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CORRIGENDUM AND ADDENDUM TO "STRUCTURE MONOIDS OF SET-THEORETIC SOLUTIONS OF THE YANG–BAXTER EQUATION"

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Abstract: One of the results in our article which appeared in *Publ. Mat.* **65(2)** (2021), 499–528, is that the structure monoid $M(X,r)$ of a left non-degenerate solution (X,r) of the Yang–Baxter equation is a left semi-truss, in the sense of Brzeziński, with an additive structure monoid that is close to being a normal semigroup. Let η denote the least left cancellative congruence on the additive monoid $M(X,r)$. It is then shown that η is also a congruence on the multiplicative monoid $M(X,r)$ and that the left cancellative epimorphic image $\bar{M} = M(X,r)/\eta$ inherits a semi-truss structure and thus one obtains a natural left non-degenerate solution of the Yang–Baxter equation on $\bar{M}$. Moreover, it restricts to the original solution r for some interesting classes, in particular if (X,r) is irretractable. The proof contains a gap. In the first part of the paper we correct this mistake by introducing a new left cancellative congruence μ on the additive monoid $M(X,r)$ and show that it also yields a left cancellative congruence on the multiplicative monoid $M(X,r)$, and we obtain a semi-truss structure on $M(X,r)/\mu$ that also yields a natural left non-degenerate solution.

In the second part of the paper we start from the least left cancellative congruence ν on the multiplicative monoid $M(X,r)$ and show that it is also a congruence on the additive monoid $M(X,r)$ in the case where r is bijective. If, furthermore, r is left and right non-degenerate and bijective, then $\nu = \eta$, the least left cancellative congruence on the additive monoid $M(X,r)$, extending an earlier result of Jespers, Kubat, and Van Antwerpen to the infinite case.

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1. Introduction

We have detected a mistake in the proof of [**3**, Lemma 5.5]. What is correctly proved is the following result for a left non-degenerate solution (X,r) of the Yang–Baxter equation (YBE) with structure monoid $M = M(X,r)$. Write $r(x,y) = (\sigma_x(y), \gamma_y(x))$. Thus, all σ_x are bijective maps. Its additive structure is denoted by $(M,+)$ and its multiplicative structure by $(M,\circ)$. The least cancellative congruence on $(M,+)$ is denoted by η. Let $\lambda' \colon (M,\circ) \to \operatorname{End}(M,+)$ denote the unique monoid homomorphism such that $\lambda'(x)(y) = \sigma_x(y)$ for $x, y \in X$ (see Proposition 3.1 in [**3**]).

Lemma 1.1. *With the same notation as in* [**3**, Lemma 5.5] *we have $\eta = \eta'$. Furthermore, for all $z \in M$,*

$$\eta = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a,b) \in \eta\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a,b) \in \eta\}.$$

We do not know whether $\lambda'_a = \lambda'_b$, for all $(a,b) \in \eta$, and whether η is a congruence on $(M,\circ)$. As a consequence [**3**, Remark 5.6, Corollaries 5.9 and 5.10] are not

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proved. Therefore [**3**, Question 5.7] and the definition of an injective left non-degenerate solution of the YBE given in [**3**] have no sense. In Section 2 we will introduce a new congruence on $(M,+)$ and prove a correct version of the listed corollaries.

In Section 3, we start from the least left cancellative congruence ν on the multiplicative monoid $(M,\circ)$ and show that it is also a congruence on the additive monoid $(M,+)$ in the case where r is bijective. If furthermore r is left and right non-degenerate and bijective, then $\nu=\eta$, the least left cancellative congruence on the additive monoid $(M,+)$, extending an earlier result of Jespers, Kubat, and Van Antwerpen to the infinite case.

2. Correction of [3, Section 5]

In this section, we shall introduce a new congruence μ on $(M,+)$ such that it is also a congruence on $(M,\circ)$, $(M,+)/\mu$ is left cancellative, and $((\lambda'_a)^{\varepsilon}(b),(\lambda'_{a'})^{\varepsilon}(b'))\in\mu$, for all $(a,a'),(b,b')\in\mu$ and $\varepsilon\in\{-1,1\}$. Furthermore, μ is the least binary relation on M with these properties.

We first recall the definition of a left semi-truss.

Definition 2.1 (Brzeziński [**1**])**.** *A left semi-truss is a quadruple $(A,+,\circ,\phi)$ such that $(A,+)$ and $(A,\circ)$ are semigroups and $\phi\colon A\times A\to A$ is a function such that*

$$a\circ(b+c)=(a\circ b)+\phi(a,c),$$

for all $a,b,c\in A$.

