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HOLOMORPHIC VECTOR FIELDS WITH A BARYCENTRIC CONDITION

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Abstract: We study the p-tuples of holomorphic vector fields $(X_1, X_2, \dots, X_p)$ satisfying the barycentric property $\sum_k \exp tX_k = p \cdot \mathrm{id}$, where $\exp tX$ denotes the flow of X.

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1. Introduction

Let $\mathcal{U}$ be a connected open subset of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). Let $X_1, X_2, \dots, X_p$ be p distinct analytic (resp. holomorphic) vector fields on $\mathcal{U}$. Denote by $\varphi_t^k = \exp(tX_k)$ the local one-parameter subgroup of X_k; it is the solution of the ordinary differential equation

$$\frac{d\varphi_t^k(x)}{dt} = X_k(\varphi_t^k(x))$$

with initial data $\varphi_0^k(x) = x$.

For any point $x \in \mathcal{U}$, $\varphi_t^k(x)$ is well defined for t sufficiently small and we assume that

$$\sum_{k=1}^{p} X_k(\varphi_t^k(x)) = 0 \quad \forall\, x \in \mathcal{U} \text{ and } t \text{ small.} \tag{1.1}$$

In particular, $\sum\limits_{k=1}^{p} \exp(tX_k) = p\,\mathrm{id}$ and by doing $t = 0$ in (1.1) we get

$$\sum_{k=1}^{p} X_k = 0.$$

Let us give an interpretation of (1.1): at any point x there are p identical particles transported by the vector fields X_k while preserving their barycenter at the initial position x. The condition (1.1) is called the *barycentric property.* A set of p vector fields $X_1, X_2, \dots, X_p$ satisfying the barycentric property is called a *p-chambar* and is denoted by $\mathrm{Ch}(X_1, X_2, \dots, X_p)$. In Section 2 we give a long list of detailed examples. The barycentric property produces interesting ordinary differential equations in dimension ≥ 1.

Remark 1.1. The barycentric property is invariant by affine transformations. Let $\mathrm{Ch}(X_1, X_2, \dots, X_p)$ be a p-chambar in some open subset $\mathcal{U} \subset \mathbb{C}^n$ and let T be an affine transformation of $\mathbb{C}^n$. Then the vector fields $T_*X_1, T_*X_2, \dots, T_*X_p$ satisfy the barycentric condition.

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In fact, if a biholomorphism $f\colon \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{C}^n$ sends any set of vector fields on $\mathcal{U}$ with the barycentric property into another set with the barycentric property, then f is an affine transformation. However, in some particular cases of p-chambars there are other types of biholomorphisms with this property (see for instance Theorem 2.13).

If X is a vector field on $\mathcal{U}$, then $\mathcal{F}_X$ denotes the (possibly singular) foliation whose leaves are the integral curves of X. Hence $\mathcal{F}_X$ is a foliation by (real or complex) curves. From now on all the vector fields X_k are not identically zero.

In the case of a 2-chambar $\mathrm{Ch}(X_1, X_2)$ condition (1.1) implies that $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$. We also have (Theorem 3.2):

Theorem A. *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). Let X_1, X_2 be two analytic (resp. holomorphic) vector fields on $\mathcal{U}$. Assume that X_1 and X_2 satisfy the barycentric property.*

Then $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$, and it is a foliation by straight lines:

- ◇ *the closure of the generic leaves is the intersection of lines with the open subset $\mathcal{U}$;*
- ◇ *on each line the flow $\varphi_t^k = \exp(tX_k)$, $k = 1, 2$, coincides with the flow of a constant vector field.*

The link between the 2-chambars and the foliations by straight lines suggests that certain special dynamics appear in dimension strictly larger than 1.

In Section 2 we will construct explicit examples satisfying Theorem A. It is sufficient to consider any (possibly singular) foliation by straight lines $\mathcal{F}$ and to take a vector field X whose restriction to each leaf is “constant”.

In the algebraic case the foliations by straight lines are classified on $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{P}^3_{\mathbb{C}}$. We will see that in this case the flows associated to a global algebraic 2-chambar are some special birational flows (Section 3).

We will consider the case of colinear vector fields (a condition satisfied by the 2-chambars), *i.e.* the case where $X_i = a_i X$ with a_i constant for any $1 \le i \le p$; such chambars are called rigid chambars. The barycentric property implies that $\mathcal{F}_X$ is a foliation by straight lines in the real case (Theorem 4.6) but not in the complex case. We will see the two following results (Theorem 4.8 and Corollary 4.11):

Theorem B. *If $\mathrm{Ch}(a_1X, a_2X, \ldots, a_pX)$, $a_k \in \mathbb{C}^*$, is a rigid p-chambar on the connected open set $\mathcal{U} \subset \mathbb{C}^n$, then the flow $\exp tX$ of X is a polynomial of degree at most $p-1$ as a function of the time t. In particular, the orbits of X are contained in some rational curves.*

Theorem C. *Let $\mathrm{Ch}(a_1X, a_2X, \ldots, a_pX)$ be a rigid p-chambar on an open set $\mathcal{U} \subset \mathbb{C}^n$. If X has a singular point, then the set $\mathrm{Sing}(X)$ of X has dimension ≥ 1.*

We will also see examples where the X_i's are polynomial vector fields, and more generally rational vector fields. In particular, in the linear case we get (Theorem 6.1):

Theorem D. *Let $X_1, X_2, \ldots, X_p$ be some linear vector fields on $\mathcal{U} \subset \mathbb{R}^n$ (resp. $\mathbb{C}^n$).*

If they satisfy the barycentric property, then they are nilpotent. In particular, the flows $\exp(tX_k)$ are polynomials in t.

In the case of 3-chambars one gets (Theorem 6.10):

Theorem E. *Let X_1, X_2, X_3 be some linear vector fields on $\mathbb{C}^n$.*

If they satisfy the barycentric property, then, up to conjugacy, they are contained in the Heisenberg Lie algebra $\mathfrak{h}_n$ (we identify X_i with its matrix).

We then give the classification of the 3-chambars in dimension 1; all chambars appearing in this classification are rigid (Theorem 5.1):

Theorem F. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a* 3*-chambar in one variable.*

In the real case $\mathrm{Ch}(X_1, X_2, X_3)$ *is constant (*i.e. *the* X_i*'s are distinct constant vector fields).*

In the complex case

- ◇ *either* $\mathrm{Ch}(X_1, X_2, X_3)$ *is constant*
- ◇ *or* $\mathrm{Ch}(X_1, X_2, X_3) = \mathrm{Ch}\big(a(x)\frac{\partial}{\partial x}, \mathbf{j}a(x)\frac{\partial}{\partial x}, \mathbf{j}^2 a(x)\frac{\partial}{\partial x}\big)$*, where* $\mathbf{j}^3 = 1$*, and* $a(x) = \sqrt{\lambda x + \mu}$ *with* $\lambda \in \mathbb{C}^*$*,* $\mu \in \mathbb{C}$*.*

Note that the classification implies that the global 3-chambars in one variable have no singularities where they are defined; this is not the case in higher dimensions (consider the nilpotent linear cases). Whereas 2-chambars and 3-chambars on an open subset of $\mathbb{C}$ are rigid the 4-chambars are not. The classification of p-chambars on $\mathbb{C}$ for $p \geq 4$ is a difficult problem in particular because of irreducibility problems. Nevertheless, we obtain interesting properties of such chambars.

In Section 7 we deal with chambars generated by homogeneous vector fields (homogeneous chambars). Among other results we will see the classification of homogeneous chambars of degree 2 (Theorem 7.6):

Theorem G. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a homogeneous* 3*-chambar of* $\mathbb{C}^2$ *of degree* 2*. Then, after a change of variables,* X_i *can be written as* $a_i y^2 \frac{\partial}{\partial x}$*, and the* a_i*'s satisfy:* $a_1 + a_2 + a_3 = 0$*. In particular, any homogeneous* 3*-chambar of* $\mathbb{C}^2$ *of degree* 2 *is rigid.*

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2. Remarks and examples

Let $\mathcal{U}$ be a connected open subset of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). Denote by $\mathcal{O}(\mathcal{U})$ the ring of analytic (resp. holomorphic) functions and by $\chi(\mathcal{U})$ the $\mathcal{O}(\mathcal{U})$-module of vector fields on $\mathcal{U}$. We also denote by $\mathcal{O}(\mathbb{C}^n, a)$, resp. by $\chi(\mathbb{C}^n, a)$, the germs of holomorphic functions, resp. of vector fields, at $a \in \mathcal{U}$. Let $X_1, X_2, \ldots, X_p$, $Y_1, Y_2, \ldots, Y_q$ be some analytic or holomorphic vector fields on $\mathcal{U}$. If the p-tuple $(X_1, X_2, \ldots, X_p)$ and the q-tuple $(Y_1, Y_2, \ldots, Y_q)$ satisfy the barycentric property, then the $(p+q)$-tuple $(X_1, X_2, \ldots, X_p, Y_1, Y_2, \ldots, Y_q)$ satisfy the barycentric property. This type of example is called a *reducible chambar*. A chambar is *irreducible* if it is not reducible.

2.1. Elementary examples and their variants. The most elementary example is the example of constant vector fields. Let $v_1, v_2, \ldots, v_p$ be p distinct constant vector fields on $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) such that

$$v_1 + v_2 + \cdots + v_p = 0.$$

The translation flows $T_t^{v_k}(x) = x + tv_k$ satisfy the barycentric property

$$\sum_{k=1}^{p} T_t^{v_k}(x) = \sum_{k=1}^{p} (x + tv_k) = \sum_{k=1}^{p} x + \sum_{k=1}^{p} tv_k = px + t \times 0 = px$$

and the vector fields $v_1, v_2, \ldots, v_p$ define a p-chambar. Such a chambar is called a *constant p-chambar*. The trajectories of the v_k are straight lines. The constant chambar $(v_1, v_2, \ldots, v_p)$ is reducible if and only if there is a subfamily $(v_{j_1}, v_{j_2}, \ldots, v_{j_\ell})$ such that $\sum_{k=1}^{\ell} v_{j_k} = 0$.

Let us give a simple variant of this example. Fix some coordinates

$$(x, y) = (x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_{n-q});$$

take p vector fields

$$X_k = f_1^k(x)\frac{\partial}{\partial y_1} + f_2^k(x)\frac{\partial}{\partial y_2} + \cdots + f_{n-q}^k(x)\frac{\partial}{\partial y_{n-q}},$$

where the f_i^k's denote some analytic functions. Assume that

$$X_1 + X_2 + \cdots + X_p = 0.$$

The X_k's satisfy the barycentric property since for any value of the parameter x the X_k's are constant vector fields in the linear subspaces $x =$ constant.

We can enrich this family of examples as follows. On the open subset $\mathcal{U}$ consider a regular foliation $\mathcal{F}$ of codimension q whose leaves are of the form $A \cap \mathcal{U}$, where the A's are affine subspaces of codimension q. Now take analytic vector fields X_k constant on any leaf of $\mathcal{F}$ and such that $X_1 + X_2 + \cdots + X_p = 0$. Then $(X_1, X_2, \ldots, X_p)$ is a p-chambar.

These examples play an important role in the article.

Another kind of construction that will be used is the formula expressing the flow of a vector field. Let $X = \sum\limits_{k=1}^{n} A_k(x)\frac{\partial}{\partial x_k}$ be an analytic vector field on an open subset $\mathcal{U}$ of $\mathbb{R}^n$ or $\mathbb{C}^n$, considered as a derivation on $\mathcal{O}(\mathcal{U})$: if $f \in \mathcal{O}(\mathcal{U})$, then

$$X(f) = \sum_{k=1}^{n} A_k \frac{\partial f}{\partial x_k}.$$

Let $(t, x) \mapsto \varphi_t(x)$ be the flow of X. For $x \in \mathcal{U}$ fixed set $h(t) = f(\varphi_t(x))$. The Taylor series of h at $t = 0$ is of the form $h(t) = h(0) + \sum\limits_{k=1}^{\infty} \frac{h^{(k)}(0)}{k!} t^k$.

On the other hand, $h(0) = x$ and $h^{(k)}(0) = X^k(f)$. In particular, we get

$$f(\varphi_t(x)) = x + \sum_{k \geq 1} \frac{1}{k!} X^k(f)(x) t^k.$$

If we specialize the above formula doing $f(x) = x_j$, the j-th coordinate of $x = (x_1, x_2, \ldots, x_n)$, then $\varphi_t(x) = (\varphi_t^1(x), \varphi_t^2(x), \ldots, \varphi_t^n(x))$, where

$$\varphi_t^j(x) = x_j + \sum_{k \geq 1} \frac{1}{k!} X^k(x_j) t^k. \tag{2.1}$$

Formula (2.1) will appear in some examples. Let us now give a consequence of (2.1):

Proposition 2.1. *Let $\mathcal{U} \subset \mathbb{C}^n$ be an open subset. Let $X_1, X_2, \ldots, X_p$ be some distinct elements of $\chi(\mathcal{U})$. Then $X_1, X_2, \ldots, X_p$ define a p-chambar if and only if for any $1 \leq j \leq n$*

$$\sum_{k=1}^{p} X_k^\ell(x_j) = 0 \quad \forall\, \ell \geq 1,$$

where x_j denotes the j-th coordinate of $x = (x_1, x_2, \ldots, x_n)$.

2.2. Barycentric property and integrability. Let $\mathrm{Ch}(X_1, X_2, \dots, X_p)$ be a p-chambar. Examples seen in Subsection 2.1 and 2-chambars may suggest that the Pfaff system generated by $X_1, X_2, \dots, X_p$ is an integrable system, *i.e.* tangent to a foliation. The following example of a 3-chambar in dimension 3 shows that this is not the case. Let us consider

$$X_1 = -2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad X_3 = \frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial x_3}.$$

The flows of the X_i are

$$\begin{aligned}
\exp tX_1 &= (x_1 - 2t, x_2, x_3 + t),\\
\exp tX_2 &= \left(x_1 + t, x_2 + tx_1 + \frac{t^2}{2}, x_3 + t\right),\\
\exp tX_3 &= \left(x_1 + t, x_2 - x_1 t - \frac{t^2}{2}, x_3 - 2t\right).
\end{aligned}$$

The barycentric property is satisfied; the leaves of X_1 are lines and the generic leaves of X_2 and X_3 are parabolas. Let $\omega = -x_1\,\mathrm{d}x_1 + \mathrm{d}x_2 - 2x_1\,\mathrm{d}x_3$. Then $\omega(X_i) = 0$, so ω defines the Pfaffian system associated to the X_i. A direct computation yields

$$\omega \wedge \mathrm{d}\omega = 2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3,$$

i.e. the 2-plane field associated to ω is a contact structure and hence is not integrable.

2.3. Fundamental example in dimension 1 and generalization. Let us consider the translation flow $\psi_t(x) = x + t$ on $\mathbb{C}$. Let ν be an integer ≥ 2. Denote by $x^{\frac{1}{\nu}}$ the principal branch of the ν-th root. Then

$$\varphi_{\nu,t}(x) = (\psi_t(x^{\frac{1}{\nu}}))^\nu = (x^{\frac{1}{\nu}} + t)^\nu$$

defines a flow, at least in a neighborhood of 1 since it is a conjugate of the translation flow. This flow is polynomial in the time t and corresponds to the vector field

$$Z_\nu = \nu x^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x} = \nu\frac{x}{x^{\frac{1}{\nu}}}\frac{\partial}{\partial x},$$

well defined at least in a neighborhood of 1.

Let σ be a primitive $(\nu+1)$-th root of unity. Then

$$\varphi_{\nu,\sigma t}(x) = (x^{\frac{1}{\nu}} + \sigma t)^\nu$$

is the flow of the vector field

$$\sigma Z_\nu = \nu\sigma\frac{x}{x^{\frac{1}{\nu}}}\frac{\partial}{\partial x}.$$

Of course $\sum\limits_{p=0}^{\nu} \sigma^p \cdot Z_\nu = 0$ and

$$\sum_{p=0}^{\nu}(x^{\frac{1}{\nu}} + \sigma^p t)^\nu = \sum_{p=0}^{\nu}\sum_{k=0}^{\nu}\binom{\nu}{k} x^{\frac{\nu-k}{\nu}}\sigma^{pk}t^k = \sum_{k=0}^{\nu}\left(\sum_{p=0}^{\nu}\sigma^{pk}\right)t^k\binom{\nu}{k} x^{\frac{\nu-k}{\nu}} = (\nu+1)x.$$

We can thus state

Proposition 2.2. *Let Z_ν be the vector field defined in a neighborhood of* 1 *by*

$$Z_\nu = \nu x^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x} = \nu\frac{x}{x^{\frac{1}{\nu}}}\frac{\partial}{\partial x}.$$

The $(\nu+1)$-tuple $(Z_\nu, \sigma Z_\nu, \dots, \sigma^\nu Z_\nu)$ is an irreducible $(\nu+1)$-chambar in a neighborhood of 1*.*

One can conjugate a chambar by an affine map; hence

$$\left((\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x}, \sigma(\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x}, \sigma^2(\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x}, \dots, \sigma^\nu(\lambda x+\mu)^{\frac{\nu-1}{\nu}}\frac{\partial}{\partial x}\right)$$

produces a $(\nu+1)$-chambar where it makes sense.

For $\nu = 2$ the previous construction gives the flow $\varphi_{2,t}(x) = x+2t\sqrt{x}+t^2$ associated to the vector field $Z_2 = 2\sqrt{x}\frac{\partial}{\partial x}$ and the 3-chambar $\mathrm{Ch}(Z_2, \mathbf{j}Z_2, \mathbf{j}^2Z_2)$, $\mathbf{j}^3 = 1$, but also its affine conjugates.

An immediate generalization in any dimension is the following. Consider $P(x) = (P_1(x), P_2(x), \dots, P_n(x))$ such that

- ⋄ $P_j \in \mathbb{C}[x_1, x_2, \dots, x_n]$, $\deg P_1 = \nu \geq 2$, and $\deg P_j \leq \nu$,
- ⋄ $P(0) = 0$,
- ⋄ and $DP(0) = \rho \cdot \mathrm{id}$, where id is the identity of $\mathbb{C}^n$ and $|\rho| > 1$.

There exists a neighborhood U of $0 \in \mathbb{C}^n$ such that $V = P(U) \supset U$ and $P_{|U}$ has an inverse $\phi\colon V \to U$. To any $a = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ we can associate a flow defined in a neighborhood of $(0,0) \in \mathbb{C}\times\mathbb{C}^n$ by

$$\varphi_t(x) = P(\phi(x)+ta).$$

The vector field associated to this flow is

$$X(x) = DP(\phi(x))\cdot a. \tag{2.2}$$

Proposition 2.3. *Let X be as in* (2.2) *and let σ be a primitive $(\nu+1)$-th root of unity. Then the $(\nu+1)$-tuple $(X, \sigma X, \dots, \sigma^\nu X)$ is an irreducible $(\nu+1)$-chambar in a neighborhood of $0 \in \mathbb{C}^n$.*

Proof: Since P has degree ν

$$\begin{aligned}\varphi_t(x) &= P(\phi(x)+ta)\\ &= P(\phi(x)) + tDP(\phi(x))\cdot a + \sum_{j=2}^{\nu}\frac{t^j}{j!}D^{(j)}P(\phi(x))\cdot a\\ &= x + tH_1(x,a) + \sum_{j=2}^{\nu} t^jH_j(x,a),\end{aligned}$$

where $H_j(x,a)$ is homogeneous of degree j with respect to $a \in \mathbb{C}^n$. Hence the flow of $\sigma^k X$ is

$$\varphi_{\sigma^k\cdot t}(x) = x + \sigma^k tH_1(x,a) + \sum_{j=2}^{\nu}\sigma^{jk}t^jH_j(x,a)$$

and so

$$\sum_{k=0}^{\nu}\varphi_{\sigma^k\cdot t}(x) = \sum_{k=0}^{\nu}\left(x + \sigma^k tH_1(x,a) + \sum_{j=2}^{\nu}\sigma^{jk}t^jH_j(x,a)\right) = (\nu+1)x$$

because $\sum\limits_{k=0}^{\nu}\sigma^{jk} = 0$ if $1 \leq j \leq \nu$. □

Remark 2.4. The construction produces vector fields X whose flow $\exp tX$ is polynomial in the variable time t.

