

ON METRIC AND COHOMOLOGICAL PROPERTIES OF OELJEKLAUS–TOMA MANIFOLDS

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Abstract: We study metric and cohomological properties of Oeljeklaus–Toma manifolds. In particular, we describe the structure of the double complex of differential forms and its Bott–Chern cohomology and we characterize the existence of pluriclosed (aka SKT) metrics in number-theoretic and cohomological terms. Moreover, we prove that they do not admit any Hermitian metric ω such that $\partial\bar{\partial}\omega^k = 0$, for $2 \leq k \leq n - 2$, and we give explicit formulas for the Dolbeault cohomology of Oeljeklaus–Toma manifolds admitting pluriclosed metrics.

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Introduction

The class of Oeljeklaus–Toma manifolds ([24]) consists of complex non-Kähler manifolds including Inoue–Bombieri surfaces ([3, 17, 33]) of type S_M . They were constructed by number field techniques in [24], where they are presented as counterexamples to a conjecture by Vaisman concerning locally conformally Kähler metrics. Because of their very construction, some of their geometric properties are encoded in the algebraic structure and can be investigated by number-theoretic techniques; see e.g. [24, 36, 8, 6, 4, 26, 18, 25, 1, 32, 34, 35, 21, 10].

In this paper, we proceed with the study of metric and cohomological properties of Oeljeklaus–Toma manifolds, after [24, 8, 10, 25, 18, 22, 26]. In what follows, $X(K, U) = \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \times U$ will denote an Oeljeklaus–Toma manifold of complex dimension $n = s + t$, associated to a number field $K \simeq \mathbb{Q}[X]/(f)$, where f has $s \geq 1$ real roots and $2t \geq 2$ complex roots, and to the admissible subgroup U of rank s of totally positive units, and \mathcal{O}_K denotes the ring of rank $s + 2t$ of algebraic integers; see Subsection 1.1 for details on the construction.

We study the existence of “special” Hermitian metrics, that is, Hermitian metrics whose associated $(1, 1)$ -form solves some PDE equation. Some results are already known by [24, 8, 10, 25]. Here, we focus in particular on *pluriclosed metrics* (aka SKT, Strong Kähler with Torsion), namely, Hermitian metrics such that the associated $(1, 1)$ -form satisfies $\partial\bar{\partial}\omega = 0$. They play a fundamental role in the Hermitian geometry of complex manifolds with applications to string theory; see e.g. [2, 11, 15, 31, 5, 19]. By [25], it is known that the existence of pluriclosed metrics is equivalent to the number-theoretic properties that $s \leq t$ and, for any $u \in U$, $|\sigma_j(u)|^2 = 1$ holds for any $j \in \{1, \dots, s\}$, and $|\sigma_{s+j}(u)|^2 = 1$ for any $j \in \{s+1, \dots, t\}$, where $\sigma_1, \dots, \sigma_s$:

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$K \rightarrow \mathbb{R}$ denote the real embeddings and $\sigma_{s+1}, \dots, \sigma_{s+t}, \sigma_{s+t+1} = \bar{\sigma}_{s+1}, \dots, \sigma_{s+2t} = \bar{\sigma}_{s+t}: K \rightarrow \mathbb{C}$ denote the complex embeddings induced by the roots of f . In Theorem 2, by number-theoretic arguments, we prove that these conditions cannot hold unless $s = t$. We then get

Corollary 3. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . It admits pluriclosed metrics if and only if $s = t$ and, for any $u \in U$,*

$$\sigma_j(u)|\sigma_{s+j}(u)|^2 = 1, \quad \text{for any } j \in \{1, \dots, s\}.$$

While the above statement can be read as a number-theoretic characterization of the existence of pluriclosed metrics, we will also provide a cohomological characterization in Theorem 17.

We also study *astheno-Kähler metrics*, namely, Hermitian metrics whose associated $(1, 1)$ -form satisfies $\partial\bar{\partial}\omega^{n-2} = 0$. They appeared in [20]; see e.g. [12] and the references therein. More generally, Hermitian metrics with $\partial\bar{\partial}\omega^k = 0$, for $k \in \{1, \dots, n-2\}$, see also [13], play a role in pluripotential theory on compact Hermitian manifolds; see [7]. In Corollary 5, we show that Oeljeklaus–Toma manifolds never admit astheno-Kähler metrics, as a consequence of a more general result on the inexistence of Hermitian metrics such that $\partial\bar{\partial}\omega^k = 0$; see Proposition 4. In Proposition 6, we also show that Oeljeklaus–Toma manifolds never admit *strongly Gauduchon* metrics in the sense of [27].

In Table 1, we summarize what is known until now concerning special metrics on Oeljeklaus–Toma manifolds. (The conditions are intended after possibly relabeling the embeddings.)

Metric (ω positive $(1, 1)$ -form; Ω non-degenerate 2-form)	Conditions	Reference
Kähler $d\omega = 0$	never	[24]
taming symplectic $d\Omega = 0, \Omega(\cdot, J\cdot) > 0$	never	[10]
balanced $d\omega^{n-1} = 0$	never	[25]
pluriclosed $\partial\bar{\partial}\omega = 0$	$s = t$ and $\sigma_j(u) \sigma_{s+j}(u) ^2 = 1$ ($\forall u \in U, \forall j \in \{1, \dots, s\}$)	[25], Corollary 3
“special” k -Gauduchon ($2 \leq k \leq n-2, n \geq 4$) $\partial\bar{\partial}\omega^k = 0$	never	Proposition 4
astheno-Kähler ($n \geq 4$) $\partial\bar{\partial}\omega^{n-2} = 0$	never	Corollary 5
Gauduchon $\partial\bar{\partial}\omega^{n-1} = 0$	always	[14]
strongly Gauduchon $\partial\omega^{n-1}$ is $\bar{\partial}$ -exact	never	Proposition 6
LCK $d\omega = \theta \wedge \omega, d\theta = 0$	$ \sigma_{s+1}(u) = \dots = \sigma_{s+t}(u) \quad (\forall u \in U)$ when $t = 1$: always when $t \geq 2$: never	[24, 36, 8, 6]
LCB $d\omega^{n-1} = \theta \wedge \omega^{n-1}, d\theta = 0$	always	[25]

TABLE 1. Metric properties of Oeljeklaus–Toma manifolds.

The cohomological properties of Oeljeklaus–Toma manifolds are studied in [24, 32, 18, 22, 26]. In particular, it is known that the first Betti number equals the number s of real places [24] and that the second Betti number equals $\binom{s}{2}$ [24, 1]. More generally, the Betti numbers ([18]) and the Hodge numbers ([26]) can be computed in terms of the non-trivial relations between the representations of U induced by the embeddings; see Theorem 7 for a precise statement. Here, we completely describe the E_1 -isomorphism type [29] of the double complex of differential forms on

Oeljeklaus–Toma manifolds, which in turn allows one to compute the dimensions of any cohomological invariant, including the Bott–Chern cohomology, the Aeppli cohomology, the Varouchas groups, any higher-page analogues, and the groups appearing in the Schweitzer complex.

More precisely, we recall that any bounded double complex A over a field K (in particular, the double complex A_X of complex differential forms on a complex manifold X) can be decomposed as a direct sum of indecomposable double complexes that are of type *square*, i.e.,

$$\begin{array}{ccc} K & \xrightarrow{\cong} & K \\ \cong \uparrow & & \cong \uparrow \\ K & \xrightarrow{\cong} & K \end{array}$$

or type *zigzag* (with length equal to the number of non-zero components), i.e.,

$$K, \quad K \xrightarrow{\cong} K, \quad \begin{array}{ccc} K & & K \\ \cong \uparrow & & \cong \uparrow \\ K & \xrightarrow{\cong} & K \end{array}, \quad K \xrightarrow{\cong} K, \quad \dots$$

See [29, Theorem A] and also [23]. One notices that squares do not contribute to any cohomology, even length zigzags correspond to differentials in the Frölicher spectral sequences, while odd length zigzags correspond to de Rham classes, and the dimension of Bott–Chern and Aeppli cohomologies equals the number of top-right, respectively bottom-left, corners in the zigzags; see [29]. In particular, Betti numbers, Hodge numbers, Bott–Chern and Aeppli numbers are linear combinations of the multiplicities $\text{mult}_Z(A)$ for zigzags Z ; see [29, Corollary B]. We also recall that an E_1 -isomorphism is a morphism between bounded double complexes inducing isomorphisms between the column (“Dolbeault”) cohomologies of source and target and between the row (“conjugate Dolbeault”) cohomologies of source and target; see [29, Definition D]. By [29, Proposition E], the E_1 -isomorphism type of a bounded double complex is completely described by the multiplicities of all non-projective indecomposable bicomplexes (i.e., all “zigzags”).

For an Oeljeklaus–Toma manifold X , we will define a graded subalgebra $VB = \bigoplus_r V^r B \subset A_X$ of the algebra of complex differential forms, see equations (15) and (16), such that the inclusion is an E_1 -isomorphism; see Lemma 9. Motivated by the above discussion, and thanks to [26, 22], we obtain the following description of the double complex of an Oeljeklaus–Toma manifold:

Theorem 10. *Let X be an Oeljeklaus–Toma manifold. For any even length zigzag Z , one has*

$$\text{mult}_{[Z]}(A_X) = 0.$$

For odd length zigzags of shape $S_d^{p,q}$, one has

$$\text{mult}_{S_d^{p,q}}(A_X) = \begin{cases} 0 & \text{if } p \neq q, \\ \text{mult}_{S_d^{p,p}}(V^p B) = h_{\bar{\partial}}^{p,d-p}(V^p B) = \sum_{q_1+q_2=d-p} \binom{s}{q_1} \rho_{p,q_2} & \text{if } p = q. \end{cases}$$

As explained above, Theorem 10 allows one to compute any cohomological invariant of X , by simply computing how this invariant evaluates odd length zigzags. In particular, we compute the Bott–Chern cohomology:

Corollary 12. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . The Bott–Chern numbers of X are given by:*

$$h_{BC}^{p,q}(X) = \sum_r h_{BC}^{p,q}(V^r B), \quad \text{where} \quad h_{BC}^{p,q}(V^r B) = \begin{cases} h_{\bar{\partial}}^{r,p+q-r} & \text{if } r \geq p, r \geq q, \\ h_{\bar{\partial}}^{r,p+q-r-1} & \text{if } r < p, r < q, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, in Subsection 3.2, we focus on Oeljeklaus–Toma manifolds admitting pluriclosed metrics and study their cohomological properties. In particular, we prove that the Dolbeault cohomology is invariant with respect to the solvmanifold structure of Oeljeklaus–Toma manifolds (see Remark 16 for more details) and we explicitly compute the Betti and Hodge numbers:

Theorem 15. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold admitting a pluriclosed metric. Then the Betti and Hodge numbers are:*

$$b_\ell = \sum_{k \leq s} \binom{s}{\ell - 3k} \cdot \binom{s}{k}, \quad h_{\bar{\partial}}^{p,q} = \begin{cases} \binom{s}{q-\frac{p}{2}} \cdot \binom{s}{\frac{p}{2}} & \text{if } p \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

We also provide the following cohomological characterization of the pluriclosed condition:

Theorem 17. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . The following statements are equivalent:*

- (i) $X(K, U)$ admits a pluriclosed metric;
- (ii) $s = t$, $h_{\bar{\partial}}^{2,1} = s$, $h_{\bar{\partial}}^{4,2} = \binom{s}{2}$, $h_{\bar{\partial}}^{1,2} = 0$, and the natural map

$$\bigwedge : H_{\bar{\partial}}^{2,1}(X) \times H_{\bar{\partial}}^{2,1}(X) \longrightarrow H_{\bar{\partial}}^{4,2}(X)$$

is surjective.

