Publ. Mat. **68** (2024), 507–544
DOI: 10.5565/PUBLMAT6822407

REGULARITY RESULTS FOR A CLASS OF NONLOCAL DOUBLE PHASE EQUATIONS WITH VMO COEFFICIENTS

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Abstract: We study a class of nonlocal double phase problems with discontinuous coefficients. A local self-improving property and a higher Hölder continuity result for weak solutions to such problems are obtained under the assumptions that the associated coefficient functions are of VMO (vanishing mean oscillation) type and that the principal coefficient depends not only on the variables but also on the solution itself.

2020 Mathematics Subject Classification: 35B65, 35J60, 35R11.

Key words: nonlocal double phase operators, self-improving property, VMO coefficients, higher Hölder regularity results.

1. Introduction

In this article, we consider the following nonlocal problem:

$$(\mathcal{P}) \qquad \mathcal{L}_{a(\cdot,u),b}u = f \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $N \geq 2$, $f \in L^{\gamma}_{\mathrm{loc}}(\Omega)$ with $\gamma > \max\{1, \frac{N}{ps}\}$, and the nonlocal operator $\mathcal{L}_{a(\cdot,),b}$ is defined as

$$\mathcal{L}_{a(\cdot,u),b}u(x) := 2 \text{ P.V.} \int_{\mathbb{R}^N} a(x,y,u(x),u(y)) \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy$$
$$+ 2 \text{ P.V.} \int_{\mathbb{R}^N} b(x,y) \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{N+qt}}\, dy, \quad \text{for } x \in \mathbb{R}^N,$$

with $1 < p \leq q < \infty$, $0 < s, t < 1$, and the kernel coefficients $a(\cdot,\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot)$ are nonnegative bounded functions. We will specify structural and regularity assumptions to be imposed on a and b later in the introduction. Specifically, the nonlocal operator in this work is motivated by the double phase equations and quasilinear equations for local cases; we refer to [**2, 12, 13, 15, 29, 40**] and [**7, 34**] for each type of problem, respectively.

The primary objective of the paper is to establish a local self-improving property and a higher Hölder regularity result for weak solutions to a class of nonlocal double phase problems with possibly discontinuous coefficients and a leading kernel coefficient depending not only on the independent variables but also on the solution. Particularly, we assume that the kernel coefficients are of VMO (vanishing mean oscillation) type. We establish the self-improving property of local weak solutions that are locally in an appropriate fractional Sobolev space which extends the results of [**28**] and [**37**]. To the best of our knowledge, there is no result to deal with a self-improving property of local weak solutions with nonzero boundary data. In this regard, our result gives a

S. Byun was supported by NRF-2022R1A2C1009312, K. Kim was supported by NRF-2017R1A2 B2003877, and D. Kumar was supported by NRF-2021R1A4A1027378.

new path to approach a local weak solution with nonzero boundary data concerning a nonlocal Calderón–Zygmund theory as in [**32, 33**]. We also complement the results of higher Hölder continuity for local weak solutions of [**4**] and [**22**] to the case of more general kernel coefficients.

We now briefly mention some recent regularity results on nonlocal problems. In the case when the kernel coefficient $a(\cdot,\cdot)$ is independent of the solution and $b \equiv 0$, i.e., the fractional p-Laplacian type equations, Di Castro, Kuusi, and Palatucci ([**18**]) proved the local Hölder regularity. Subsequently, for the case when the coefficients $a \equiv 1$ and $b \equiv 0$, Brasco, Lindgren, and Schikorra ([**4**]) obtained a higher Hölder regularity for the weak solutions to the problem $(\mathcal{P})$ for the superquadratic case. For $p = 2$, Nowak in [**31**] established a similar result as in [**4**] for problems involving irregular kernel coefficients. For additional regularity results of nonlocal equations, we refer to [**9, 10, 14, 19, 26, 39**] and references therein.

Concerning the nonlocal double phase type problems, we refer to [**17**] for Hölder regularity results for bounded viscosity solutions to the problem $(\mathcal{P})$ for the case when a is independent of the solution and $qt \le ps$. Later, Fang and Zhang in [**21**] and Byun, Ok, and Song in [**6**] obtained the Hölder continuity results for weak solutions to a similar problem when $tq \le ps$ and $ps < tq$ (with the coefficient b being Hölder continuous in the latter), respectively. Recently, Giacomoni, Kumar, and Sreenadh in [**22**] obtained higher Hölder continuity results with an explicit Hölder exponent for weak solutions to the problem $(\mathcal{P})$ for the case $qt \le ps$ and the coefficient b being locally continuous only along the diagonals in $\Omega \times \Omega$. For some other regularity results of problems with nonstandard growth nonlocal operators, we refer to [**5, 11, 16, 23, 24**].