Example 2.5. A global example of this kind (Proposition 2.3) can be given by a polynomial diffeomorphism $P\colon \mathbb{C}^n \to \mathbb{C}^n$. For instance,

$$P(x_1, x_2, \dots, x_n) = (x_1, x_2 + q_2(x_1), x_3 + q_3(x_1, x_2), \dots, x_n + q_n(x_1, x_2, \dots, x_{n-1})),$$

where $q_j \in \mathbb{C}[x_1, x_2, \dots, x_{j-1}]$, $2 \leq j \leq n$.

As a particular example, consider the polynomial diffeomorphism of $\mathbb{C}^2$

$$\phi(x, y) = (x + y^2, y).$$

Conjugating the flow

$$(x + a_k t, y + b_k t), \quad a_k, b_k \in \mathbb{C},$$

with ϕ we get the flow

$$\phi_k^t = (x + a_k t + 2b_k ty + b_k^2 t^2, y + b_k t);$$

one can check that it is the flow of the affine vector field

$$X_k = (a_k + 2b_k y)\frac{\partial}{\partial x} + b_k \frac{\partial}{\partial y}.$$

Note that this flow is polynomial in the time t.

As soon as $b_k \neq 0$ the trajectories are the parabola

$$f_k = a_k y + b_k y^2 - b_k x = \text{constant}.$$

For $p \geq 3$ if we choose $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_p$ such that

$$a_1 + a_2 + \dots + a_p = b_1 + b_2 + \dots + b_p = b_1^2 + b_2^2 + \dots + b_p^2 = 0, \tag{2.3}$$

then the X_k's satisfy the barycentric property and produce a p-chambar. For a generic choice of the parameters a_k and b_k the X_k's are not $\mathbb{C}$-colinear. Note that for $p = 3$ if (2.3) holds, then the web $\mathrm{W}(X_1, X_2, X_3)$ is a hexagonal web (see for instance [**5**]) since $f_1 + f_2 + f_3 = 0$.

2.4. Polynomial vector fields that satisfy the barycentric property.

Proposition 2.6. *In dimension* 1 *the polynomial vector fields that satisfy the barycentric property are the constant vector fields*

$$a_k \frac{\partial}{\partial x}$$

with $a_k \in \mathbb{C}^*$ *and* $\sum_{k=1}^{p} a_k = 0$.

Proof: The proof is based on Proposition 2.1. Let $X = P(x)\frac{\partial}{\partial x}$, where $P \in \mathcal{O}(\mathbb{C})$ is viewed as a derivation on $\mathcal{O}(\mathbb{C})$. According to (2.1) the flow φ_t of X is

$$\varphi_t(x) = x + \sum_{k \geq 1} \frac{1}{k!} X^k(x) t^k.$$

If $P \in \mathbb{C}[x]$ is a polynomial of degree $d \geq 1$, then $X^k(x)$ is also a polynomial for any $k \geq 1$. Let us write $X^k(x)$ as $X^k(x) = \sum_{j=0}^{d(k)} a_j^k x^j$. If we set $d(\ell) := \deg(X^\ell(x))$, then

(1) since $\deg(X) = d$, then $a_d^1 \neq 0$;
(2) $d(\ell) = (d-1)\ell + 1$ because $d(\ell + 1) = \deg(X(x)) + d(\ell) - 1 = d + d(\ell) - 1$;
(3) the equality $a_{d(\ell+1)}^{\ell+1} = d(\ell) a_d^1 a_{d(\ell)}^\ell$ holds.

By recurrence we get from (3) that $a^{\ell}_{d(\ell)} = A(\ell)(a^1_d)^{\ell}$, where

(4) $A(1) = 1$ and $A(\ell+1) = d(\ell)A(\ell)$ for $\ell \geq 1$.

On the one hand if $d = 0$, then $X(x) \neq 0$ and $X^{\ell}(x) = 0$ for all $\ell \geq 2$. On the other hand it follows from (2), (3), and (4) that if $d \geq 1$, then $d(\ell) \geq 1$ and $A(\ell) \geq 1$ for all $\ell \geq 1$.

Now assume that $(X_1, X_2, \dots, X_p)$ is a polynomial p-chambar on $\mathbb{C}$. Let $d = \max\limits_{1 \leq j \leq p} \deg(X_j)$. Suppose by contradiction that $d \geq 1$. Without loss of generality we can assume that

$$\{j \mid \deg(X_j) = d\} = \{1, 2, \dots, q\} \subset \{1, 2, \dots, p\}.$$

Set $X_k = P_k(x)\frac{\partial}{\partial x}$, where $P_k(x) = \sum\limits_{j=0}^{d} a_{kj}x^j$, $1 \leq k \leq p$, where

$$a_{jd} \neq 0 \text{ if } 1 \leq j \leq q \quad \text{and} \quad a_{jd} = 0 \text{ if } q < j \leq p.$$

Claim 1. *For any $\ell \geq 1$ we have*

$$(a_{1d})^{\ell} + (a_{2d})^{\ell} + \cdots + (a_{qd})^{\ell} = 0.$$

The statement follows from Claim 1 (indeed, if $(a_{1d})^{\ell} + (a_{2d})^{\ell} + \cdots + (a_{qd})^{\ell} = 0$ for any $\ell \geq 1$, then $a_{1d} = a_{2d} = \cdots = a_{qd} = 0$). Let us now justify it:

Proof of Claim 1: Set $d(k,\ell) = \deg(X^{\ell}_k(x))$, $1 \leq k \leq p$. Note that:

- $\diamond$ if $d = 1$, then $d(k,\ell) = 1$ for all $1 \leq k \leq q$ and all $\ell \geq 1$; furthermore, if $q < k \leq p$, then $X^{\ell}_k(x) = 0$ for all $\ell \geq 2$,
- $\diamond$ if $d > 1$ and $1 \leq k \leq q$, then $d \leq d(k,\ell) = (d-1)k+1$ and so $d(k,\ell) < d(k,\ell+1)$ for all $\ell \geq 1$. Moreover, if $q < k \leq p$, then either $d(k,\ell) < (d-1)k+1$ or $X^{\ell}_k(x) = 0$ for all $\ell \geq 2$.

Given $1 \leq k \leq p$ let $a(k,\ell)$ be the coefficient of $x^{d(k,\ell)}$ in the polynomial $X^{\ell}_k(x)$. It follows from the above computations that

- $\diamond$ if $1 \leq k \leq q$, then $a(k,\ell) = A(\ell)(a_{kd})^{\ell}$, where $A(\ell) \neq 0$,
- $\diamond$ if $q < k \leq p$, then $a(k,\ell) = 0$.

According to Proposition 2.1 we get that

$$X^{\ell}_1(x) + X^{\ell}_2(x) + \cdots + X^{\ell}_p(x) = 0$$

implies

$$A(\ell)((a_{1d})^{\ell} + (a_{2d})^{\ell} + \cdots + (a_{qd})^{\ell} = 0) = 0;$$

as $A(\ell) \neq 0$ we finally obtain that $(a_{1d})^{\ell} + (a_{2d})^{\ell} + \cdots + (a_{qd})^{\ell} = 0$. □

Remark 2.7. If $p = 3$, then Proposition 2.6 is a consequence of Theorem 5.1.

Remark 2.8. If X is a holomorphic vector field on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then in the affine chart $\mathbb{C}$ there exists a polynomial function a of degree ≤ 2 such that $X = a(x)\frac{\partial}{\partial x}$. The only p-tuples of global vector fields that satisfy the barycentric property in this chart are the constant vector fields.

2.5. Examples produced by those of dimension 1. We need a definition:

Definition 2.9. A p-chambar of the form $\mathrm{Ch}(a_1X, a_2X, \dots, a_pX)$, with a_i constant, is called *rigid*.

Propositions 2.2 and 2.3 give examples of rigid p-chambars.

Let us give a construction presented in dimension 2 for simplicity but that can be generalized in any dimension n and for any p.

Consider the vector field $X(x) = 2\sqrt{x}\frac{\partial}{\partial x}$ that induces the flow $\varphi_t(x) = x+2t\sqrt{x}+t^2$, a special case of Subsection 2.3. A first 3-chambar in dimension 2 is

$$\mathrm{Ch}(X(x)+X(y), \mathbf{j}(X(x)+X(y)), \mathbf{j}^2(X(x)+X(y))),$$

which is rigid. Similarly, one can consider

$$\mathrm{Ch}(X(x)+X(y), \mathbf{j}X(x)+\mathbf{j}^2X(y), \mathbf{j}^2X(x)+\mathbf{j}X(y)),$$

which is non-rigid. These examples are well defined on any simply connected open subset that does not intersect the axis $x=0$ and $y=0$.

Let us now give an example of a non-rigid irreducible 4-chambar still in dimension 2,

$$\mathrm{Ch}(X(x), \mathbf{j}X(x)+X(y), \mathbf{j}^2X(x)+\mathbf{j}X(y), \mathbf{j}^2X(y)),$$

that can be generalized to a 5-chambar as follows:

$$\mathrm{Ch}(X(x), \mathbf{j}X(x), \mathbf{j}^2X(x)+X(y), \mathbf{j}X(y), \mathbf{j}^2X(y)).$$

Example 2.10. Another way to obtain examples is by taking the real part of a complex p-chambar on $\mathbb{C}^n$. For instance, if we set $z = x+\mathbf{i}y$, then $\frac{d}{dz} = \frac{1}{2}\big(\frac{d}{dx} - \mathbf{i}\frac{d}{dy}\big)$,

$$\sqrt{2}\sqrt{z} = \underbrace{\sqrt{\sqrt{x^2+y^2}+x}}_{A(x,y)} + \mathbf{i}\underbrace{\sqrt{\sqrt{x^2+y^2}-x}}_{B(x,y)}$$

and

$$\mathrm{Re}\left(\sqrt{z}\frac{d}{dz}\right) = \frac{1}{2\sqrt{2}}\left(A(x,y)\frac{d}{dx} + B(x,y)\frac{d}{dy}\right).$$

The three vector fields $\mathrm{Re}\big(\sqrt{z}\frac{d}{dz}\big)$, $\mathrm{Re}\big(\mathbf{j}\sqrt{z}\frac{d}{dz}\big)$, $\mathrm{Re}\big(\mathbf{j}^2\sqrt{z}\frac{d}{dz}\big)$ give a real 3-chambar but if we consider x, y as complex variables, we get a 3-chambar on a suitable open set of $\mathbb{C}^2$.

Let us point out that we can iterate this process: take a chambar on $\mathbb{C}^n$, its real part gives a chambar on $\mathbb{R}^{2n}$ whose complexification is a chambar on $\mathbb{C}^{2n}$, and so on.

2.6. Examples associated to some polynomial flows in *t*.

2.6.1. Polynomial examples. Let $P = p_0 + p_1x + \dots + p_Nx^\nu$ be a polynomial of degree ν. Consider the vector field

$$X = a\frac{\partial}{\partial x} + P(x)\frac{\partial}{\partial y},$$

where $a \in \mathbb{C}^*$. Its flow is polynomial in t:

$$\varphi_t(x,y) = \left(x+at, y+\sum_{k=0}^{\nu} p_k\left(\frac{(x+at)^{k+1}}{a(k+1)} - \frac{x^{k+1}}{a(k+1)}\right)\right),$$

which can be rewritten

$$\varphi_t(x,y) = (x+at, y+\widetilde{P}_a(x+at) - \widetilde{P}_a(x)),$$

where $\widetilde{P}_a(y) = \sum\limits_{k=0}^{\nu} p_k\frac{y^{k+1}}{a(k+1)}$.

Let us consider p vector fields $X_1, X_2, \ldots, X_p$ of the form

$$X_k = a_k \frac{\partial}{\partial x} + P_k(x)\frac{\partial}{\partial y}.$$

The barycentric property is equivalent to

$$\sum_{k=1}^{p} a_k = 0 \tag{2.4}$$

and

$$\sum_{k=1}^{p} \widetilde{P}_{k,a_k}(x+a_k t) - \widetilde{P}_{k,a_k}(x) = 0. \tag{2.5}$$

Note that (2.5) holds if and only if

$$\frac{\partial}{\partial t}\left(\sum_{k=1}^{p} \widetilde{P}_{k,a_k}(x+a_k t)\right) = 0$$

if and only if

$$\sum_{k=1}^{\nu} P_k(x+a_k t) = 0. \tag{2.6}$$

As soon as we have fixed the constants $a_1, a_2, \ldots, a_p$ the equality (2.6) is a linear system in the coefficients of the polynomials P_k, a system that sometimes has non-trivial solutions.

Consider for instance the case $p = 3$ and $\nu = 2$. Set

$$P_1 = \alpha_0 + \alpha_1 x + \alpha_2 x^2, \quad P_2 = \beta_0 + \beta_1 x + \beta_2 x^2, \quad P_3 = \gamma_0 + \gamma_1 x + \gamma_2 x^2.$$

Conditions (2.4) and (2.6) are equivalent to

$$(I)\begin{cases} a_1 + a_2 + a_3 = 0 \\ \alpha_0 + \beta_0 + \gamma_0 = 0 \\ \alpha_1 + \beta_1 + \gamma_1 = 0 \\ \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 a_3 = 0, \end{cases} \qquad (II)\begin{cases} \alpha_2 + \beta_2 + \gamma_2 = 0 \\ \alpha_2 a_1 + \beta_2 a_2 + \gamma_2 a_3 = 0 \\ \alpha_2 a_1^2 + \beta_2 a_2^2 + \gamma_2 a_3^2 = 0. \end{cases}$$

In other words (2.4) and (2.6) give seven equations in the parameter space α, β, γ, a of dimension 12. The set of solutions is not irreducible. Assume that the parameters $a = \underline{a}$ satisfy $\underline{a_1} \neq \underline{a_2} \neq \underline{a_3}$. Then in a neighborhood of $a = \underline{a}$ the system (II) is a Vandermonde one so $\alpha_2 = \beta_2 = \gamma_2 = 0$ is a solution of (II). Then (I) and (II) are equivalent to

$$\begin{cases} a_1 + a_2 + a_3 = 0 \\ \alpha_0 + \beta_0 + \gamma_0 = 0 \\ \alpha_1 + \beta_1 + \gamma_1 = 0 \\ \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 a_3 = 0 \\ \alpha_2 = \beta_2 = \gamma_2 = 0, \end{cases}$$

which defines a quadric of dimension $12 - 7 = 5$. But there are solutions such that two of the $\underline{a_i}$'s are equal. For instance if $\underline{a_1} = \underline{a_2} = \underline{a_3} = 0$, then (I) and (II) are equivalent to

$$\underline{a_1} = \underline{a_2} = \underline{a_3} = \alpha_0 + \beta_0 + \gamma_0 = \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 0,$$

which is a linear space of dimension $12 - 6 = 6$.

Hence the set Σ of vector fields of this type satisfying the barycentric property is not irreducible.

2.6.2. Birational examples. Take $(a_1, a_2, \ldots, a_p)$ a p-tuple of $\mathbb{C}^n$ and set for $1 \leq k \leq p$

$$a_k = (a_{k,1}, a_{k,2}, \ldots, a_{k,n}).$$

Consider the translation flow

$$T_t^{a_k}(x_1, x_2, \ldots, x_n) = (x_1 + a_{k,1}t, x_2 + a_{k,2}t, \ldots, x_n + a_{k,n}t).$$

Denote by ψ the blow-up

$$\psi\colon (x_1, x_2, \ldots, x_n) \dashrightarrow (x_1, x_1x_2, \ldots, x_1x_n).$$

The lift F_t^k of $T_t^{a_k}$ by ψ can be written

$$\begin{aligned} F_t^k(x) &= \psi \circ T_t^{a_k} \circ \psi^{-1}(x) \\ &= \left(x_1 + a_{k,1}t, (x_1 + a_{k,1}t)\left(\frac{x_2}{x_1} + a_{k,2}t\right), \ldots, (x_1 + a_{k,1}t)\left(\frac{x_n}{x_1} + a_{k,n}t\right)\right). \end{aligned}$$

The condition $\sum_{k=1}^{p} F_t^k(x) = px$ is satisfied if

- ⋄ for any $1 \leq \ell \leq n$
$$\sum_{k=1}^{p} a_{k,\ell} = 0$$
- ⋄ and for any $2 \leq \ell \leq n$
$$\sum_{k=1}^{p} a_{k,1}a_{k,\ell} = 0.$$

Remark 2.11. In the previous examples we assume that the a_k's are not all zero. Up to a linear conjugation (such a conjugation preserves a barycentric property) we can assume that $a_1 = (1, 0, 0, \ldots, 0)$. The previous conditions can be rewritten

$$\begin{cases} \displaystyle\sum_{k=1}^{p} a_{k,\ell} = 0, & 1 \leq \ell \leq n, \\ a_{1,\ell} = 0, & 2 \leq \ell \leq n, \end{cases}$$

which thus form a linear subspace of the space of coefficients $a_{j,i}$. These examples of p-chambars are given by birational flows quadratic in the time t (see [**3**] for other examples).

2.7. Examples of chambars whose flows are non-algebraic/non-polynomial in *t*. Let k be an integer; consider q_k vector fields of the form

$$X_k^j = a_k \frac{\partial}{\partial x} + b_{k,j}\mathrm{e}^{\lambda_k x}\frac{\partial}{\partial y}, \quad 1 \leq j \leq q_k,$$

where a_k, $b_{k,j}$, and λ_k belong to $\mathbb{C}^*$. The flows of X_k^j are

$$(\exp tX_k^j)(x, y) = \left(x + a_kt, y + \frac{b_{k,j}}{\lambda_k a_k}\mathrm{e}^{\lambda_k x}(\mathrm{e}^{\lambda_k a_k t} - 1)\right).$$

Set $\ell = \sum_{k=1}^{p} q_k$. The ℓ vector fields X_k^j form an ℓ-chambar if and only if for any $1 \leq k \leq p$ the following equalities hold:

$$\sum_{k=1}^{p} q_k a_k = 0, \quad \sum_{j=1}^{q_k} b_{k,j} = 0.$$

Contrary to the previous example the flows $\exp tX_k^j$ are non-polynomial: their orbits are the levels of the functions

$$\lambda_k a_k y - b_{k,j}\mathrm{e}^{\lambda_k x}.$$

This construction starts with $\ell = 4$ and produces global chambars on $\mathbb{C}^2$. It can be generalized to higher dimensions.

2.8. Compatible diffeomorphisms. The concept of a p-chambar is an affine one, that is, the barycentric property is invariant under the action of the group of affine transformations; if $\mathcal{C}$ is a local p-chambar and ϕ a diffeomorphism, then, in general, $\phi_*\mathcal{C}$ is not a chambar.

Problem 2.12. *Let* Ch_c *be a constant chambar; what are the diffeomorphisms* ϕ *such that* $\phi_* \mathrm{Ch}_c$ *is a p-chambar? What is the structure of such a set of diffeomorphisms?*

Let us give an answer to this problem in the special case $p = 3$, $n = 2$. Let $\mathrm{Ch}(X_1, X_2, X_3)$ be a constant 3-chambar in $\mathbb{C}^2$. We say that $\mathrm{Ch}(X_1, X_2, X_3)$ is *generic* if the X_i's are linearly independent. We immediately notice that a generic constant 3-chambar is linearly conjugate to the "standard" 3-chambar

$$\mathrm{Ch}_0 = \mathrm{Ch}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\right).$$

Let ϕ be a local diffeomorphism; we say that ϕ is *compatible with* Ch_0 if $\phi_* \mathrm{Ch}_0$ is a 3-chambar. We have the following statement (recall that $\mathbf{j}$, $\mathbf{j}^2$ are the roots of t^2+t+1):

Theorem 2.13. *A local diffeomorphism of* $\mathbb{C}^2$ *is compatible with* Ch_0 *if and only if it can be written* $L + F$*, where* L *denotes an affine inversible transformation and* $F = (f, g)$ *with*

$$f, g \in \langle (y + \mathbf{j}x)^2, (y + \mathbf{j}^2 x)^2, xy(y - x)\rangle_{\mathbb{C}}.$$

Remark 2.14. A local compatible diffeomorphism is in fact a global application, but not in general a global diffeomorphism.