Example 2.2 ([**3**, Example 5.2])**.** *Let (X,r) be a left non-degenerate set-theoretic solution of the YBE (not necessarily bijective). Again write $r(x,y)=(\sigma_x(y),\gamma_y(x))$, for $x,y\in X$. As stated in* [**3**, Section 3]*, and with the same notation, the map*

$$r'(x,y)=(y,\sigma_y\gamma_{\sigma_x^{-1}(y)}(x))$$

defines the left derived solution on X. Let $M=M(X,r)$ and $M'=A(X,r)=M(X,r')$ be the structure monoids of the solutions (X,r) and (X,r') respectively. From [**3**, Corollary 3.9 and Proposition 3.1] *we obtain a left action $\lambda'\colon(M,\circ)\to\mathrm{Aut}(M',+)$ and a bijective 1-cocycle $\pi\colon M\to M'$ with respect to λ' satisfying $\lambda'(x)(y)=\sigma_x(y)$ and $\pi(x)=x$, for all $x,y\in X$. We identify M and M' via π, that is, $a=\pi(a)$ for all $a\in M$. With this identification, we obtain the operation $+$ on M, and $a\circ b=a+\lambda'_a(b)$, for all $a,b\in M$. Put $\phi(a,b)=\lambda'_a(b)$, for all $a,b\in M$. Then,*

$$a\circ(b+c)=a+\lambda'_a(b+c)=a+\lambda'_a(b)+\lambda'_a(c)=(a\circ b)+\phi(a,c),$$

for all $a,b\in M$. Furthermore, $M+a\subseteq a+M$, for all $a\in M$. Hence $(M,+,\circ,\phi)$ is a left semi-truss. Note that if, furthermore, r is bijective, then it can easily be verified that (X,r') is a right non-degenerate solution and thus $M+a=a+M$ for all $a\in M$; that is, $(M,+)$ consists of normal elements. As shown in [**4**]*, this property is fundamental in the study of the associated structure algebra $KM(X,r)$, where K is a field.*

We will use the assumptions and notations as in Example 2.2.

Let

$$\mu_0=\{(a,b)\in M^2\mid\exists\, c\in M\text{ such that }c+a=c+b\}.$$

Note that μ_0 is a reflexive and symmetric binary relation on M. Let μ_1 be its transitive closure, that is,

$$\mu_1 = \{(a,b) \in M^2 \mid \exists\, a_1,\dots,a_n \in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b) \in \mu_0\}.$$

Thus μ_1 is an equivalence relation on M. Let

$$\mu_2 = \{((\lambda'_z)^{\varepsilon}(a\circ c),(\lambda'_z)^{\varepsilon}(b\circ c)) \in M^2 \mid z,c \in M,\ \varepsilon \in \{-1,1\},\ \text{and } (a,b) \in \mu_1\},$$
$$\mu_3 = \{(a,b) \in M^2 \mid \exists\, a_1,\dots,a_n \in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b) \in \mu_2\},$$
$$\mu_4 = \{(c+a+d,c+b+d) \in M^2 \mid c,d \in M \text{ and } (a,b) \in \mu_3\}$$
$$\cup\,\{(a,b) \in M^2 \mid \exists\, c \in M \text{ such that } (c+a,c+b) \in \mu_3\},$$

and for every $m \geq 1$ we define

$$\mu_{4m+1} = \{(a,b) \in M^2 \mid \exists\, a_1,\dots,a_n \in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b) \in \mu_{4m}\},$$
$$\mu_{4m+2} = \{((\lambda'_z)^{\varepsilon}(a\circ c),(\lambda'_z)^{\varepsilon}(b\circ c)) \in M^2 \mid z,c \in M,\ \varepsilon \in \{-1,1\},\ \text{and } (a,b) \in \mu_{4m+1}\},$$
$$\mu_{4m+3} = \{(a,b) \in M^2 \mid \exists\, a_1,\dots,a_n \in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b) \in \mu_{4m+2}\},$$
$$\mu_{4(m+1)} = \{(c+a+d,c+b+d) \in M^2 \mid c,d \in M \text{ and } (a,b) \in \mu_{4m+3}\}$$
$$\cup\,\{(a,b) \in M^2 \mid \exists\, c \in M \text{ such that } (c+a,c+b) \in \mu_{4m+3}\}.$$

Note that $\mu_n \subseteq \mu_{n+1}$, for all $n \geq 0$. Let $\mu = \bigcup_{n=0}^{\infty} \mu_n$.

Lemma 2.3. *With the above notation, we have that μ is a congruence on $(M,+)$ and it is also a congruence on $(M,\circ)$. Furthermore, $(M,+)/\mu$ and $(M,\circ)/\mu$ are left cancellative monoids, and*

$$(\lambda'_c(a),\lambda'_d(b)),((\lambda'_c)^{-1}(a),(\lambda'_d)^{-1}(b)) \in \mu,$$

for all $(a,b),(c,d) \in \mu$.

Proof: First we shall prove that μ is a congruence on $(M,+)$. Clearly μ is reflexive and symmetric because so is each μ_n. Let $a,b,c \in M$ be such that $(a,b),(b,c) \in \mu$. There exists a positive integer m such that $(a,b),(b,c) \in \mu_{2m}$. Since μ_{2m+1} is the transitive closure of μ_{2m}, we have that $(a,c) \in \mu_{2m+1} \subseteq \mu$. Hence μ is an equivalence relation.