The paper is organized as follows. In Section 1, we recall the construction of Oeljeklaus–Toma manifolds and their solvmanifold structure. In Section 2, we investigate metric properties of Oeljeklaus–Toma manifolds, in particular, the existence of pluriclosed metrics (Theorem 2 and Corollary 3) and astheno–Kähler metrics (Proposition 4 and Corollary 5). In Section 3, we study the double complex of an Oeljeklaus–Toma manifold (Theorem 10) and its Bott–Chern cohomology (Corollary 12), and then focus on the cohomologies of pluriclosed Oeljeklaus–Toma manifolds (Theorems 15 and 17).

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1. The class of Oeljeklaus–Toma manifolds

1.1. Construction. We briefly recall their construction. Fix $s, t \in \mathbb{N} \setminus \{0\}$. Let K be an algebraic number field, say,

$$K \simeq \mathbb{Q}[X]/(f),$$

where $f \in \mathbb{Q}[X]$ is a monic irreducible polynomial of degree $n = [K : \mathbb{Q}]$ with s real roots, and $2t$ complex roots. Such a field always exists [24, Remark 1.1]. Denote by a_1, \dots, a_s the real roots of f , and by $a_{s+1}, \dots, a_{s+t}, a_{s+t+1} = \bar{a}_{s+1}, \dots, a_{s+2t} = \bar{a}_{s+t}$ the complex non-real roots of f . By considering $X \mapsto a_j$, the field K admits $s + 2t$ embeddings in \mathbb{C} by the conjugate roots, more precisely, s real embeddings and $2t$ complex embeddings:

$$\begin{aligned} \sigma_1, \dots, \sigma_s : K &\longrightarrow \mathbb{R}, \\ \sigma_{s+1}, \dots, \sigma_{s+t}, \sigma_{s+t+1} = \bar{\sigma}_{s+1}, \dots, \sigma_{s+2t} = \bar{\sigma}_{s+t} : K &\longrightarrow \mathbb{C}. \end{aligned}$$

The ring \mathcal{O}_K of algebraic integers of K is a finitely generated free Abelian group of rank n . The multiplicative group \mathcal{O}_K^* of units of \mathcal{O}_K is a finitely generated free Abelian group of rank $s + t - 1$, by the Dirichlet unit theorem. The group $\mathcal{O}_K^{*,+}$ of totally positive units is a finite index subgroup of \mathcal{O}_K^* .

Denote by $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$ the upper half-plane. By [24, p. 162], one can always choose an *admissible subgroup* $U \subset \mathcal{O}_K^{*,+}$, of rank s , such that the fixed-point-free action

$$\mathcal{O}_K \rtimes U \curvearrowright \mathbb{H}^s \times \mathbb{C}^t$$

induced by

$$\begin{aligned} T : \mathcal{O}_K \curvearrowright \mathbb{H}^s \times \mathbb{C}^t, \\ T_a(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) &:= (w_1 + \sigma_1(a), \dots, z_{s+t} + \sigma_{s+t}(a)), \\ R : \mathcal{O}_K^{*,+} \curvearrowright \mathbb{H}^s \times \mathbb{C}^t, \\ R_u(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) &:= (w_1 \cdot \sigma_1(u), \dots, z_{s+t} \cdot \sigma_{s+t}(u)), \end{aligned}$$

is properly discontinuous and co-compact. One defines the *Oeljeklaus–Toma manifold* of type (s, t) associated to the algebraic number field K and to the admissible subgroup U of $\mathcal{O}_K^{*,+}$ as

$$X(K, U) := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \rtimes U.$$

In particular, for an algebraic number field K with $s = 1$ real embedding and $2t = 2$ complex embeddings, $X(K, \mathcal{O}_K^{*,+})$ is an Inoue–Bombieri surface of type S_M [17, 3].

The Oeljeklaus–Toma manifold $X(K, U)$ is called *of simple type* if there exists no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subseteq \mathcal{O}_{K'}^{*,+}$. This is equivalent to saying that there exists no holomorphic foliation of $X(K, U)$ with a leaf isomorphic to $X(K', U)$ [24, Remark 1.7].

1.2. Solvmanifold structure. Following [21], we can endow Oeljeklaus–Toma manifolds with a solvmanifold structure (namely, a compact quotient of a solvable Lie group by a co-compact discrete subgroup) in such a way that the complex structure is left-invariant (namely, it descends from a complex structure on the Lie group that is invariant by left-translations). For later use, we recall here the construction of a left-invariant coframe of $(1, 0)$ -forms and its structure equations, following the description in [21, Section 6] and [25, Section 2].

On $\mathbb{H}^s \times \mathbb{C}^t$, take coordinates $(w^1, \dots, w^s, z^1, \dots, z^t)$. By the very construction,

$$\{(\log \sigma_1(u), \dots, \log \sigma_s(u)) : u \in U\}$$

yields a lattice in \mathbb{R}^s of rank s . In particular, fix any generators $\{u_1, \dots, u_s\}$ for U , then there exist real numbers $b_{k,i} \in \mathbb{R}$ and $c_{k,i} \in \mathbb{R}$, for $k \in \{1, \dots, s\}$, $i \in \{1, \dots, t\}$, such that, for any $i \in \{1, \dots, t\}$,

$$(b_{1,i} \cdots b_{s,i}) \cdot \begin{pmatrix} \log \sigma_1(u_1) & \cdots & \log \sigma_1(u_s) \\ \vdots & \ddots & \vdots \\ \log \sigma_s(u_1) & \cdots & \log \sigma_s(u_s) \end{pmatrix} = (2 \log |\sigma_{s+i}(u_1)| \cdots 2 \log |\sigma_{s+i}(u_s)|)$$

and

$$(c_{1,i} \cdots c_{s,i}) \cdot \begin{pmatrix} \log \sigma_1(u_1) & \cdots & \log \sigma_1(u_s) \\ \vdots & \ddots & \vdots \\ \log \sigma_s(u_1) & \cdots & \log \sigma_s(u_s) \end{pmatrix} = (\arg \sigma_{s+i}(u_1) \cdots \arg \sigma_{s+i}(u_s));$$

equivalently, for any $u \in U$ and any $i \in \{1, \dots, t\}$, we have

$$\sigma_{s+i}(u) = \prod_{k=1}^s \sigma_k(u)^{\frac{1}{2}b_{k,i}} \cdot \exp \left(\sqrt{-1} \sum_{k=1}^s c_{k,i} \log \sigma_k(u) \right).$$

The manifold $\mathbb{H}^s \times \mathbb{C}^t$ is endowed with the product

$$\begin{aligned} (\dots, w^k, \dots, z^i, \dots) * (\dots, \tilde{w}^k, \dots, \tilde{z}^i, \dots) \\ = \left(\dots, \operatorname{Re} w^k + \operatorname{Im} w^k \cdot \operatorname{Re} \tilde{w}^k + \sqrt{-1} \operatorname{Im} w^k \cdot \operatorname{Im} \tilde{w}^k, \dots, z^i \right. \\ \left. + \prod_{k=1}^s (\operatorname{Im} w^k)^{\frac{1}{2}b_{k,i}} \exp \left(\sqrt{-1} \sum_{k=1}^s c_{k,i} \log (\operatorname{Im} w^k) \right) \tilde{z}^i, \dots \right). \end{aligned}$$

It gives $\mathbb{H}^s \times \mathbb{C}^t$ a solvable Lie group structure. One can show that

$$\mathcal{O}_K \rtimes U \subset \mathbb{H}^s \times \mathbb{C}^t, \quad (u, a) \longmapsto (\dots, \sigma_k(a) + \sqrt{-1} \sigma_k(u), \dots, \sigma_{s+i}(a), \dots)$$

is a discrete subgroup of $(\mathbb{H}^s \times \mathbb{C}^t, *)$.

One can check that the following $(1, 0)$ -forms yield a left-invariant coframe for $(\mathbb{H}^s \times \mathbb{C}^t, *)$:

$$(1) \begin{cases} \omega^k := \frac{1}{\operatorname{Im} w^k} dw^k & \text{for } k \in \{1, \dots, s\}, \\ \gamma^i := \prod_{k=1}^s (\operatorname{Im} w^k)^{-\frac{1}{2}b_{k,i}} \exp(-\sqrt{-1} \sum_{k=1}^s c_{k,i} \log (\operatorname{Im} w^k)) dz^i & \text{for } i \in \{1, \dots, t\}, \end{cases}$$

with structure equations

$$(2) \begin{cases} d\omega^k = \frac{\sqrt{-1}}{2} \omega^k \wedge \bar{\omega}^k & \text{for } k \in \{1, \dots, s\}, \\ d\gamma^i = \sum_{k=1}^s \left(\frac{\sqrt{-1}}{4} b_{k,i} - \frac{1}{2} c_{k,i} \right) \omega^k \wedge \gamma^i \\ \quad + \sum_{k=1}^s \left(-\frac{\sqrt{-1}}{4} b_{k,i} + \frac{1}{2} c_{k,i} \right) \bar{\omega}^k \wedge \gamma^i & \text{for } i \in \{1, \dots, t\}. \end{cases}$$

2. Metric properties of Oeljeklaus–Toma manifolds

On a complex manifold X of complex dimension n endowed with a Hermitian metric g , denote by J the integrable almost complex structure and by $\omega := g(J-, -)$ the associated $(1, 1)$ -form. We recall some special classes of Hermitian metrics [16]:

- *Kähler* metrics satisfy $d\omega = 0$;
- *taming symplectic* (aka *Hermitian symplectic*) structures Ω are symplectic forms satisfying $\Omega(-, J-) > 0$;

- pluriclosed (aka *SKT*) metrics satisfy $\partial\bar{\partial}\omega = 0$;
- if $n \geq 3$, *astheno-Kähler* metrics satisfy $\partial\bar{\partial}\omega^{n-2} = 0$;
- *balanced* metrics satisfy $d\omega^{n-1} = 0$;
- *Gauduchon* metrics satisfy $\partial\bar{\partial}\omega^{n-1} = 0$;
- *strongly Gauduchon* metrics satisfy $\partial\omega^{n-1}$ is $\bar{\partial}$ -exact;
- for $k \in \{1, \dots, n-1\}$, *k-Gauduchon* metrics satisfy $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$; notice that the case $k = n-1$ coincides with Gauduchon metrics, the case $k = 1$ includes pluriclosed metrics, and the case $k = n-2$ includes *astheno-Kähler* metrics;
- *locally conformally Kähler* (for short, *LCK*) metrics satisfy $d\omega = \theta \wedge \omega$ for some 1-form θ such that $d\theta = 0$;
- *locally conformally balanced* (for short, *LCB*) metrics satisfy $d\omega^{n-1} = \theta \wedge \omega^{n-1}$ for some 1-form θ such that $d\theta = 0$.