Let us first state and prove the following result we use in the proof of Theorem 2.13:

Lemma 2.15. *If* h *is a holomorphic function satisfying the PDE's*

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial x \partial y} + \frac{\partial^2 h}{\partial y^2} = 0, \qquad \frac{\partial^3 h}{\partial x^2 \partial y} + \frac{\partial^3 h}{\partial x \partial y^2} = 0,$$

then h *is a polynomial of degree* 3 *of the form*

$$h(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 (x + \mathbf{j}y)^2 + \alpha_4 (x + \mathbf{j}^2 y)^2 + \alpha_5 xy(y - x)$$

with $\alpha_0, \alpha_1, \ldots, \alpha_5 \in \mathbb{C}$.

Proof: To simplify the notations let us consider the differential operators

$$S = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}, \quad T = \frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial x \partial y^2}.$$

The inclusion $\langle 1, x, y, (y+\mathbf{j}x)^2, (y+\mathbf{j}^2 x)^2, xy(y-x)\rangle_{\mathbb{C}} \subset \ker(S) \cap \ker(T)$ is straightforward.

Note that

$$\frac{\partial}{\partial x}\cdot S=\frac{\partial^3}{\partial x^3}+\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}=\frac{\partial^3}{\partial x^3}+T$$

so $\ker(S)\cap\ker(T)\subset\ker\big(\frac{\partial^3}{\partial x^3}\big)$.

Similarly, $\frac{\partial}{\partial y}\cdot S=\frac{\partial^3}{\partial y^3}+T$ and thus $\ker(S)\cap\ker(T)\subset\ker\big(\frac{\partial^3}{\partial y^3}\big)$.

As a result, $\ker(S)\cap\ker(T)\subset\ker\big(\frac{\partial^3}{\partial x^3}\big)\cap\ker\big(\frac{\partial^3}{\partial y^3}\big)$. In particular, if h belongs to $\ker(S)\cap\ker(T)$, then $\frac{\partial^3 h}{\partial x^3}=\frac{\partial^3 h}{\partial y^3}=0$.

Let $h=\sum\limits_{k,\ell}h_{k,\ell}x^k y^\ell$ be the Taylor series of h at $(0,0)$. If $\frac{\partial^3 h}{\partial x^3}=\frac{\partial^3 h}{\partial y^3}=0$, then $h_{k,\ell}\neq 0$ if and only if $k,\ell\leq 2$. However, if $k=\ell=2$, then we have $S(x^2y^2)=2y^2+2x^2+4xy\neq 0$ and so

$$\ker(S)\cap\ker(T)=\langle 1,x,y,(y+\mathbf{j}x)^2,(y+\mathbf{j}^2x)^2,xy(y-x)\rangle_{\mathbb{C}}.\qquad\square$$

Proof of Theorem 2.13: If ϕ is a local diffeomorphism of $\mathbb{C}^2$ compatible with Ch_0, then the barycentric condition asserts that

$$\phi(x+t,y)+\phi(x,y+t)+\phi(x-t,y-t)=3\phi(x,y).\tag{2.7}$$

We can assume that ϕ is defined in a neighborhood of $(0,0)$. Let us write ϕ as $L+(f,g)$, where L is affine and $f,g\in\mathcal{O}(\mathbb{C}^2,0)$ satisfy $(f,g)(0,0)=D(f,g)(0,0)=(0,0)$. By differentiating (2.7) twice with respect to t, we get that both components f and g satisfy the PDE

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial x\partial y}+\frac{\partial^2 h}{\partial y^2}=0.$$

The solutions of such PDE are of the type

$$h=\varphi_+(y+\mathbf{j}x)+\varphi_-(y+\mathbf{j}^2x),$$

with $\mathbf{j}$, $\mathbf{j}^2$ the roots of t^2+t+1 and φ_+, φ_- holomorphic in one variable defined on suitable domains.

A third derivation with respect to t shows that f and g also satisfy the PDE

$$0=\frac{\partial^3 h}{\partial x^2\partial y}+\frac{\partial^3 h}{\partial x\partial y^2}=\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}\right).$$

Lemma 2.15 allows us to conclude (note that, with the notations of Lemma 2.15, an element of $\ker S\cap\ker T$ satisfies relation (2.7)). $\square$

More generally, one can state:

Theorem 2.16. *Let $f\colon\mathcal{U}\to f(\mathcal{U})\subset\mathbb{C}^n$ be a biholomorphism from the open set $\mathcal{U}\subset\mathbb{C}^n$ to $f(\mathcal{U})$, $n\geq 2$. Assume that the vector fields*

$$f_*\frac{\partial}{\partial x_1},f_*\frac{\partial}{\partial x_2},\dots,f_*\frac{\partial}{\partial x_n},f_*\left(-\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}-\dots-\frac{\partial}{\partial x_n}\right)$$

satisfy the barycentric property. Then all the components f_j of f are polynomial.

Lemma 2.17. *Let $h \in \mathcal{O}(\mathcal{U})$ be a holomorphic function with the property that*

$$(2.8)\quad \sum_{j=1}^{n} h(x_1, x_2, \ldots, x_{j-1}x_j + t, x_{j+1}, x_{j+2}, \ldots, x_n) + h(x_1 - t, x_2 - t, \ldots, x_n - t) = (n+1)h(x_1, x_2, \ldots, x_n)$$

for all $x \in \mathcal{U}$ and $t \in \mathbb{C}$ with $|t|$ small enough. Then h satisfies the system of PDE's

$$\begin{cases} T_2(h) = 0, \\ T_3(h) = 0, \\ \quad \vdots \end{cases}$$

where T_k is the differential operator

$$T_k = \frac{\partial^k}{\partial x_1^k} + \frac{\partial^k}{\partial x_2^k} + \cdots + \frac{\partial^k}{\partial x_n^k} + (-1)^k \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n} \right)^k.$$

Proof: Let $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$ and $v = -\sum_{j=1}^{n} e_j$. The idea is to prove by induction on $k \geq 1$ that for any $t \in (\mathbb{C}, 0)$

$$(2.9)\qquad \sum_{j=1}^{n} \frac{\partial^k}{\partial x_j} h(x + te_j) + (-1)^k \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n} \right)^k h(x + tv) = 0;$$

indeed, if $t = 0$ in (2.9), then we get (2.8).

Let $\varphi(t, x) = \sum_{j=1}^{n} h(x + te_j) + h(x + tv)$. According to (2.8) the function $\varphi(t, x)$ depends only on x. In particular, differentiating k times with respect to t we get

$$\frac{\partial^k \varphi(t, x)}{\partial t^k} = \sum_{j=1}^{n} \frac{\partial^k}{\partial x_j} h(x + te_j) + (-1)^k \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n} \right)^k h(x + tv) = 0.$$

Furthermore, doing $t = 0$ we get $T_k(h) = 0$. □

Proof of Theorem 2.16: Now suppose that $f \colon \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{C}^n$ is a biholomorphism such that the vector fields $f_* \frac{\partial}{\partial x_1}, f_* \frac{\partial}{\partial x_2}, \ldots, f_* \frac{\partial}{\partial x_n}, f_* \big(-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \cdots - \frac{\partial}{\partial x_n} \big)$ satisfy the barycentric property. Setting $f = (f_1, f_2, \ldots, f_n)$ we see that it is equivalent to

$$\sum_{j=1}^{n} f_\ell(x + te_j) + f_\ell(x + tv) = (n+1) f_\ell(x) \quad \forall\, 1 \leq \ell \leq n.$$

Therefore each component f_ℓ of f satisfies (2.8) so that f_ℓ belongs to $\bigcap_{k \geq 2} \ker(T_k)$ for any $1 \leq \ell \leq n$ (Lemma 2.17). The idea is to prove that $\bigcap_{k \geq 2} \ker(T_k) \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$: if $h \in \bigcap_{k \geq 2} \ker(T_k)$, then h is a polynomial.

Let $\mathcal{P}$ be the Noetherian ring of linear differential operators on $\mathcal{O}(\mathcal{U})$ with constant coefficients

$$\mathcal{P} = \left\{ P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) \mid P \in \mathbb{C}[z_1, z_2, \ldots, z_n] \right\}$$

and let $\mathcal{I} = \langle T_k \mid k \geq 2 \rangle$ be the ideal of $\mathcal{P}$ generated by all the operators T_k, $k \geq 2$. Note that if S belongs to $\mathcal{I}$, then $\bigcap_{k \geq 2} \ker(T_k)$ is contained in $\ker(S)$.

Claim 2. *There exists* $p \in \mathbb{N}$ *such that* $\frac{\partial^p}{\partial x_j^p}$ *belongs to* $\mathcal{I}$ *for all* $1 \leq j \leq n$.

Claim 2 implies that if h belongs to $\bigcap_{k\geq 2} \ker(T_k)$, then h is a polynomial of degree at most $n(p-1)$.

Proof of Claim 2: Let $\Phi\colon \mathcal{P} \to \mathcal{O}_n$ be the unique ring homomorphism satisfying

$$\Phi\left(\frac{\partial}{\partial x_j}\right) = z_j \quad \forall\, 1 \leq j \leq n.$$

Note that $\Phi(T_k) = z_1^k + z_2^k + \cdots + z_n^k + (-1)^k(z_1 + z_2 + \cdots + z_n)^k$. Let us set

$P_k(z) = z_1^k + z_2^k + \cdots + z_n^k + (-1)^k(z_1 + z_2 + \cdots + z_n)^k, \quad \widetilde{\mathcal{I}} = \langle P_k \mid k \geq 2\rangle, \quad \Phi(\mathcal{I}) = \widetilde{\mathcal{I}}.$

Claim 3. *One has*

$$Z(\widetilde{\mathcal{I}}) = \{z \in \mathbb{C}^n \mid P_k(z) = 0 \quad \forall\, k \geq 2\} = \{0\}.$$

From $Z(\widetilde{\mathcal{I}}) = \{0\} = Z(\mathfrak{m}_n)$ one gets (using the definition of $\sqrt{\widetilde{\mathcal{I}}}$) that $\sqrt{\widetilde{\mathcal{I}}} = \mathfrak{m}_n$. According to Hilbert's theorem (Nullstellensatz) one obtains that $\widetilde{\mathcal{I}} \supset \mathfrak{m}_n^p$ for some p. As a result, z_j^p belongs to $\widetilde{\mathcal{I}}$ for all $1 \leq j \leq n$ and so $\frac{\partial^p}{\partial z_j^p}$ belongs to $\mathcal{I}$ for all $1 \leq j \leq n$. □

Proof of Claim 3: Define $S := -(z_1 + z_2 + \cdots + z_n)$ so that $P_k = z_1^k + z_2^k + \cdots + z_n^k + S^k$. Therefore if z belongs to $Z(\widetilde{\mathcal{I}})$, then

$$(**)\begin{cases} z_1 + z_2 + \cdots + z_n + S = 0 \\ z_1^2 + z_2^2 + \cdots + z_n^2 + S^2 = 0 \\ \qquad\vdots \\ z_1^n + z_2^n + \cdots + z_n^n + S^n = 0 \\ z_1^{n+1} + z_2^{n+1} + \cdots + z_n^{n+1} + S^{n+1} = 0. \end{cases}$$

Doing $S = z_{n+1}$ system $(**)$ is equivalent to $Q_{n+1}v^t = 0$, where Q_{n+1} is the matrix

$$Q_{n+1}(z) = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n+1} \\ z_1^2 & z_2^2 & \cdots & z_{n+1}^2 \\ \vdots & & & \vdots \\ z_1^{n+1} & z_2^{n+1} & \cdots & z_{n+1}^{n+1} \end{pmatrix}$$

and $v = (1, 1, \ldots, 1)$. Finally it can be checked by induction on $n \geq 0$ that if $Q_{n+1}(z)u^t = 0$ for some $u = (u_1, u_2, \ldots, u_{n+1})$, where $u_j > 0$ for all $1 \leq j \leq n+1$, then $z = 0$. □

3. Description of the 2-chambars

3.1. Examples coming from foliations by straight lines. In order to make the previous statements precise we recall the classification of foliations by straight lines on $\mathbb{P}^3_{\mathbb{C}}$ that can be found in [**2**] (according to Jorge Pereira this classification was already known to Kummer). We do not know if such a classification exists on $\mathbb{P}^3_{\mathbb{R}}$.

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}$. Chow's theorem asserts that $\mathcal{F}$ is algebraic; such a foliation $\mathcal{F}$ has singularities. We say that $\mathcal{F}$ is a *foliation by straight lines* if the generic leaf is contained in a line (in fact a line without a few points). Let us mention the difference from the real case: foliations by straight lines of $\mathbb{P}^3_{\mathbb{R}}$ without singularities exist. The typical example is produced by Hopf fibration: the real projectivization of complex vector lines of $\mathbb{C}^2 \simeq \mathbb{R}^4$ gives such a foliation $\mathcal{H}$. Setting $z = x_1 + \mathbf{i}x_2$ and $w = x_3 + \mathbf{i}x_4$ these foliations have the first integral

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{x_1x_3 - x_2x_4 + \mathbf{i}(x_1x_4 + x_2x_3)}{x_3^2 + x_4^2}.$$

In particular, $\frac{x_1x_3-x_2x_4}{x_3^2+x_4^2}$ and $\frac{x_1x_4+x_2x_3}{x_3^2+x_4^2}$ are real first integrals of $\mathcal{H}$.

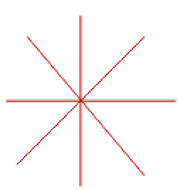
Let us recall the classification of foliations by straight lines of $\mathbb{P}^3_{\mathbb{C}}$:

Theorem 3.1 ([**2**]). *Every holomorphic foliation by straight lines in $\mathbb{P}^3_{\mathbb{C}}$ is, up to linear equivalence, of one of the following types:*

(1) *a radial foliation at a point;*
(2) *a radial foliation "in the pages of an open book",* i.e. *a family of radial foliations of dimension* 2 *each contained in a plane of the family of planes containing a fixed line;*
(3) *a foliation associated to the twisted cubic $t \mapsto (t, t^2, t^3)$; here the (closure of the) leaves of the foliation are the chords and the lines tangent to the twisted cubic.*

Foliations of the first type correspond to foliations by parallel lines in a well-chosen affine chart (singular point at infinity).

To construct a foliation of the second type we consider an open book, *i.e.* a pencil of hyperplanes, for instance $\frac{x_1}{x_2}$ = constant; in any page $\frac{x_1}{x_2} = c$ we fix a point $(\underline{x_1}, c\underline{x_2}, \underline{x_3})$ and ask that any leaf of $\mathcal{F}$ be a line contained in a page $\frac{x_1}{x_2} = c$ and pass through the prescribed point $(c\underline{x_2}, \underline{x_2}, \underline{x_3})$ (see [**2**] for further details).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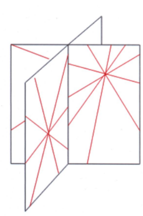
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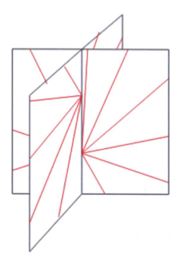
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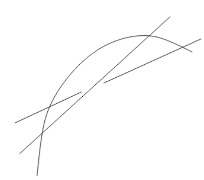

Type (3)

Note that Theorem 3.1 gives the description of algebraic foliations by straight lines in the affine space $\mathbb{C}^3$.

Let us now explain how we can construct a 2-chambar from a foliation $\mathcal{F}$ by lines defined on an open subset $\mathcal{U}$ of $\mathbb{C}^n$. For a good choice of the affine coordinates x_i the foliation $\mathcal{F}$ is defined by a vector field

$$X = \frac{\partial}{\partial x_1} + \alpha_2\frac{\partial}{\partial x_2} + \alpha_3\frac{\partial}{\partial x_3} + \cdots + \alpha_n\frac{\partial}{\partial x_n}$$

on $\mathcal{U}$. Of course, in general, the α_i's are meromorphic and we consider $\mathcal{U}^* = \mathcal{U} \setminus \bigcup_{i=2}^{n}(\text{poles of } \alpha_i)$. Then if m belongs to $\mathcal{U}^*$, the trajectory of X passing through m is a line D_m and $\exp(tX)_{|D_m}$ is a translation flow on D_m. The pair $(X, -X)$ thus defines a 2-chambar.

One can next consider $f \cdot X$, where f is any meromorphic first integral of X, instead of X. Since f is constant on the trajectories of X, $f \cdot X$ still defines a translation flow on any trajectory of X, and $(f \cdot X, -f \cdot X)$ is also a 2-chambar.

3.2. Some properties. The barycentric property for a 2-chambar $\mathrm{Ch}(X_1, X_2)$ implies that $X_1 + X_2 = 0$ and can be rewritten as

$$\varphi_t(x) + \varphi_{-t}(x) = 2x \quad \forall\, x \in \mathcal{U},$$

where φ_t denotes the flow of $X = X_1$.

Differentiating the previous equality with respect to time t, we get

$$\dot{\varphi}_t(x) - \dot{\varphi}_{-t}(x) = X(\varphi_t(x)) - X(\varphi_{-t}(x)) = 0;$$

differentiating a second time with respect to t, we obtain

$$DX(\varphi_t(x))\dot{\varphi}_t(x) + DX(\varphi_{-t}(x))\dot{\varphi}_{-t}(x) = 0,$$

where $DX : \mathcal{U} \to \mathbb{R}^n$ (or $DX : \mathcal{U} \to \mathbb{C}^n$) denotes the differential of X.

If $X = \sum\limits_{i=1}^{n} \alpha_i(x)\frac{\partial}{\partial x_i}$, the above relation is equivalent to

$$DX(X) = \sum_{i=1}^{n} X(\alpha_i)\frac{\partial}{\partial x_i} = 0.$$

In particular, the coefficients α_k are first integrals of X, $2 \leq k \leq n$. As a result, the α_k's are constant along the trajectories of X; these trajectories are thus (contained in) lines.

Note that in dimension 1 we can write $X = \alpha\frac{\partial}{\partial x}$ and the above relation is equivalent to $\alpha\frac{\partial \alpha}{\partial x} = 0$; hence α is constant. On any of its trajectories the flow of X thus coincides with the flow of a constant vector field. As a result, one can state:

Theorem 3.2. *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). Let X_1, X_2 be two analytic (resp. holomorphic) vector fields on $\mathcal{U}$. Assume that X_1 and X_2 satisfy the barycentric property.*

Then the leaves of $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$ are contained in lines; on each of these lines the flows $\exp(tX_k)_{|D}$ are translation flows.

In particular, in dimension 1 any 2-chambar $(X, -X)$ is produced by a constant vector field. Note also that any local 2-chambar in one variable can be globalized.

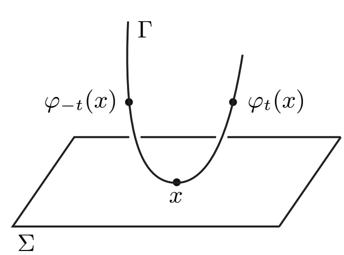
Corollary 3.3. *Let X be a rational vector field on $\mathbb{C}^n$. Assume that $(X, -X)$ defines a 2-chambar. Then $\exp(tX) = \mathrm{id} + tX^0$ defines a flow of birational maps of $\mathbb{C}^n$.*

Note that in $\exp(tX) = \mathrm{id} + tX^0$ the letter X^0 denotes the map whose components are the components of the vector field X, a system of coordinates having been chosen.