Let $(a,b) \in \mu$ and $c,d \in M$. There exists a positive integer k such that $(a,b) \in \mu_{4k+3}$. Thus, $(c+a+d,c+b+d) \in \mu_{4(k+1)} \subseteq \mu$. Hence, μ is a congruence on $(M,+)$.

Let $(c,c') \in \mu$ and $a,b \in M$ be such that $(c+a,c'+b) \in \mu$. Since μ is a congruence on $(M,+)$, we have that $(c'+a,c+a) \in \mu$. Hence, $(c'+a,c'+b) \in \mu$. There exists a positive integer m such that $(c'+a,c'+b) \in \mu_{4m+3}$. Hence $(a,b) \in \mu_{4(m+1)} \subseteq \mu$. Therefore, $(M,+)/\mu$ is a left cancellative monoid.

Let $(a,b) \in \mu$ and $c,d \in M$. There exists a positive integer k such that $(a,b) \in \mu_{4k+1}$. It follows that $(\lambda'_d(a\circ c),\lambda'_d(b\circ c)) \in \mu_{4k+2}$ and $(d\circ a\circ c, d\circ b\circ c) = (d+\lambda'_d(a\circ c), d+\lambda'_d(b\circ c)) \in \mu_{4(k+1)} \subseteq \mu$. Hence, μ is a congruence on $(M,\circ)$.

Let $(c,c') \in \mu$ and $a,b \in M$ be such that $(c\circ a,c'\circ b) \in \mu$. Since μ is a congruence on $(M,\circ)$, we have that $(c'\circ a,c\circ a) \in \mu$. Hence $(c'+\lambda_{c'}(a),c'+\lambda_{c'}(b)) = (c'\circ a,c'\circ b) \in \mu$. Since $(M,+)/\mu$ is a left cancellative monoid we get that $(\lambda_{c'}(a),\lambda_{c'}(b)) \in \mu$. Now there exists a positive integer m such that $(\lambda_{c'}(a),\lambda_{c'}(b)) \in \mu_{4m+1}$, and thus $(a,b) \in \mu_{4m+2} \subseteq \mu$. Therefore $(M,\circ)/\mu$ is a left cancellative monoid.

Let $(a,b),(c,d) \in \mu$. Since μ is a congruence on $(M,\circ)$, we have that

$$(c+\lambda'_c(x),d+\lambda'_d(x)) = (c\circ x,d\circ x) \in \mu,$$

for all $x \in M$. Since $(M,+)/\mu$ is a left cancellative monoid, we get that $(\lambda'_c(x), \lambda'_d(x)) \in \mu$, for all $x \in M$. For $x = (\lambda'_c)^{-1}(y)$, we have that

$$(y, \lambda'_d(\lambda'_c)^{-1}(y)) \in \mu,$$

for all $y \in M$. Thus, there exists a positive integer m such that $(y, \lambda'_d(\lambda'_c)^{-1}(y)) \in \mu_{4m+1}$. Hence $((\lambda'_d)^{-1}(y), (\lambda'_c)^{-1}(y)) \in \mu_{4m+2}$. Therefore,

$$((\lambda'_d)^{-1}(y), (\lambda'_c)^{-1}(y)) \in \mu,$$

for all $y \in M$. Now there exists a positive integer k such that

$$((\lambda'_d)^{-1}(a), (\lambda'_c)^{-1}(a)), (\lambda'_d(a), \lambda'_c(a)), (a, b) \in \mu_{4k+1}.$$

Hence,

$$((\lambda'_c)^{-1}(a), (\lambda'_d)^{-1}(a)), ((\lambda'_d)^{-1}(a), (\lambda'_d)^{-1}(b)), (\lambda'_c(a), \lambda'_d(a))(\lambda'_d(a), \lambda'_d(b)) \in \mu_{4k+2},$$

and thus,

$$(\lambda'_c(a), \lambda'_d(b)), ((\lambda'_c)^{-1}(a), (\lambda'_d)^{-1}(b)) \in \mu_{4k+3}.$$

Therefore,

$$(\lambda'_c(a), \lambda'_d(b)), ((\lambda'_c)^{-1}(a), (\lambda'_d)^{-1}(b)) \in \mu,$$

for all $(a, b), (c, d) \in \mu$, and the result follows. $\square$

With the assumptions and notations as in Example 2.2, let $\bar{M} = M/\mu$ and let $M \to \bar{M} \colon a \mapsto \bar{a}$ be the natural map. Let $\bar{\lambda} \colon (\bar{M}, \circ) \to \operatorname{Aut}(\bar{M}, +)$ be the map defined by $\bar{\lambda}(\bar{a}) = \bar{\lambda}_{\bar{a}}$ and $\bar{\lambda}_{\bar{a}}(\bar{b}) = \overline{\lambda'_a(b)}$, for all $a, b \in M$.

