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ON THE DUALS OF SMOOTH PROJECTIVE COMPLEX HYPERSURFACES

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Abstract: We first show that a generic hypersurface V of degree $d \geq 3$ in the projective complex space $\mathbb{P}^n$ of dimension $n \geq 3$ has at least one hyperplane section $V \cap H$ containing exactly n ordinary double points, alias A_1 singularities, in general position, and no other singularities. Equivalently, the dual hypersurface $V^\vee$ has at least one normal crossing singularity of multiplicity n. Using this result, we show that the dual of any smooth hypersurface with $n, d \geq 3$ has at least a very singular point q, in particular a point q of multiplicity $\geq n$.

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Key words: hypersurface, dual hypersurface, Lefschetz properties, hyperplane section, singularities.

1. Introduction

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the graded polynomial ring in $n+1$ variables with complex coefficients, with $n \geq 2$. Let $f \in S_d$ be a homogeneous polynomial such that the hypersurface $V = V(f) : f = 0$ in the projective space $\mathbb{P}^n$ is smooth. Our main result is the following.

Theorem 1.1. *For any $n \geq 3$ and $d \geq 3$, a generic hypersurface $V \subset \mathbb{P}^n$ of degree d has at least one hyperplane section $V \cap H$, which has exactly n nodal singularities in general position.*

Here a node is a non-degenerate quadratic singularity, in Arnold notation an A_1 singularity. A nodal hypersurface is a hypersurface having only such nodes as singularities. As usual, a property is generic if it holds for a Zariski open and dense subset of the parameter space, which is in this case $\mathbb{P}(S_d)$. For any smooth hypersurface V and any hyperplane H, the singularities of the hyperplane section $V \cap H$ are exactly the points where H is tangent to V. The fact that a generic hypersurface has the tangency points in general position was established in [**3**]. This means that for a generic, smooth hypersurface $V \subset \mathbb{P}^n$ and any hyperplane $H \subset \mathbb{P}^n$, the singularities of the section $V \cap H$ are points in general position in H, i.e., the corresponding vectors in the vector space associated to H are linearly independent. In particular, there are at most n singularities in any such section $V \cap H$ when V is generic.

For a smooth hypersurface $V(f)$, consider the corresponding dual mapping

$$\phi_f \colon V(f) \to (\mathbb{P}^n)^\vee, \quad x \mapsto (f_0(x) : f_1(x) : \cdots : f_n(x)),$$

where we set

$$f_j = \frac{\partial f}{\partial x_j} \text{ for } j = 0, \dots, n.$$

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Then the dual hypersurface

$$V(f)^\vee = \phi_f(V(f))$$

has a normal crossing singularity of multiplicity n at the point of the dual projective space $(\mathbb{P}^n)^\vee$ corresponding to the hyperplane H if and only if $V(f) \cap H$ has exactly n singularities of type A_1 in general position. To prove this, note that $V(f) \cap H$ has a node A_1 at a point p if and only if the dual mapping ϕ_f is an immersion at p; see for instance the equivalences (11.33) in [**6**]. Moreover, the tangent space to the corresponding branch $\phi_f(V(f), p)$ of $V(f)^\vee$ at $H = \phi_f(p)$ is given by

$$p \in \mathbb{P}^n = ((\mathbb{P}^n)^\vee)^\vee.$$

It follows that Theorem 1.1 can be reformulated as follows.

Theorem 1.2. *For any dimension $n \geq 3$ and degree $d \geq 3$, the dual hypersurface $V^\vee$ of a generic hypersurface $V \subset \mathbb{P}^n$ of degree d has at least one normal crossing singularity of multiplicity n.*

Remark 1.3. (i) For any smooth hypersurface V and any hyperplane H, the hyperplane section $V \cap H$ has only isolated singularities. Conversely, any hypersurface $W \subset H$ with only isolated singularities may occur as a section $W = V \cap H$ for a certain smooth hypersurface V; see [**6**, Proposition (11.6)].