By [14, Théorème 1], Gauduchon metrics always exist on compact complex manifolds. In what follows, we collect what is known about the existence of special Hermitian metrics on Oeljeklaus–Toma manifolds, after Oeljeklaus, Toma, Vuletescu, Dubickas, Otiman, Fino, Kasuya, and Vezzoni:

Theorem 1 ([24, 8, 10, 25]). *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . Then:*

- *It never admits Kähler metrics, [24, Proposition 2.5].*
- *It never admits taming symplectic forms, [10, Theorem 1.1].*
- *It admits pluriclosed metrics if and only if $s \leq t$ and, for any $u \in U$,*

$$\begin{aligned} \sigma_j(u)|\sigma_{s+j}(u)|^2 &= 1, \quad \text{for any } j \in \{1, \dots, s\}, \\ |\sigma_{s+j}(u)|^2 &= 1, \quad \text{for any } j \in \{s+1, \dots, t\}, \end{aligned}$$

after possibly relabeling the embeddings, [25, Theorem 1.1].

- *It never admits balanced metrics, [25, Theorem 1.2].*
- *It admits an LCK metric if and only if $|\sigma_{s+1}(u)| = \dots = |\sigma_{s+t}(u)|$ for any $u \in U$, [24, Proposition 2.9], [8, Theorem 8].*
 - *If $s \geq 1$ and $t = 1$, then there is an admissible subgroup U with this property, [24, Proposition 2.9].*
 - *If $t \geq 2$, then there is no admissible subgroup U with this property, [6].*
- *It always admits an LCB metric, [25, Theorem 1.3].*

By methods similar to those used in [8] and [9], we shall prove:

Theorem 2. *Let K be a number field with s real embeddings and $2t$ complex embeddings such that*

$$1 \leq s < t.$$

Then, the smallest $d = s + 2t$ for which there is a unit $u \in K$ of degree d satisfying

$$(3) \quad \sigma_j(u)|\sigma_{s+j}(u)|^2 = 1$$

for each $j \in \{1, \dots, s\}$ and

$$(4) \quad |\sigma_{s+j}(u)| = 1$$

for each $j \in \{s+1, \dots, t\}$ is $d = 12$. Moreover, there is no group of units $U \subset \mathcal{O}_K^{,+}$ of rank s such that for each $u \in U$ (not necessarily of degree d) we have (3) and (4).*

Proof: Assume that a unit $u \in K$ satisfies (4). Then, u must be a reciprocal unit. (An algebraic number α is called *reciprocal* if α^{-1} is a conjugate of α over \mathbb{Q} .) We claim that

$$(5) \quad \prod_{j=1}^s \sigma_j(u) = 1$$

for each reciprocal unit $u \in \mathcal{O}_K^*$ satisfying (3) and (4).

Note that $\sigma_j(u) > 0$ for $j = 1, \dots, s$ by (3), and hence $u \in \mathcal{O}_K^{*,+}$. Clearly, (5) is true for $u = 1$. Otherwise, $u \in \mathcal{O}_K^{*,+}$ is irrational. Since u is reciprocal and $u \notin \mathbb{Q}$, it must be of even degree. Since $\mathbb{Q}(u)$ is a subfield of K , the degree $n = s + 2t$ of K must be even. Consequently, s is even.

Recall that $\sigma_j(u)$ are real and positive for all $j = 1, \dots, s$. We claim that $\sigma_{s+j}(u)$ are non-real for all $j > 0$. Indeed, if $\sigma_{s+j}(u)$ were real for some $j > 0$, then $\bar{\sigma}_{s+j}(u) = \sigma_{s+j}(u)$. So, by (3), we must have

$$u''u'^2 = 1$$

for some two conjugates $u'' = \sigma_j(u)$, $u' = \sigma_{s+j}(u)$ of u over \mathbb{Q} . Take the smallest $\rho \geq 1$ such that all the conjugates of u lie in the annulus $\rho^{-1} \leq |z| \leq \rho$, with at least one conjugate lying on the boundary. By Kronecker's theorem (see e.g. [28, Theorem 4.5.4]), since u is an algebraic integer which is not a root unity, we must have $\rho > 1$.

Consider an automorphism $\tau \in \{\sigma_1, \dots, \sigma_{s+2t}\}$ which maps u' to a conjugate lying on the boundary of the annulus, say u . Then,

$$1 = \tau(u''u'^2) = \tau(u'')\tau(u')^2 = u^2\tau(u''),$$

where $|u| \in \{\rho, \rho^{-1}\}$. However, if $|u| = \rho$, then

$$1 = |u|^2|\tau(u'')| = \rho^2|\tau(u'')| \geq \rho^2\rho^{-1} = \rho,$$

which is impossible. Likewise, for $|u| = \rho^{-1}$ we obtain $1 = \rho^{-2}|\tau(u')| \leq \rho^{-2}\rho = \rho^{-1}$, which is impossible too. Consequently, all $\sigma_{s+j}(u)$ for $j > s$ are non-real.

In particular, this implies that the list $\sigma_1(u), \dots, \sigma_s(u)$ contains all the real conjugates of u , possibly with repetitions. Suppose the reciprocal unit u is of degree 2ℓ , where $\ell \in \mathbb{N}$, and its conjugates are

$$u_1, u_1^{-1}, \dots, u_\ell, u_\ell^{-1}.$$

Then s must be a multiple of 2ℓ , and the product in (5) contains $s/2\ell$ copies of each real conjugate of u . This clearly yields (5).

Suppose now that U is admissible for K , and for each $u \in U$ the conditions (3) and (4) are satisfied. Since U is of rank s , it must contain s units, say v_1, \dots, v_s for which the matrix $\|\log \sigma_j(v_i)\|_{i,j=1, \dots, s}$ is non-degenerate. However, as we have already shown above, each $v_i \in U$ is reciprocal, and hence $\sum_{j=1}^s \log \sigma_j(v_i) = 0$ by (5). It follows that the rank of U is less than s , which proves the second assertion of the theorem.

In order to prove the first assertion we first show that no K of degree less than 12 exists for which a unit $u \in \mathcal{O}_K^{*,+}$ satisfies (3) and (4). Assume such a unit of degree $d = s + 2t$ exists in the number field K with signature (s, t) . Then, as s is even and $s < t$, we have $d \geq 12$, unless $(s, t) = (2, 3)$ or $(s, t) = (2, 4)$.

We first investigate the case $(s, t) = (2, 3)$. Suppose that the real conjugates of u are $u_1 > 1$ and $u_2 = u_1^{-1} < 1$. By (3), it also has conjugates $u_3, u_4 = \bar{u}_3, u_5, u_6$, where $u_1|u_3|^2 = 1$ and $\{u_5, u_6\} = \{u_3^{-1}, u_4^{-1}\}$. Without loss of generality, we may assume that $u_5 = u_3^{-1}$ and $u_6 = u_4^{-1} = \bar{u}_3^{-1} = \bar{u}_5$. Also, by (4), it has two conjugates u_7 and $u_8 = \bar{u}_7 = u_7^{-1}$ on the unit circle. For convenience we present them in a table.

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	u_1^{-1}		\bar{u}_3	u_3^{-1}	$\bar{u}_5 = u_4^{-1}$		$\bar{u}_7 = u_7^{-1}$

TABLE 2. Conjugates of u .

Recall that $u_1 > 1$ and

$$u_1|u_3|^2 = u_1u_3u_4 = 1.$$

Take an automorphism $\sigma \in \{\sigma_1, \dots, \sigma_8\}$ which maps u_1 to u_7 . Then, from $u_1 = u_3^{-1}u_4^{-1} = u_5u_6$ we obtain $u_7 = \sigma(u_5)\sigma(u_6)$. Since $|\sigma(u_5)\sigma(u_6)| = |u_7| = 1$ and u_7 is non-real, the only possibilities for $\{\sigma(u_5), \sigma(u_6)\}$ are $\{u_4, u_5\}$ and $\{u_3, u_6\}$. Hence, $u_7 = u_4u_5$ or $u_7 = u_3u_6$. Since $u_2 = u_3u_4$, multiplying it by $u_7 = u_4u_5$ we get $u_2u_7 = u_3u_4^2u_5 = u_4^2$. This, mapping u_4 to u_1 , gives a contradiction by the modulus consideration. Likewise, in the case when $u_7 = u_3u_6$, we deduce $u_2u_7 = u_3u_4u_3u_6 = u_3^2$, which leads to the same contradiction by mapping u_3 to u_1 . This shows that for K with signature $(2, 3)$ no unit $u \in \mathcal{O}_K^{*,+}$ of degree 8 satisfying (3) and (4) exists.

Now, we investigate the case $(s, t) = (2, 4)$. Then, without loss of generality we can label the conjugates u_1, \dots, u_8 as in Table 2. In addition, by (4), there are two more conjugates u_9 and $u_{10} = \bar{u}_9 = u_9^{-1}$ on the unit circle. From $u_1 = u_5u_6$, as above we obtain $u_7 = \sigma(u_5)\sigma(u_6)$. This time, the possibilities for $\{\sigma(u_5), \sigma(u_6)\}$ are not only $\{u_4, u_5\}$, $\{u_3, u_6\}$, but also $\{u_8, u_9\}$ and $\{u_8, u_{10}\}$. In the first two cases the argument is exactly the same as above. Assume that $\{\sigma(u_5), \sigma(u_6)\} = \{u_8, u_9\}$. Then, from $u_7 = u_8u_9$ we find that $u_7^2 = u_7u_8u_9 = u_9$. Now, mapping u_7 to u_1 , we arrive at a contradiction by the modulus consideration. (The case $\{\sigma(u_5), \sigma(u_6)\} = \{u_8, u_{10}\}$ can be treated in the same fashion.) Therefore, for K with signature $(2, 4)$ no unit $u \in \mathcal{O}_K^{*,+}$ of degree 10 satisfying (3) and (4) exists.