Note that $\bar{\lambda}$ is well defined, because if $\bar{c} = \bar{a}$ and $\bar{d} = \bar{b}$, then, by Lemma 2.3,

$$\overline{\lambda'_a(b)} = \overline{\lambda'_c(d)}.$$

Now it is easy to check that $\bar{\lambda}_{\bar{a}} \in \operatorname{Aut}(\bar{M}, +)$; in fact $(\bar{\lambda}_{\bar{a}})^{-1} \colon \bar{M} \to \bar{M}$ is the map defined by $(\bar{\lambda}_{\bar{a}})^{-1}(\bar{b}) = \overline{(\lambda'_a)^{-1}(b)}$, which is also well defined by Lemma 2.3. Furthermore, by Lemma 2.3, $(\bar{M}, \circ)$ is left cancellative and $\bar{\lambda}$ is a homomorphism such that $\bar{a} \circ \bar{b} = \bar{a} + \bar{\lambda}_{\bar{a}}(\bar{b})$, for all $a, b \in M$.

Let $\bar{\phi} \colon \bar{M} \times \bar{M} \to \bar{M}$ be the map defined by $\bar{\phi}(\bar{a}, \bar{b}) = \bar{\lambda}_{\bar{a}}(\bar{b})$, for all $a, b \in M$. Then $(\bar{M}, +, \circ, \bar{\phi})$ is a left semi-truss.

By [**3**, Lemma 5.8], the left cancellative monoid $(\bar{M}, +)$ satisfies that for all $\bar{a}, \bar{b} \in \bar{M}$ there exists a unique $\bar{c} \in \bar{M}$ (denoted as $c(\bar{a}, \bar{b})$) such that $\bar{a} + \bar{b} = \bar{b} + \bar{c}$. Hence, from [**3**, Proposition 5.4], we have the following corollary.

Corollary 2.4. *Let (X, r) be a left non-degenerate set-theoretic solution of the YBE. Let μ be the congruence on $M = (M(X, r'), +)$ defined above. Then $(\bar{M}, +, \circ, \bar{\phi})$ is a left semi-truss with $\bar{M} + \bar{a} \subseteq \bar{a} + \bar{M}$ for all $\bar{a} \in \bar{M}$ and with $\bar{\phi}(\bar{a}, \bar{b}) = \bar{\lambda}_{\bar{a}}(\bar{b})$, for all $\bar{a}, \bar{b} \in \bar{M}$. Furthermore, $(\bar{M}, \bar{r})$, where*

$$\bar{r}(\bar{a}, \bar{b}) = (\bar{\lambda}_{\bar{a}}(\bar{b}), \bar{\lambda}^{-1}_{\bar{\lambda}_{\bar{a}}(\bar{b})}(c(\bar{a}, \bar{\lambda}_{\bar{a}}(\bar{b})))),$$

for all $\bar{a}, \bar{b} \in \bar{M}$, is a left non-degenerate set-theoretic solution of the YBE. In particular, $(\bar{X}, \bar{r}_{|\bar{X}^2})$ is a left non-degenerate solution on the image $\bar{X}$ of X in $\bar{M}$.

3. Addendum

In this section, we will generalize the first part of [**4**, Proposition 4.2]. Let η be the left cancellative congruence on $(M,+)$, defined in [**3**]. For a left non-degenerate solution (X,r), we will define the (least) left cancellative congruence on $(M,\circ)$, say ν, and show that $\eta=\nu$ and $\lambda'_a=\lambda'_b$, for all $(a,b)\in\eta$, in the case where the solution is bijective and (left and right) non-degenerate. Again we will follow the notation of [**3**].

Let ν be the left cancellative congruence on $(M,\circ)$, that is, ν is the smallest congruence such that $\bar{M}=(M,\circ)/\nu$ is a left cancellative monoid.

We shall give a description of the elements in ν. Let

$$\nu_0=\{(a,b)\in M^2 \mid \exists\, c\in M \text{ such that } c\circ a=c\circ b\}.$$

Note that ν_0 is a reflexive and symmetric binary relation on M. Let ν_1 be its transitive closure, that is,

$$\nu_1=\{(a,b)\in M^2 \mid \exists\, a_1,\dots,a_n\in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b)\in\nu_0\}.$$

Thus, ν_1 is an equivalence relation on M. Let

$$\begin{aligned}\nu_2=\{&(c\circ a,c\circ b)\in M^2 \mid c\in M \text{ and } (a,b)\in\nu_1\}\\ &\cup\{(a,b)\in M^2 \mid \exists\, c\in M \text{ such that } (c\circ a,c\circ b)\in\nu_1\},\end{aligned}$$

and for every $m\geq 1$ we define

$$\nu_{2m+1}=\{(a,b)\in M^2 \mid \exists\, a_1,\dots,a_n\in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b)\in\nu_{2m}\}$$

and

$$\begin{aligned}\nu_{2m+2}=\{&(c\circ a,c\circ b)\in M^2 \mid c\in M \text{ and } (a,b)\in\nu_{2m+1}\}\\ &\cup\{(a,b)\in M^2 \mid \exists\, c\in M \text{ such that } (c\circ a,c\circ b)\in\nu_{2m+1}\}.\end{aligned}$$

Note that $\nu_n\subseteq\nu_{n+1}\subseteq\nu$ for all $n\geq 0$. Let $\nu'=\bigcup_{n=0}^{\infty}\nu_n$.

