} \right) \xleftarrow{\simeq} \Phi \left(\begin{array}{c} \mathbf{n} \\ \parallel \\ \mathbf{n} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{c} \mathbf{n} \\ \downarrow \\ \mathbf{1} \end{array} \right)$$

and

$$\prod_{i=1}^n \Phi \left(\begin{array}{ccc} \mathbf{n}_i & \xrightarrow{\phi} & \mathbf{1} \\ & \searrow & \parallel \\ & & \mathbf{1} \end{array} \right) \xleftarrow{\simeq} \Phi \left(\begin{array}{ccc} \mathbf{n} & \xrightarrow{\phi} & \mathbf{m} \\ & \searrow & \parallel \\ & & \mathbf{m} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{ccc} \mathbf{n} & \xrightarrow{\phi} & \mathbf{m} \\ & \searrow & \swarrow \\ & & \mathbf{1} \end{array} \right),$$

respectively. Here we have exactly the same maps as in Lemma 3.4.6, so that the conditions there precisely correspond to those of Remark 2.4.8, as required. \square

Theorem 3.4.5 has the following immediate corollary:

Corollary 3.4.7. *The ∞ -category Opd_∞ is presentable.*

The following observation will be useful later:

Lemma 3.4.8. *An object $\Phi \in \text{Seg}'_{\mathcal{F}}(\mathcal{S})$ lies in $\text{CSeg}'_{\mathcal{F}}(\mathcal{S})$ if and only if the Segal space $\Phi_{(1), \mathbf{1}}$ is complete.*

Proof: For $\mathbf{n} \in \mathbb{F}$ we have a natural zig-zag of simplicial spaces

$$\prod_{i=1}^n \Phi \left(\begin{array}{c} \mathbf{1} = \cdots = \mathbf{1} \\ \swarrow \quad \searrow \\ \mathbf{1} \end{array} \right) \longleftarrow \Phi \left(\begin{array}{c} \mathbf{n} = \cdots = \mathbf{n} \\ \swarrow \quad \searrow \\ \mathbf{n} \end{array} \right) \longrightarrow \Phi \left(\begin{array}{c} \mathbf{n} = \cdots = \mathbf{n} \\ \swarrow \quad \searrow \\ \mathbf{1} \end{array} \right),$$

where both maps are equivalences for Φ in $\text{Seg}'_{\mathcal{F}}(\mathcal{S})$. Thus we have an equivalence between the Segal spaces $\Phi_{(1),1}^{\times n}$ and $\Phi_{(1),\mathbf{n}}$. Since complete Segal spaces are closed under limits, this implies that if $\Phi_{(1),1}$ is complete, then so is $\Phi_{(1),\mathbf{n}}$, and hence by definition Φ lies in $\text{CSeg}'_{\mathcal{F}}(\mathcal{S})$. \square

4. From Barwick's ∞ -operads to symmetric monoidal ∞ -categories

In this section we first review Barwick's model of ∞ -operads in Subsection 4.1. Then in Subsection 4.2 we use our work in the previous section to give a new proof of the equivalence between Barwick's and Lurie's models by passing through the equivalence of Theorem 3.4.5.

4.1. Barwick's ∞ -operads. Here we recall Barwick's definition of ∞ -operads from [3] (there called *complete Segal operads*). This definition can be phrased as complete Segal spaces for a certain algebraic pattern, which we introduce first:

Definition 4.1.1. The category $\Delta_{\mathbb{F}}$ has as objects pairs $([n] \in \Delta, f: [n] \rightarrow \mathbb{F})$ interpreted as chains (of length n) of composable arrows in \mathbb{F} , and morphisms $([n], f) \rightarrow ([m], g)$ are given by morphisms $\phi: [n] \rightarrow [m]$ in Δ together with a natural transformation $\eta: f \rightarrow g \circ \phi$ such that

- (1) for every $i \in [n]$, the map $\eta_i: f(i) \rightarrow g(i)$ is an injection,
- (2) for every i, j in $[n]$ with $i \leq j$, the commutative square

$$\begin{array}{ccc} f(i) & \longrightarrow & f(j) \\ \downarrow \eta_i & & \downarrow \eta_j \\ g(i) & \longrightarrow & g(j) \end{array}$$

is a pullback.

For small values of n , we shall also write out an object $([n], f)$ as a chain

$$f(0) \longrightarrow \cdots \longrightarrow f(n).$$

Note that the projection $\Delta_{\mathbb{F}} \rightarrow \Delta$ is a cartesian fibration. We can lift the active-inert factorization system on Δ to one on $\Delta_{\mathbb{F}}$ by declaring a map $(\phi, \eta): ([n], f) \rightarrow ([m], g)$ to be

- *active* if ϕ is active in Δ and $\eta_i: f(i) \rightarrow g(i)$ is an isomorphism for all i ,
- *inert* if ϕ is inert in Δ .