Remark 3.4. In the real case there is another proof of Theorem 3.2 which is geometric.

Let Γ be a generic leaf of $\mathcal{F}_{X_1} = \mathcal{F}_{X_2}$. Assume that Γ is not (contained in) a line. If $x \in \Gamma$ is a generic point, then there exists a hyperplane Σ tangent to Γ at x such that

- ⋄ the germ Γ, x is contained in one of the half spaces delimited by Σ,
- ⋄ $\Gamma, x \cap \Sigma = \{x\}$.



If we set $\varphi_t = \exp tX_1$, then $\varphi_t(x) - x + \varphi_{-t}(x) - x \not\equiv 0$: a contradiction.

Let $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$ be a germ of vector fields at the origin of $\mathbb{C}^n$. Denote by $\mathrm{Sing}(X) = \{\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0\}$ the singular set of X.

The following statement is a special case of Theorem 4.10; its proof is algebraic in contrast with the geometric proof of Theorem 4.10.

Theorem 3.5. *Let* $\mathrm{Ch}(X, -X)$ *be a* 2*-chambar at* $0 \in \mathbb{C}^n$*. Assume that* X *is singular at* 0*, that is,* $\{0\} \subset \mathrm{Sing}(X)$*.*

Then $\dim \mathrm{Sing}(X) \geq 1$*.*

Proof: The condition $X(\alpha_k) = 0$, $1 \leq k \leq n$, is equivalent to

$$\sum_{i=1}^{n} \alpha_i \frac{\partial \alpha_k}{\partial x_i} = 0, \quad 1 \leq k \leq n.$$

Hence the partial derivatives $\left(\frac{\partial \alpha_k}{\partial x_1}, \frac{\partial \alpha_k}{\partial x_2}, \dots, \frac{\partial \alpha_k}{\partial x_n}\right)$ are relations of the ideal $(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Assume by contradiction that $\dim \mathrm{Sing}(X) = 0$. Then according to [**6**] the relations are generated by the trivial relations

$$(0, 0, \dots, 0, \underbrace{\alpha_j}_{i\text{th coordinate}}, 0, \dots, 0, \underbrace{-\alpha_i}_{j\text{th coordinate}}, 0, 0, \dots, 0).$$

This gives a contradiction with the following fact: the algebraic multiplicity at 0 of one of the $\frac{\partial \alpha_k}{\partial x_i}$ is less than the algebraic multiplicity at 0 of α_k. □

Remark 3.6. Let $u \in \mathcal{O}^*(\mathbb{C}^n, 0)$ be a unit. Then the vector field $u \cdot \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ which has linear trajectories cannot belong to a 2-chambar; but the rational vector field $\frac{1}{x_1} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ can.

4. Rigid chambars

4.1. Flows which are polynomial in the time t.

Definition 4.1. Let X be a holomorphic vector field on the open set $\mathcal{U} \subset \mathbb{C}^n$. We say that X is a *t-polynomial* vector field if $t \mapsto \exp tX$ is polynomial. The *t-degree* of X is the usual degree in the variable t and is denoted by $t \cdot d(X) \in \mathbb{N} \cup \{\infty\}$.

We have seen a lot of examples of t-polynomial vector fields: constant vector fields, nilpotent vector fields, the vector field $2\sqrt{x}\frac{\partial}{\partial x}$, ...

If $\mathcal{U} = \mathbb{C}^n$, then the trajectories of a t-polynomial vector field are points or rational curves.

Proposition 4.2. *Let* X *be a* t*-polynomial vector field of* t*-degree* ν *on the open set* $\mathcal{U} \subset \mathbb{C}^n$*. Write* $\exp tX$ *as* $\mathrm{Id} + tF_1 + t^2F_2 + \cdots + t^\nu F_\nu$*, with* $F_k \in \mathcal{O}(\mathcal{U})$*. Then the components* $F_{\nu,1}, F_{\nu,2}, \dots, F_{\nu,n}$ *of* F_ν *are first integrals of* X*.*

In particular in the 1*-dimensional case,* F_ν *is a non-zero constant.*

Proof: It is a direct consequence of the identity $\exp tX \circ \exp sX = \exp(s+t)X$: the coefficient of t^ν in that identity is exactly

$$F_\nu(\exp sX) = F_\nu.$$

This implies the statement. □

Note the $F_{\nu,k}$ may be constant; this is the case for the flow of $X = 2\sqrt{x}\frac{\partial}{\partial x}$. *A contrario* if a t-polynomial vector field X of degree ν is singular at a point, say 0 (*i.e.* $X(0) = 0$), then obviously some of the $F_{\nu,k} = \frac{X^{\nu}(x_k)}{\nu!}$ are not constant. In particular in dimension 2, a t-polynomial vector field X singular at the origin $0 \in \mathbb{C}^2$, $X(0) = 0$, has a non-constant holomorphic first integral f. The generic leaves of X are the levels of f; note that since the flow is polynomial one has the following important property: $X_{|f^{-1}(0)} \equiv 0$.

The t-polynomial vector fields produce examples of p-chambars as we have seen previously. Typically if σ is a primitive ν-th root of unity and $t \cdot d(X) = \nu$, then $X, \sigma X, \dots, \sigma^{\nu-1}X$ defines a (rigid) ν-chambar.

If $t \cdot d(X) = 1$, then $\exp tX = \mathrm{Id} + tF_1$ and the foliation associated to X is a foliation by straight lines. Conversely to a foliation by straight lines we can associate a (meromorphic) t-polynomial vector field X such that $t \cdot d(X) = 1$.

In dimension 2, consider a foliation given by the vector field $X = f\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Then X is a t-polynomial vector field of degree 1 if and only if the foliation $\mathcal{F}_X$ is a foliation by straight lines; this means that f satisfies the non-linear PDE

$$0 = X(f) = f\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y};$$

note that this PDE is the famous inviscid Burgers' equation, a well-known PDE in fluid mechanics. Similarly, t-polynomial vector fields of degree 2 on an open set of $\mathbb{C}^2$ correspond to foliations in parabolas, etc. In that case generalizations of Burgers' equation appear, as the reader can see.

The following result gives the classification of the t-polynomial vector field on the complex line.

Theorem 4.3. *Let $X(x) = a(x)\frac{\partial}{\partial x}$ be a germ at $0 \in \mathbb{C}$ of a holomorphic vector field. Assume that the flow of X is polynomial in t of t-degree ℓ. Then $a = f' \circ \phi$, where*

- ⋄ *f is a polynomial of degree ℓ with $f(0) = 0$ and $f'(0) = a(0) \neq 0$;*
- ⋄ *$\phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a local inverse of f: $f \circ \phi(x) = x$.*

In other words X is conjugate to the constant vector field $\frac{\partial}{\partial x}$ via a polynomial (local) diffeomorphism.

Proof: Suppose that $a(0) \neq 0$. In this case the vector field X is conjugate to a constant vector field, say $Y = \frac{\partial}{\partial x}$. Let f be an element of $\mathrm{Diff}(\mathbb{C}, 0)$ such that $f_*Y = X$. The flow φ_t of X can be written as

$$\varphi_t(x) = f(f^{-1}(x) + t),$$

where $f^{-1} \in \mathrm{Diff}(\mathbb{C}, 0)$ is the local inverse of f. We thus have $a(x) = f' \circ f^{-1}(x)$. As we have seen in (2.1),

$$\varphi_t(x) = x + \sum_{k \geq 1} \frac{1}{k!} X^k(x) t^k;$$

since $t \cdot d(X) = d$ we must have $X^k(x) = 0$ for all $k \geq d + 1$. Note that the functions $f_k(x) = X^k(x)$, $k \geq 1$, satisfy the recurrence rule:

(1) $f_1 = a$,
(2) $f_{k+1} = af'_k \ \forall\, k \geq 1$.

Let us define another sequence of germs at $0 \in \mathbb{C}$ as $g_k = f_k \circ f$, $k \geq 1$. This new sequence satisfies the recurrence rule:

(1') $g_1 = f_1 \circ f = a \circ f = f'$,
(2') $g_{k+1} = f_{k+1} \circ f = a \circ f \cdot f'_k \circ f = f'_k \circ f \cdot f' = (f_k \circ f)' = g'_k \ \forall\, k \geq 1$.

Therefore from (1') and (2') we get for all $k \geq 1$

$$g_k = \frac{\partial^k f}{\partial x^k}.$$

Now, as $f_{\ell+1} \equiv 0$ we have $g_{\ell+1} \equiv 0$ and so f is a polynomial of degree at most ℓ. But since the flow φ_t has degree ℓ, f must be of degree exactly ℓ.

Suppose by contradiction that $a(0) = 0$. In this case we can write $a(x) = x^\ell h(x)$, where $\ell \geq 1$ and $h(0) \neq 0$. But using the recurrence rule (2) it is possible to prove that $f_k(x) = x^{\ell k - k + 1} h_k(0)$, where $h_k(0) \neq 0$ for all $k \geq 1$. As a consequence, the flow cannot be polynomial in t. □

Remark 4.4. Fixing $x = 0$ in the third line of the proof we immediately get that f is polynomial; we followed a longer process because it is essential in the study of the case $a(0) = 0$.

Theorem 4.3 implies that a germ of a holomorphic t-polynomial vector field in one variable has no singularities. This is not the case in $n \geq 2$ variables (consider for instance $x_2 \frac{\partial}{\partial x_1}$). Nevertheless, Theorem 4.3 has a natural generalization in $n \geq 2$ variables, but with an additional assumption of "non-singularities":

Theorem 4.5. *Let* $X = \sum\limits_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$ *be a germ at* 0 *of a non-singular* t*-polynomial vector field,* $a_1(0) \neq 0$*, for fixing ideas.*

There exists $f \in \mathrm{Diff}(\mathbb{C}^n, 0)$ *a germ of a diffeomorphism which is polynomial in the variable* x_1 *such that* $X = f_* \frac{\partial}{\partial x_1}$, i.e. $\varphi_t(x) = f(f^{-1}(x) + te_1)$*, where* φ_t *is the flow of* X *and* f^{-1} *is the local inverse of* f *at* 0.

Proof: Let f be a local conjugacy between X and $\frac{\partial}{\partial x_1}$ satisfying $f(0, x_2, x_3, \ldots, x_n) = (0, x_2, x_3, \ldots, x_n)$ (it is well known that such a conjugacy exists). In particular, $\varphi_t(x) = f(f^{-1}(x) + te_1)$ and

$$\varphi_t(0, x_2, x_3, \ldots, x_n) = f(t, x_2, x_3, \ldots, x_n);$$

in particular, f is thus polynomial in the variable x_1. □

4.2. Rigid chambars on $\mathbb{R}^n$ and foliations by straight lines. The following statement generalizes to the real case the property satisfied by the 2-chambars:

Theorem 4.6. *If* $\mathrm{Ch}(a_1 X, a_2 X, \ldots, a_p X)$ *is a rigid* p*-chambar on an open subset of* $\mathbb{R}^n$*, then the foliation* $\mathcal{F}_X$ *associated to* X *is a foliation by straight lines.*

Proof: As in the proof of Theorem 3.2, we get by successive derivations the equalities

$$\begin{cases} \displaystyle\sum_{k=1}^{p} a_k = 0 \\ \displaystyle\Big(\sum_{k=1}^{p} a_k^2\Big) DX \cdot X = 0. \end{cases}$$

Since $a_k \neq 0$ for any $1 \leq k \leq p$ one has $DX \cdot X = 0$. As a result, all the non-singular trajectories of X are straight lines. □

Theorem 4.6 cannot be generalized to the complex case. Let us give a counterexample of Theorem 4.6 in the complex case in dimension 2. Consider on $\mathbb{C}^2$ the linear vector field

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}.$$

The closure of its trajectories is the parabola $y = cx^2$ with $c \in \mathbb{P}^1_{\mathbb{C}}$ (if $c \in \{0, \infty\}$, then the trajectory is a line). Let us consider the vector field

$$Y = \frac{1}{x}X = \frac{\partial}{\partial x} + \frac{2y}{x}\frac{\partial}{\partial y},$$

which is holomorphic outside $x = 0$. Its 1-parameter group is the group of birational maps

$$(\exp tY)(x, y) = \left(x + t, \left(\frac{x+t}{x}\right)^2 y\right).$$

Hence if a_k belongs to $\mathbb{C}^*$, then one has

$$(\exp ta_kY)(x, y) = \left(x + a_kt, \left(\frac{x+a_kt}{x}\right)^2 y\right).$$

Take some non-zero constants $a_1, a_2, \ldots, a_p$, $p \geq 3$, such that

$$\sum_{k=1}^{p} a_k = \sum_{k=1}^{p} a_k^2 = 0.$$

Then the vector fields $Y_k = a_kY$, $1 \leq k \leq p$, form a p-chambar on the open set $\mathcal{U} = \mathbb{C}^2 \setminus \{x = 0\}$. But the trajectories of Y, which are almost the trajectories of X, are not straight lines.

Remark 4.7. Let X be a germ at $0 \in \mathbb{C}^n$ of a holomorphic vector field. Suppose that there exist some constants $a_1, a_2, \ldots, a_p$ such that the $X_k = a_kX$ generate a p-chambar. If X is not singular at 0, $X(0) \neq 0$, then $\mathrm{Ch}(a_1X, a_2X, \ldots, a_pX)$ is locally conjugate to the constant p-chambar $\mathrm{Ch}\big(a_1\frac{\partial}{\partial x}, a_2\frac{\partial}{\partial x}, \ldots, a_p\frac{\partial}{\partial x}\big)$. Indeed, if ϕ is a local diffeomorphism that conjugates X to $\frac{\partial}{\partial x}$ and if a belongs to $\mathbb{C}$, then ϕ conjugates aX to $a\frac{\partial}{\partial x}$. Take care to note that it does not mean that the image of a constant p-chambar via a diffeomorphism is a p-chambar (see Theorem 2.13).

4.3. Rigid and semi-rigid chambars on $\mathbb{C}^n$.

4.3.1. Rigid chambars on $\mathbb{C}^n$ and t-polynomial vector fields.

Theorem 4.8. *Let $\mathrm{Ch}(X, a_1X, a_2X, \ldots, a_{p-1}X)$ be a germ at $0 \in \mathbb{C}^n$ of a rigid p-chambar.*

Then the flow φ_t of X is a polynomial of degree at most $p-1$, as a function of the time t.

If $t \cdot d(X) = d$, then $a_1, a_2, \ldots, a_p$ satisfy

$$a_1^\ell + a_2^\ell + \cdots + a_p^\ell = 0 \quad \forall\, 1 \leq \ell \leq d.$$

In particular, if $d = p-1$, then $a_1^p = a_2^p = \cdots = a_p^p$.

Moreover, if the p-chambar $(a_1X, a_2X, \ldots, a_pX)$ is irreducible, then $\frac{a_k}{a_1}$ is a primitive p-th root of unity for some $1 \leq k \leq p$.

Proof: Write X as $\sum\limits_{k=1}^{n} X_k\frac{\partial}{\partial x_k}$; the barycentric condition is the following:

$$px_j = px_j + t(a_1 + a_2 + \cdots + a_p)X_j + \frac{t^2}{2}(a_1^2 + a_2^2 + \cdots + a_p^2)X(X_j) + \cdots + \frac{t^k}{k!}(a_1^k + a_2^k + \cdots + a_p^k)X^{k-1}(X_j) + \cdots$$

for $j = 1, 2, \ldots, n$.

The fact that the coefficients a_k are different from zero implies that a Newton formula

$$a_1^\ell + a_2^\ell + \cdots + a_p^\ell$$

is non-zero for an $\ell \leq p$. As a consequence, $X^m(X_j) \equiv 0$ for all $m \geq \ell - 1$ and $1 \leq j \leq n$. This implies that the flow of X, and the flows of the $a_k X$, are polynomial in t.

The other facts can be checked by the reader. □

4.3.2. A property of the singular set. Let X be a holomorphic vector field defined on an open subset $\mathcal{U}$ of $\mathbb{C}^n$. Denote by $\mathcal{F}_X$ the singular 1-dimensional foliation defined by X on $\mathcal{U}$. A *separatrix* γ of X through $x_0 \in \mathrm{Sing}(X)$ is a germ of an analytic curve at x_0 such that

- ◇ $X \not\equiv 0$ on $\gamma \setminus \{x_0\}$,
- ◇ x_0 belongs to γ,
- ◇ $\gamma \setminus \{x_0\}$ is a leaf of the germ of $\mathcal{F}_X$ at x_0.

This means that x_0 belongs to γ and if x belongs to $\gamma \setminus \{x_0\}$, then $X(x) \neq 0$ and $T_x\gamma = \mathbb{C} \cdot X(x)$.

Let X be a holomorphic vector field defined on a closed ball $B = \overline{B(0,r)}$ with $X(0) = 0$. We suppose that X is a t-polynomial vector field, that is, $t \mapsto \varphi_t(x)$ is polynomial in t, $x \in B$, $\varphi_t = \exp tX$. Note that for any $x \in B$, $t \mapsto \varphi_t(x)$ can be extended along the whole line $\mathbb{C}$. As a consequence, if $x \in B$, the leaf $\mathcal{L}_x$ of $\mathcal{F}_X$ in B is

- ◇ either the point x (case $x \in \mathrm{Sing}(X)$)
- ◇ or the connected component of $\mathcal{L}'_x \cap B$ containing x, where $\mathcal{L}'_x$ is the rational curve image of $t \mapsto \varphi_t(x)$.

Lemma 4.9. *Suppose that x does not belong to* $\mathrm{Sing}(X)$*; then* 0 *does not belong to the closure $\overline{\mathcal{L}_x}$ of $\mathcal{L}_x$ in B.*

Proof: Assume by contradiction that 0 belongs to $\overline{\mathcal{L}_x}$. Then there is a sequence $(t_n)_n$ of complex numbers such that $\lim_{n\to+\infty} \varphi_{t_n}(x) = 0$. Since $0 \in \mathrm{Sing}(X)$ one has $\lim_{n\to+\infty} |t_n| = +\infty$, and as $t \mapsto \varphi_t(x)$ is polynomial (non-constant) $\lim_{n\to+\infty} |\varphi_{t_n}(x)| = +\infty$: a contradiction. □

Theorem 4.10. *Let $X \in \chi(\mathbb{C}^n, 0)$ be a germ of a t-polynomial vector field at the origin of $\mathbb{C}^n$.*

Assume that $\mathrm{Sing}(X) \neq \emptyset$. *Then* $\dim \mathrm{Sing}(X) \geq 1$.

Moreover, X has no separatrices through a singularity.

Proof: Assume that X is defined on the ball $B = \overline{B(0,r)}$ and that 0 is an isolated singularity of X. Let $(x_n)_n$ be a sequence of points of B such that $\lim_{n\to+\infty} x_n = 0$. The leaf $\mathcal{L}_{x_n}$ is closed in B and cuts the sphere $S(0,r) = B \setminus B(0,r)$. Let y_n be a point in $\mathcal{L}_{x_n} \cap S(0,r)$ and y_0 a limit point of y_n, up to extraction $y_0 = \lim_{n\to+\infty} y_n$. According to Lemma 4.9 the point 0 does not belong to $\overline{\mathcal{L}_{y_0}}$ and $\mathcal{L}_{y_0}$ can be seen as the leaf of the restriction of $\mathcal{F}_{X|B\setminus B(0,r')}$ for r' sufficiently small. The fact that $y_0 = \lim_{n\to+\infty} y_n$ implies that $\mathcal{L}_{y_n}$ is contained in $B \setminus B(0,r')$ for n sufficiently large: a contradiction with $\lim_{n\to+\infty} x_n = 0$. □

Corollary 4.11. *Let* $\mathrm{Ch}(a_1X, a_2X, \ldots, a_pX)$ *be a rigid* p*-chambar on an open set* $\mathcal{U}$ *of* $\mathbb{C}^n$*. Then*

- ⋄ *either* $\mathrm{Sing}(X) = \emptyset$*, that is,* X *is regular*
- ⋄ *or* $\dim \mathrm{Sing}(X) \geq 1$*.*

Example 4.12. Let X be a linear nilpotent vector field on $\mathbb{C}^n$. Then the flow $\exp tX$ is a polynomial of degree $d = \mathrm{rk}\, X$. Moreover, $\dim \mathrm{Sing}(X) = n - d$. For instance if $X^{n-1} \neq 0$, then $\dim \mathrm{Sing}(X) = 1$.