(ii) The fact that a generic curve C in $\mathbb{P}^2$ of degree $d \geq 4$ has only simple tangents, bitangents, and simple flexes is known classically, and corresponds to the claim that the dual curve $C^\vee$ has only nodes A_1 and cusps A_2 as singularities. The number of bitangents of C, which is also the number of nodes A_1 of the dual curve, as a function of d is also known; see [**8**, p. 277], for the classical approach using Plücker formulas, or [**2**] for a modern view-point. It follows from [**9**, Proposition 2.1] that *any smooth quartic curve has at least 16 bitangents* (there called simple bitangents) which correspond to the nodes of the dual curve.

(iii) The fact that a surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$ admits tritangent planes H is well known, and there are formulas for the number of these planes in terms of the degree d; see for instance [**13**, Section (8.3)]. The fact that, for S generic of degree $d \geq 5$, the singularities of $S \cap H$ are exactly three nodes follows from [**14**, Proposition 3]. For a generic hypersurface $V \subset \mathbb{P}^n$, with $n \geq 4$ and $d = \deg V \geq n+2$, it follows from [**14**, Proposition 4] that all the singularities of a hyperplane section $V \cap H$ are double points (not necessarily A_1 singularities) in number *at most n*.

Using the above results for generic hypersurfaces, one can prove the following result, *holding for any smooth hypersurface*.

Theorem 1.4. *For any dimension $n \geq 3$ and degree $d \geq 3$, the dual hypersurface $V^\vee$ of a smooth hypersurface $V \subset \mathbb{P}^n$ of degree d has either a singularity of multiplicity n with the corresponding tangent cone a union of hyperplanes, or a singularity of multiplicity $> n$. Moreover, a smooth hypersurface $V \subset \mathbb{P}^n$ of degree d, where $n, d \geq 3$, has at least one hyperplane section $V \cap H$ whose total Tjurina number $\tau(V \cap H)$ is at least n.*

We recall that for an isolated hypersurface singularity $(X, 0) : g = 0$ defined by a germ $g \in R = \mathbb{C}[[y_1, \ldots, y_n]]$, we define its Milnor number $\mu(X, 0)$ and its Tjurina number $\tau(X, 0)$ by the formulas

$$\mu(X, 0) = \dim R/J_g \quad \text{and} \quad \tau(X, 0) = \dim R/(J_g + (g)),$$

where J_g is the Jacobian ideal of g in R. For a projective hypersurface W having only isolated singularities, we define its total Milnor number $\mu(W)$ and its total Tjurina number $\tau(W)$ by the formulas

$$\mu(W) = \sum_p \mu(W,p) \quad \text{and} \quad \tau(W) = \sum_p \tau(W,p),$$

where both sums are over all the singular points $p \in W$. For any point $H \in V^\vee$ it is known that

$$\operatorname{mult}_H(V^\vee) = \mu(V \cap H),$$

where $\operatorname{mult}_p(Y)$ denotes the multiplicity of a variety Y at a point $p \in Y$; see [**5**]. Hence the first claim in Theorem 1.4 implies that

$$\mu(V \cap H) \geq n$$

if $\operatorname{mult}_H(V^\vee) \geq n$. However, our second claim in Theorem 1.4 is a stronger version of this inequality, since $\mu(X,0) \geq \tau(X,0)$, with equality exactly when the singularity $(X,0)$ is weighted homogeneous; see [**12**].

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2. Proof of Theorem 1.1

The starting point is Remark 1.3(i) above. We first consider the projective space $\mathbb{P}^{n-1}$ and the subset $Z_n \subset \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ given by the classes p_i of the canonical basis e_i, $i = 1, \dots, n$, of the vector space $\mathbb{C}^n$.