Lemma 3.1. *With the above notation we have that $\nu'=\nu$ and $\lambda'_a=\lambda'_b$, for all $(a,b)\in\nu$. Furthermore, if r is bijective, then for all $z\in M$,*

$$\nu\supseteq\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b)) \mid (a,b)\in\nu\},$$

and ν is also a congruence on $(M,+)$.

Proof: First we shall prove that ν' is a congruence on $(M,\circ)$. Clearly ν' is reflexive and symmetric because so is each ν_n. Let $a,b,c\in M$ be such that $(a,b),(b,c)\in\nu'$. There exists a positive integer m such that $(a,b),(b,c)\in\nu_{2m}$. Since ν_{2m+1} is the transitive closure of ν_{2m}, we have that $(a,c)\in\nu_{2m+1}\subseteq\nu'$. Hence ν' is an equivalence relation. Note that every ν_n satisfies that $(x\circ z,y\circ z)\in\nu_n$, for all $(x,y)\in\nu_n$. Thus $(a\circ c,b\circ c)\in\nu_{2m}\subseteq\nu'$. Since $(a,b)\in\nu_{2m}\subseteq\nu_{2m+1}$, we have that $(c\circ a,c\circ b)\in\nu_{2m+2}\subseteq\nu'$. Therefore, ν' is a congruence.

Let $a,b,c,c'\in M$ be elements such that $(c,c'),(c\circ a,c'\circ b)\in\nu'$. Since ν' is a congruence on $(M,\circ)$, $(c'\circ b,c\circ b)\in\nu'$. Hence $(c\circ a,c\circ b)\in\nu'$. There exists a positive integer t such that $(c\circ a,c\circ b)\in\nu_{2t+1}$. Thus $(a,b)\in\nu_{2t+2}\subseteq\nu'$. Hence $(M,\circ)/\nu'$ is a left cancellative monoid. Since $\nu'\subseteq\nu$, we have $\nu'=\nu$ by the definition of ν.

Let $(a,b)\in\nu_0$. Then there exists $c\in M$ such that $c\circ a=c\circ b$. Hence,

$$\lambda'_c\lambda'_a=\lambda'_c\lambda'_b$$

and thus $\lambda'_a=\lambda'_b$, for all $(a,b)\in\nu_0$. Let $n>0$ and suppose that $\lambda'_a=\lambda'_b$, for all $(a,b)\in\nu_{n-1}$. If $n-1$ is even, then for every $(a,b)\in\nu_n$ there exist $(a,c_1),(c_1,c_2),\dots,(c_k,b)\in\nu_{n-1}$. By the induction hypothesis $\lambda'_a=\lambda'_{c_1}=\cdots=\lambda'_{c_k}=\lambda'_b$. If $n-1$ is odd and $(a,b)\in\nu_n$, then either $(a,b)=(c\circ a',c\circ b')$, for some $c\in M$ and $(a',b')\in\nu_{n-1}$, or there exists $c \in M$ such that $(c \circ a, c \circ b) \in \nu_{n-1}$. In the first case, by the induction hypothesis, we have that

$$\lambda'_a = \lambda'_{c\circ a'} = \lambda'_c \lambda'_{a'} = \lambda'_c \lambda'_{b'} = \lambda'_{c\circ b'} = \lambda'_b.$$

In the second case, by the induction hypothesis, we have that

$$\lambda'_c \lambda'_a = \lambda'_{c\circ a} = \lambda'_{c\circ b} = \lambda'_c \lambda'_b,$$

and thus $\lambda'_a = \lambda'_b$. Hence, we get that $\lambda'_a = \lambda'_b$, for all $(a,b) \in \nu_n$. Hence, by induction, we have that $\lambda'_a = \lambda'_b$, for all $(a,b) \in \nu$.

Suppose that r is bijective. By Example 2.2, we have that $M + a = a + M$, for all $a \in M$. Let $(a,b) \in \nu_0$. Then there exists $c \in M$ such that $c \circ a = c \circ b$. Let $y \in M$. We have that there exists $z \in M$ such that $z + c = c + y$. Hence,

$$\begin{aligned}
(\lambda'_z)^{-1}(c \circ a) &= (\lambda'_z)^{-1}(c + \lambda'_c(a)) \\
&= (\lambda'_z)^{-1}(c) + (\lambda'_z)^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_z)^{-1}(c)})^{-1}(\lambda'_z)^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{z\circ(\lambda'_z)^{-1}(c)})^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{z+c})^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{c+y})^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{c\circ(\lambda'_c)^{-1}(y)})^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(\lambda'_c)^{-1}\lambda'_c(a) \\
&= (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a).
\end{aligned}$$

Since $c \circ a = c \circ b$, we have that

$$(\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a) = (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(b).$$