Regarding a self-improving property of weak solutions to the nonlocal equations, Kuusi, Mingione, and Sire ([**28**]) proved this property for fractional Laplacian type problems with linear growth by introducing the notion of dual pairs. Subsequently, Scott and Mengesha ([**37**]) extended this result to bounded weak solutions of $(\mathcal{P})$ with $a(x,y,u(x),u(y)) = a(x,y)$ when $\frac{p-1}{p} \le \frac{tq}{sp} \le 1$. On the other hand, in [**1, 35**], the authors employed different techniques such as functional analysis and harmonic analysis tools to obtain similar self-improving properties. Moreover, we refer to [**3, 20, 30, 32, 33**] for Sobolev regularity results for nonlocal problems involving fractional Laplacian type operators (or their nonlinear versions).

Motivated by the above discussion, in this article, we consider the problem $(\mathcal{P})$ with the coefficient functions $a\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $b\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following:

(A1) the functions a and b are symmetric; that is, $a(x,y,z,w) = a(y,x,w,z)$ and $b(x,y) = b(y,x)$ for all $x, y \in \mathbb{R}^N$ and $z, w \in \mathbb{R}$;

(A2) for all $x, y \in \mathbb{R}^N$ and $z, w \in \mathbb{R}$, there hold

$$0 < \Lambda^{-1} \le a(x,y,z,w) \le \Lambda \quad \text{and} \tag{1.1}$$

$$0 \le b(x,y) \le \Lambda; \tag{1.2}$$

(A3) the function a is locally uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$; that is, for any $M > 0$, there is a nondecreasing function $\omega_{a,M}\colon [0,\infty) \to [0,\infty)$ with $\omega_{a,M}(0) = 0$ and $\lim\limits_{t\downarrow 0} \omega_{a,M}(t) = 0$ such that

$$|a(x,y,w,z) - a(x,y,w',z')| \le \omega_{a,M}\left(\frac{|w-w'| + |z-z'|}{2}\right)$$

for all $z, z', w, w' \in [-M, M]$ uniformly in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$;

(A4) the function $a(\cdot,\cdot,z,w)$ is in VMO on $\Omega\times\Omega$ locally uniformly in z, w and the function $b(\cdot,\cdot)$ is in VMO on $\Omega\times\Omega$, in the sense of Definition 2.2 below.

To prove our higher Hölder continuity result, we first obtain a self-improving identity for the weak solution to the problem $(\mathcal{P})$ much in the spirit of [**28**] and [**37**]. It is worth mentioning that, unlike the previously mentioned works, for our case, solutions considered here are assumed to be locally bounded and locally in an appropriate fractional Sobolev space (see later, in Section 2). However, this requires careful handling of the nonlocal tail terms. More precisely, we replace the notion of the standard nonlocal tail with a refined version as in (3.14) so that only the local behavior of the solution with respect to the fractional Sobolev space is taken into account. As a consequence, our self-improving identity holds for all $tq \le ps$ without requiring any lower bound on the quantity $\frac{tq}{sp}$, unlike in [**37**, (A2)]. Subsequently, we use a suitable approximation technique for VMO coefficients to establish an appropriate comparison result, which finally yields the Hölder continuity estimates for weak solutions to the problem $(\mathcal{P})$.

For the sake of completeness, we prove the existence of a weak solution to the problem $(\mathcal{P})$ with prescribed exterior data (see problem $(\mathcal{G})$). For this, we use the theory of M-type operators (as described in [**38**, Chapter II]) defined on a suitable separable reflexive Banach space. The main difficulty in this regard lies in the lack of monotonicity caused by the fact that the kernel coefficient a depends on the solution.

Before introducing our main results, we give a definition of a local weak solution. See Section 2 for a precise definition of the terms involved.