In order to complete the proof of the theorem we will give an example of a number field K with signature $(2, 5)$ and a positive unit $u \in K$ of degree 12 satisfying (3) and (4). Select the numbers

$$S := -12 + 9\sqrt{2} = 0.727922\dots \quad \text{and} \quad P := 33 - 22\sqrt{2} = 1.887301\dots$$

Let R and T be two complex conjugate numbers satisfying $R + T = S$, $RT = P$, and $\text{Im}(R) > 0$. Consider the polynomial

$$x^3 + Rx^2 - Tx - 1$$

and its reciprocal

$$x^3 + Tx^2 - Rx - 1.$$

Their product is a reciprocal polynomial

$$\begin{aligned} G(x) &:= x^6 + Sx^5 + (P - S)x^4 + (2P - S^2 - 2)x^3 + (P - S)x^2 + Sx + 1 \\ &= x^6 - 12x^5 + 45x^4 - 242x^3 + 45x^2 - 12x + 1 + (9x^5 - 31x^4 + 172x^3 - 31x^2 + 9x)\sqrt{2}. \end{aligned}$$

We claim that it has six roots on the unit circle.

Indeed, setting $y = x + x^{-1}$ and using the identities $x^3 + x^{-3} = y^3 - 3y$, $x^2 + x^{-2} = y^2 - 2$ we derive that

$$\begin{aligned} x^{-3}G(x) &= y^3 - 3y + S(y^2 - 2) + (P - S)y + 2P - S^2 - 2 \\ &= y^3 + Sy^2 + (P - S - 3)y + 2P - S^2 - 2S - 2 \\ &= y^3 + (-12 + 9\sqrt{2})y^2 + (42 - 31\sqrt{2})y - 218 + 154\sqrt{2}. \end{aligned}$$

The latter polynomial (in y) has three roots

$$y_1 = -1.724350\dots, \quad y_2 = -0.110593\dots, \quad y_3 = 1.107021\dots$$

in the interval $(-2, 2)$, so the six roots of G coming from $x + x^{-1} = y_i$ are all on $|z| = 1$.

Similarly, setting

$$S' := -12 - 9\sqrt{2} = -24.727922\dots \quad \text{and} \quad P' := 33 + 22\sqrt{2} = 64.112698\dots,$$

we define two negative real numbers $R' = -21.784939\dots$ and $T' = -2.942982\dots$ which satisfy $R' + T' = S'$, $R'T' = P'$. The product of two reciprocal polynomials

$$x^3 + R'x^2 - T'x - 1 \quad \text{and} \quad x^3 + T'x - R'x - 1$$

equals

$$\begin{aligned} \bar{G}(x) &:= x^6 + S'x^5 + (P' - S')x^4 + (2P' - S'^2 - 2)x^3 + (P' - S')x^2 + S'x + 1 \\ &= x^6 - 12x^5 + 45x^4 - 242x^3 + 45x^2 - 12x + 1 - (9x^5 - 31x^4 + 172x^3 - 31x^2 + 9x)\sqrt{2}. \end{aligned}$$

This time, for $y = x + x^{-1}$ we find that

$$x^{-3}\bar{G}(x) = y^3 + (-12 - 9\sqrt{2})y^2 + (42 + 31\sqrt{2})y - 218 - 154\sqrt{2},$$

and the latter polynomial (in y) has one real root $21.697332\dots$ and two complex conjugate roots. It follows that $\bar{G} \in \mathbb{R}[x]$ has two real roots, say $u_1 = u = 21.651145\dots$ and $u_2 = u^{-1}$, and two pairs of complex conjugate roots, $u_3, u_4 = \bar{u}_3$ and $u_5 = u_3^{-1}$, $u_6 = \bar{u}_5 = u_4^{-1}$. Relabeling the conjugates if necessary, we can assume that

$$x^3 + R'x^2 - T'x - 1 = (x - u_1)(x - u_3)(x - u_4)$$

and

$$x^3 + T'x^2 - R'x - 1 = (x - u_2)(x - u_5)(x - u_6).$$

Here, we clearly have

$$(6) \quad u_1u_3u_4 = 1 \quad \text{and} \quad u_2u_5u_6 = 1.$$

Suppose u_7, \dots, u_{12} are the six unimodular roots of G , so that

$$(7) \quad |u_7| = \dots = |u_{12}| = 1.$$

Multiplying G and \bar{G} we find the reciprocal polynomial

$x^{12} - 24x^{11} + 72x^{10} - 448x^9 - 191x^8 - 440x^7 - 432x^6 - 440x^5 - 191x^4 - 448x^3 + 72x^2 - 24x + 1$ in $\mathbb{Z}[x]$ with 12 roots u_1, \dots, u_{12} satisfying (6) and (7). One can easily verify with `Maple` that this polynomial is irreducible over \mathbb{Q} , so its root $u = 21.651145\dots$ is a reciprocal unit of degree 12 satisfying the conditions (3) and (4). This completes the proof of the first claim of the theorem. \square

As an immediate corollary, we get:

Corollary 3. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . It admits pluriclosed metrics if and only if $s = t$ and, for any $u \in U$,*

$$(8) \quad \sigma_j(u)|\sigma_{s+j}(u)|^2 = 1, \quad \text{for any } j \in \{1, \dots, s\}.$$

For each $s \geq 1$ the construction of such fields K with signature (s, s) and such admissible group of units $U \subset \mathcal{O}_K^{*,+}$ of rank s for which (8) holds for any $u \in U$ was given in [9].

As for astheno-Kähler metrics, it is easy to prove that Oeljeklaus–Toma manifolds do not admit *left-invariant* Hermitian metrics of this type. We notice here that, as opposed to the balanced, pluriclosed, or taming symplectic cases, the averaging trick on solvmanifolds does not apply for astheno-Kähler metrics. Moreover, we will prove that astheno-Kähler metrics never exist on Oeljeklaus–Toma manifolds.

This will follow by considering a more general metric condition. We recall that *k-Gauduchon metrics* ([13]) are defined by the property $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$. In what follows, we prove that *k-Gauduchon metrics* of “special type”, including astheno-Kähler metrics, never exist on Oeljeklaus–Toma manifolds:

Proposition 4. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) and complex dimension $n \geq 4$. Fix an arbitrary integer k such that $2 \leq k \leq n - 2$. Then there is no Hermitian metric ω such that $\partial\bar{\partial}\omega^k = 0$.*

Proof: Consider the coframe of left-invariant $(1, 0)$ -forms given by $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^t\}$, as presented in Subsection 1.2.

Case $s \geq 2$ and $t \leq k \leq n - 2$: Consider the $(1, 1)$ -form $\eta := -\sqrt{-1} \sum_{k, \ell=1}^s \omega^k \wedge \bar{\omega}^\ell$. Following (2), we get $\partial\bar{\partial}\eta = \frac{\sqrt{-1}}{2} \sum_{k, \ell} \omega^k \wedge \bar{\omega}^k \wedge \omega^\ell \wedge \bar{\omega}^\ell$, which is a non-zero, strongly positive form of type $(2, 2)$. For $I = \{i_1, \dots, i_{n-k-2}\}$, denote $\omega_I := \omega_{i_1} \wedge \dots \wedge \omega_{i_{n-k-2}}$. Let

$$\omega_0 := \sum_{|I|=n-k-2} \omega_I \wedge \bar{\omega}_I \wedge \eta.$$

Then $\partial\bar{\partial}\omega_0 = \frac{\sqrt{-1}}{2} \sum_{|I|=n-k} \omega_I \wedge \bar{\omega}_I$ is a strongly positive $(n-k, n-k)$ form. If ω such that $\partial\bar{\partial}\omega^k = 0$ existed, we would get the following contradiction:

$$(9) \quad 0 < \int_X \partial\bar{\partial}\omega_0 \wedge \omega^k = \int_X \omega_0 \wedge \partial\bar{\partial}\omega^k = 0.$$

Case $s = 1$: Let us fix $1 \leq k \leq n - 2$. We shall prove that there exists a positive $\sqrt{-1}\partial\bar{\partial}$ -exact $(n-k, n-k)$ -form. To this end, we claim that there exist $1 \leq i_1 < \dots < i_{n-k-1} \leq t$ such that

$$\begin{aligned} \rho := & (b_{1, i_1} + \dots + b_{1, i_{n-k-1}})(b_{1, i_1} + \dots + b_{1, i_{n-k-1}} + 1) \\ & \cdot \sqrt{-1}^{n-k-1} \gamma_{i_1} \wedge \bar{\gamma}_{i_1} \wedge \dots \wedge \gamma_{i_{n-k-1}} \wedge \bar{\gamma}_{i_{n-k-1}} \end{aligned}$$

satisfies $\sqrt{-1}\partial\bar{\partial}\rho \geq 0$ and it is not 0. Using the structure equations given in (2), one can easily compute that:

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\rho = & \frac{1}{4} (b_{1, i_1} + \dots + b_{1, i_{n-k-1}})^2 (b_{1, i_1} + \dots + b_{1, i_{n-k-1}} + 1)^2 \\ & \cdot \sqrt{-1}^{n-k} \omega \wedge \bar{\omega} \wedge \bigwedge_{\ell=1}^{n-k-1} \gamma_{i_\ell} \wedge \bar{\gamma}_{i_\ell}. \end{aligned}$$

If there were no choice of $\{i_1, \dots, i_{n-k-1}\}$ such that $\sqrt{-1}\partial\bar{\partial}\rho$ is not 0, it would further imply that for any subset $\{i_1, \dots, i_{n-k-1}\} \subset \{1, \dots, t\}$ we have

$$(10) \quad b_{1, i_1} + \dots + b_{1, i_{n-k-1}} \in \{0, -1\}.$$

Assume for now that this is the case. If $n-k-1 \in \{1, 2\}$, it is straightforward to see that (10) combined with the fact that $\sum_{j=1}^t b_{1, j} = -1$ amounts to $b_{1, i_0} = -1$ for some $1 \leq i_0 \leq t$ and $b_{1, i} = 0$ for $i \neq i_0$. If $t > 1$, this further leads to $\sigma_1(u)\sigma_{1+i_0}(u)\bar{\sigma}_{1+i_0}(u) \equiv 1$ and $|\sigma_{1+i}(u)| \equiv 1$, for any $i \neq i_0$, which, however, is impossible, due to Theorem 2.

Let us suppose now that $n-k-1 \geq 3$. If there exist i_m, i_p, i_ℓ such that $b_{1, i_m} \neq b_{1, i_p}$, $b_{1, i_m} \neq b_{1, i_\ell}$, and $b_{1, i_\ell} \neq b_{1, i_p}$, then, since $n-k-1 < t$, one can take $\{j_1, \dots, j_{n-k-3}\} \in \{1, \dots, t\} \setminus \{i_m, i_\ell, i_p\}$ and therefore the sums

$$b_{1, i_m} + b_{1, i_\ell} + \sum_{t=1}^{n-k-3} b_{1, j_t}, \quad b_{1, i_p} + b_{1, i_\ell} + \sum_{t=1}^{n-k-3} b_{1, j_t} \quad \text{and} \quad b_{1, i_m} + b_{1, i_p} + \sum_{t=1}^{n-k-3} b_{1, j_t}$$

are all different and cannot attain only the values 0 and -1 . Moreover, it is impossible to have $b_{1, 1} = \dots = b_{1, t}$, since this would contradict (10) and $\sum_{j=1}^t b_{1, j} = -1$.