Proposition 2.1. *For any degree $d \geq 3$, there is a hypersurface $Y \subset \mathbb{P}^{n-1}$, with $n \geq 3$, of degree d having as singularities n nodes A_1, located at the points in Z_n.*

Proof: Let $y = (y_1, \dots, y_n)$ be the coordinates on $\mathbb{P}^{n-1}$. We first consider the case $d = 3$ and take Y to be the hypersurface $g(y) = 0$, where

$$g(y) = \sum_{1 \leq i < j < k \leq n} y_i y_j y_k.$$

It is easy to see that Y has an A_1-singularity at each point p_i, for $i = 1, \dots, n$. Now we show that there are no other singularities. Note that for the partial derivative g_i of g with respect to y_i we have

$$g_i(y) = \sum_{1 \leq j < k \leq n,\, j \neq i,\, k \neq i} y_j y_k. \tag{2.1}$$

Assume that $g_i(y) = 0$ for $i = 1, \dots, n$ and take the sum of all these equations. In this way we get

$$\sum_{1 \leq j < k \leq n} y_j y_k = 0. \tag{2.2}$$

Subtracting the equation (2.1) from (2.2) we get

$$y_i \sum_{1 \leq j \leq n,\, j \neq i} y_j = 0. \tag{2.3}$$

If we assume that $y_{i_1} \neq 0$ and $y_{i_2} \neq 0$ for some indices $1 \leq i_1 < i_2 \leq n$, the equation (2.3) implies that $y_{i_1} = y_{i_2}$. Hence, for any singular point y^0 of Y, there is an integer a with $1 \leq a \leq n$ such that a coordinates of y^0 are equal to 1, and the remaining $n - a$ coordinates are 0. The equation (2.3) implies that only the case $a = 1$ is possible, and hence y^0 is one of the points p_i. This completes the proof in the case $d = 3$.

Next we look at the case $d = 4$ and take Y to be the hypersurface $g(y) = 0$, where

$$g(y) = \sum_{1 \leq i < j \leq n} y_i^2 y_j^2.$$

It is easy to see that Y has an A_1-singularity at each point p_i, for $i = 1, \ldots, n$. Now we show that there are no other singularities. In this case we have

$$g_i(y) = 2y_i \sum_{1 \leq j \leq n,\, j \neq i} y_j^2. \tag{2.4}$$

If we assume $g_i(y) = 0$ for all i, and that $y_{i_1} \neq 0$ and $y_{i_2} \neq 0$ for some indices $1 \leq i_1 < i_2 \leq n$, the equation (2.4) implies that $y_{i_1}^2 = y_{i_2}^2$. Hence, for any singular point y^0 of Y, there is an integer a with $1 \leq a \leq n$ such that a coordinates of y^0 are equal to ± 1, and the remaining $n - a$ coordinates are 0. The equation (2.4) implies that only the case $a = 1$ is possible, and hence y^0 is one of the points p_i. This completes the proof in the case $d = 4$.

Finally, to treat the case $d > 4$, let

$$h_i(y) = y_i^{d-2} \sum_{1 \leq j \leq n,\, j \neq i} y_j^2.$$

Note that h_i has a singularity of type A_1 at p_i and vanishes of order $d - 2 > 2$ at the other points p_j, for $j \neq i$. Consider the linear system spanned by $h_1, \ldots, h_n$. It is easy to see, repeating the argument already used twice above, that the base locus $h_1 = \cdots = h_n = 0$ of this linear system is exactly the set Z_n. It follows, by Bertini's theorem, that a generic member Y of this linear system is smooth except possibly at the points of Z_n. The choice of the h_i implies that Y has an A_1 singularity at each point in Z_n. □

Now we give a proof of Theorem 1.1 stated in the introduction. Using Remark 1.3(i) and Proposition 2.1, it follows that, in any dimension $n \geq 3$ and degree $d \geq 3$, there are smooth hypersurfaces $V(f) \subset \mathbb{P}^n$ of degree d which have at least one nodal hyperplane section $V(f) \cap H$, with exactly n singularities in general position.