We get that

$$((\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a), (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(b)) \in \nu_0,$$

for all $y \in M$. Hence,

$$\nu_0 \supseteq \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a,b) \in \nu_0\},$$

for all $z \in M$. Let n be a positive integer and suppose that

$$\nu_{n-1} \supseteq \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a,b) \in \nu_{n-1}\},$$

for all $z \in M$. Let $(a,b) \in \nu_n$. If n is odd, then there exist $(a,c_1),(c_1,c_2),\dots,(c_k,b) \in \nu_{n-1}$. By the induction hypothesis,

$$((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(c_1)), ((\lambda'_z)^{-1}(c_1), (\lambda'_z)^{-1}(c_2)), \dots, ((\lambda'_z)^{-1}(c_k), (\lambda'_z)^{-1}(b)) \in \nu_{n-1}.$$

Hence $((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \in \nu_n$, in this case. If n is even, then either $(a,b) = (c \circ a', c \circ b')$, for some $c \in M$ and $(a',b') \in \nu_{n-1}$, or there exists $c \in M$ such that $(c \circ a, c \circ b) \in \nu_{n-1}$. In the first case,

$$(\lambda'_z)^{-1}(a) = (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a'),$$

and

$$(\lambda'_z)^{-1}(b) = (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(b'),$$

where $z + c = c + y$. Hence, by the induction hypothesis, $((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \in \nu_n$, in this case. In the second case, by the induction hypothesis,

$$((\lambda'_z)^{-1}(c \circ a), (\lambda'_z)^{-1}(c \circ b)) \in \nu_{n-1}.$$

Since $(\lambda'_z)^{-1}(c \circ a) = (\lambda'_z)^{-1}(c) \circ (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a)$, we have that

$$((\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a), (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(b)) \in \nu_n.$$

Since $M + c = c + M$,

$$((\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(a), (\lambda'_{(\lambda'_c)^{-1}(y)})^{-1}(b)) \in \nu_n,$$

for all $y \in M$. Hence,

$$\nu_n \supseteq \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \nu_n\},$$

for all $z \in M$. By induction, we get that

$$\nu \supseteq \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \nu\},$$

for all $z \in M$.

Let $(a, b) \in \nu$. Then for every $c \in M$, we have that

$$(c + a, c + b) = (c \circ (\lambda'_c)^{-1}(a), c \circ (\lambda'_c)^{-1}(b)) \in \nu.$$

Since $\lambda'_a = \lambda'_b$, we have that

$$(a + c, b + c) = (a \circ (\lambda'_a)^{-1}(c), b \circ (\lambda'_b)^{-1}(c)) = (a \circ (\lambda'_a)^{-1}(c), b \circ (\lambda'_a)^{-1}(c)) \in \nu.$$

Hence ν is a congruence on $(M, +)$, and the result follows. $\square$

In order to prove the main result of this section, we first show that, for left non-degenerate set-theoretic solutions of the YBE, the maps λ and λ' are equal. Here λ is the unique monoid homomorphism $M \to \mathrm{Map}(M, M)\colon a \mapsto \lambda_a$ defined in [**3**, Theorem 2.1] such that $\lambda_b(a \circ c) = \lambda_b(a) \circ \lambda_{\rho_a(b)}(c)$ and $\rho_b(c \circ a) = \rho_{\lambda_a(b)}(c) \circ \rho_b(a)$, where also $\rho\colon M \to \mathrm{Map}(M, M)$ is the monoid anti-homomorphism defined in [**3**, Theorem 2.1]. This result comes from [**2**], but for completeness' sake we include a proof.

Lemma 3.2. *Let (X, r) be a set-theoretic solution of the YBE. Let $M = M(X, r)$ and $M' = A(X, r)$. As usual, write $r(x, y) = (\sigma_x(y), \gamma_y(x))$. Then, $\lambda'_a(\pi(b)) = \pi(\lambda_a(b))$, for all $a, b \in M$, where $\pi\colon M \to M'$ is the unique 1-cocycle with respect to the left action λ' such that $\pi(x) = x$, for all $x \in X$. Furthermore, if (X, r) is left non-degenerate, then, with the identification of M and M' in Example 2.2, $\lambda'_a(b) = \lambda_a(b)$, for all $a, b \in M$. In particular,*

$$\lambda'_x(x_1 \circ \cdots \circ x_k \circ a) = \lambda'_x(x_1 \circ \cdots \circ x_k) \circ \lambda'_{\gamma_{x_k} \cdots \gamma_{x_1}(x)}(a), \tag{1}$$

for all $x, x_1, \dots, x_k \in X$ and $a \in M$.

Proof: The existence and uniqueness of π is proved in [**3**, Proposition 3.2]. Let $b \in M$. There exist a non-negative integer k and $x_1, \dots, x_k \in X$ such that $b = x_1 \circ \cdots \circ x_k$. We first prove that $\lambda'_x(\pi(b)) = \pi(\lambda_x(b))$, for all $x \in X$, by induction on k. If $k = 0$, then $\pi(1) = 0$ and by the definition of λ, $\lambda_x(1) = 1$. Hence $\lambda'_x(\pi(1)) = \lambda'_x(0) = 0 = \pi(1) = \pi(\lambda_x(1))$. For $k = 1$,

$$\pi(\lambda_x(x_1)) = \sigma_x(x_1) = \sigma_x(\pi(x_1)) = \lambda'_x(\pi(x_1)).$$