This gives a factorization system on $\Delta_{\mathbb{F}}$ compatible with that on Δ .

Remark 4.1.2. Given $(\phi, \eta): ([n], f) \rightarrow ([m], g)$, where $\phi: [n] \rightarrow [m]$ factors as $[n] \xrightarrow{a} [k] \xrightarrow{i} [m]$ with a active and i inert, to find the active-inert factorization in $\Delta_{\mathbb{F}}$ we first take a factorization of (ϕ, η) as

$$([n], f) \longrightarrow ([k], gi) \longrightarrow ([m], g)$$

with the second morphism cartesian, and then a factorization of $([n], f) \rightarrow ([k], gi)$ as $([n], f) \rightarrow ([k], g') \rightarrow ([k], gi)$, where g' is given by taking pullbacks along $f(n) \rightarrow gi(a(n)) = gi(k)$.

Definition 4.1.3. We give $\Delta_{\mathbb{F}}^{\text{op}}$ the structure of an algebraic pattern using the inert-active factorization system we have just defined, and with the elementary objects being the 1-chains $\mathbf{n} \rightarrow \mathbf{1}$ for all \mathbf{n} in \mathbb{F} as well as the 0-chain $\mathbf{1}$.

Remark 4.1.4. A functor $F: \Delta_{\mathbb{F}}^{\text{op}} \rightarrow \mathcal{S}$ is a Segal $\Delta_{\mathbb{F}}^{\text{op}}$ -space if and only if the following three conditions hold:

- (1) $F([n], f) \xrightarrow{\sim} F([1], f_{01}) \times_{F(f(1))} \cdots \times_{F(f(n-1))} F([1], f_{(n-1)n})$,
- (2) $F(\mathbf{a} \rightarrow \mathbf{b}) \xrightarrow{\sim} \prod_{i \in \mathbf{b}} F(\mathbf{a}_i \rightarrow \mathbf{1})$,
- (3) $F(\mathbf{b}) \xrightarrow{\sim} \prod_{i \in \mathbf{b}} F(\mathbf{1})$.

Remark 4.1.5. Segal $\Delta_{\mathbb{F}}^{\text{op}}$ -objects describe the algebraic structure of ∞ -operads:

- (1) $F(\mathbf{1})$ is the space of objects,
- (2) $F(\mathbf{n} \rightarrow \mathbf{1})$ is the space of n -ary operations, with the map

$$F(\mathbf{n} \rightarrow \mathbf{1}) \longrightarrow F(\mathbf{1})^{\times n} \times F(\mathbf{1})$$

coming from the $n + 1$ inclusions $(\mathbf{1}) \rightarrow (\mathbf{n} \rightarrow \mathbf{1})$ assigning to each operation its sources and target,

- (3) $F(\mathbf{n} \rightarrow \mathbf{m} \rightarrow \mathbf{1})$ decomposes under the Segal condition as the space of n_i -ary operations that can be composed with an m -ary operation,
- (4) the map $F(\mathbf{n} \rightarrow \mathbf{m} \rightarrow \mathbf{1}) \rightarrow F(\mathbf{n} \rightarrow \mathbf{1})$ induced by the inner face map d_1 encodes composition,
- (5) and the remaining data encodes the homotopy-coherent associativity and unitality of this composition operation.

To complete the definition we also need to add a completeness condition:

Definition 4.1.6. Let $u: \Delta^{\text{op}} \rightarrow \Delta_{\mathbb{F}}^{\text{op}}$ be the functor given by

$$[n] \longmapsto ([n], \mathbf{1} \rightrightarrows \mathbf{1} \rightrightarrows \cdots \rightrightarrows \mathbf{1}).$$

Composition with u gives a functor $u^*: \text{Seg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S}) \rightarrow \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})$, and we say $F \in \text{Seg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S})$ is *complete* if u^*F is a complete Segal space. We write $\text{CSeg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S})$ for the full subcategory of $\text{Seg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S})$ spanned by the complete Segal objects.