Therefore, $b_{1,j}$, for any $1 \leq j \leq t$, can attain precisely two values. Let $m < M$ be these two values. Let x denote the number of $b_{1,j}$ that are equal to m and y the number of $b_{1,j}$ that are equal to M . Then $x + y = t$ and $x \cdot m + y \cdot M = -1$. If $x < n - k - 1$ and $y < n - k - 1$, then we must have $2 \leq x$ and $2 \leq y$, since $n - k - 1 < t$. However, we cannot have the situation $x = y = 2$, because this leads to the contradiction $2(m + M) = -1$, $m + 2M = 0$, and $2m + M = -1$. Therefore, in both cases $x \geq 3$ and $y \geq 3$, we can construct three different sums that cannot cover only the two values 0 and -1 . Finally, this gives that necessarily $x \geq n - k - 1$ or $y \geq n - k - 1$. Nevertheless, $x \geq n - k - 1$ leads easily to a contradiction with (10) and the case $y \geq n - k - 1$ is the only one not conflicting with (10) and giving $b_{1,i_0} = -1$ for precisely one $1 \leq i_0 \leq t$ and $b_{1,j} = 0$ for $j \neq i_0$. If $t > 1$, we further have again $\sigma_1(u)\sigma_{1+i_0}(u)\bar{\sigma}_{1+i_0}(u) \equiv 1$ and $|\sigma_{1+i}(u)| \equiv 1$, for any $i \neq i_0$, which is impossible by Theorem 2.

The contradiction therefore stems from assumption (10), which means that there exists a choice of $\{i_1, \dots, i_{n-k-1}\}$ such that $\sqrt{-1}\partial\bar{\partial}\rho$ is non-zero and positive. From here we argue as in (9) to deduce the inexistence of a Hermitian metric ω with $\sqrt{-1}\partial\bar{\partial}\omega^k = 0$.

Case $s \geq 2$ and $2 \leq k < t$: Assume there exists a Hermitian metric ω such that $\partial\bar{\partial}\omega^k = 0$. Now define

$$\omega_0(X_1, \dots, X_{2k}) := \int_X \omega^k(X_1, \dots, X_{2k}) d\text{vol},$$

where X_1, \dots, X_{2k} are any left-invariant vector fields and $d\text{vol}$ is a bi-invariant volume form. Then ω_0 is a strictly positive left-invariant (k, k) -form that is $\partial\bar{\partial}$ -closed. Note that ω_0 is not necessarily the k -th power of a strictly positive $(1, 1)$ -form. We consider the frame $\mathcal{B} := (\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^t)$ and we denote by α^j its elements and by $\bar{\beta}^j$ their conjugates; we also denote by $(\omega_1, \dots, \omega_s, \gamma_1, \dots, \gamma_t)$ the dual basis. We have a decomposition of ω_0 over the basis \mathcal{B} as follows:

$$\omega_0 = \sum_{|I|=k, |J|=k} A(\alpha^I, \bar{\beta}^J) \alpha^{i_1} \wedge \bar{\beta}^{j_1} \wedge \dots \wedge \alpha^{i_k} \wedge \bar{\beta}^{j_k},$$

where $A(\alpha^I, \bar{\beta}^J) \in \mathbb{C}$. Since ω_0 is strictly positive, we get that, for any $I := \{i_1, \dots, i_k\} \subseteq \{1, \dots, t\}$, $A(\gamma^I, \bar{\gamma}^I) > 0$ holds, where $\gamma^I := \gamma_{i_1} \wedge \bar{\gamma}_{i_1} \wedge \dots \wedge \gamma_{i_k} \wedge \bar{\gamma}_{i_k}$. A straightforward computation using (2) shows that, for any $I \subseteq \{1, \dots, t\}$ and $h \neq \ell$,

$$\begin{aligned} 0 &= i_{\omega_\ell \wedge \bar{\omega}_h \wedge \gamma_I \wedge \bar{\gamma}_I} \partial\bar{\partial}\omega_0 = A(\gamma^I, \bar{\gamma}^I) i_{\omega_\ell \wedge \bar{\omega}_h \wedge \gamma_I \wedge \bar{\gamma}_I} \partial\bar{\partial}(\gamma^I \wedge \bar{\gamma}^I) \\ (11) \quad &= \frac{1}{4} A(\gamma^I, \bar{\gamma}^I) (b_{\ell, i_1} + \dots + b_{\ell, i_k}) (b_{h, i_1} + \dots + b_{h, i_k}) \end{aligned}$$

and

$$\begin{aligned} 0 &= i_{\omega_\ell \wedge \bar{\omega}_\ell \wedge \gamma_I \wedge \bar{\gamma}_I} \partial\bar{\partial}\omega_0 = A(\gamma^I, \bar{\gamma}^I) i_{\omega_\ell \wedge \bar{\omega}_\ell \wedge \gamma_I \wedge \bar{\gamma}_I} \partial\bar{\partial}(\gamma^I \wedge \bar{\gamma}^I) \\ &= \frac{1}{2} A(\gamma^I, \bar{\gamma}^I) (b_{\ell, i_1} + \dots + b_{\ell, i_k}) (b_{\ell, i_1} + \dots + b_{\ell, i_k} + 1). \end{aligned}$$

This further implies that $b_{\ell, i_1} + \dots + b_{\ell, i_k} \in \{0, -1\}$, for any $1 \leq \ell \leq s$ and for any $\{i_1, \dots, i_k\} \subseteq \{1, \dots, t\}$. Now following the same argument as presented in the previous case, we obtain that, for any $1 \leq \ell \leq s$, there exists $1 \leq j_\ell \leq t$ such that $b_{\ell, j_\ell} = -1$ and $b_{\ell, j} = 0$, for any $j \neq j_\ell$. We now take $\ell \neq h$ and choose $\{i_1, \dots, i_k\}$ such that $j_\ell, j_h \in \{i_1, \dots, i_k\}$. This is possible since $2 \leq k$. Then we further have $(b_{\ell, i_1} + \dots + b_{\ell, i_k}) (b_{h, i_1} + \dots + b_{h, i_k}) = -1$, which is not possible by (11). Therefore, there exists no metric ω such that $\partial\bar{\partial}\omega^k = 0$, with $2 \leq k < t$. \square

By Proposition 4 for $k = n - 2$, we immediately get the following

Corollary 5. *Oeljeklaus–Toma manifolds of complex dimension $n \geq 4$ never admit astheno-Kähler metrics.*

We further investigate the existence of *strongly Gauduchon metrics*, which are by definition Hermitian metrics ω such that $\partial\omega^{n-1}$ is $\bar{\partial}$ -exact. They were introduced in [27] and possess the remarkable property of openness under complex analytic deformation. One special subclass of such metrics are the balanced ones. We prove the following:

Proposition 6. *Oeljeklaus–Toma manifolds $X(K, U)$ of any type do not admit strongly Gauduchon metrics.*

Proof: Assume there exists a strongly Gauduchon metric ω . Then, by definition, there exists an $(n, n - 2)$ -form α such that $\partial\omega^{n-1} = \bar{\partial}\alpha$. Let $\tau := \frac{1}{2} \sum_{i=1}^s \bar{\omega}^i$, with the notation of (1). By Stokes, we have the following:

$$(12) \quad \int_X \bar{\partial}\tau \wedge \alpha = \int_X \tau \wedge \bar{\partial}\alpha = \int_X \tau \wedge \partial\omega^{n-1} = \int_X \partial\tau \wedge \omega^{n-1}.$$

However, by (2), $\bar{\partial}\tau = 0$ and $\partial\tau$ is a (strongly) positive $(1, 1)$ -form. Then the first term of (12) vanishes, but the last term is strictly positive and hence, there cannot exist any strongly Gauduchon metric on $X(K, U)$. \square

3. Cohomological properties of Oeljeklaus–Toma manifolds

In what follows, we collect what is known about the cohomologies of Oeljeklaus–Toma manifolds, after Oeljeklaus, Toma, Tomassini, Torelli, Kasuya, Istrati, and the third named author. We use the following notation: for any multi-index $I = (1 \leq i_1 < \dots < i_m \leq n)$ of length $|I| = m$, we consider $\sigma_I: U \rightarrow \mathbb{C}^*$ the representation $\sigma_I(u) := \sigma_{i_1}(u) \cdot \dots \cdot \sigma_{i_m}(u)$. We denote by ρ_m the number of non-trivial relations among the representations σ_i with fixed length m , respectively, by $\rho_{p,m}$ the number of non-trivial relations among the σ_i with fixed contribution from real and complex parts:

$$(13) \quad \rho_m := \#\{I : |I| = m, \sigma_I = 1\},$$

$$(14) \quad \rho_{p,m} := \#\{I \subseteq \{1, \dots, s+t\}, J \subseteq \{s+1, \dots, s+t\} : |I| = p, |J| = m, \sigma_I \bar{\sigma}_J = 1\}.$$

With this notation, we have the following:

Theorem 7 ([24, 18, 22, 26]). *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . Then:*

- *The Betti numbers are computed in [18, Theorem 3.1]:*

$$b_k = \sum_{\ell+m=k} \binom{s}{\ell} \cdot \rho_m.$$

In particular:

- $b_1 = s$, [24, Proposition 2.3].
- $b_2 = \binom{s}{2}$, [24, Proposition 2.3], [1, Proposition 2.4].

- *The Hodge numbers are computed in [26, Theorem 4.5] and also [22, Corollary 3.5]:*

$$h_{\bar{\partial}}^{p,q} = \sum_{\ell+m=q} \binom{s}{\ell} \rho_{p,m}.$$

In particular:

- $h_{\bar{\partial}}^{0,1} = s$, [26, Corollary 4.6].

- *The Hodge–Frölicher spectral sequence degenerates at the first page, [26, Theorem 4.5].*

3.1. The E_1 -isomorphism type and Bott–Chern cohomology. Let $X := X(K, U)$ be an Oeljeklaus–Toma manifold associated to the number field K with s real places and t complex places and to the admissible subgroup U . Denote by $n = s + t$ its complex dimension. We order the embeddings σ_i s.t. $\sigma_i = \bar{\sigma}_i$ for $i \in \{1, \dots, s\}$, and $\bar{\sigma}_j = \sigma_{t+j}$ for $j \in \{s+1, \dots, s+t\}$. With respect to the solvmanifold presentation of $X = G/\Gamma$ in Subsection 1.2, denote the G -invariant $(1, 0)$ -forms by ω^k, γ^i , for $k \in \{1, \dots, s\}, i \in \{1, \dots, t\}$.