Problem 4.13. *Does there exist a vector field with an isolated singularity belonging to a* p*-chambar?*

Remark 4.14. Recall that the Camacho–Sad theorem ([**1**]) says that a holomorphic foliation $\mathcal{G}$ by curves at the origin 0 of $\mathbb{C}^2$ has an invariant curve passing through 0. As a consequence, if X is a t-polynomial vector field at the origin 0 of $\mathbb{C}^2$, with $X(0) = 0$, then the invariant curves of the foliation associated to X are contained in the singular set $\mathrm{Sing}(X)$.

The previous considerations suggest in dimension ≥ 3 the following question:

Question 4.1. Let X be a germ at $0 \in \mathbb{C}^n$ of a holomorphic vector field. Assume that the closure of the integral curves is analytic. Does X preserve an invariant curve passing through 0?

4.3.3. Semi-rigid chambars on $\mathbb{C}^n$.

Definition 4.15. A p-chambar $\mathrm{Ch}(X_1, X_2, \ldots, X_p)$ on an open subset of $\mathbb{C}^n$ is *semi-rigid* if the X_k's are colinear, that is, if $X_1 \wedge X_k = 0$ for any $2 \leq k \leq p$.

In dimension 1 all chambars are semi-rigid.

Example 4.16. The 3-chambar $\mathrm{Ch}\big(\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, -(y+1)\frac{\partial}{\partial x}\big)$ on $\mathbb{C}^2$ is semi-rigid but not rigid.

Example 4.17. The 4-chambar $\mathrm{Ch}\big(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, -y\frac{\partial}{\partial x}\big)$ on $\mathbb{C}^2$ is semi-rigid but not rigid. Note that it is a non-irreducible chambar.

Proposition 4.18. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a semi-rigid* 3*-chambar on an open subset of* $\mathbb{C}^n$*. Then one of the following holds:*[1]

- ⋄ $\mathcal{F}_{X_1} = \mathcal{F}_{X_2} = \mathcal{F}_{X_3}$ *and* $\mathcal{F}_{X_i}$ *is a foliation by straight lines;*
- ⋄ $\mathrm{Ch}(X_1, X_2, X_3)$ *is a rigid chambar.*

Proof: Let $\mathcal{U}$ be an open subset of $\mathbb{C}^n$ where the X_i's are defined. Set $X_1 = X$; then $X_2 = fX$, where f denotes a meromorphic function defined on $\mathcal{U}$. The barycentric condition implies that $X_3 = -(1 + f)X$. The equality

$$\sum_{k=1}^{3} DX_k \cdot X_k = 0$$

obtained by derivation from the barycentric property can be rewritten as

$$2(1 + f + f^2)DX \cdot X + (1 + 2f)X(f) \cdot X = 0,$$

which implies that

$$(1 + f + f^2)X \wedge DX \cdot X = 0.$$

If $1 + f + f^2 = 0$, then f is constant and $\mathrm{Ch}(X_1, X_2, X_3)$ is rigid. Otherwise, we have $X \wedge DX \cdot X = 0$ and so $\mathcal{F}_X$ is a foliation by lines. □

[1]Note that the two properties are not mutually exclusive.

Question 4.2. Does there exist a generalization of Proposition 4.18 for p-chambars, $p \geq 3$?

The answer is positive in the real case:

Proposition 4.19. *Let* $\mathrm{Ch}(X_1, X_2, \ldots, X_p)$ *be a semi-rigid* p*-chambar on an open subset* $\mathcal{U} \subset \mathbb{R}^n$, $n \geq 2$. *Then* $\mathcal{F}_{X_1} = \mathcal{F}_{X_2} = \cdots = \mathcal{F}_{X_p}$ *is a foliation by straight lines.*

Proof: Since the chambar is semi-rigid we can write $X_j = f_j \cdot X$, where X is a vector field on $\mathcal{U}$ and $f_j : \mathcal{U} \to \mathbb{R}$, $1 \leq j \leq p$. Note that

$$DX_j \cdot X_j = D(f_j \cdot X) \cdot (f_j X) = f_j \cdot X(f_j) \cdot X + f_j^2 \cdot DX \cdot X.$$

In particular, we get

$$0 = \sum_{k=1}^{p} DX_k \cdot X_k = \left(\sum_{k=1}^{p} f_k \cdot X(f_k) \right) \cdot X + \left(\sum_{k=1}^{p} f_k^2 \right) \cdot DX \cdot X.$$

Taking the wedge product with X in the above relation, we get

$$\left(\sum_{k=1}^{p} f_k^2 \right) X \wedge DX \cdot X = 0.$$

Since the f_k's are not identically zero, we get $X \wedge DX \cdot X \equiv 0$. Therefore, $\mathcal{F}_X$ is a foliation by straight lines. □

5. Description of 3-chambars and 4-chambars in one variable

5.1. Description of 3-chambars in one variable.

Theorem 5.1. *Let* $\mathcal{B}$ *be a holomorphic* 3*-chambar on some connected open subset of* $\mathbb{C}$. *Then*

- ◇ *either* $\mathcal{B}$ *is a constant* 3*-chambar*
- ◇ *or* $\mathcal{B} = \mathrm{Ch}\big(a(x)\frac{\partial}{\partial x}, \mathbf{j}a(x)\frac{\partial}{\partial x}, \mathbf{j}^2 a(x)\frac{\partial}{\partial x}\big)$, *where* $a(x) = \sqrt{\lambda x + \mu}$ *with* $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C}$.

In particular, $\mathcal{B}$ *is a rigid chambar.*

Remark 5.2. In a certain sense Theorem 5.1 shows that the set of 3-chambars on a connected set of $\mathbb{C}$ has two "irreducible components".

Proof: Set $\mathcal{B} = \mathrm{Ch}(X_1, X_2, X_3)$. We can write $X_k = a_k(x)\frac{\partial}{\partial x}$, where $a_k \in \mathcal{O}_1$, $1 \leq k \leq 3$. The barycentric property implies that $\sum_{i=1}^{3} X_i^k(x) = 0$ for any $k \geq 1$.

Assume that the a_i's are non-constant and that $X_i^2(x) \neq 0$ for any $1 \leq i \leq 3$. Furthermore, $X_i^{k+1}(x) = a_i(X_i^k(x))'$ thus

$$\begin{cases} a_1' + a_2' + a_3' = 0 \\ a_1 a_1' + a_2 a_2' + a_3 a_3' = 0 \\ (X_1^k(x))' + (X_2^k(x))' + (X_3^k(x))' = 0 \\ a_1 (X_1^k(x))' + a_2 (X_2^k(x))' + a_3 (X_3^k(x))' = 0. \end{cases} \tag{5.1}$$

As a consequence, for any $k \geq 2$, there exists a meromorphic function f_k such that $(X_i^k(x))' = f_k a_i'$ for any $1 \leq i \leq 3$, where $f_2 \neq 0$. This yields

$$X_i^{k+1}(x) = a_i(X_i^k(x))' = f_k a_i a_i' = f_k X_i^2(x) \quad \forall\, 1 \leq i \leq 3, \forall\, k \geq 2$$

and

$$f_k (X_i^2(x))' + f_k' X_i^2(x) = (X_i^{k+1}(x))' = f_{k+1} a_i' \quad \forall\, 1 \leq i \leq 3, \forall\, k \geq 2.$$

In particular,

$$f_2(X_i^2(x))' + f_2' X_i^2(x) = f_3 a_i', \quad f_k(X_i^2(x))' + f_k' X_i^2(x) = f_{k+1} a_i' \quad \forall\, k \geq 3$$

and so $X_i^2(x)$ satisfies an equation of the form $F_k(X_i^2(x))' + G_k X_i^2(x) = 0$, where $F_k = f_2 f_{k+1} - f_3 f_k$ and $G_k = f_2' f_{k+1} - f_3 f_k'$ $\forall\, k \geq 3$.

(1) Let us assume first that $F_k \neq 0$ for some $k \geq 3$. In this case, for any $1 \leq i \leq 3$ there exist constants c_i such that $X_i^2(x) = c_i^2 H$, where $H = \exp\bigl(-\int G_k/F_k \,\mathrm{d}x\bigr)$. As a result, the equality $a_i a_i' = c_i^2 H$ holds for any $1 \leq i \leq 3$, and $a_i^2 = c_i^2 K + d_i$ for some complex numbers d_i. At a generic point x_0 the function K is holomorphic and (by implicit function theorem) conjugate to $\varepsilon + x$, $\varepsilon = K(x_0)$. The barycentric property

$$\sum_{i=1}^{3} a_i = \sum_{i=1}^{3} c_i \left(K + \frac{d_i}{c_i^2}\right)^{1/2} = 0$$

implies

$$\sum_{i=1}^{3} c_i \left(x + \varepsilon + \frac{d_i}{c_i^2}\right)^{1/2} = 0,$$

which is a global identity between multivaluate elementary functions. By looking at the roots of $x + \varepsilon + \frac{d_i}{c_i^2}$ we see that

$$\frac{d_1}{c_1^2} = \frac{d_2}{c_2^2} = \frac{d_3}{c_3^2} := \mu.$$

As a result, $a_i = c_i(K + \ \mu)^{1/2}$, which implies $a_i a_i' = c_i^2 K'$. According to the second equation of (5.1) we have

$$(c_1^2 + c_2^2 + c_3^2) K' = 0.$$

- ◇ If $c_1^2 + c_2^2 + c_3^2 \neq 0$, then K is constant.
- ◇ If $c_1^2 + c_2^2 + c_3^2 = 0$, then up to multiplication by a constant either $(c_1, c_2, c_3) = (1, \mathbf{j}, \mathbf{j}^2)$ or $(c_1, c_2, c_3) = (1, \mathbf{j}^2, \mathbf{j})$.

Let us recall that if $X = b(x)\frac{\partial}{\partial x}$, then by formula (2.1):

$$(5.2) \qquad \begin{aligned}(\exp tX)(x) = x + tb(x) + \frac{t^2}{2} b(x)b'(x) + \frac{t^3}{3!}(b(x)b'(x)^2 + b^2(x)b''(x)) \\ + \frac{t^4}{4!} b(x)(b(x)b'(x)^2 + b^2(x)b''(x))' + \cdots\end{aligned}$$

From (5.2) we get $\bigl(\sum_{k=1}^{3} c_k^3\bigr) \cdot (K''K^2 + K'^2 K) = 0$ and

$$K''K^2 + K'^2 K = 0$$

since $c_1^3 + c_2^3 + c_3^3 = 3$. Therefore

$$0 = K''K^2 + K'^2 K = K(K''K + K'^2) = K(KK')'$$

and $KK' = \frac{\lambda}{2}$ for some λ in $\mathbb{C}$. As a result, $K^2 = \lambda x + \mu$ for some $\mu \in \mathbb{C}$.

(2) If $F_k = 0$ for all $k \geq 3$, but $G_\ell \neq 0$ for some ℓ, then $a_i a_i' = 0$, $1 \leq i \leq 3$, so that the a_i's are constant.

(3) If $F_k = G_k = 0$ for all $k \geq 3$, we get $f_2 f_{k+1} = f_k f_3$ and $f_2' f_{k+1} = f_k' f_2$ for any $k \geq 3$, which implies the following possibilities:

◇ If $f_3 = 0$, then $a_i a_i' = k_i$, where the k_i's are constant. Hence, we get $a_i(x) = (k_i x + m_i)^{1/2}$. From the first equation of (5.1) we get $\sum_{i=1}^{3}(k_i x + m_i)^{1/2} = 0$. Therefore, $\frac{m_1}{k_1} = \frac{m_2}{k_2} = \frac{m_3}{k_3} = m$ and $a_i = k_i^{1/2}(x+m)^{1/2}$.

◇ If $f_3 \neq 0$, then $f_2 f_k' = f_2' f_k$ for any $k \geq 3$, and thus there exists a constant c_k such that $f_k = c_k f_2$.

In particular,

$$(X_i^k(x))' = c_k f_2 a_i' = c_k (X_i^2(x))'$$

so $X_i^{k+1}(x) = c_k X_i^3(x)$ for any $k \geq 3$ and $1 \leq i \leq 3$. But $(X_i^3(x))' = c_3(X_i^2(x))'$ implies $f_2 X_i^2(x) = X_i^3(x) = c_3 X_i^2(x) + d_3$, where d_3 denotes a complex number. From $(f_2 - c_3)X_i^2(x) = d_3$, we get $d_3 = 0$ and $f_2 = c_3$; as a result, for any $k \geq 2$ there exists $\alpha_k \in \mathbb{C}$ such that

$$(X_i^k(x))' = f_k a_i' = c_k c_3 a_i' = \alpha_k a_i' \quad \forall\, 1 \leq i \leq 3.$$

Consequently, for any $k \geq 2$ there exist α_k and β_k in $\mathbb{C}$ such that

$$X_i^k(x) = \alpha_k a_i(x) + \beta_k \quad \forall\, 1 \leq i \leq 3$$

and

$$\varphi_t^i(x) = x + F(t) a_i(x) + G(t),$$

where φ_t^i is the flow of X_i and

$$F(t) = t + \sum_{k \geq 2} \frac{\alpha_k}{k!} t^k, \quad G(t) = \sum_{k \geq 2} \frac{\beta_k}{k!} t^k.$$

Recall that if X is a vector field and if φ_t is its flow, then the derivation of a holomorphic function f by X satisfies

$$X(f) = \frac{\partial}{\partial s} f \circ \varphi_s \Big|_{s=0}.$$

In particular in one variable, by taking $f(x) = \varphi_t^i(x)$ (holomorphic function with parameter t) we get

$$\begin{aligned} F'(t) a_i(x) + G'(t) &= a_i(x) \frac{\partial}{\partial x}(x + F(t) a_i(x) + G(t)) \\ &= a_i(x) + F(t) a_i(x) a_i'(x). \end{aligned}$$

Looking at the coefficients of t in both sides of the equality we get $2\alpha_2 a(x) = a(x)a'(x)$, that is, $2\alpha_2 a(x) = a(x)a'(x)$ and so $2\alpha_2 = a'(x)$. Therefore, $a_i(x) = 2\alpha_2(x - x_i)$, and

$$\varphi_t^i(x) = x_i + e^{2\alpha_2 t}(x - x_i).$$

However, this is not possible for a chambar, unless $\alpha_2 = 0$, and the a_i's are constant. □

Corollary 5.3. *Let $\mathcal{B} = \mathrm{Ch}(X_1, X_2, X_3)$ be a local 3-chambar on $\mathbb{R}$. Then $\mathcal{B}$ is a constant 3-chambar $\mathrm{Ch}\big(c_1 \frac{\partial}{\partial x}, c_2 \frac{\partial}{\partial x}, c_3 \frac{\partial}{\partial x}\big)$ with c_i non-zero real numbers such that $c_1 + c_2 + c_3 = 0$.*

5.2. p-chambars with weights.

Definition 5.4. Let us consider p analytic vector fields $X_1, X_2, \dots, X_p$, defined on some open subset $\mathcal{U}$ of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), with flows $t \mapsto \varphi_t^\ell$, $1 \le \ell \le p$. Consider also non-zero real (resp. complex) numbers $\alpha_1, \alpha_2, \dots, \alpha_p$ and $\alpha = \sum_\ell \alpha_\ell$.

We say that $X_1, X_2, \dots, X_p$ define a *holomorphic p-chambar with weights* $\alpha_1, \alpha_2, \dots, \alpha_p$ if

$$\alpha_1 \varphi_t^1(x) + \alpha_2 \varphi_t^2(x) + \cdots + \alpha_p \varphi_t^p(x) = \alpha x, \tag{5.3}$$

for all (t, x) where the above formula makes sense.

Remark 5.5. This definition is equivalent to

$$\alpha_1 X_1^k(x_\ell) + \alpha_2 X_2^k(x_\ell) + \cdots + \alpha_p X_p^k(x_\ell) = 0 \quad \forall k \ge 1, \forall 1 \le \ell \le n.$$

We note that the condition is not equivalent to considering the flows of the vector fields $\alpha_\ell X_\ell$, $1 \le \ell \le n$.

The classification of 3-chambars (Theorem 5.1) can be extended to this type of chambars with an adaptation in the second case:

Theorem 5.6. *Assume that X_1, X_2, and X_3 define a holomorphic 3-chambar $\mathcal{B}$ with weights α_1, α_2, and α_3 on some connected open subset of $\mathbb{C}$. Then*

- ◇ *either $\mathcal{B}$ is a constant 3-chambar*
- ◇ *or $\mathcal{B} = \mathrm{Ch}\big(\beta_1 a(x)\frac{\partial}{\partial x}, \beta_2 a(x)\frac{\partial}{\partial x}, \beta_3 a(x)\frac{\partial}{\partial x}\big)$, where $a(x) = \sqrt{\lambda x + \mu}$ with $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C}$, and*

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = \alpha_1\beta_1^2 + \alpha_2\beta_2^2 + \alpha_3\beta_3^2 = 0.$$

In particular, $\mathcal{B}$ is a rigid chambar.

5.3. Almost p-chambars.

Definition 5.7. Let X be a vector field. We say that X is *almost a p-chambar* if there exist non-zero vector fields $X_2, X_3, \dots, X_p$ such that $(X, X_2, X_3, \dots, X_p)$ is a p-chambar.

We say that X is *almost a chambar* if there exists an integer p such that X is almost a p-chambar.

Remark 5.8. If X is almost a p-chambar, then X is almost a $(p+q)$-chambar for any $q \ge 2$.

Example 5.9. The constant vector fields are almost p-chambars for any $p \ge 2$.

Example 5.10. Let X be a nilpotent linear vector field, and let p be its index of nilpotency. Then X is almost a p-chambar.

We suspect that most vector fields are not almost chambars. Let us give an explicit example in (real or complex) dimension 1:

Proposition 5.11. *If λ is a non-zero constant, then the vector field $\lambda x \frac{\partial}{\partial x}$ is*

- ◇ *not almost a 2-chambar in a neighborhood of 0;*
- ◇ *not almost a 3-chambar in a neighborhood of 0.*

Remark 5.12. The first assertion of the statement is clear.

The second one is a direct consequence of the classification of the 3-chambars (Theorem 5.1). Note that the argument does not use the property of nilpotency of linear chambars; indeed, if $(X_1, X_2, \dots, X_p)$ is a p-chambar containing $X = \lambda x \frac{\partial}{\partial x}$, then it is possible that one of the $X_k(0)$ is non-zero. We conjecture that any semi-simple linear vector field $\sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, $\lambda_i \neq 0$, is not almost a p-chambar.

5.4. Some remarks on 4-chambars in one variable. The 2-chambars and 3-chambars on an open subset of $\mathbb{C}$ are rigid. This property is not satisfied by all the 4-chambars. Consider the vector fields $X = 2\sqrt{x}\frac{\partial}{\partial x}$ and $Y = 2\sqrt{x+\varepsilon}\frac{\partial}{\partial x}$, $\varepsilon \neq 0$, on a suitable domain of $\mathbb{C}$. As we know, the flows of X and Y are

$$\exp tX = x + 2t\sqrt{x} + t^2, \quad \exp tY = x + 2t\sqrt{x+\varepsilon} + t^2$$

and it is easy to see that the 4-chambar $\mathrm{Ch}(X, -X, \mathbf{i}Y, -\mathbf{i}Y)$ is irreducible and non-rigid. Such a 4-chambar is said to be *special*.