Suppose that $k>1$ and we have proved the result for words in $M(X,r)$ of length at most $k-1$. By the definition of λ, [**3**, Theorem 2.1], and the induction hypothesis, we have

$$\begin{aligned}\pi(\lambda_x(b)) &= \pi(\lambda_x(x_1\circ\cdots\circ x_k))\\ &= \pi(\lambda_x(x_1)\circ\lambda_{\rho_{x_1}(x)}(x_2\circ\cdots\circ x_k))\\ &= \pi(\lambda_x(x_1))+\lambda'_{\lambda_x(x_1)}(\pi(\lambda_{\rho_{x_1}(x)}(x_2\circ\cdots\circ x_k)))\\ &= \lambda'_x(\pi(x_1))+\lambda'_{\lambda_x(x_1)}(\lambda'_{\rho_{x_1}(x)}(\pi(x_2\circ\cdots\circ x_k)))\\ &= \lambda'_x(\pi(x_1))+\lambda'_x(\lambda'_{x_1}(\pi(x_2\circ\cdots\circ x_k)))\\ &= \lambda'_x(\pi(x_1)+\lambda'_{x_1}(\pi(x_2\circ\cdots\circ x_k)))\\ &= \lambda'_x(\pi(x_1\circ\cdots\circ x_k))\\ &= \lambda'_x(\pi(b)).\end{aligned}$$

Hence, by induction $\lambda'_x(\pi(b))=\pi(\lambda_x(b))$, for all $x\in X$ and $b\in M$. Using that both λ and λ' are homomorphisms, we obtain $\lambda'_a(\pi(b))=\pi(\lambda_a(b))$ for all $a,b\in M$.

Suppose that (X,r) is left non-degenerate. Then with the identification of M and M' in Example 2.2, we have that $\lambda'_a(b)=\lambda_a(b)$, for all $a,b\in M$. In this case, by [**3**, Theorem 2.1],

$$\begin{aligned}\lambda'_x(x_1\circ\cdots\circ x_k\circ a) &= \lambda_x(x_1\circ\cdots\circ x_k\circ a)\\ &= \lambda_x(x_1\circ\cdots\circ x_k)\circ\lambda_{\rho_{x_1\circ\cdots\circ x_k}(x)}(a)\\ &= \lambda_x(x_1\circ\cdots\circ x_k)\circ\lambda_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a)\\ &= \lambda'_x(x_1\circ x_2\circ\cdots\circ x_k)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a),\end{aligned}$$

for all $x,x_1,\dots,x_k\in X$ and $a\in M$. Hence, (1) follows. ☐

Proposition 3.3. *Let (X,r) be a bijective (left and right) non-degenerate set-theoretic solution of the YBE. Let $M=M(X,r)$. As usual, write $r(x,y)=(\sigma_x(y),\gamma_y(x))$. Let ν be the left cancellative congruence on $(M,\circ)$, and let η be the left cancellative congruence on $(M,+)$. Then $\eta=\nu$ and thus, for every $z\in M$,*

$$\nu=\{(\lambda'_z(a),\lambda'_z(b))\mid(a,b)\in\nu\}=\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid(a,b)\in\nu\}.$$

Furthermore $\lambda'_a=\lambda'_b$, for all $(a,b)\in\eta$.

Proof: From the proof of Lemma 3.1, we know that for all $z\in M$,

$$\nu_0\supseteq\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid(a,b)\in\nu_0\},$$

and ν is also a congruence on $(M,+)$.

Let $(a,b)\in\nu_0$. Then there exists $c\in M$ such that $c\circ a=c\circ b$. There exist $x_1,\dots,x_k\in X$ such that $c=x_1\circ\cdots\circ x_k$. Let $x\in X$. By (1) (in Lemma 3.2), we have that

$$\lambda'_x(c\circ a)=\lambda'_x(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a).$$

Hence

$$\lambda'_x(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a)=\lambda'_x(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(b),$$

for all $x\in X$. Hence, $(\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a),\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(b))\in\nu_0$, for all $x\in X$. Since (X,r) is right non-degenerate, and thus all γ_{x_i} are bijective, we obtain that $(\lambda'_y(a),\lambda'_y(b))\in\nu_0$, for all $y\in X$. Therefore $(\lambda'_z(a),\lambda'_z(b))\in\nu_0$, for all $z\in M$. Since $((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\in\nu_0$, for all $z\in M$, we get that

$$\nu_0=\{(\lambda'_z(a),\lambda'_z(b))\mid (a,b)\in\nu_0\}=\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid (a,b)\in\nu_0\},$$

for all $z\in M$. We shall prove by induction on n that

$$\nu_n=\{(\lambda'_z(a),\lambda'_z(b))\mid (a,b)\in\nu_n\}=\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid (a,b)\in\nu_n\}, \tag{2}$$

for all $z\in M$ and all non-negative integers n. Suppose that $n>0$ and (2) is true for $n-1$. From the proof of Lemma 3.1, we know that for all $z\in M$,

$$\nu_n\supseteq\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid (a,b)\in\nu_n\}.$$