Definition 1.1 (Local weak solution)**.** Let $f \in (\mathcal{W}(\Omega))^*$. Then we say that $u \in \mathcal{W}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt,b}(\mathbb{R}^N)$ is a local weak solution to the problem $(\mathcal{P})$ if, for all $\phi \in \mathcal{W}(\Omega)$ with compact support contained in Ω, there holds

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} a(x,y,u(x),u(y))\frac{[u(x)-u(y)]^{p-1}}{|x-y|^{N+ps}}(\phi(x)-\phi(y))\,dx\,dy\\ +\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} b(x,y)\frac{[u(x)-u(y)]^{q-1}}{|x-y|^{N+qt}}(\phi(x)-\phi(y))\,dx\,dy = \langle f,\phi\rangle_{\mathcal{W},\mathcal{W}^*}.\end{aligned} \tag{1.3}$$

A local weak subsolution (resp. supersolution) is defined similarly by replacing the sign "=" with "≤" (resp. "≥") in (1.3) for all nonnegative test functions.

We now introduce our main results. The first one is the following local self-improving property of a weak solution to $(\mathcal{P})$.

Theorem 1.1 (A priori estimate)**.** *Suppose that* $2 \le p \le q \le \frac{ps}{t}$ *and that the assumptions* (A1) *and* (A2) *hold. Let* $u \in W^{s,p}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)\cap L^{p-1}_{ps}(\mathbb{R}^N)\cap L^{q-1}_{qt}(\mathbb{R}^N)$ *be a local weak solution to* $(\mathcal{P})$ *with the nonhomogeneous term satisfying*

$$f \in L^{p_*+\delta_0}_{\mathrm{loc}}(\Omega) \quad \textit{for some small } \delta_0 > 0, \tag{1.4}$$

where

$$p_* = \frac{Np'}{N+sp'} \textit{ if } sp < N \quad \textit{and} \quad p_* = 1 \textit{ if } sp \ge N.$$

Then, for all $\widetilde{\Omega} \Subset \Omega$, there is a constant $\delta = \delta(N, s, t, p, q, \Lambda, \delta_0, \|u\|_{L^\infty(\widetilde{\Omega})}) > 0$ such that $u \in W^{s+\frac{N\delta}{p(1+\delta)}, p(1+\delta)}_{\mathrm{loc}}(\widetilde{\Omega})$. In particular, there exists a constant c depending only on N, s, t, p, q, Λ, δ_0, and $\|u\|_{L^\infty(B_{2\rho_0}(x_0))}$ such that

$$\left(\int_{B_{\frac{\rho_0}{2}}(x_0)} \fint_{B_{\frac{\rho_0}{2}}(x_0)} \left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\right)^{(1+\delta)} dx\,dy\right)^{\frac{p-1}{p(1+\delta)}}$$

$$\leq c\Bigg[\left(\int_{B_{2\rho_0}(x_0)} \fint_{B_{2\rho_0}(x_0)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p'}} + \rho_0^s\left(\fint_{B_{2\rho_0}(x_0)} |f(x)|^{p_*+\delta_0}\,dx\right)^{\frac{1}{p_*+\delta_0}}$$

$$+ \rho_0^{s-tq} T_{tq}^{q-1}(u; x_0, 2\rho_0) + \rho_0^{-s(p-1)} T_{ps}^{p-1}(u; x_0, 2\rho_0) + \rho_0^{-s(p-1)}\Bigg], \tag{1.5}$$

whenever $B_{2\rho_0}(x_0) \Subset \widetilde{\Omega}$ with $\rho_0 \in (0, 1]$.

Remark 1. We observe that if $u \in W^{s,p}(\mathbb{R}^N)$, then

$$u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{ps}(\mathbb{R}^N),$$

but the converse is not true. Therefore, as we pointed out earlier, our result generalizes the previous works in [**28**] and [**37**]. On the other hand, if we consider the case for $b = 0$, then it suffices to take a weak solution

$$u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{ps}(\mathbb{R}^N).$$

We next describe the second main result, which is the higher Hölder regularity.

Theorem 1.2. *Suppose that $2 \leq p \leq q < \min\{p_s^*, ps/t\}$. Let the kernel coefficients satisfy the assumptions* (A1) *through* (A4) *and let u be a local weak solution to the problem $(\mathcal{P})$ such that $u \in L^{q-1}_{qt}(\mathbb{R}^N)$. Then, $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for all $\alpha \in (0, \Theta)$, where*

$$\Theta := \min\left\{\frac{ps - N/\gamma}{p-1}, \frac{qt}{q-1}, 1\right\}. \tag{1.6}$$

Before ending the section, we mention the layout of the rest of the paper. Section 2 deals with some preliminaries related to the paper. Section 3 corresponds to the self-improving property and we prove Theorem 1.1. Section 4 contains the proof of the higher Hölder regularity result of Theorem 1.2. Section 5 deals with the existence result for the problem $(\mathcal{P})$. Finally, the appendix is devoted to some boundedness results.