We define a subalgebra of the algebra $A_X := \mathcal{C}^\infty(X; \wedge^\bullet T^*X)$ of differential forms by

$$B := \Lambda \langle \omega^k, \bar{\omega}^k \rangle_{k \in \{1, \dots, s\}}.$$

Further consider the space

$$\begin{aligned} V &:= \text{span}_{\mathbb{C}} \{ dz^K \wedge d\bar{z}^L \mid K, L \subseteq \{1, \dots, t\}, \exists J \subseteq \{1, \dots, s\} \text{ s.t. } \sigma_{JKL}|_U \equiv 1 \} \\ (15) \quad &= \text{span}_{\mathbb{C}} \{ \exp(\Psi_{JKL}(x)) \gamma^K \wedge \bar{\gamma}^L \mid J \subseteq \{1, \dots, s\}, K, L \subseteq \{1, \dots, t\} \\ &\quad \text{s.t. } \sigma_{JKL}|_U \equiv 1 \}, \end{aligned}$$

where here we put $\sigma_{JKL} := \prod_{j \in J} \sigma_j \prod_{k \in K} \sigma_{s+k} \prod_{\ell \in L} \sigma_{s+t+\ell}$, and

$$\begin{aligned} \Psi_{JKL} &:= - \sum_{j \in J} \log \text{Im } w^j - \sum_{k \in K} \sum_{h=1}^s \left(\frac{1}{2} b_{h,k} - \sqrt{-1} c_{h,k} \right) \log \text{Im } w^h \\ &\quad - \sum_{\ell \in L} \sum_{h=1}^s \left(\frac{1}{2} b_{h,\ell} + \sqrt{-1} c_{h,\ell} \right) \log \text{Im } w^h \\ &= - \sum_{j \in J} \log \text{Im } w^j - \frac{1}{2} \sum_{k \in K} \sum_{h=1}^s b_{h,k} \log \text{Im } w^h - \frac{1}{2} \sum_{\ell \in L} \sum_{h=1}^s b_{h,\ell} \log \text{Im } w^h \end{aligned}$$

is as in [22], and the simplification follows by $\sigma_{JKL}|_U \equiv 1$. The space V carries a grading by

$$V^r := \text{span}_{\mathbb{C}} \{ \exp(\Psi_{JKL}(x)) \gamma^K \wedge \bar{\gamma}^L \in V \mid |J| + |K| = r \} \subseteq V.$$

We denote by VB the bigraded vector space with basis all wedges of elements in V and B , i.e.,

$$(16) \quad VB := V \wedge B \subseteq A_X.$$

The grading of V^r induces a grading on VB via $V^r B := V^r \wedge B$.

Lemma 8. *The subspace $VB \subseteq A_X$ is a bigraded, bidifferential subalgebra stable under conjugation. The grading of V induces a direct sum decomposition $VB = \bigoplus_r V^r B$ into sub-double complexes which are also stable under conjugation.*

Proof: The statements that the space VB is a bigraded subalgebra of A_X and that $VB = \bigoplus_r V^r B$ as vector spaces are clear from the definitions. From the equations for the differentials, see (2), it follows that VB and $V^r B$ are sub-double complexes. It only remains to show the statement about real structures, which follows from the relation $\sigma_{JKL} = \bar{\sigma}_{JLK}$. \square

Lemma 9. *The inclusion $\iota: VB \hookrightarrow A_X$ is an E_1 -isomorphism.*

Proof: The statement will follow from the following steps.

Step 1: VB is a direct summand in A_X . We show the existence of a map of double complexes $r: A_X \rightarrow VB$ s.t. $r \circ \iota = \text{Id}$. Namely, consider the left-invariant Hermitian metric

$$h := \sum_{i=1}^s \omega_i \cdot \bar{\omega}_i + \sum_{i=1}^t \gamma_i \cdot \bar{\gamma}_i.$$

We obtain an induced L^2 -metric on A_X . The basis for VB , given by all elements of the type $\exp(\Psi_{JKL}(x))\omega_H\bar{\omega}_I\gamma_K\bar{\gamma}_L$ with $\sigma_{JKL}|_U \equiv 1$, is orthonormal with respect to this metric (this follows from the same calculations as in the proof of [22, Lemma 2.3]). Now define r to be the orthogonal projection to VB . Because h is Hermitian, r respects the bigrading and since VB is closed under d and d^* , one checks that r is a chain map.

Step 2: The inclusion is an E_1 -isomorphism. Since ι is compatible with the real structure, we only have to check that the inclusion induces an isomorphism in Dolbeault cohomology. By Step 1, $H_{\bar{\partial}}(VB) \subseteq H_{\bar{\partial}}(X)$. On the other hand, the complex $(B_{\Lambda}^{*,*}, \bar{\partial})$ defined in [22] is a subcomplex of $(VB, \bar{\partial})$ and it is shown in [22] that $H_{\bar{\partial}}(B_{\Lambda}) \cong H_{\bar{\partial}}(X)$, hence we obtain surjectivity. \square

We now describe the E_1 -isomorphism type of A_X completely. This is equivalent to describing the multiplicities of all non-projective indecomposable bicomplexes (i.e., all “zigzags”) in A_X . We use the notation in [29] (in particular, an *odd length zigzag of shape $S_d^{p,q}$* has endpoints $(p, d - p)$ and $(d - q, q)$, length $2|p + q - d| + 1$, and is concentrated in degrees $d, d - \text{sgn}(p + q - d)$):

Theorem 10. *Let X be an Oeljeklaus–Toma manifold. For any even length zigzag Z , one has*

$$\text{mult}_{[Z]}(A_X) = 0.$$

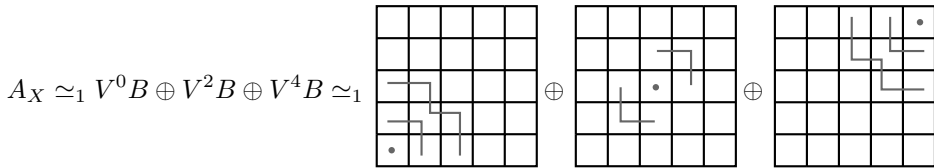
For odd length zigzags of shape $S_d^{p,q}$, one has

$$\text{mult}_{S_d^{p,q}}(A_X) = \sum_r \text{mult}_{S_d^{p,q}}(V^r B)$$

and

$$\text{mult}_{S_d^{p,q}}(V^r B) = \begin{cases} h_{\bar{\partial}}^{p,d-p}(V^r B) & \text{if } p = r, q = r, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if X is a pluriclosed Oeljeklaus–Toma manifold with $s = t = 2$, the only non-trivial relations are $\sigma_1|\sigma_3|^3 = \sigma_2|\sigma_4|^2 = 1$ and one may picture the non-zero zigzags in A_X as follows (we are only depicting those zigzags with non-zero multiplicity, but not writing the multiplicity):



We will study the cohomologies of pluriclosed Oeljeklaus–Toma manifolds in more detail in Subsection 3.2.

We note that the Dolbeault cohomology of X is known explicitly [26, 22] and from this we have [22, Theorem 3.3 and Corollary 3.5]:

Lemma 11. *Let X be an Oeljeklaus–Toma manifold of type (s, t) . Consider the number $\rho_{p,m}$ of non-trivial relations among the σ_i with fixed contributions from real and complex parts as defined in (14). Then,*

$$h_{\bar{\partial}}^{p,q}(V^r B) = \begin{cases} \sum_{q_1+q_2=q} \binom{s}{q_1} \rho_{p,q_2} & \text{if } p = r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 10: The statement about even length zigzags is equivalent to the degeneration of the Frölicher spectral sequence of X on the first page, which is known by [26]. By Lemmas 8 and 9, we have

$$\text{mult}_{[Z]}(A_X) = \sum_r \text{mult}_{[Z]}(V^r B)$$

for any zigzag Z . The multiplicities of odd length zigzags can be computed from the refined Betti numbers, i.e., $\text{mult}_{[S_d^{p,q}]}(V^r B) = b_d^{p,q}(V^r B) := \text{gr}_F^p \text{gr}_{\bar{F}}^q H_{dR}^d(V^r B)$ (see [29, 30]), where F denotes the Hodge filtration induced by the Frölicher filtration $F^p A_X := \bigoplus_{r \geq p} A_X^{r,\bullet}$ and \bar{F} its conjugate. By Lemma 11, there is just one vertical strip of non-zero entries in $H^{\bullet,\bullet}(V^r B)$. Therefore, the Hodge filtration on $H_{dR}^d(V^r B)$ has exactly one breakpoint (if $b_d(V^r B) \neq 0$), namely,

$$\{0\} = F^{r+1} H_{dR}^d(V^r B) \subseteq F^r H_{dR}^d(V^r B) = H_{dR}^d(V^r B).$$

As $V^r B$ is stable under conjugation, the same holds for the conjugate Hodge filtration. Hence we get that for each d there is at most one non-zero refined Betti number $b_d^{p,q}(V^r B)$, i.e.,

$$b_d^{p,q}(V^r B) = \text{mult}_{S_d^{p,q}}(V^r B) = \begin{cases} h_{\bar{\partial}}^{p,d-p}(V^r B) & \text{if } p = r, q = r, \\ 0 & \text{otherwise,} \end{cases}$$

completing the proof. □

From Theorem 10, we may compute the dimensions of any cohomological invariant, by simply computing how this invariant evaluates odd length zigzags. We carry this out for Bott–Chern cohomology, but the reader will have no difficulty in doing the analogous procedure for Aeppli cohomology (which also follows by duality), the Varouchas groups, any higher-page analogues or the groups appearing in the Schweitzer complex.