Let $(a,b)\in\nu_n$, $z\in M$. If n is odd, then there exist $(a,c_1),(c_1,c_2),\dots,(c_k,b)\in\nu_{n-1}$. Hence,

$$(\lambda'_z(a),\lambda'_z(c_1)),(\lambda'_z(c_1),\lambda'_z(c_2)),\dots,(\lambda'_z(c_k),\lambda'_z(b))\in\nu_{n-1},$$

and thus $(\lambda'_z(a),\lambda'_z(b))\in\nu_n$, in this case. If n is even, then either $(a,b)=(c\circ a',c\circ b')$, for some $c\in M$ and $(a',b')\in\nu_{n-1}$, or there exists $c\in M$ such that $(c\circ a,c\circ b)\in\nu_{n-1}$. Put $c=x_1\circ\cdots\circ x_k$. In the first case, by the previous lemma, we get $(\lambda'_z(a),\lambda'_z(b))=(\lambda'_z(c\circ a'),\lambda'_z(c\circ b'))\overset{(1)}{=}(\lambda'_z(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(z)}(a'),\lambda'_z(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(z)}(b'))$. By the induction hypothesis, and since $(a',b')\in\nu_{n-1}$, also $(\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(z)}(a'),\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(z)}(b'))\in\nu_{n-1}$, and then $(\lambda'_z(a),\lambda'_z(b))=(\lambda'_z(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(z)}(a'),\lambda'_z(c)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(z)}(b'))\in\nu_n$ (because n is even). In the second case, by (1),

$$\lambda'_x(c\circ a)=\lambda'_x(x_1\circ\cdots\circ x_k\circ a)=\lambda'_x(x_1\circ\cdots\circ x_k)\circ\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a),$$

for all $x\in X$. By the induction hypothesis,

$$(\lambda'_x(c\circ a),\lambda'_x(c\circ b))\in\nu_{n-1},$$

for all $x\in X$. Hence,

$$(\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(a),\lambda'_{\gamma_{x_k}\cdots\gamma_{x_1}(x)}(b))\in\nu_n,$$

for all $x\in X$. Since (X,r) is right non-degenerate, we have that

$$(\lambda'_y(a),\lambda'_y(b))\in\nu_n,$$

for all $y\in X$. Hence, $(\lambda'_z(a),\lambda'_z(b))\in\nu_n$, for all $z\in M$. Since

$$\nu_n\supseteq\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid (a,b)\in\nu_n\},$$

we get that

$$\nu_n=\{(\lambda'_z(a),\lambda'_z(b))\mid (a,b)\in\nu_n\}=\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid (a,b)\in\nu_n\},$$

for all $z\in M$. Hence, by induction,

$$\nu=\{(\lambda'_z(a),\lambda'_z(b))\mid (a,b)\in\nu\}=\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b))\mid (a,b)\in\nu\},$$

for all $z\in M$.

Let $a,b,c,c'\in M$ be such that $(c,c'),(c+a,c'+b)\in\nu$. Since ν is a congruence on $(M,+)$, $(c'+b,c+b)\in\nu$. Hence $(c+a,c+b)\in\nu$. Then, $(c\circ(\lambda'_c)^{-1}(a),c\circ(\lambda'_c)^{-1}(b))=(c+a,c+b)\in\nu$. Hence, $((\lambda'_c)^{-1}(a),(\lambda'_c)^{-1}(b))\in\nu$ and thus $(a,b)\in\nu$. Therefore, $(M,+)/\nu$ is left cancellative and thus clearly $\eta\subseteq\nu$.

By Lemma 3.1, $\lambda'_a=\lambda'_b$, for all $(a,b)\in\eta\subseteq\nu$. Let $(a,b)\in\eta$ and let $c\in M$. By Lemma 1.1, we have that $(c\circ a,c\circ b)=(c+\lambda'_c(a),c+\lambda'_c(b))\in\eta$, and since $\lambda'_a=\lambda'_b$, we have that $(a\circ c,b\circ c)=(a+\lambda'_a(c),b+\lambda'_b(c))=(a+\lambda'_a(c),b+\lambda'_a(c))\in\eta$. Hence,

η is a congruence on $(M, \circ)$. Let $a, b, c, c' \in M$ be such that $(c, c'), (c \circ a, c' \circ b) \in \eta$. Then, $(c + \lambda'_c(a), c' + \lambda'_{c'}(b)) = (c \circ a, c' \circ b) \in \eta$. Since $\lambda'_c = \lambda'_{c'}$, we have that

$$(c + \lambda'_c(a), c' + \lambda'_c(b)), (c' + \lambda'_c(b), c + \lambda'_c(b)) \in \eta$$

and then $(c+\lambda'_c(a), c+\lambda'_c(b)) \in \eta$. Hence, $(\lambda'_c(a), \lambda'_c(b)) \in \eta$. By Lemma 1.1, $(a, b) \in \eta$. Therefore, $(M, \circ)/\eta$ is left cancellative and $\nu \subseteq \eta$. So, $\eta = \nu$ and the result follows. $\square$

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