2. Preliminaries

In this section, we give some notations and introduce related function spaces. Here, we will also recall some of the well-known results.

2.1. Notation. For $1 < p < \infty$, we set $[\xi]^{p-1} = |\xi|^{p-2}\xi$, for all $\xi \in \mathbb{R}$. We abbreviate

$$(-\Delta)^s_{p,a(\cdot,u)} u(x) := 2\ \mathrm{P.V.} \int_{\mathbb{R}^N} a(x, y, u(x), u(y)) \frac{[u(x)-u(y)]^{p-1}}{|x-y|^{N+ps}}\,dy, \quad x \in \mathbb{R}^N,$$

and analogously $(-\Delta)^t_{q,b}$ is defined. The number p' denotes the Hölder conjugate of p; that is, $1/p + 1/p' = 1$. Additionally, for $1 < p < \infty$ and $s \in (0,1)$, we define the Sobolev conjugate of p by

$$p^*_s := \begin{cases} Np/(N-ps) & \text{if } N > ps, \\ \mathring{\mathrm{p}} & \text{if } N \le ps, \end{cases}$$

where $\mathring{\mathrm{p}}$ is an arbitrarily large number. For $x_0 \in \mathbb{R}^N$ and $v \in L^1(B_r(x_0))$, we set

$$(v)_{r,x_0} := \fint_{B_r(x_0)} v(x)\,dx = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} v(x)\,dx$$

and when the center is clear from the context, we will denote it as $(v)_r$. In several places, we will use

$$d\mu_1(x,y) := \frac{dx\,dy}{|x-y|^{N+ps}} \quad \text{and} \quad d\mu_2(x,y) := \frac{dx\,dy}{|x-y|^{N+qt}}.$$

For a Banach space $(X, \|\cdot\|)$, we denote its topological dual by X^* and $\langle\cdot,\cdot\rangle_{X,X^*}$ denotes the duality pairing. The constant c appearing in the proofs may vary from line to line and is always greater than or equal to 1. In particular, we write the relevant dependencies on parameters using parentheses; e.g., $c = c(N,s)$. On the other hand, we write

$$\mathsf{data} \equiv \mathsf{data}(N,s,t,p,q,\Lambda).$$

2.2. Function spaces and definitions. For an open set $\Omega \subset \mathbb{R}^N$, we define the space $\mathcal{W}_b(\Omega)$ as below:

$$\mathcal{W}_b(\Omega) := \{u \in W^{s,p}(\Omega) : [u]_{W^{t,q}_b(\Omega)} + \|u\|_{L^q(W_b,\,\Omega)} < \infty\},$$

equipped with the norm

$$\|u\|_{\mathcal{W}_b(\Omega)} := \|u\|_{L^p(\Omega)} + \|u\|_{L^q(W_b,\Omega)} + [u]_{W^{s,p}(\Omega)} + [u]_{W^{t,q}_b(\Omega)},$$

where

$$\|u\|^q_{L^q(W_b,\Omega)} := \int_\Omega W_b(x)|u(x)|^q\,dx \quad \text{with } W_b(x) := \int_{\mathbb{R}^N\setminus\Omega} \frac{b(x,y)}{|x-y|^{N+qt}}\,dy \tag{2.1}$$

and

$$[u]^q_{W^{t,q}_b(\Omega)} := \int_\Omega\int_\Omega b(x,y)\frac{|u(x)-u(y)|^q}{|x-y|^{N+tq}}\,dx\,dy \quad \text{with } [u]_{W^{s,p}(\Omega)} = [u]_{W^{s,p}_1(\Omega)}.$$

Note that $C^\infty_c(\Omega)$ is obviously contained in $\mathcal{W}_b(\Omega)$. It is not difficult to verify that $\mathcal{W}_b(\Omega)$ is a uniformly convex Banach space. Moreover, $\mathcal{W}_b(\Omega)$ is continuously embedded into $W^{s,p}(\Omega)$.