Conjecture 5.13. *Up to affine conjugacy a* 4*-chambar on an open subset of* $\mathbb{C}$ *is of one of the following types:*

- *constant* $\mathrm{Ch}\big(a_1\frac{\partial}{\partial x}, a_2\frac{\partial}{\partial x}, a_3\frac{\partial}{\partial x}, a_4\frac{\partial}{\partial x}\big)$*,* $a_k \in \mathbb{C}^*$*;*
- *rigid of t-degree* 2*:* $\mathrm{Ch}(a_1X, a_2X, a_3X, a_4X)$ *with* $X = 2\sqrt{x}\frac{\partial}{\partial x}$ *and* a_k *constants satisfying* $a_1 + a_2 + a_3 + a_4 = a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0$*;*
- *rigid of t-degree* 3*:* $\mathrm{Ch}(X, \sigma X, \sigma^2X, \sigma^3X)$ *with* X *of t-degree* 3 *and* σ *a root of unity of order* 4*;*
- *special* $\mathrm{Ch}(X, -X, Y, -Y)$ *with* X *and* Y *of t-degree* 2*.*

Remark 5.14. The classification of p-chambars on $\mathbb{C}$ for $p \geq 4$ is a difficult problem in particular because of irreducibility problems. Indeed, if $p = 6$, for instance, one can consider the vector field $Z_5 = 5x^{\frac{4}{5}}\frac{\partial}{\partial x}$ to which one can associate the 6-chambar

$$\mathrm{Ch}(Z_5, \sigma Z_5, \sigma^2 Z_5, \sigma^3 Z_5, \sigma^4 Z_5, \sigma^5 Z_5),$$

which is irreducible. But one can also consider the non-irreducible 6-chambar obtained as follows:

$$\mathrm{Ch}(X_1, \mathbf{j}X_1, \mathbf{j}^2X_1, X_2, \mathbf{j}X_2, \mathbf{j}^2X_2),$$

where $X_k = \sqrt{\lambda_k x + \mu_k}\frac{\partial}{\partial x}$ and λ_k, μ_k are complex numbers such that $\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$.

Problem 5.15. *Classify irreducible p-chambars in dimension* 1*, for* $p \geq 4$*.*

Theorem 5.16. *Let* $\mathrm{Ch}(X_1, X_2, X_3, X_4)$ *be a holomorphic* 4*-chambar on some open set* $\mathcal{U} \subset \mathbb{C}$*. Set* $X_k = y_k(x)\frac{\partial}{\partial x}$ *with* $y_k \in \mathcal{O}(\mathcal{U})$ *for* $1 \leq k \leq 4$*.*

Then there exists a polynomial $P\colon \mathbb{C}^3 \to \mathbb{C}^4$ *independent of the* y_k*'s such that the vector* $y = (y_1, y_2, y_3, y_4)$ *satisfies a differential equation of the form*

$$\Delta(y) \cdot y''' = P(y, y', y''), \tag{5.4}$$

where $\Delta(y) = \prod_{i<j}(y_j - y_i)$*.*

Furthermore, the polynomial P *is homogeneous of degree* 7*.*

Proof: Let us recall some basic facts. The operator X_k on $\mathcal{O}(\mathcal{U})$ acts as $X_k(f) = y_k \cdot f'$. In particular,

$$X_k(x) = y_k, \quad X_k^2(x) = y_k y_k', \quad X_k^3(x) = p(y_k, y_k') + y_k^2 y_k'', \quad X_k^4(x) = q(y_k, y_k', y_k'') + y_k^3 y_k''',$$

where $p(y,z) = yz^2$ and $q(y,z,w) = yz^3 + 4y^2zw$. More generally we have

$$X_k^\ell(x) = P_\ell(y_k, y_k', \dots, y_k^{(\ell-2)}) + y_k^{\ell-1} \cdot y_k^{(\ell-1)}, \tag{5.5}$$

where P_ℓ denotes a homogeneous polynomial of degree ℓ.

Using (5.5) we get by an induction argument

$$\frac{\partial^n X_k^\ell(x)}{\partial x^n} = P_{\ell,n}(y_k, y_k', \dots, y_k^{(\ell+n-2)}) + y_k^{\ell-1} \cdot y_k^{(\ell+n-1)}, \tag{5.6}$$

where $P_{\ell,n}$ is homogeneous of degree ℓ and $P_{\ell,0} = P_\ell$. Note that $P_{\ell,n}$ is independent of the open set $\mathcal{U}$ and of the function $y\colon \mathcal{U} \to \mathbb{C}^4$.

Since the X_k's satisfy the barycentric condition, we have $\sum_{k=1}^4 X_k^\ell(x) = 0$, $1 \le k \le 3$, and so

$$\sum_{k=1}^4 \frac{\partial^n X_k^\ell(x)}{\partial x^n} = 0 \quad \forall\, 1 \le \ell \le 4, \forall\, n \ge 0.$$

From the above relations and (5.5) we get the following system of equations:

$$\begin{cases} y_1''' + y_2''' + y_3''' + y_4''' = 0 \\ y_1 y_1''' + y_2 y_2''' + y_3 y_3''' + y_4 y_4''' = Q_2(y, y', y'') \\ y_1^2 y_1''' + y_2^2 y_2''' + y_3^2 y_3''' + y_4^2 y_4''' = Q_3(y, y', y'') \\ y_1^3 y_1''' + y_2^3 y_2''' + y_3^3 y_3''' + y_4^3 y_4''' = Q_4(y, y', y'') \end{cases}$$

with

$$\begin{cases} Q_2(y, y', y'') = -3 \sum_{i=1}^4 y_i' y_i'' \\ Q_3(y, y', y'') = -\sum_{i=1}^4 ((y_i')^3 + 4 y_i y_i' y_i'') \\ Q_4(y, y', y'') = -\sum_{i=1}^4 (y_i (y_i')^3 + 4 y_i^2 y_i' y_i''). \end{cases}$$

Writing the above system in the matrix form we get $W(y) \cdot {}^{\mathrm{t}}(y''') = {}^{\mathrm{t}}Q(y, y', y'')$, where ${}^{\mathrm{t}}v$ denotes the transpose of v and

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 \end{pmatrix}.$$

Solving (5.6) we get that the vector function y satisfies the ODE

$$\Delta\, {}^{\mathrm{t}}(y''') = \operatorname{adj}(W)(y) \cdot {}^{\mathrm{t}}Q(y, y', y''), \tag{5.7}$$

where $\operatorname{adj}(W)$ is the adjoint of the matrix W, $\Delta = \det(W) = \prod_{i<j} (y_j - y_i)$, and $Q = (0, Q_2, Q_3, Q_4)$. Set $P(y, y', y'') = \operatorname{adj}(W)(y) \cdot {}^{\mathrm{t}}Q(y, y', y'')$. By looking carefully at the right-hand side of the above relation, we see that P is homogeneous of degree 7. □

Remarks 5.17. Let us fix three (constant) vectors α_0, α_1, and α_2 in $\mathbb{C}^4$ and assume that the components of α_0 are two by two different. Then there exists a unique germ $y = (y_1, y_2, y_3, y_4) \in \mathcal{O}(\mathbb{C}^4, 0)$ satisfying (5.4) with initial conditions $y(0) = \alpha_0$, $y'(0) = \alpha_1$, and $y''(0) = \alpha_2$.

Since the differential equation (5.4) is meromorphic on $\mathbb{C}^4$ the solution $x \mapsto y(x)$ can be extended until it reaches the codimension 1 submanifold $\bigcup_{i<j} (y_i = y_j)$ of $\mathbb{C}^4$.

For instance, the constant vectors $y = (a_1, a_2, a_3, a_4)$ are solutions of the ODE (5.7). In fact, if y is a constant vector, then $y' = y'' = 0$ and $Q(y, y', y'') = 0$.

Next we will study the solutions with initial condition of the form $y_i(0) = y_j(0)$, $i \neq j$. The idea is to lift the ODE to a first-order ODE on $\mathbb{C}^{12}$.

Consider the ODE (5.4) of order 3 on $\mathcal{U} \subset \mathbb{C}^4$. Introducing new variables $z = y'$ and $w = z' = y''$, this ODE can be lifted to a system of meromorphic ODE's of order 1 on $\mathcal{V} = \mathcal{U} \times \mathbb{C}^4 \times \mathbb{C}^4$ as

$$\begin{cases} y' = z \\ z' = w \\ w' = \Delta^{-1} \cdot P(y, z, w). \end{cases} \tag{5.8}$$

Multiplying (5.8) by Δ we obtain a tangent holomorphic vector field on $\mathcal{V}$

$$\chi(y, z, w) = \Delta \sum_{j=1}^{4} z_j \frac{\partial}{\partial y_j} + \Delta \sum_{j=1}^{4} w_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{4} P_j(y, z, w) \frac{\partial}{\partial w_j}.$$

Theorem 5.18. *The following submanifolds of $\mathbb{C}^{12}$ are χ-invariant:*

- $\diamond$ $\Sigma_{ij} := \mathcal{Z}(\langle y_j - y_i \rangle)$ *for any* $1 \leq i < j \leq 4$*;*
- $\diamond$ $\Sigma_1 := \mathcal{Z}\big(\big\langle \sum\limits_j y_j, \sum\limits_j z_j, \sum\limits_j w_j \big\rangle\big)$*;*
- $\diamond$ $\Sigma_2 := \mathcal{Z}\big(\big\langle \sum\limits_j y_j z_j, \sum\limits_j (z_j^2 + y_j w_j) \big\rangle\big)$*;*
- $\diamond$ $\Sigma_3 := \mathcal{Z}\big(\big\langle \sum\limits_j (y_j z_j^2 + y_j^2 w_j) \big\rangle\big)$*.*

The notation $\mathcal{Z}(\mathcal{J})$ stands for the zeroes of the ideal $\mathcal{J}$.

All these submanifolds are complete intersections and the codimensions coincide with the number of generators of the ideal. Furthermore, the submanifolds Σ_i, $1 \leq i \leq 3$, coincide with the initial conditions corresponding to the barycentric conditions

$$\sum_{k=1}^{4} \frac{\partial^n X_k^\ell}{\partial x^n} = 0 \quad \forall\, 1 \leq n + \ell \leq 4, \, \forall\, n \geq 0.$$

Let us now give a lemma that will be useful for the proof of Theorem 5.18.

Lemma 5.19. *The components P_1, P_2, P_3, P_4 of χ satisfy the following relations:*

- $\diamond$ $\sum\limits_i P_i = 0$,
- $\diamond$ $\sum\limits_i y_i P_i = \Delta Q_2(y, z, w) = -3\Delta \sum\limits_i z_i w_i$,
- $\diamond$ $\sum\limits_i y_i^2 P_i = \Delta Q_3(y, z, w) = -\Delta \sum\limits_i (z_i^3 + 4 y_i z_i w_i)$,
- $\diamond$ $\sum\limits_i y_i^3 P_i = \Delta Q_4(y, z, w) = -\Delta \sum\limits_i (y_i z_i^3 + 4 y_i^2 z_i w_i)$.

Proof: Recall that on the one hand

$${}^{\mathrm{t}}P(y, y', y'') = \mathrm{adj}(W)(y)\,{}^{\mathrm{t}}Q(y, y', y'')$$

so

$${}^{\mathrm{t}}P(y, z, w) = \mathrm{adj}(W)(y)\,{}^{\mathrm{t}}Q(y, z, w).$$

On the other hand the relations in the statement of the lemma are equivalent to $W(y)\,{}^{\mathrm{t}}P(y, z, w) = \Delta\,{}^{\mathrm{t}}Q(y, z, w)$. Finally, if id is the identity matrix, we know from linear algebra that $W(y)\,\mathrm{adj}(W)(y) = \Delta \cdot \mathrm{id}$. As a consequence,

$$W(y)\,{}^{\mathrm{t}}P(y, z, w) = W(y)\,\mathrm{adj}(W)(y)\,{}^{\mathrm{t}}Q(y, z, w) = \Delta\,{}^{\mathrm{t}}Q(y, z, w). \qquad \square$$

Proof of Theorem 5.18: Let $\mathcal{J}$ be an ideal of $\mathbb{C}[y,z,w]$. Recall that the submanifold $\mathcal{Z}(\mathcal{J})$, defined by $\mathcal{J}$, is χ-invariant if, and only if, $\chi(\mathcal{J}) \subset \mathcal{J}$. So, for instance,

$$\chi(y_k - y_\ell) = (z_k - z_\ell)\prod_{i<j}(y_j - y_i)$$

and $\chi(y_k - y_\ell)$ belongs to $\langle y_k - y_\ell\rangle$; in particular, $\Sigma_{k\ell}$ is χ-invariant.

Consider the ideal $\mathcal{J}_1 = \big\langle \sum\limits_j y_j, \sum\limits_j z_j, \sum\limits_j w_j\big\rangle$. We have

$$\chi\Big(\sum_i y_i\Big) = \sum_i \chi(y_i) = \Delta \sum_i z_i \in \mathcal{J}_1,$$
$$\chi\Big(\sum_i z_i\Big) = \sum_i \chi(z_i) = \Delta \sum_i w_i \in \mathcal{J}_1,$$
$$\chi\Big(\sum_i w_i\Big) = \sum_i \chi(w_i) = \sum_i P_i = 0 \in \mathcal{J}_1 \text{ by the first assertion of Lemma 5.19.}$$

With a similar computation, using the other assertions of Lemma 5.19 it is possible to prove that Σ_1, Σ_2, and Σ_3 are χ-invariant. □

Corollary 5.20. *Let* $\mathrm{Ch}(X_1, X_2, X_3, X_4)$ *be a* 4*-chambar on an open set* $\mathcal{U} \subset \mathbb{C}$*, with* $X_j = y_j \frac{\partial}{\partial x}$*,* $y_j \in \mathcal{O}(\mathcal{U})$*,* $1 \le j \le 4$*.*

Suppose that $y_k(x_0) = y_\ell(x_0)$ *and that* $P(y_0, z_0, w_0) \neq 0$ *for some initial condition and* $k \neq \ell$*. Then* $y_k(x) = y_\ell(x)$ *for all* $x \in \mathcal{U}$*. Moreover, if* $k = 1$ *and* $\ell = 2$*, for instance, then either the chambar is constant and* $2a_1 + a_3 + a_4 = 0$ *or* $y_j(x) = a_j\sqrt{\lambda x + \mu}$ *with* $\lambda \neq 0$*,* $a_1 = a_2 = -\frac{1}{3}$*, and* a_3 *and* a_4 *the roots of* $3z^2 + 2z + 3 = 0$*.*

Proof: According to Theorem 5.16 if $y_1\frac{\partial}{\partial x_1}, y_2\frac{\partial}{\partial x_2}, \dots, y_4\frac{\partial}{\partial x_4}$ are holomorphic vector fields that define a 4-chambar on an open set $\mathcal{U} \subset \mathbb{C}$, then the vector function $x \in \mathcal{U} \mapsto y(x) = (y_1(x), y_2(x), \dots, y_4(x))$ satisfies an ODE of the form

$$\Delta\, y''' = P(y, y', y''),$$

where $\Delta = \prod\limits_{i<j}(y_j - y_i)$.

Assume that $y_1(x_0) = y_2(x_0)$. Since $P(y(x_0), z(x_0), w(x_0)) \neq 0$, we see that the point $(y(x_0), z(x_0), w(x_0))$ is not a singularity of the vector field χ, and there is only one solution through this point. Using that the set $\{y_1 = y_2\}$ is χ-invariant, we get $y_1(x) = y_2(x)$ for any $x \in \mathcal{U}$.

The condition on the flows is now

$$2\varphi_t^1(x) + \varphi_t^3(x) + \varphi_t^4(x) = 4x,$$

which is a particular case of (5.3). □

A natural question is the following:

Question 5.1. What could happen in the case $P(y(x_0), z(x_0), w(x_0)) = 0$ and $y_k(x_0) = y_\ell(x_0)$? Are there solutions with these conditions and $y_k(x) \not\equiv y_\ell(x)$, but $(y(x), z(x), w(x)) \in \Sigma = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ for all $x \in \mathcal{U}$?

Let us denote by $\mathrm{Ch}(4,1)$ the set of 4-tuples (X_1, X_2, X_3, X_4) of germs at $0 \in \mathbb{C}$ of holomorphic vector fields whose flows satisfy the barycentric conditions.

Corollary 5.21. *The set* $\mathrm{Ch}(4,1)$ *is isomorphic to an algebraic submanifold of* $\mathbb{C}^{12}$ *whose irreducible components have dimension at most* 6*.*

Proof: According to Theorems 5.16 and 5.18 any 4-chambar on $\mathbb{C}$ gives origin to a trajectory $(y,z,w)\colon (\mathbb{C},0)\to\mathbb{C}^{12}$ tangent to the χ-invariant submanifold $\Sigma=\Sigma_1\cap\Sigma_2\cap\Sigma_3$ of $\mathbb{C}^{12}$. The initial condition $(y(0),z(0),w(0))$ characterizes the trajectory (y,z,w) and defines an embedding of $\mathrm{Ch}(4,1)$ on Σ. □

6. Linear chambars

Theorem 6.1. *Let $X_1,X_2,\dots,X_p$ be some linear vector fields on $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). If they satisfy the barycentric property, then they are nilpotent.*

Proof: The flow φ_t^k of X_k can be written

$$\varphi_t^k(x)=(\exp tA_k)(x),$$

where the A_k's belong to $\mathrm{End}(\mathbb{R}^n)$ or $\mathrm{End}(\mathbb{C}^n)$. We identify the A_k's with some matrices. The barycentric property is equivalent to

$$\sum_{k=1}^{p}\sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}A_k^\ell=p\,\mathrm{Id},$$

which implies $\sum\limits_{k=1}^{p}A_k^\ell=0$ for any $\ell\geq 1$. Let $\lambda_{k,j}$ be the eigenvalues of A_k, $1\leq j\leq n$. We get for all $\ell\geq 1$

$$0=\mathrm{Tr}\left(\sum_{k=1}^{p}A_k^\ell\right)=\sum_{k=1}^{p}\sum_{j=1}^{n}\lambda_{k,j}^n.$$

As a result, all the $\lambda_{k,j}$ are equal to zero. □

Remark 6.2. The φ_t^k are polynomial in x and t.

Remark 6.3. If $p=2$, then the indices of nilpotency are 2 (*i.e.* $A^2=0$) and we recover the fact that the trajectories are straight lines. Note also that if X is a nilpotent vector field of index 2, then the pair $(X,-X)$ is a 2-chambar.

Example 6.4. Let X be a nilpotent linear vector field of order p. Let $\sigma=\exp\left(\frac{2\mathbf{i}\pi}{p}\right)$ be a primitive p-th root of unity. Then the vector fields $X,\sigma X,\sigma^2X,\dots,\sigma^{p-1}X$ satisfy the barycentric property.

Remark 6.5. Let $\mathrm{Ch}(X_1,X_2,\dots,X_p)$ be a linear p-chambar. Denote by k the maximal order of nilpotency of the X_i's. Take $\ell<k$ an integer. Then $\mathrm{Ch}(X_1^\ell,X_2^\ell,\dots,X_p^\ell)$ is a q-chambar for some $q\leq p$. The inequality comes from the fact that two X_k^ℓ's can be equal or X_k^ℓ can be zero. The fact that $q<p$ measures some degeneration and if $q=p$ for any $\ell<k$, it gives some condition of transversality.

Remark 6.6. Let $\mathrm{Ch}(X_1,X_2,\dots,X_p)$ be a singular p-chambar such that $X_k(0)=0$. Denote by A_i the linear part of X_i for $1\leq i\leq p$.