Let $B = \mathbb{P}(S_d)_0$ be the set of points in $\mathbb{P}(S_d)$ corresponding to polynomials $f \in S_d$ such that the hypersurface $V(f) : f = 0$ is smooth. Let $\mathcal{A}(n, d) \subset B$ be the subset of such hypersurfaces $V(f)$, which have at least one hyperplane section $V(f) \cap H$, with exactly n singularities A_1 in general position. We know already that $\mathcal{A}(n, d) \neq \emptyset$. It is easy to see that $\mathcal{A}(n, d)$ is a constructible (or semialgebraic) subset in B. Indeed, consider the subset

$$\Gamma \subset B \times (\mathbb{P}^n)^n$$

consisting of pairs (f, q), where $f \in B$, $q = (q^1, \ldots, q^n) \in (\mathbb{P}^n)^n$, such that the points

$$q^j = (q_0^j : q_1^j : \cdots : q_n^j) \in \mathbb{P}^n$$

are linearly independent, that is, they span a hyperplane $H(q)$ in $\mathbb{P}^n$, and the following conditions hold:

$$\sum_{i=0}^{n} q_i^j f_i(q^k) = 0 \text{ for any } j, k = 1, \ldots, n \tag{2.5}$$

and

$$\mathrm{Hess}(f)(q^j) \neq 0 \text{ for any } j = 1, \ldots, n, \tag{2.6}$$

where Hess(f) is the Hessian polynomial of f. In fact, the equation (2.5) for $k = j$ tells us that $q^j \in V(f)$ for any $j = 1, \dots, n$. Moreover, it says that the point q^j is on the tangent space $T_{q^k}V(f)$. This implies that

$$T_{q^k}V(f) = H(q) \text{ for any } k = 1, \dots, n.$$

The equation (2.6) tells us that the singularity of $V(f) \cap H(q)$ at the point q^j is a node; see for instance [**6**, equivalence (11.33)]. It is clear that Γ is a constructible set in $B \times (\mathbb{P}^n)^n$, since it is defined by finitely many algebraic equalities and inequalities. Let $p_1 \colon B \times (\mathbb{P}^n)^n \to B$ be the first projection and note that $\mathcal{A}(n, d) = p_1(\Gamma)$. Using the Chevalley theorem, see for instance [**10**, p. 395], we conclude that the set $\mathcal{A}(n, d)$ is constructible in B.

We now show that $\mathcal{A}(n, d)$ is a non-empty Zariski open subset in B, and hence it is dense in B and in $\mathbb{P}(S)_d$; see [**11**, Theorem 2.33]. Let $Z = B \setminus \mathcal{A}(n, d)$. Then Z is also a constructible set, and [**10**, Proposition 2, p. 394] implies that the closure of Z in B in the Zariski topology coincides with its closure in the strong complex topology. Hence, to show that $\mathcal{A}(n, d)$ is a Zariski open subset in B, it is enough to show that $\mathcal{A}(n, d)$ is an open subset of B in the strong complex topology.

We now fix one element $f \in \mathcal{A}(n, d)$ and show that $\mathcal{A}(n, d)$ contains a neighborhood of f in B. The set B is open, hence there are arbitrarily small open neighborhoods U of f with $f \in U \subset B$. For any polynomial $f' \in U$, we consider the gradient map

$$\Phi_{f'} \colon \mathbb{P}^n \to (\mathbb{P}^n)^\vee \text{ given by } x \mapsto (f'_0(x) : f'_1(x) : \cdots : f'_n(x))$$

and the corresponding dual mapping

$$\phi_{f'} = \Phi_{f'}|V(f') \colon V(f') \to (\mathbb{P}^n)^\vee.$$

Since $f \in \mathcal{A}(n, d)$, there is a hyperplane H such that $V(f) \cap H$ has n singularities A_1 in general position, say at the points $p_j \in \mathbb{P}^n$, for $j = 1, \dots, n$. It follows that the Hessian polynomial Hess(f) of f satisfies $\mathrm{Hess}(f)(p_j) \neq 0$ for $j = 1, \dots, n$ and hence each analytic germ