Corollary 12. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . The Bott–Chern numbers of X are given by:*

$$h_{BC}^{p,q}(X) = \sum_r h_{BC}^{p,q}(V^r B)$$

and

$$h_{BC}^{p,q}(V^r B) = \begin{cases} h_{\bar{\partial}}^{r,p+q-r} & \text{if } r \geq p, r \geq q, \\ h_{\bar{\partial}}^{r,p+q-r-1} & \text{if } r < p, r < q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We have $A_X \simeq_1 \bigoplus_r V^r B$ and hence the first equality follows. For the second, by Theorem 10, we have

$$h_{BC}(V^r B) = \sum_d h_{\bar{\partial}}^{r,d-r}(V^r B) h_{BC}^{p,q}([S_d^{r,r}]).$$

Now, use the formulas

$$h_{BC}^{a,b}([S_d^{p,q}]) = \begin{cases} 1 & \text{if } a + b = d, a \leq p, b \leq q, \\ 0 & \text{otherwise,} \end{cases}$$

in the case $p + q \geq d$ and

$$h_{BC}^{a,b}([S_d^{p,q}]) = \begin{cases} 1 & \text{if } a + b = d + 1, a > p, b > q, \\ 0 & \text{otherwise,} \end{cases}$$

for $p + q < d$. These are obtained from [29] and easily seen directly as Bott–Chern cohomology counts top-right corners of zigzags, e.g.:

$$H_{BC} \begin{pmatrix} \langle a_1 \rangle & \longrightarrow & \langle \partial a_1 \rangle \\ & & \uparrow \\ & & \langle a_2 \rangle \end{pmatrix} = \begin{pmatrix} 0 & \langle \partial a_1 \rangle \\ 0 & 0 \end{pmatrix}$$

and

$$H_{BC} \begin{pmatrix} \langle \bar{\partial} a \rangle \\ \uparrow \\ \langle a \rangle \longrightarrow \langle \partial a \rangle \end{pmatrix} = \begin{pmatrix} \langle \bar{\partial} a \rangle & 0 \\ 0 & \langle \partial a \rangle \end{pmatrix}. \quad \square$$

Remark 13. One can in principle also argue via explicit representatives, i.e., explicitly compute the Bott–Chern cohomology of $V^\bullet B$. However, such representatives are not necessarily of a simple form; in particular, they are generally not elementary wedges, but rather sums of such. For example, if $s = t = 2$, the natural map

$$H_{BC}^{2,2}(X) \rightarrow H_{\bar{\partial}}^{2,2}(X)$$

is an isomorphism, but the preimage of $[\gamma_{1\bar{1}}\omega_{1\bar{2}}]$ cannot be represented by the same form as it is not ∂ -closed. Rather, one has to modify it by

$$\bar{\partial}(\gamma_{1\bar{1}}\omega_2) = \frac{\sqrt{-1}}{2}(\omega_{2\bar{1}}\gamma_{1\bar{1}} - \gamma_{1\bar{1}}\omega_{2\bar{2}})$$

to obtain something in the kernel of ∂ and $\bar{\partial}$.

3.2. Cohomological properties of pluriclosed Oeljeklaus–Toma manifolds.

In this section, we restrict to Oeljeklaus–Toma manifolds admitting pluriclosed metrics, as characterized in Corollary 3, and we study their cohomological properties. In particular, we prove that the Dolbeault cohomology is invariant (see Remark 16) and we explicitly compute the Betti and Hodge numbers (see Theorem 15): note that the cohomologies are not zero most of the times.

Recall that $\rho_m, \rho_{p,m}$ were introduced in (13), respectively (14). It is clear that $\rho_0 = 1$ and $\rho_1 = 0$ always; one can prove that $\rho_2 = 0$ always [1, Proposition 2.4]. When $X(K, U)$ admits pluriclosed metrics, then it is proven that $\rho_3 = s$ and $\rho_{2,1} = s$, therefore $b_3 = \binom{s}{3} + s$ and $h_{\bar{\partial}}^{2,1} = s$ [25, Proposition 3.2.1]. Moreover, for $n = 4$, namely, $s = t = 2$, the converse holds true: X admits a pluriclosed metric if and only if $b_3 = 2$ if and only if $h_{\bar{\partial}}^{2,1} = 2$ [25, Proposition 3.2.2]. In this section, we generalize the above results.

Meanwhile, we notice the following:

Proposition 14. *Oeljeklaus–Toma manifolds admitting pluriclosed metrics are of simple type.*

Proof: Assume there exists a finite extension $\mathbb{Q} \subseteq K' \subseteq K$ such that $U \subseteq K'$; we take $K' = \mathbb{Q}(U)$. The embeddings of K' are simply the restrictions of the embeddings of K , namely $\sigma_1|_{K'}, \dots, \bar{\sigma}_{2s}|_{K'}$. Since U is admissible and $U \subseteq K'$, $\sigma_i|_{K'} \neq \sigma_j|_{K'}$, for any $1 \leq i < j \leq s$. However, as the pluriclosed condition is satisfied, $\sigma_i|_U \sigma_{s+i}|_U \bar{\sigma}_{s+i}|_U \equiv 1$ and therefore $\sigma_i|_{K'} \sigma_{s+i}|_{K'} \bar{\sigma}_{s+i}|_{K'} \equiv 1$, which further implies that $\sigma_{s+i}|_{K'} \neq \sigma_{s+j}|_{K'}$, for any $1 \leq i < j \leq s$. Moreover, $\sigma_{s+i}|_{K'} \neq \sigma_j|_{K'}$, for any $1 \leq i, j \leq s$, otherwise this would imply non-trivial multiplicative relations between $\sigma_1|_{K'}, \dots, \sigma_s|_{K'}$, which is impossible. Hence, the degree $[K' : \mathbb{Q}] \geq 2s$. As $[K : \mathbb{Q}] = [K : K'] \cdot [K' : \mathbb{Q}]$ and $[K : \mathbb{Q}] = 3s$, we infer that $K = K'$ and conclude that $X(K, U)$ is of simple type. \square

We now compute the de Rham and Dolbeault cohomologies of pluriclosed Oeljeklaus–Toma manifolds in the following:

Theorem 15. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold admitting a pluriclosed metric. Then the Betti and Hodge numbers are:*

$$b_\ell = \sum_{k \leq s} \binom{s}{\ell - 3k} \cdot \binom{s}{k}, \quad h_{\bar{\partial}}^{p,q} = \begin{cases} \binom{s - \frac{p}{2}}{q - \frac{p}{2}} \cdot \binom{s}{\frac{p}{2}} & \text{if } p \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Step 1: By Corollary 3, we have $s = t$ and $|\sigma_{s+i}(u)|^2 \sigma_i(u) = 1$, for any $1 \leq i \leq s$, for any $u \in U$.

By [18, Theorem 3.1]:

$$(17) \quad b_\ell = \rho_\ell + \binom{s}{1} \rho_{\ell-1} + \binom{s}{2} \rho_{\ell-2} + \dots + \binom{s}{\ell-2} \rho_2 + \binom{s}{\ell},$$

where ρ_m is defined in (13). We now prove that $\rho_{3k} = \binom{s}{k}$ and $\rho_m = 0$, if $m \neq 3k$. Indeed, let $I = \{i_1, \dots, i_m\}$ be such that $\sigma_I(u) = 1, \forall u \in U$. Then

$$(18) \quad \sigma_I(u) \cdot \sigma_I \bar{u} = 1, \quad \forall u \in U.$$

If $I \neq \bigcup_{j \in J} \{j, s + j, 2s + j\}$, where $J \subseteq \{1, \dots, s\}$, then, since for any $i_r \geq s + 1$, $\sigma_{i_r}(u) \cdot \sigma_{i_r} \bar{u} = \frac{1}{\sigma_{i_r-s}(u)}$ thanks to the pluriclosed condition, we get that (18) would further imply a multiplicative relation between $\sigma_1|_U, \dots, \sigma_s|_U$, which is impossible by the admissibility of U .

Therefore, if $\sigma_I|_U \equiv 1$, then $I = \bigcup_{j \in J} \{j, s + j, 2s + j\}$ and consequently $\rho_m = 0$, if $m \neq 3k$ and

$$\rho_{3k} = \#\{J \mid |J| = k, J \subseteq \{1, \dots, s\}\} = \binom{s}{k}.$$

Applying (17), we get the formula for the Betti numbers.

Step 2: By [26, Relation (4.11)] we have the following formula for the Hodge numbers:

$$(19) \quad h_{\bar{\partial}}^{p,q} = \sum_{\ell+m=q} \binom{s}{\ell} \rho_{p,m},$$

where $\rho_{p,m}$ is defined in (14).

By the previous characterization of non-trivial multiplicative relations between the embeddings, we obtain that $\sigma_{I \cup J}|_U \equiv 1$ (where I and J are as in (19)) if and only if $p = 2j$ and $I \cup J = \bigcup_{k \in K} \{k, s + k, 2s + k\}$, where $K \subseteq \{1, \dots, s\}$. This further means that:

$$\#\{I \subseteq \{1, \dots, s + t\}, J \subseteq \{s + t + 1, \dots, s + 2t\} \mid |I| = p, |J| = j, \sigma_{I \cup J}|_U \equiv 1\} = \binom{s}{\frac{p}{2}}$$

and we obtain the formulas for the Hodge numbers for pluriclosed $X(K, U)$. \square

Remark 16. Notice that pluriclosed Oeljeklaus–Toma manifolds $X(K, U)$ have invariant de Rham and Dolbeault cohomology, in the sense that $H^\bullet(X(K, U)) \simeq H^\bullet(\mathfrak{g})$ and $H_{\bar{\partial}}^{\bullet, \bullet}(X(K, U)) \simeq H_{\bar{\partial}}^{\bullet, \bullet}(\mathfrak{g})$, where \mathfrak{g} is the Lie algebra of the Lie group $(\mathbb{H}^s \times \mathbb{C}^t, *)$. Indeed, this is an immediate consequence of the fact that

$$H_{\bar{\partial}}^{\bullet, \bullet}(X(K, U)) = \left\langle \frac{d\bar{w}_1}{\operatorname{Im} w_1}, \dots, \frac{d\bar{w}_s}{\operatorname{Im} w_s}, dw_1 \wedge dz_1 \wedge d\bar{z}_1, \dots, dw_s \wedge dz_s \wedge d\bar{z}_s \right\rangle_{\mathbb{C}}$$

and all the listed forms are left-invariant with respect to the solvmanifold structure described in Subsection 1.2.

Generalizing [25, Proposition 3.2.2], we have the following cohomological characterization of the pluriclosed condition:

Theorem 17. *Let $X(K, U)$ be an Oeljeklaus–Toma manifold of type (s, t) . The following statements are equivalent:*

- (i) $X(K, U)$ admits a pluriclosed metric;
- (ii) $s = t$, $h_{\bar{\partial}}^{2,1} = s$, $h_{\bar{\partial}}^{4,2} = \binom{s}{2}$, $h_{\bar{\partial}}^{1,2} = 0$, and the natural map

$$\bigwedge : H_{\bar{\partial}}^{2,1}(X) \times H_{\bar{\partial}}^{2,1}(X) \longrightarrow H_{\bar{\partial}}^{4,2}(X)$$

is surjective.

Proof: The implication (i) \Rightarrow (ii) is clear by Theorem 15 and the fact that $H_{\bar{\partial}}^{2,1}(X)$ is generated by $\{[dw_i \wedge dz_i \wedge d\bar{z}_i]\}_i$ and $H_{\bar{\partial}}^{4,2}(X)$ by $\{[dw_i \wedge dw_j \wedge dz_i \wedge dz_j \wedge d\bar{z}_i \wedge d\bar{z}_j]\}_{ij}$; see Remark 16.