Assume that the A_i's generate a linear p-chambar $\mathrm{Ch}(A_1,A_2,\dots,A_p)$.

Consider the homothety $h_s\colon x\mapsto sx$, $s\in\mathbb{C}^*$, and

$$X_k^s=h_{s*}X_k=A_k+s(\dots).$$

We construct in this way a family $\mathrm{Ch}^s=\mathrm{Ch}(X_1^s,X_2^s,\dots,X_p^s)$ of p-chambars, all conjugate for $s\neq 0$, and that joins the initial chambar $\mathrm{Ch}^1=\mathrm{Ch}(X_1,X_2,\dots,X_p)$ to the linear chambar $\mathrm{Ch}^0=\mathrm{Ch}(A_1,A_2,\dots,A_p)$.

6.1. Linear p-chambars in dimension 2.

Lemma 6.7. *Let B be a (2×2)-matrix with complex coefficients. If $\operatorname{Tr}(B)=0$, then B is the sum of two nilpotent matrices.*

Proof: If $B=0$, then the result holds.

Let us now assume that $B\neq 0$. Let us write B as $\left(\begin{smallmatrix} a & b \\ c & -a\end{smallmatrix}\right)$. We are looking for two nilpotent matrices

$$A=\begin{pmatrix} x & y \\ z & -x\end{pmatrix},\quad A'=\begin{pmatrix} x' & y' \\ z' & -x'\end{pmatrix}$$

such that $B=A+A'$. We thus have to solve the system

$$\begin{cases} x+x'=a\\ y+y'=b\\ z+z'=c\\ x^2+yz=0\\ {x'}^2+y'z'=0\end{cases}$$

(the last two conditions guaranteeing nilpotency). After elimination of x', y', and z' we get

$$\begin{cases} x^2+yz=0\\ (a-x)^2+(b-y)(c-z)=0,\end{cases}$$

that is,

$$\begin{cases} x^2+yz=0\\ 2ax+bz+cy-a^2-bc=0,\end{cases}$$

which is the non-trivial intersection of a quadric and a plane. These two sets intersect along a plane conic. □

Second proof: Since $\operatorname{Tr}(B)=0$, then B is conjugate to $\left(\begin{smallmatrix} 0 & x \\ y & 0\end{smallmatrix}\right)$ for some x, y in $\mathbb{C}$ (note that if B is nilpotent, then $xy=0$). We conclude using the fact that

$$\begin{pmatrix} 0 & x \\ y & 0\end{pmatrix}=\underbrace{\begin{pmatrix} 0 & x \\ 0 & 0\end{pmatrix}}_{\text{nilpotent}}+\underbrace{\begin{pmatrix} 0 & 0 \\ y & 0\end{pmatrix}}_{\text{nilpotent}}.$$ □

Corollary 6.8. *Let $A_3,A_4,\dots,A_p$ be $(p-2)$ nilpotent (2×2)-matrices.*

There exist two nilpotent (2×2)-matrices A_1, A_2 such that the flows $\varphi_t^k=\exp tA_k$, $1\le k\le p$, satisfy the barycentric property.

Proof: Let A_1 and A_2 be two nilpotent matrices such that

$$A_1+A_2+A_3+\cdots+A_p=0.$$

As $\exp tA_k=\operatorname{Id}+tA_k$ in dimension 2, the p-tuple $(A_1,A_2,\dots,A_p)$ satisfies the required condition. □

Remark 6.9. If A_1, A_2, A_3 are nilpotent (2×2)-matrices that satisfy the barycentric property, then the A_i's are $\mathbb{C}$-colinear, *i.e.* $\operatorname{Ch}(A_1,A_2,A_3)$ is rigid. Indeed, the nilpotent (2×2)-matrices form a quadratic cone.

6.2. Linear 3-chambars. The following example illustrates that we can find solutions to the barycentric property in some Lie algebras of vector fields. In the particular case $n = 3$ one can find p-chambars in the Heisenberg Lie algebra $\mathfrak{h}_3$ formed by matrices

$$M(\alpha, \beta, \gamma) = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

One has $M^2(\alpha, \beta, \gamma) = M(0, 0, \alpha\beta)$. The barycentric property for the vector fields X_k corresponding to the matrices $M(\alpha_k, \beta_k, \gamma_k)$, $k = 1, \ldots, p$, is equivalent to the equalities

$$\sum_{k=1}^{p} \alpha_k = \sum_{k=1}^{p} \beta_k = \sum_{k=1}^{p} \gamma_k = \sum_{k=1}^{p} \alpha_k \beta_k = 0.$$

In the coefficient space $(\mathbb{C}^3)^p$ the barycentric property is the intersection of three hyperplanes and one quadric which thus has dimension $3p - 4$.

Theorem 6.10. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a linear* 3*-chambar on* $\mathbb{C}^3$*. Then, up to conjugacy, the* X_i*'s (identified with their matrices) are contained in the Heisenberg Lie algebra* $\mathfrak{h}_3 \subset \mathrm{gl}(3, \mathbb{C})$*.*

Proof: Let us identify X_i with its matrix.

We will distinguish two cases according to the rank of the X_i's.

- ◇ If one of the X_i's has rank 2, for instance X_1, then up to conjugacy one can assume that $X_1 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. We are now looking for X_2 and X_3 such that X_2 and X_3 are nilpotent (in particular their traces are zero) and $X_1 + X_2 + X_3 = X_1^2 + X_2^2 + X_3^2 = 0$. A straightforward computation implies that X_2 and X_3 belong to $\mathfrak{h}_3$.
- ◇ It suffices now to deal with the case where the three nilpotent matrices X_1, X_2, and X_3 have rank 1. Up to conjugacy one can suppose that $X_1 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. As X_2 has rank 1 the three columns of X_2 are colinear, *i.e.* $X_2 = (\lambda E, \mu E, \nu E)$, where $E = \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right) \neq 0$. Then $X_3 = -X_1 - X_2 = \left(-\lambda E, -\mu E, -\nu E - \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)\right)$. Let us distinguish three cases:
  - First assume that $\lambda = \mu = 0$. Changing the notations if needed, let us take $\nu = 1$. Then

    $$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \quad X_3 = -\begin{pmatrix} 0 & 0 & a+1 \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}.$$

    Since X_1 and X_2 are nilpotent, c has to be 0; but $c = 0$ leads to $X_2^2 = X_3^2 = 0$, and the X_i belong to $\mathfrak{h}_3$.
  - Now suppose $\lambda \neq 0$, *i.e.* $\lambda = 1$. Then

    $$X_2 = \begin{pmatrix} a & \mu a & \nu a \\ b & \mu b & \nu b \\ c & \mu c & \nu c \end{pmatrix}, \quad X_3 = -\begin{pmatrix} a & \mu a & \nu a + 1 \\ b & \mu b & \nu b \\ c & \mu c & \nu c \end{pmatrix}.$$

    As X_3 has rank 1, the coefficients b and c are zero. Therefore $X_2 = \left(\begin{smallmatrix} a & \mu a & \nu a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$; since X_2 is nilpotent, a has to be 0. As a consequence, $X_2 = 0$, which is impossible (the matrices are implicitly assumed to be non-zero).

- Finally assume that $\lambda = 0$ and $\mu \neq 0$, that is, $\lambda = 0$ and $\mu = 1$ and

$$X_2 = \begin{pmatrix} 0 & a & \nu a \\ 0 & b & \nu b \\ 0 & c & \nu c \end{pmatrix}, \quad X_3 = -\begin{pmatrix} 0 & a & \nu a + 1 \\ 0 & b & \nu b \\ 0 & c & \nu c \end{pmatrix}.$$

The fact that $\operatorname{rk} X_3 = 1$ leads to $b = c = 0$ and

$$X_2 = \begin{pmatrix} 0 & a & \nu a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = -\begin{pmatrix} 0 & a & \nu a + 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

belong to $\mathfrak{h}_3$. □

In fact the statement holds in any dimension but we keep the previous result and its proof because this last one is much easier. Let us start with some definitions, notations and intermediate results of non-commutative algebra.

A *monomial* of k-variables on $\operatorname{End}(\mathbb{C}^n)$ is a map $f\colon \operatorname{End}(\mathbb{C}^n)^k \to \operatorname{End}(\mathbb{C}^n)$ of the form

$$f(X_1, X_2, \dots, X_k) = X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_r}^{k_r},$$

where $r \geq 1$, $i_j \in \{1, 2, \dots, k\}$, and $k_j \geq 0$ for any $1 \leq j \leq r$. By convention $X_i^0 = 1$.

We say that the monomial is *reduced* if

◇ $k_j \geq 1$ for any $1 \leq j \leq r$;

◇ $i_j \neq i_{j+1}$ for any $1 \leq j \leq r - 1$.

The *degree* of f is $\deg f = \sum_{i=1}^{r} k_i$. A *polynomial of k variables* on $\operatorname{End}(\mathbb{C}^n)$ is a linear combination of monomials of k variables on $\operatorname{End}(\mathbb{C}^n)$:

$$P(X_1, X_2, \dots, X_k) = \sum_{j=1}^{s} a_j F_j(X_1, X_2, \dots, X_k)$$

with $a_1, a_2, \dots, a_s$ in $\mathbb{C}$. The *degree* of P is $\deg P = \max\{\deg(F_j) \mid a_j \neq 0\}$. If $\deg F_j \geq 1$ for any $1 \leq j \leq s$, then we say that P is *without constant term.*

If $\operatorname{Ch}(X_1, X_2, X_3)$ is a 3-linear chambar on $\mathbb{C}^n$, we denote by $\mathcal{G} = \langle X_1, X_2, X_3 \rangle \subset \operatorname{End}(\mathbb{C}^n) \simeq \operatorname{gl}(n, \mathbb{C})$ the subalgebra generated by X_1, X_2, and X_3. As previously, we identify the linear vector fields X_j with elements of $\operatorname{End}(\mathbb{C}^n)$.

We can now state the result:

Theorem 6.11. *Let $\operatorname{Ch}(X_1, X_2, X_3)$ be a linear 3-chambar on $\mathbb{C}^n$. Let $\mathcal{G} = \langle X_1, X_2, X_3 \rangle$ be the algebra of linear transformations generated by X_1, X_2, and X_3.*

If $Y_1, Y_2, \dots, Y_n$ belong to $\mathcal{G}$, then $Y_1 Y_2 \cdots Y_n = 0$.

In particular, up to conjugacy, the X_i's (identified with their matrices) are contained in the Heisenberg Lie algebra $\mathfrak{h}_n \subset \operatorname{gl}(n, \mathbb{C})$.

Proof: Let us start the proof with the following statement:

Lemma 6.12. *Let $\operatorname{Ch}(X_1, X_2, X_3)$ be a linear 3-chambar on $\mathbb{C}^n$.*

Let f be a monomial of two variables on $\operatorname{End}(\mathbb{C}^n)$.

There exists $n(f) \in \mathbb{Z}$ such that

$$f(X_1, X_2) + f(X_2, X_1) = n(f) \cdot X_3^{\deg f}.$$

Proof: For instance, from

$$X_1^k + X_2^k = -X_3^k \quad \forall\, k \geq 1$$

we get

$$X_3^{k+j} = (X_1^k + X_2^k)(X_1^j + X_2^j) = X_1^{k+j} + X_2^{k+j} + X_1^k X_2^j + X_2^k X_1^j = -X_3^{k+j} + X_1^k X_2^j + X_2^k X_1^j$$

and so $X_1^k X_2^j + X_2^k X_1^j = 2X_3^{k+j}$.

A reduced monomial g of two variables on $\mathrm{End}(\mathbb{C}^n)$ can be written as

$$g(X,Y) = X^{k_1} Y^{j_1} X^{k_2} \cdots Y^{j_r},$$

where $k_1 \geq 0$, $j_r \geq 0$, $k_2, k_3, \ldots, k_r \geq 1$, and $j_1, j_2, \ldots, j_{r-1} \geq 1$. Note that $\deg g = \sum_{i=1}^{r} (k_i + j_i)$. Let us introduce the following definitions:

- the X-length of g is $\ell_X(g) = \#\{i \mid k_i > 0\}$;
- the Y-length of g is $\ell_Y(g) = \#\{i \mid j_i > 0\}$;
- the length of g is $\ell(g) = \ell_X(g) + \ell_Y(g)$.

The proof is by induction on $\ell(f)$. Let us state the induction assumption: given $m \in \mathbb{N}$ the assertion of the lemma is true for any reduced monomial g with $\ell(g) \leq m$.

The induction assumption is true if $m \leq 2$:

- for $\ell(f) = 1$ it is a consequence of the equality $X_1^k + X_2^k = -X_3^k$;
- for $\ell(f) = 2$ it is a consequence of the equality $X_1^k X_2^j + X_2^k X_1^j = 2X_3^{k+j}$.

Assume that the assertion of the lemma is true for $m \geq 2$ and let us prove that it is true for $m+1$. Let f be a monomial with length $m+1 \geq 3$. Without loss of generality we can assume that $f(X,Y) = X^k Y^j X^m g(X,Y)$; note that $\ell(g) = \ell(f) - 3 = m - 2$. Using that $X_1^k X_2^j + X_2^k X_1^j = 2X_3^{k+j}$ we have

$$\begin{aligned}
&f(X_1,X_2) + f(X_2,X_1) \\
&\quad = X_1^k X_2^j X_1^m g(X_1,X_2) + X_2^k X_1^j X_2^m g(X_2,X_1) \\
&\quad = (2X_3^{k+j} - X_2^k X_1^j) X_1^m g(X_1,X_2) + (2X_3^{k+j} - X_1^k X_2^j) X_2^m g(X_2,X_1) \\
&\quad = 2X_3^{k+j}(X_1^m g(X_1,X_2) + X_2^m g(X_2,X_1)) - X_2^k X_1^{j+m} g(X_1,X_2) - X_1^k X_2^{j+m} g(X_2,X_1) \\
&\quad = 2X_3^{k+j}(g_1(X_1,X_2) + g_1(X_2,X_1)) - (g_2(X_1,X_2) + g_2(X_2,X_1)),
\end{aligned}$$

where $g_1(X,Y) = X^m g(X,Y)$ and $g_2(X,Y) = Y^k Y^{j+m} g(X,Y)$. Note that

$$\ell(g_1) = 1 + \ell(g) = m - 1 \quad \text{and} \quad \ell(g_2) = \ell(g) + 2 = m.$$

Therefore the induction assumption implies that for $i \in \{1,2\}$

$$g_i(X_1,X_2) + g_i(X_2,X_1) = \ell(g_i) X_3^{\deg f}.$$

Hence

$$f(X_1,X_2) + f(X_2,X_1) = \ell(f) X_3^{\deg f},$$

where $\ell(f) = 2\ell(g_1) - \ell(g_2)$. □

Lemma 6.13. *Let* $\mathrm{Ch}(X_1,X_2,X_3)$ *be a linear* 3*-chambar on* $\mathbb{C}^n$*.*

Let $P(X,Y)$ *be a polynomial of two variables on* $\mathrm{End}(\mathbb{C}^n)$*. Assume that* P *is without constant term.*

Then $P(X_1,X_2)$ *is nilpotent, that is,* $P(X_1,X_2)^n = 0$*.*

Proof: Assume first that P is a reduced monomial. Set $d = \deg P$. Denote by $\lambda_1, \lambda_2, \dots, \lambda_n$ (resp. by $\mu_1, \mu_2, \dots, \mu_n$) the eigenvalues of $P(X_1, X_2)$ (resp. $P(X_2, X_1)$). It follows from Lemma 6.12 that

$$\sum_j \lambda_j + \sum_j \mu_j = \operatorname{tr}(P(X_1, X_2) + P(X_2, X_1)) = \operatorname{tr}(n(P)X_3^d) = 0.$$

Given any $m \in \mathbb{N}$, since $P(X, Y)^m$ is a monomial we have

$$\sum_j \lambda_j^m + \sum_j \mu_j^m = \operatorname{tr}(P(X_1, X_2)^m + P(X_2, X_1)^m) = 0 \quad \forall\, m \in \mathbb{N}.$$

This implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ and so $P(X_1, X_2)$ is nilpotent. In particular, we get $\operatorname{tr}(P(X_1, X_2)) = 0$.

Suppose now that P is a polynomial of two variables on $\operatorname{End}(\mathbb{C}^n)$ without constant term. Since P is a linear combination of non-constant monomials we get $\operatorname{tr}(P(X_1, X_2)) = 0$. Similarly, given $m \in \mathbb{N}$ then $P(X, Y)^m$ is also a polynomial without constant term and so $\operatorname{tr}(P(X_1, X_2)^m) = 0$. Therefore $P(X_1, X_2)$ is nilpotent and as $P(X_1, X_2)$ belongs to $\operatorname{End}(\mathbb{C}^n)$ we get $P(X_1, X_2)^n = 0$. □

Let $\mathfrak{g}$ be any Lie algebra. Recall some classical well-known facts. If x belongs to $\mathfrak{g}$, $y \mapsto [x, y]$ is an endomorphism of $\mathfrak{g}$, which we denote by $\operatorname{ad} x$. We say that x is *ad-nilpotent* if $\operatorname{ad} x$ is a nilpotent endomorphism. If $\mathfrak{g}$ is nilpotent, then all elements of $\mathfrak{g}$ are ad-nilpotent. The converse is also true; it is the Engel theorem ([**4**]). If now $\mathfrak{g}$ is a matrix algebra all of whose elements are nilpotent (for the multiplication), then the algebra is, up to conjugacy, contained in the Heisenberg Lie algebra $\mathfrak{h}_n$. This ends the proof of the theorem. □

6.3. Some remarks on linear 4-chambars. As previously, we will identify the vector field X_i with its matrix.

Definition 6.14. A p-chambar $\operatorname{Ch}(X_1, X_2, \dots, X_p)$ has *rank* r if r is the maximal rank of the X_i.

Let us start with the following property:

Proposition 6.15. *Let* $\operatorname{Ch}(X_1, X_2, X_3, X_4)$ *be a linear* 4*-chambar that has rank* 2. *Then it is irreducible.*

Proof: Suppose, by contradiction, that $\operatorname{Ch}(X_1, X_2, X_3, X_4)$ is reducible. Then it consists of two pairs of 2-chambars: the trajectories are thus lines and the X_i's (identified with their matrices) have rank 1. □

6.3.1. A first family of examples. Consider the four matrices

$$X_1 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & a & -\frac{ab}{c} \\ 0 & b & -\frac{b^2}{c} \\ 0 & c & -b \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & d & \frac{db}{e} \\ 0 & -b & -\frac{b^2}{e} \\ 0 & e & b \end{pmatrix},$$

where α, β, γ, δ, a, b, c, d, e are complex numbers satisfying the conditions

$$\gamma + a + d = 0, \quad \alpha - \frac{ab}{c} + \frac{db}{e} = 0, \quad \beta - \frac{b^2}{c} - \frac{b^2}{e^2} = 0, \quad \delta + c + e = 0.$$

These matrices define a generically irreducible 4-chambar whose elements are not contained in a nilpotent algebra. Indeed,

- ⋄ on the one hand the nilpotent algebras of matrices are triangularizable; in particular, the eigenvalues of a commutator are zero;
- ⋄ on the other hand the eigenvalues of the commutator $[X_1, X_2] = \begin{pmatrix} 0 & \alpha\delta & -\beta\gamma \\ 0 & \beta\delta & 0 \\ 0 & 0 & -\beta\delta \end{pmatrix}$ are non-zero as soon as $\beta\delta \neq 0$.

Note that the X_i's have a common kernel for generic values of the parameters.

6.3.2. A second family of examples. Let us consider

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ b & -c-2 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & -a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2b & 1 & 0 \end{pmatrix}.$$

Then (X_1, X_2, X_3, X_4) is a linear 4-chambar of rank 2 in $\mathbb{C}^3$ and the X_i's (identified with their matrices) are not contained in a nilpotent algebra of matrices.