$$\Phi_f \colon (\mathbb{P}^n, p_j) \to ((\mathbb{P}^n)^\vee, H)$$

is invertible; see for instance [**6**, equation (11.10)]. It follows that there is a neighborhood N of H in $(\mathbb{P}^n)^\vee$ and neighborhoods N_j of p_j in $\mathbb{P}^n$ for $j = 1, \dots, n$ such that the restrictions

$$\Phi_f^{(j)} = \Phi_f|N_j \colon N_j \to N$$

are analytic isomorphisms, with corresponding inverse mappings

$$\Psi_f^{(j)} = (\Phi_f^{(j)})^{-1} \colon N \to N_j.$$

Any polynomial $f' \in U$ can be regarded as a deformation of f, the parameters being the coefficients of f'. Since f depends analytically on these parameters, the inverse mapping $\Psi_f^{(j)}$ also depends analytically on these parameters. It follows that, by choosing small enough neighborhoods U, N, and N_j for $j = 1, \dots, n$, we have inverse mappings as above

$$\Psi_{f'}^{(j)} \colon N \to N_j$$

for all $f' \in U$. Choose $g \in S_d$ a polynomial such that $g(p_j) \neq 0$ for $j = 1, \dots, n$. Then

$$h_{f'} = \frac{f'}{g}$$

is an analytic function defined on all N_j's, if they are chosen small enough. Now define

$$\alpha_{f'} \colon N \to \mathbb{C}^n \text{ given by } y \mapsto (h_{f'}(\Psi_{f'}^{(1)}(y)), \dots, h_{f'}(\Psi_{f'}^{(n)}(y))).$$

Notice that we have an obvious equality of (possibly non-reduced) analytic spaces $\alpha_f^{-1}(0) = \{H\}$, where H is regarded as a point with its reduced structure. Indeed, $h_f = 0$ in N_j defines the intersection $V(f) \cap N_j$, and $\Phi_f^{(j)}(V(f) \cap N_j)$ is the trace on N of the irreducible smooth branch of the dual variety $V(f)^\vee$ at the point $H \in (\mathbb{P}^n)^\vee$, whose tangent space at $H = \Phi_f^{(j)}(p_j)$ corresponds to the point $p_j \in \mathbb{P}^n$. The intersection of these n smooth branches, meeting transversally at H, is exactly the simple point H. Let D be a small closed ball in N, centered at H, and consider the restricted mapping

$$\beta_{f'} = \alpha_{f'}|\partial D \colon \partial D = S^{2n-1} \to \mathbb{C}^n \setminus \{0\}.$$

Here ∂D is the boundary of the closed ball D. If D is small, it is clear by the above discussion that α_f has no zeros on the compact set ∂D. By continuity, the same is true for $\alpha_{f'}$, and hence $\beta_{f'}$ is correctly defined for f' close to f. Notice that the mapping β_f has degree 1; see for instance [**1**, Section 5.4] or the topological interpretation of intersection multiplicity of n divisors in [**8**, p. 670]. By continuity, it follows that $\deg \beta_{f'} = 1$ for any $f' \in U$. Therefore, for any $f' \in U$ there is a unique point $H' \in (\mathbb{P}^n)^\vee$ such that $\alpha_{f'}^{-1}(0) = \{H'\}$. As explained above, this is equivalent to the fact that the dual hypersurface $V(f')^\vee$ has a normal crossing singularity of multiplicity n at the point H'. Hence H' corresponds to a hyperplane section of $V(f')$ with n nodes, that is, $f' \in \mathcal{A}(n,d)$. Therefore $U \subset \mathcal{A}(n,d)$ and this completes our proof.