We now prove (ii) \Rightarrow (i). By (19) and $h_{\bar{\partial}}^{2,1} = s$, we have that there are precisely s non-trivial multiplicative relations between the embeddings:

$$\begin{aligned} \sigma_{i_{11}}(u) \cdot \sigma_{i_{12}}(u) \cdot \sigma_{i_{13}}(u) &= 1, \quad \forall u \in U, \\ &\dots \\ \sigma_{i_{s1}}(u) \cdot \sigma_{i_{s2}}(u) \cdot \sigma_{i_{s3}}(u) &= 1, \quad \forall u \in U, \end{aligned}$$

where $i_{l1} < i_{l2} < i_{l3}$, for any $1 \leq l \leq s$. This further implies that $H_{\bar{\partial}}^{2,1}(X)$ is generated by $\{[dz_{i_{l1}} \wedge dz_{i_{l2}} \wedge d\bar{z}_{i_{l3}}]\}_{1 \leq l \leq s}$. Now take

$$\bigwedge : H_{\bar{\partial}}^{2,1}(X) \times H_{\bar{\partial}}^{2,1}(X) \longrightarrow H_{\bar{\partial}}^{4,2}(X)$$

given by $[\alpha] \wedge [\beta] \mapsto [\alpha \wedge \beta]$. If for some $k \neq l$ we had $\{i_{k1}, i_{k2}, i_{k3}\} \cap \{i_{l1}, i_{l2}, i_{l3}\} \neq \emptyset$, then $[dz_{i_{k1}} \wedge dz_{i_{k2}} \wedge dz_{i_{k3}}] \wedge [dz_{i_{l1}} \wedge dz_{i_{l2}} \wedge dz_{i_{l3}}] = 0$ and therefore $\dim_{\mathbb{C}} \operatorname{Im} \bigwedge < \binom{s}{2}$. However, \bigwedge is surjective, hence $\{i_{k1}, i_{k2}, i_{k3}\} \cap \{i_{l1}, i_{l2}, i_{l3}\} = \emptyset$ for different k and l and

$$\bigcup_{1 \leq k \leq s} \{i_{k1}, i_{k2}, i_{k3}\} = \{1, \dots, s + 2t\}.$$

Moreover, each of the s multiplicative relations contains exactly one real embedding (i.e., $i_{k1} \leq s$, for all $1 \leq k \leq s$). Indeed, if this were not true, then a certain multiplicative relation $\sigma_{i_{k1}} \cdot \sigma_{i_{k2}} \cdot \sigma_{i_{k3}}|_U \equiv 1$ would have $s < i_{k1} < i_{k2}$. Since in this case the conjugate relation $\bar{\sigma}_{i_{k1}} \cdot \bar{\sigma}_{i_{k2}} \cdot \bar{\sigma}_{i_{k3}}|_U \equiv 1$ would produce a non-zero element in $H_{\bar{\partial}}^{1,2}(X)$, which is impossible, we get our conclusion. Now, since $i_{k1} \leq s$, for every $1 \leq k \leq s$, we get that necessarily $\sigma_{i_{k2}}$ and $\sigma_{i_{k3}}$ are conjugate to each other, otherwise $\bar{\sigma}_{i_{k1}} \cdot \bar{\sigma}_{i_{k2}} \cdot \bar{\sigma}_{i_{k3}}|_U \equiv 1$ would be a different relation, still involving the term $\sigma_{i_{k1}}$, which is again impossible. Therefore the s multiplicative relations are exactly the numerical condition described in [25, Theorem 3.2], see Corollary 3, which is equivalent to the existence of a pluriclosed metric. \square

References

- [1] D. ANGELLA, M. PARTON, AND V. VULETESCU, Rigidity of Oeljeklaus–Toma manifolds, *Ann. Inst. Fourier (Grenoble)* **70(6)** (2020), 2409–2423. DOI: 10.5802/aif.3387.
- [2] J.-M. BISMUT, A local index theorem for non Kähler manifolds, *Math. Ann.* **284(4)** (1989), 681–699. DOI: 10.1007/BF01443359.
- [3] E. BOMBIERI, Letter to Kodaira (1973).
- [4] O. BRAUNLING, Oeljeklaus–Toma manifolds and arithmetic invariants, *Math. Z.* **286(1-2)** (2017), 291–323. DOI: 10.1007/s00209-016-1763-1.
- [5] G. R. CAVALCANTI, Hodge theory of SKT manifolds, *Adv. Math.* **374** (2020), 107270, 42 pp. DOI: 10.1016/j.aim.2020.107270.
- [6] Ș. DEACONU AND V. VULETESCU, On locally conformally Kähler metrics on Oeljeklaus–Toma manifolds, *Manuscripta Math.* **171(3-4)** (2023), 643–647. DOI: 10.1007/s00229-022-01403-0.
- [7] S. DINEW, Pluripotential theory on compact Hermitian manifolds, *Ann. Fac. Sci. Toulouse Math. (6)* **25(1)** (2016), 91–139. DOI: 10.5802/afst.1488.
- [8] A. DUBICKAS, Nonreciprocal units in a number field with an application to Oeljeklaus–Toma manifolds, *New York J. Math.* **20** (2014), 257–274.
- [9] A. DUBICKAS, Units in number fields satisfying a multiplicative relation with application to Oeljeklaus–Toma manifolds, *Results Math.* **76(2)** (2021), Paper no. 78, 12 pp.
- [10] A. FINO, H. KASUYA, AND L. VEZZONI, SKT and tamed symplectic structures on solvmanifolds, *Tohoku Math. J. (2)* **67(1)** (2015), 19–37. DOI: 10.2748/tmj/1429549577.
- [11] A. FINO AND A. TOMASSINI, A survey on strong KT structures, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **52(100)**, no. 2 (2009), 99–116.
- [12] A. FINO AND A. TOMASSINI, On astheno-Kähler metrics, *J. Lond. Math. Soc. (2)* **83(2)** (2011), 290–308. DOI: 10.1112/jlms/jdq066.
- [13] J. FU, Z. WANG, AND D. WU, Semilinear equations, the γ_k function, and generalized Gauduchon metrics, *J. Eur. Math. Soc. (JEMS)* **15(2)** (2013), 659–680. DOI: 10.4171/JEMS/370.
- [14] P. GAUDUCHON, Le théorème de l’excentricité nulle, *C. R. Acad. Sci. Paris Sér. A-B* **285(5)** (1977), A387–A390.
- [15] D. GRANTCHAROV, G. GRANTCHAROV, AND Y. S. POON, Calabi–Yau connections with torsion on toric bundles, *J. Differential Geom.* **78(1)** (2008), 13–32. DOI: 10.4310/jdg/1197320602.
- [16] A. GRAY AND L. M. HERVELLA, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl. (4)* **123** (1980), 35–58. DOI: 10.1007/BF01796539.
- [17] M. INOUE, On surfaces of Class VII₀, *Invent. Math.* **24** (1974), 269–310. DOI: 10.1007/BF01425563.
- [18] N. ISTRATI AND A. OTIMAN, De Rham and twisted cohomology of Oeljeklaus–Toma manifolds, *Ann. Inst. Fourier (Grenoble)* **69(5)** (2019), 2037–2066. DOI: 10.5802/aif.3288.
- [19] S. IVANOV AND G. PAPADOPOULOS, Vanishing theorems and string backgrounds, *Classical Quantum Gravity* **18(6)** (2001), 1089–1110. DOI: 10.1088/0264-9381/18/6/309.
- [20] J. JOST AND S.-T. YAU, Correction to: “A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry” [*Acta Math.* **170(2)** (1993), 221–254; MR1226528], *Acta Math.* **173(2)** (1994), 307. DOI: 10.1007/BF02398438.
- [21] H. KASUYA, Vaisman metrics on solvmanifolds and Oeljeklaus–Toma manifolds, *Bull. Lond. Math. Soc.* **45(1)** (2013), 15–26. DOI: 10.1112/blms/bds057.
- [22] H. KASUYA, Remarks on Dolbeault cohomology of Oeljeklaus–Toma manifolds and Hodge theory, *Proc. Amer. Math. Soc.* **149(7)** (2021), 3129–3137. DOI: 10.1090/proc/15436.
- [23] M. KHOVANOV AND Y. QI, A faithful braid group action on the stable category of tricomplexes, *SIGMA Symmetry Integrability Geom. Methods Appl.* **16** (2020), Paper no. 019, 32 pp. DOI: 10.3842/SIGMA.2020.019.
- [24] K. OELJEKLAUS AND M. TOMA, Non-Kähler compact complex manifolds associated to number fields, *Ann. Inst. Fourier (Grenoble)* **55(1)** (2005), 161–171. DOI: 10.5802/aif.2093.
- [25] A. OTIMAN, Special Hermitian metrics on Oeljeklaus–Toma manifolds, *Bull. Lond. Math. Soc.* **54(2)** (2022), 655–667. DOI: 10.1112/blms.12590.
- [26] A. OTIMAN AND M. TOMA, Hodge decomposition for Cousin groups and for Oeljeklaus–Toma manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **22(2)** (2021), 485–503. DOI: 10.2422/2036-2145.201812_001.
- [27] D. POPOVICI, Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics, *Invent. Math.* **194(3)** (2013), 515–534. DOI: 10.1007/s00222-013-0449-0.
- [28] V. V. PRASOLOV, *Polynomials*, Translated from the 2001 Russian second edition by Dimitry Leites, Algorithms Comput. Math. **11**, Springer-Verlag, Berlin, 2010.

- [29] J. STELZIG, On the structure of double complexes, *J. Lond. Math. Soc. (2)* **104(2)** (2021), 956–988. DOI: 10.1112/jlms.12453.
- [30] J. STELZIG, On linear combinations of cohomological invariants of compact complex manifolds, *Adv. Math.* **407** (2022), Paper no. 108560. DOI: 10.1016/j.aim.2022.108560.
- [31] J. STREETS AND G. TIAN, A parabolic flow of pluriclosed metrics, *Int. Math. Res. Not. IMRN* **2010(16)** (2010), 3101–3133. DOI: 10.1093/imrn/rnp237.
- [32] A. TOMASSINI AND S. TORELLI, On the cohomology of Oeljeklaus–Toma manifolds, Preprint (2015).
- [33] F. TRICERRI, Some examples of locally conformal Kähler manifolds, *Rend. Sem. Mat. Univ. Politec. Torino* **40(1)** (1982), 81–92.
- [34] S. M. VERBITSKAYA, Curves on Oeljeklaus–Toma manifolds (Russian), *Funktsional. Anal. i Prilozhen.* **48(3)** (2014), 84–88; translation in: *Funct. Anal. Appl.* **48(3)** (2014), 223–226. DOI: 10.1007/s10688-014-0063-y.
- [35] S. VERBITSKY, Surfaces on Oeljeklaus–Toma manifolds, Preprint (2013). arXiv:1306.2456v1.
- [36] V. VULETESCU, LCK metrics on Oeljeklaus–Toma manifolds versus Kronecker’s theorem, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **57(105)**, no. 2 (2014), 225–231.

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