More generally for $1 \leq j \leq 4$ set

$$X_j = \begin{pmatrix} A_j & 0 \\ B_j & 0 \end{pmatrix},$$

where A_j is a (2×2)-matrix and B_j is a (1×2)-matrix such that

$$\begin{cases} A_j^2 = 0 \\ \displaystyle\sum_{j=1}^{4} B_j A_j = 0. \end{cases}$$

Then (X_1, X_2, X_3, X_4) is a linear 4-chambar of rank 2 in $\mathbb{C}^3$ and the X_i's (identified with their matrices) are not contained in a nilpotent algebra of matrices.

6.3.3. A third family of examples. Consider

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ -c & \gamma & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & -b \\ c & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & -\alpha & 0 \\ 0 & 0 & 0 \\ -c & -\beta & 0 \end{pmatrix},$$

where a, b, c, α, β denote some complex numbers. Note that

$$X_1 + X_2 = \begin{pmatrix} 0 & \alpha & 0 \\ a & 0 & b \\ 0 & \beta & 0 \end{pmatrix}, \quad X_1 + tX_2 = \begin{pmatrix} 0 & t\alpha & 0 \\ a & 0 & b \\ (1-t)c & t\beta & 0 \end{pmatrix}$$

so that

- ⋄ $X_1 + X_2$ has rank 2 generically on a, b, α, and β,
- ⋄ $X_1 + tX_2$ has rank 3 generically on t.

The eigenvalues of the commutator $[X_1, X_2] = \begin{pmatrix} -a\alpha & 0 & -b\alpha \\ -bc & a\alpha+b\beta & 0 \\ -a\beta & \alpha c & -b\beta \end{pmatrix}$ are non-zero as soon as $\alpha bc \neq 0$. As a consequence, $\mathrm{Ch}(X_1, X_2, X_3, X_4)$ is a generically irreducible 4-chambar and the matrices associated to the X_i's are not contained in a nilpotent algebra of matrices.

Note that for generic values of parameters the X_i's do not all have the same kernel. As a consequence, examples of Subsections 6.3.1 and 6.3.3 are not conjugate.

Finally one can state:

Proposition 6.16. *There exist linear, irreducible 4-chambars on $\mathbb{C}^3$ with the two following properties:*

- ◇ *their flows are generically quadratic in t;*
- ◇ *the associated matrices are not contained in a nilpotent algebra of matrices.*

7. Homogeneous chambars

7.1. First properties. Let $\mathcal{B} = \mathrm{Ch}(X_1, X_2, \ldots, X_p)$ be a p-chambar at $0 \in \mathbb{C}^n$. We say that $\mathcal{B}$ is *homogeneous of degree ν* if any X_i is homogeneous of degree ν.

Remark 7.1. Let $\mathrm{Ch}(X, -X)$ be a homogeneous 2-chambar on $\mathbb{C}^2$. Then up to linear conjugacy $X = x^\nu \frac{\partial}{\partial y}$ (the proof is an exercise).

Given two holomorphic vector fields X and Y on $\mathbb{C}^n$, we define the set of colinearity between X and Y as

$$\mathrm{Col}(X, Y) := \{m \in \mathbb{C}^n \mid X(m) \wedge Y(m) = 0\}.$$

Remarks 7.2. We would like to point out the following facts:

- ◇ $\mathrm{Col}(X, Y)$ is an analytic set;
- ◇ if $\mathrm{Col}(X, Y) \neq \emptyset$, then $\dim_{\mathbb{C}}(\mathrm{Col}(X, Y)) \geq 1$;
- ◇ if X and Y are homogeneous vector fields, then $\dim_{\mathbb{C}}(\mathrm{Col}(X, Y)) \geq 1$;
- ◇ if X is homogeneous and $Y = R$ is the radial vector field of $\mathbb{C}^n$, then $\mathrm{Col}(X, R)$ is a union of straight lines through the origin $0 \in \mathbb{C}^n$. If $X \wedge R \neq 0$, then the vector fields X and R generate a singular foliation $\mathcal{F}$ of dimension 2 of $\mathbb{C}^n$. There is a holomorphic foliation $\widetilde{\mathcal{F}}$ on $\mathbb{P}^{n-1}_{\mathbb{C}}$ such that $\mathcal{F} = \pi^*(\widetilde{\mathcal{F}})$. It is possible to prove that
$$\mathrm{Col}(X, R) = \pi^{-1}(\mathrm{Sing}(\widetilde{\mathcal{F}})) = \mathrm{Sing}(\mathcal{F}).$$

The various previous examples suggest the following conjecture:

Conjecture 7.3. *Let $\mathrm{Ch}(X_1, X_2, \ldots, X_p)$ be a homogeneous p-chambar of degree $\nu \geq 1$ on $\mathbb{C}^n$, where $p \geq 2$. Then, for any $k \geq 1$, $\mathrm{Col}(X_k, R) = \mathrm{Sing}(X_k)$.[2] In particular,* $\dim \mathrm{Sing}(X_k) \geq 1$.

In the same spirit we have the following problem:

Problem 7.4. *Let $\mathrm{Ch}(X_1, X_2, \ldots, X_p)$ be a (non-homogeneous) p-chambar such that $X_k(0) = 0$. Do the inequalities* $\dim \mathrm{Sing}(X_k) \geq 1$ *hold?*

[2]Recall that $\mathrm{Sing}(X_k)$ is the singular set of X_k:
$$\mathrm{Sing}(X_k) = \{m \in \mathbb{C}^n \mid X_k(m) = 0\}.$$

Remark 7.5. The problem is solved in the following cases:

- $\nu = 1$ (Theorem 6.1);
- $p = 2$ (Theorem 3.5);
- rigid chambars (Corollary 4.11).

We proved the conjecture in the special case of a homogeneous 3-chambar on $\mathbb{C}^2$ of degree 2. In fact we will prove the following:

Theorem 7.6. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a homogeneous* 3*-chambar on* $\mathbb{C}^2$ *of degree* 2*. Then, after a change of variables,* X_j *can be written as* $a_j y^2 \frac{\partial}{\partial x}$*, where* $a_1+a_2+a_3 = 0$*. In particular, any homogeneous* 3*-chambar on* $\mathbb{C}^2$ *of degree* 2 *is rigid.*

Let X be a homogeneous vector field of degree d on $\mathbb{C}^2$. Then X has $d+1$ invariant straight lines through $0 \in \mathbb{C}^2$, counted with multiplicity. These lines are the solutions of $f(x, y) = 0$, where f is the homogeneous polynomial of degree $d + 1$ defined by

$$R \wedge X = f(x, y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \tag{7.1}$$

that is, $f(x, y) = \det\left(\begin{smallmatrix} x & y \\ X(x) & X(y)\end{smallmatrix}\right)$. We will assume that $f \not\equiv 0$ (if $f \equiv 0$, then X is colinear to the radial vector field R).

Since $f = 0$ is X-invariant, then $X(f) = h\cdot f$, where h is a homogeneous polynomial of degree $d - 1$. Moreover, $h = 0$ if and only if f is a first integral of X. In this case, the foliations defined by X and by f must coincide: the relation $X(f) = 0$ gives $X(x)\frac{\partial f}{\partial x} = -X(y)\frac{\partial f}{\partial y}$. Since the degrees of $X(x)$, $X(y)$, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are equal, we obtain that

$$X = \alpha\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right).$$

Using that $R(f) = (d + 1)f$ and (7.1) we get $\alpha = \frac{1}{d+1}$ in the above relation.

In general, we have

$$(d + 1)X - hR = H(f), \tag{7.2}$$

where $H(f) = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}$.

Another relation that we will use is

$$X(f) = X\left(\det\begin{pmatrix} x & y \\ X(x) & X(y)\end{pmatrix}\right) = \det\begin{pmatrix} x & y \\ X^2(x) & X^2(y)\end{pmatrix}. \tag{7.3}$$

Lemma 7.7. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a homogeneous* 3*-chambar of degree* d *on* $\mathbb{C}^2$*. For* $1 \leq j \leq 3$ *define* f_j *by* $R \wedge X_j = f_j(x, y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$*. Suppose that the* f_j *are not identically* 0*. Then*

- *either* f_1*,* f_2*, and* f_3 *have two common linear factors*
- *or* f_j *is a first integral of* X_j*,* $1 \leq j \leq 3$*.*

Proof: First of all, using relations (7.1), (7.3), and both

$$\sum_j X_j(x) = \sum_j X_j(y) = 0, \quad \sum_j X_j^2(x) = \sum_j X_j^2(y) = 0$$

we obtain $\sum_j f_j = 0$ and $\sum_j X_j(f_j) = 0$. If we set $X_j(f_j) = h_j \cdot f_j$, $1 \le j \le 3$, then $\sum_j h_j \cdot f_j = 0$. On the other hand, since $\sum_j X_j = 0$ and $\sum_j f_j = 0$, we get from (7.2) that

$$0 = \sum_j ((d+1)X_j - h_j R - H(f_j)) = -\sum_j h_j R$$

and so $\sum_j h_j = 0$.

Let us assume that $h_j \not\equiv 0$ for some $1 \le j \le 3$. In this case, from $\sum_j h_j = 0$ there are $i \ne j$ such that $h_i \ne h_j$. Suppose for instance that $h_1 \ne h_2$. Then the equalities

$$\begin{cases} f_1 + f_2 + f_3 = 0 \\ h_1 f_1 + h_2 f_2 + h_3 f_3 = 0 \end{cases}$$

imply

$$(h_1 - h_3) f_1 = (h_3 - h_2) f_2. \tag{7.4}$$

In particular, both members of relation (7.4) are not identically zero. Since $h_1 - h_3$ and $h_3 - h_2$ have degree $d-1$, and f_1 and f_2 degree $d+1$, f_1 and f_2 must have two common factors. As $f_3 = -f_1 - f_2$ these factors are also factors of f_3. □

Remark 7.8. Lemma 7.7 implies that for a homogeneous 3-chambar on $\mathbb{C}^2$ Problem 7.4 has a positive answer, possibly except when the f_i's are first integrals.

Lemma 7.9. *Let* $\mathrm{Ch}(X_1, X_2, X_3)$ *be a homogeneous* 3*-chambar of degree* 2 *on* $\mathbb{C}^2$*, and let* f_ℓ *be as in Lemma 7.7.*

Then the f_ℓ*'s are not identically zero.*

Proof: Suppose that $f_1 \equiv 0$; up to a linear change of coordinates we can assume that $X_1 = xR = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$. Let $\ell = 0$ be an X_2-invariant line; then $\ell = 0$ is X_1-invariant, and also X_3-invariant since $X_1 + X_2 + X_3 = 0$. These facts imply that the restriction of X_1, X_2, X_3 to $\ell = 0$ define a 3-chambar on the line $\ell = 0$. The classification of 3-chambars on $\mathbb{C}$ (Theorem 5.1) implies that the X_i are 0 on $\ell = 0$. In particular, $\ell = 0 = (x = 0)$ and $X_1 = xR$, $X_2 = xL_2$, $X_3 = xL_3$, with L_i a linear vector field, and $R + L_2 + L_3 = 0$. The same argument as before implies that the invariant lines of L_2, L_3 are necessarily $x = 0$, *i.e.*:

$$L_2 = a_2 x \frac{\partial}{\partial x} + (b_2 x + c_2 y)\frac{\partial}{\partial y}, \quad L_3 = a_3 x \frac{\partial}{\partial x} + (b_3 x + c_3 y)\frac{\partial}{\partial y}.$$

The first components of the flows of X_1, X_2, X_3 are respectively $\frac{x}{1-tx}$, $\frac{x}{1-ta_2 x}$, $\frac{x}{1-ta_3 x}$; the sum of these three homographies cannot be $3x$: a contradiction. □

Problem 7.10. *Is Lemma 7.9 true in any degree?*

Assume that $\mathrm{Ch}(X_1, X_2, X_3)$ is homogeneous of degree 2, and that the f_j's have two common factors. Let ℓ_1 and ℓ_2 be the two linear common factors of the f_j's. We have the following two possibilities:

(1) $\ell_1 \ne \ell_2$: we can thus assume that xy is a factor of the f_j's;
(2) $\ell_1 = \ell_2$: we can thus suppose that y^2 is a factor of the f_j's.

Another fact is that a polynomial p-chambar in dimension 1 is constant (Proposition 2.6). Therefore, if a straight line $\ell = 0$ is invariant for all vector fields of the chambar, then $X_{j_{|_\ell}} = 0$, and ℓ is a factor of X_j. In dimension 2 this implies that $X_j = \ell \cdot L_j$, where L_j is a linear vector field, $1 \le j \le 3$.

In particular, (1) and (2) imply the following possibilities:

(1') if $\ell_1 = x$ and $\ell_2 = y$, then we must have $X_j = xyV_j$, where V_j is a constant vector field;

(2') if $\ell_1 = \ell_2 = y$, then $X_j = yL_j$, where $R \wedge L_j = ym_j \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, $m_j = a_j x + b_j y$. In particular, we must have

$$L_j = (\alpha_j x + \beta_j y)\frac{\partial}{\partial x} + \gamma_j y \frac{\partial}{\partial y},$$

where $a_j = \gamma_j - \alpha_j$ and $b_j = -\beta_j$.

Let us check that (1') cannot happen. In fact, let $X = xyV$, where $V = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$. By a direct computation we find

$$\begin{cases} \dfrac{X(x)}{xy} = a, \quad \dfrac{X^2(x)}{xy} = a^2y + abx, \quad \dfrac{X^3(x)}{xy} = a^3y^2 + \alpha xy + \beta x^2 \\ \dfrac{X(y)}{xy} = b, \quad \dfrac{X^2(x)}{xy} = aby + b^2x, \quad \dfrac{X^3(y)}{xy} = b^3x^2 + \gamma xy + \delta y^2. \end{cases}$$

This implies, with obvious notations, that for any $1 \leq k \leq 3$

$$a_1^k + a_2^k + a_3^k = 0 \quad \text{and} \quad b_1^k + b_2^k + b_3^k = 0$$

so $V_1 = V_2 = V_3 = 0$.

In situation (2') the vector fields $X_j = yL_j$ are of the form $X = y\big((ax+by)\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}\big)$, and a direct computation shows that $X(y) = cy^2$, $X^2(y) = 2c^2y^3$, and $X^3(y) = 6c^3y^4$. This implies $\sum\limits_j c_j^k = 0$ for $1 \leq k \leq 3$, so that $c_1 = c_2 = c_3 = 0$, and $X_j = y\ell_j\frac{\partial}{\partial x}$, where $\ell_j = a_jx + b_jy$ is linear. In particular, we get

$$X_j(x) = y\ell_j, \quad X_j^2(x) = y^2\frac{\partial \ell_j}{\partial x}\ell_j = a_jy^2\ell_j, \quad X_j^3(x) = a_j^2y^3\ell_j;$$

as a consequence, $a_1^k + a_2^k + a_3^k = 0$ for any $1 \leq k \leq 3$. This yields $a_1 = a_2 = a_3 = 0$, and $X_j = b_jy^2\frac{\partial}{\partial x}$ for any $1 \leq j \leq 3$. Note that the f_j's are first integrals of X_j.

It remains to consider the case where $h_1 = h_2 = h_3 = 0$ and f_j is a first integral of X_j, $1 \leq j \leq 3$. Let us come back to the definition of $f_j := xX_j(y) - yX_j(x)$, so that $X_j(f_j) = 0$. Note first that X is a constant multiple of the Hamiltonian of f

$$H(f_j) = \frac{\partial f_j}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f_j}{\partial x}\frac{\partial}{\partial y};$$

it can be checked that this follows from $X_j(f_j) = 0$. From the definition of f_j and Euler's identity we get $X_j = -\frac{1}{3}H(f_j)$. Let ℓ be a straight line invariant for X_1 and passing through 0. Suppose by contradiction that it is not invariant by X_2. We assert that either $X_{1_{|\ell}} = 0$ or the trajectories of X_2 and X_3 are parallel straight lines. Assume that $X_{1_{|\ell}} \neq 0$; we will see that f_2 is a perfect cube, *i.e.* $f_2 = h^3$, where h is linear, so that the trajectories of X_2 are the levels of h. Without loss of generality we can suppose that $\ell = (y = 0)$. We can write $f_2(x,y) = ax^3 + yq(x,y)$, where q is homogeneous of degree 2 and $a \neq 0$ because $y = 0$ is not X_2-invariant. If $c \neq 0$, then the level $f_2 = c$ cuts ℓ at three points $z_j := (x_j, 0)$, $1 \leq j \leq 3$, where the x_j's are the roots of $x^3 = \frac{c}{a}$. If f_2 is not a perfect cube, then the level $f_2 = c$ is irreducible, and so it is connected. Denote by φ_t^j the flow of X_j, $1 \leq j \leq 3$. Let us point out the following facts:

(a) $\varphi_t^1(x,0)+\varphi_t^2(x,0)+\varphi_t^3(x,0)=3(x,0)$ for all $x\in\mathbb{C}$, for all t where the flows are defined (barycentric property);

(b) $X_{1|_{y=0}}=\alpha x^2\frac{\partial}{\partial x}$ so $\varphi_t^1(x,0)=\frac{x}{1-\alpha t x}$ and since we are assuming $X_{1|_{y=0}}\neq 0$, α is non-zero;

(c) as $(f_2=c)\cap(y=0)=\{(x_j,0)\mid 1\leq j\leq 3\}$ and $f_2=c$ is connected, there exists $\tau\neq 0$ such that $\varphi_\tau^2(x_1,0)=(x_2,0)$.

It is possible to prove that $\varphi_{3\tau}^2(x_1,0)=(x_1,0)$, and more generally $\varphi_{3k\tau}^2(x_i,0)=(x_i,0)$ for all $k\in\mathbb{Z}$, $i=1,2,3$. Since f_3 is a first integral of X_3, the leaf of the foliation generated by X_3 through $(x_1,0)$ must cut ℓ at no more than three points. However, (a) and (b) imply that

$$\varphi_{3k\tau}^3(x_1,0)=\left(2x_1-\frac{x_1}{1-3k\tau x_1},0\right),$$

contradicting that the number is finite. As a result,

(i) either f_2 and f_3 are perfect cubes

(ii) or $X_{1|_{y=0}}=0$.

Let us deal with these two possibilities.

(i) Assume that $f_2=\ell_2^3$ and $f_3=\ell_3^3$, where ℓ_2 and ℓ_3 are linear. In this case, the trajectories of X_2, and also of X_3, are parallel lines. We have the alternatives
   (i1) either $d\ell_2\wedge d\ell_3=0$
   (i2) or $d\ell_2\wedge d\ell_3\neq 0$.

   In case (i1), we have $\ell_3=\alpha\ell_2$, $\alpha\neq 0$, and ℓ_2 is a line invariant for the chambar. After a linear change of variables we can suppose that $X_j=a_jy^2\frac{\partial}{\partial x}$, and the statement is proved. Note that in this case $X_{j_{|\ell}}=0$ for $1\leq j\leq 3$.

   In case (i2), after a linear change of variables, we can suppose that $f_2=-x^3$ and $f_3=-y^3$, which implies $X_2=-x^2\frac{\partial}{\partial y}$, and $X_3=-y^2\frac{\partial}{\partial x}$. However, in this case we would have $X_1=y^2\frac{\partial}{\partial x}+x^2\frac{\partial}{\partial y}$. This is not a 3-chambar because

$$X_2^2(x)=X_3^2(x)=0\quad\text{and}\quad X_1^2(x)\neq 0.$$

(ii) Suppose that $X_{1|_{y=0}}=0$. From the above we have the following consequences: the Hamiltonian $H(f_j)=X_j$ is identically zero on the lines $f_j=0$. In particular, all the irreducible components of f_j have multiplicity. Since the f_j's have degree 3, the f_j's are perfect cubes and we conclude as previously.

This ends the proof of Theorem 7.6.

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