3. Proof of Theorem 1.4

Fix $V(f) : f = 0$ a smooth hypersurface of degree d in $\mathbb{P}^n$, with $d, n \geq 3$. In view of Theorem 1.1, there is a sequence of pairs $(V(f_m), H_m)$ such that f_m converges to f in the projective space $\mathbb{P}(S_d)$ and $V(f_m) \cap H_m$ has n nodes in general position. By passing to a subsequence, we can assume that $q_m = H_m$ converges to a hyperplane $q = H$ in $(\mathbb{P}^n)^\vee$. By passing to the dual hypersurfaces, we get a sequence of hypersurfaces $V(f_m)^\vee$ converging to $V(f)^\vee$, and a sequence of points $q_m \in V(f_m)^\vee$ converging to the point $q \in V(f)^\vee$ such that $(V(f_m)^\vee, q_m)$ is a normal crossing singularity of multiplicity n. If we consider the n-jet at q_m of a reduced defining equation F_m for $V(f_m)^\vee$ in $(\mathbb{P}^n)^\vee$, we see that

$$h_m = j^n_{q_m} F_m$$

is a degree-n homogeneous polynomial which splits as a product of n linearly independent linear forms. Let F be a reduced defining equation for $V(f)^\vee$ in $(\mathbb{P}^n)^\vee$. Then F_m converges to F in $\mathbb{P}(S_D)$, where $D = d(d-1)^{n-1}$; see for instance [**7**, Theorem 1.2.5]. Hence for the n-jet

$$h = j^n_q F$$

there are the following two possibilities. Either $h \neq 0$, and then $h = \lim h_m$ in the corresponding projective space, and so h is a product of n (maybe non-distinct) linear forms, or $h = 0$ and then $\operatorname{mult}_q V(f)^\vee \geq n+1$. This proves the first claim in Theorem 1.4. To prove the second claim, note that for m large, we can identify the hyperplane H_m with H using a linear projection, and in this way $V(f_m) \cap H_m$ give rise to a sequence of hypersurfaces W_m in H converging to the intersection $W = V(f) \cap H$. Let G_m (resp. G) be the reduced defining equation of the hypersurface W_m (resp. W) in $H = \mathbb{P}^{n-1}$. If we choose a system of coordinates $y = (y_1 : \cdots : y_n)$ and set

$S' = \mathbb{C}[y_1, \ldots, y_n]$, let $M(G_m) = S'/J(G_m)$ and $M(G) = S'/J(G)$ denote the corresponding Milnor (or Jacobian) algebras of G_m and G. Here $J(G_m)$ (resp. $J(G)$) denotes the Jacobian ideal of G_m (resp. G) spanned by all the first-order partial derivatives of G_m (resp. G) with respect to the y_j's. For $k > 0$ an integer, we set

$$M(G_m)^k = \frac{S'}{J(G_m) + M^{k+1}} \quad \text{and} \quad M(G)k = \frac{S'}{J(G) + M^{k+1}},$$

where M is the maximal ideal $(y_1, \ldots, y_n) \subset S'$. Let $T = n(d-2)$ and recall that the homogeneous components of the Milnor algebras $M(G_m)$ and $M(G)$ satisfy

$$\dim M(G_m)_j = \tau(W_m) \quad \text{and} \quad \dim M(G)_j = \tau(W),$$

for any $j > T$; see [**4**, Corollary 9]. It follows that

$$\dim M(G_m)^k = \dim M(G_m)^T + (k - T)\tau(W_m)$$

and

$$\dim M(G)^k = \dim M(G)^T + (k - T)\tau(W),$$

for any $k > T$. Using the semicontinuity of the dimension of a quotient space, we get

$$\dim M(G)^k \geq \dim M(G_m)^k,$$

for all $k > T$. This clearly implies

$$\tau(W) \geq \tau(W_m) = n$$

and this proves our second claim.

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