4.2. Comparison. Our goal is now to show that $\text{CSeg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S})$ is equivalent to the ∞ -category $\text{CSeg}'_{\mathcal{F}}(\mathcal{S})$ considered in the previous section, where $\pi: \mathcal{F} \rightarrow \mathbb{F}_* \times \Delta^{\text{op}}$ is the left fibration corresponding to the symmetric monoidal category \mathbb{F}^{II} viewed as a commutative monoid in Segal spaces. As a first step, we see that there is a functor relating \mathcal{F} to $\Delta_{\mathbb{F}}^{\text{op}}$:

Definition 4.2.1. We define $P: \mathcal{F} \rightarrow \Delta_{\mathbb{F}}^{\text{op}}$ on the object

$$\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow & \cdots \longrightarrow & \mathbf{a}_m \\ & \searrow & & \swarrow \\ & & \mathbf{n}, & \end{array}$$

by forgetting the “augmentation” to \mathbf{n} , so that P takes this object to $\mathbf{a}_0 \rightarrow \cdots \rightarrow \mathbf{a}_m$. Comparing the definitions of the morphisms in \mathcal{F} and $\Delta_{\mathbb{F}}^{\text{op}}$, we see that a morphism in \mathcal{F} restricts to a morphism in $\Delta_{\mathbb{F}}^{\text{op}}$ when we forget the augmentations, which gives the action of P on morphisms.

Remark 4.2.2. The functor P fits into a commutative triangle

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{P} & \Delta_{\mathbb{F}}^{\text{op}} \\ & \searrow & \swarrow \\ & \Delta^{\text{op}} & \end{array}$$

where both maps to Δ^{op} are cocartesian fibrations. From the definitions of the cocartesian morphisms we also see that P preserves these.

The key observation is the following:

Proposition 4.2.3. *The functor $P: \mathcal{F} \rightarrow \Delta_{\mathbb{F}}^{\text{op}}$ is a localization, and composition with it gives an equivalence*

$$\text{Fun}(\Delta_{\mathbb{F}}^{\text{op}}, \mathcal{S}) \xrightarrow{\sim} \text{Fun}'(\mathcal{F}, \mathcal{S}).$$

We begin by looking at P on each fibre over Δ^{op} :

Definition 4.2.4. For $[m] \in \Delta$, let $S_m: \Delta_{\mathbb{F}, [m]}^{\text{op}} \rightarrow \mathcal{F}_{[m]}$ be the functor given by taking the object $\mathbf{a}_0 \rightarrow \cdots \rightarrow \mathbf{a}_m$ to

$$\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow \cdots \longrightarrow & \mathbf{a}_m \\ & \searrow & \swarrow \\ & \mathbf{a}_m & \end{array}$$

and a morphism $(\mathbf{a}_0 \rightarrow \cdots \rightarrow \mathbf{a}_m) \rightarrow (\mathbf{b}_0 \rightarrow \cdots \rightarrow \mathbf{b}_m)$ in $\Delta_{\mathbb{F}}$ given by $\eta: \mathbf{a}_{(-)} \rightarrow \mathbf{b}_{(-)}$ to the morphism in \mathcal{F} given by pulling back along $\mathbf{a}_m \hookrightarrow \mathbf{b}_m$.

Lemma 4.2.5. *Let P_m be the restriction of P to the fibre over $[m] \in \Delta^{\text{op}}$.*

- (1) *The functor S_m is left adjoint to P_m .*
- (2) *P_m is a localization.*

Proof: We have $P_m S_m = \text{id}$ by inspection. We can define a natural transformation $\alpha: S_m P_m \rightarrow \text{id}_{\mathcal{F}}$ given at the object

$$\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow \cdots \longrightarrow & \mathbf{a}_m \\ & \searrow & \swarrow \\ & \mathbf{n} & \end{array}$$

by the map from

$$\begin{array}{ccc} \mathbf{a}_0 & \longrightarrow \cdots \longrightarrow & \mathbf{a}_m \\ & \searrow & \swarrow \\ & \mathbf{a}_m & \end{array}$$

lying over $\mathbf{a}_m \xleftarrow{=} \mathbf{a}_m \rightarrow \mathbf{n}$ (given by composing with $\mathbf{a}_m \rightarrow \mathbf{n}$). Then αS_m and $P_m \alpha$ are clearly both the respective identity transformations, so this indeed exhibits S_m as the left adjoint of P_m . This proves (1). Moreover, since $P_m \alpha$ is the identity we see that α becomes a natural isomorphism after we invert the morphisms in $\mathcal{F}_{[m]}$ that are taken to isomorphisms by P_m . This means that after localizing, S_m is an inverse of P_m , which proves (2). \square

To prove Proposition 4.2.3 we use the following criterion:

Proposition 4.2.6 (Hinich). *Suppose we have a commutative triangle*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{E}' \\ & \searrow p & \swarrow p' \\ & & \mathcal{B}, \end{array}$$

where p and p' are cocartesian fibrations and f preserves cocartesian morphisms. If for every $b \in \mathcal{B}$ the functor on fibres $f_b: \mathcal{E}_b \rightarrow \mathcal{E}'_b$ is a localization, then so is f .