In what follows, the subscript b from the definitions of $\mathcal{W}_b(\Omega)$ and W_b will be suppressed if it has no relevance to the context. We call a function $u \in \mathcal{W}_{\mathrm{loc}}(\Omega)$ if $u \in \mathcal{W}(\widetilde{\Omega})$, for all $\widetilde{\Omega} \Subset \Omega$. Now we give definitions of the tail space and kernel coefficients.

Definition 2.1. Let $0 < m, \alpha < \infty$ and $b(\cdot,\cdot) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ be a nonnegative function. Then, we define the tail space as below:

$$L^m_{\alpha,b}(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable function} : \sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N} b(x,y)\frac{|u(y)|^m}{(1+|y|)^{N+\alpha}}\,dy < \infty\right\}.$$

And for $b(\cdot,\cdot) \equiv 1$, we denote it by $L^m_\alpha(\mathbb{R}^N)$. In particular, we write

$$\|u\|_{L^m_{\alpha,b}(\mathbb{R}^N)} = \left(\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N} b(x,y)\frac{|u(y)|^m}{(1+|y|)^{N+\alpha}}\,dy\right)^{\frac{1}{m}}.$$

For $0 < \alpha < 1 < m < \infty$ and a measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$, the nonlocal tail centered at $x_0 \in \mathbb{R}^N$ with radius $R > 0$ is defined as

$$T_{m\alpha,b}(u; x_0, R) = \left(R^{m\alpha} \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B_R(x_0)} b(x,y) \frac{|u(y)|^{m-1}}{|x_0 - y|^{N+m\alpha}}\, dy \right)^{\frac{1}{m-1}}.$$

For $b(\cdot,\cdot) \equiv 1$, we follow the convention $T_{m\alpha,1}(u; x_0, R) = T_{m\alpha}(u; x_0, R)$.

Remark 2. Using Minkowski's inequality, we check the following algebraic fact:

$$T_{m\alpha,b}(u + v; x_0, R) \le T_{m\alpha,b}(u; x_0, R) + T_{m\alpha,b}(v; x_0, R)$$

for any $u, v \in L^m_{\alpha,b}(\mathbb{R}^N)$ with $m \ge 2$. We will often use this inequality when we deal with a tail estimate in Section 4.

Definition 2.2 (VMO functions). (1) For any $M > 0$ and any ball $B_R \subset \Omega$, we say that the function a is (δ_M, R)-vanishing in $B_R \times B_R$, if for all $x_0, y_0 \in B_R$ and $r \in (0, R]$ such that $B_r(x_0), B_r(y_0) \subset B_R$,

$$\fint_{B_r(x_0)} \fint_{B_r(y_0)} |a(x,y,w,z) - (a)_{r,x_0,y_0}(w,z)|\, dx\, dy \le \delta_M, \quad \text{for all } w, z \in [-M, M],$$

where $(a)_{r,x_0,y_0}(w,z) = \fint_{B_r(x_0)} \fint_{B_r(y_0)} a(x,y,w,z)\, dx\, dy$.

(2) We say that the function a is in VMO on $\Omega \times \Omega$ locally uniformly in (w,z) if for any $M > 0$ and $B_\rho(x), B_\rho(y) \subset \Omega$,

$$\nu_{a,M}(\rho) := \sup_{|w|,|z|\le M} \sup_{0<r\le\rho} \sup_{x,y\in\Omega} \fint_{B_r(x)} \fint_{B_r(y)} |a(x',y',w,z) - (a)_{r,x,y}(w,z)|\, dx'\, dy' \tag{2.2}$$

tends to 0 as $\rho \downarrow 0$.

Especially, if the function a is independent of (w,z), then we say that a is in VMO on $\Omega \times \Omega$ and the VMO modulus of a is denoted by ν_a:

$$\nu_a(\rho) = \sup_{0<r\le\rho} \sup_{x,y\in\Omega} \fint_{B_r(x)} \fint_{B_r(y)} |a(x',y') - (a)_{r,x,y}|\, dx'\, dy'.$$

We recall the following inequalities (see [**27**]):

- for $\ell \ge 2$, there exists a constant $c(\ell) > 0$ such that

$$c(\ell)|\xi - \zeta|^\ell \le ([\xi]^{\ell-1} - [\zeta]^{\ell-1})(\xi - \zeta) \quad \text{for all } \xi, \zeta \in \mathbb{R}; \tag{2.3}$$

- for $\ell \ge 2$, there exists a constant $c = c(\ell) > 0$ such that for all $\xi, \zeta \in \mathbb{R}$,

$$|[\xi - w]^{\ell-1} - [\zeta - w]^{\ell-1}| \le c|\xi - \zeta|^{\ell-1} + c|\xi - \zeta||\xi - w|^{\ell-2}. \tag{2.4}$$

Before ending this section, we mention the following iteration lemma, which will be used in the proof of Lemma 3.6.