Proof: Let W_b denote the collection of morphisms in \mathcal{E}_b that are taken to equivalences by p_b ; since f preserves cocartesian morphisms we have for every map $\beta: b \rightarrow b'$ a commutative square

$$\begin{array}{ccc} \mathcal{E}_b & \xrightarrow{\beta_1} & \mathcal{E}_{b'} \\ \downarrow & & \downarrow \\ \mathcal{E}'_b & \xrightarrow{\beta_1} & \mathcal{E}'_{b'}, \end{array}$$

from which it is immediate that $\beta_1 W_b \subseteq W_{b'}$. Unstraightening, we see that f corresponds to the natural localization maps $\mathcal{E}_b \rightarrow \mathcal{E}_b[W_b^{-1}]$. It follows from Hinich's work on localizations of fibrations in [14] that \mathcal{E}' is then the localization of \mathcal{E} at the union of the W_b 's, which is to say at the maps that f takes to equivalences. (More precisely, we apply [14, Proposition 2.1.4] in the form [12, Proposition 4.2.5].) \square

Proof of Proposition 4.2.3: We saw in Remark 4.2.2 that P preserves cocartesian morphisms over Δ^{op} and in Lemma 4.2.5 that fibrewise P_m is a localization for every $[m] \in \Delta^{\text{op}}$. Proposition 4.2.6 then implies that P is also a localization. If W denotes the collection of morphisms in \mathcal{F} that are taken to isomorphisms in $\Delta_{\mathbb{F}}^{\text{op}}$, then it follows that composition with P gives a fully faithful functor

$$P^*: \text{Fun}(\Delta_{\mathbb{F}}^{\text{op}}, \mathcal{S}) \longrightarrow \text{Fun}(\mathcal{F}, \mathcal{S})$$

whose image is spanned by the functors $\mathcal{F} \rightarrow \mathcal{S}$ that take the morphisms in W to equivalences in \mathcal{S} . We can identify the morphisms in W as those morphisms in \mathcal{F} that lie over an identity in Δ^{op} and over a map of the form $\mathbf{n} \xleftarrow{\mathbb{F}} \mathbf{n} \rightarrow \mathbf{n}''$ in \mathbb{F}_* . By definition, $\text{Fun}'(\mathcal{F}, \mathcal{S})$ is the full subcategory of functors that take these morphisms to equivalences, and so it is precisely the image of P^* , as required. \square

Corollary 4.2.7. *Composition with P induces equivalences*

$$P^*: \text{Seg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S}) \xrightarrow{\sim} \text{Seg}'_{\mathcal{F}}(\mathcal{S}),$$

$$P^*: \text{CSeg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S}) \xrightarrow{\sim} \text{CSeg}'_{\mathcal{F}}(\mathcal{S}).$$

Proof: We want to show that these subcategories correspond to each other under the equivalence of Proposition 4.2.3. In other words, we must show that a functor $\Phi: \Delta_{\mathbb{F}}^{\text{op}} \rightarrow \mathcal{S}$ lies in $\text{Seg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S})$ if and only if $P^*\Phi$ lies in $\text{Seg}'_{\mathcal{F}}(\mathcal{S})$, and similarly for completeness. For the Segal conditions this is clear since the conditions in Remark 3.4.2 applied to $P^*\Phi$ give precisely the Segal conditions in Remark 4.1.4, while for completeness this follows similarly using the simplified condition from Lemma 3.4.8. \square

Combining Corollary 4.2.7 with Theorem 3.4.5 we have a zig-zag of equivalences

$$\text{Opd}_{\infty} \xrightarrow{\sim} \text{CSeg}'_{\mathcal{F}}(\mathcal{S}) \xleftarrow{\sim} \text{CSeg}_{\Delta_{\mathbb{F}}^{\text{op}}}(\mathcal{S}),$$

which gives:

Corollary 4.2.8. *There is an equivalence of ∞ -categories*

$$\mathrm{Opd}_{\infty} \simeq \mathrm{CSeg}_{\Delta_{\mathbb{F}}^{\mathrm{op}}}(\mathcal{S})$$

between Lurie’s and Barwick’s models for ∞ -operads.

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Received on December 7, 2021.

Accepted on September 2, 2022.