Lemma 2.1 (see [**25**, Lemma 6.1]). *Let φ be a bounded nonnegative function in $[t_1, t_2]$. For $t_1 \le r < \rho \le t_2$,*

$$\varphi(r) \le \eta\varphi(\rho) + \frac{M}{(\rho - r)^\alpha},$$

with $\eta \in (0,1)$, $M > 0$, and $\alpha > 0$. Then we have

$$\varphi(t_1) \le c\frac{M}{(t_2 - t_1)^\alpha},$$

for some constant $c = c(\eta, \alpha)$.

3. Self-improving properties

Throughout this section, we assume that the local weak solution u to the problem $(\mathcal{P})$ satisfies the following:

$$u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt}(\mathbb{R}^N)$$

with $2 \le p \le q \le \frac{sp}{t}$ and that (1.4) holds. We now fix

$$\widehat{\Omega} \Subset \widetilde{\Omega} \Subset \Omega.$$

In what follows, we write

$$\mathsf{data}_1 = \mathsf{data}_1(\mathsf{data}, \|u\|_{L^\infty(\widetilde{\Omega})}).$$

For any $v \in L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt}(\mathbb{R}^N)$, we denote

$$T(v; x_0, R) = \int_{\mathbb{R}^N \setminus B_R(x_0)} \left(\frac{|v(y)|^{p-1}}{|x_0 - y|^{N+sp}} + \|b\|_{L^\infty} \frac{|v(y)|^{q-1}}{|x_0 - y|^{N+tq}} \right) dy, \quad B_R(x_0) \subset \widetilde{\Omega}.$$

For a unified approach to handle the forcing term, we set a nonnegative number $\mathfrak{A}$ such that

$$\begin{cases} \mathfrak{A} = 0 & \text{if } sp < N, \\ \mathfrak{A} = \frac{1}{2}\min\{\delta_0, 1/p\} & \text{if } sp \ge N. \end{cases} \tag{3.1}$$

Before going further, we first give the following Caccioppoli-type estimate.

Lemma 3.1. *Let u be a local weak solution to $(\mathcal{P})$. Let $B \equiv B_R(x_0) \subset \Omega$ with $R \le \frac{1}{8}$, and let $\psi \in C^\infty_c(B)$ be a cutoff function such that $0 \le \psi \le 1$, $\psi \equiv 1$ in $B_r(x_0)$, and $|\nabla \psi| \le \frac{4}{R-r}$ with $r \in (0, R)$. Then we have*

$$\begin{aligned}
&\int_B \fint_B \frac{|\psi^{\frac{q}{p}}(x)u(x) - \psi^{\frac{q}{p}}(y)u(y)|^p}{|x-y|^{N+sp}} + \int_B \fint_B b(x,y) \frac{|\psi(x)u(x) - \psi(y)u(y)|^q}{|x-y|^{N+tq}}\, dx\, dy \\
&\le \frac{cR^{p(1-s)}}{(R-r)^p} \fint_B |u(x)|^p\, dx + \frac{cR^{q(1-t)}}{(R-r)^q} \fint_B |u(x)|^q\, dx + cR^{sp'} \left(\fint_B |f(x)|^{p_*+\mathfrak{A}}\, dx \right)^{\frac{p'}{p_*+\mathfrak{A}}} \\
&\quad + c\frac{R^{N+sp}}{(R-r)^{N+sp}} \int_{\mathbb{R}^N \setminus B} \frac{|u(y)|^{p-1}}{|x_0 - y|^{N+sp}}\, dy \fint_B \psi^q(x)|u(x)|\, dx \\
&\quad + c\frac{R^{N+tq}}{(R-r)^{N+tq}} \int_{\mathbb{R}^N \setminus B} \|b\|_{L^\infty} \frac{|u(y)|^{q-1}}{|x_0 - y|^{N+tq}}\, dy \fint_B \psi^q(x)|u(x)|\, dx,
\end{aligned}$$

for some constant $c = c(\mathsf{data})$.

Proof: Note that Hölder's inequality, Sobolev's embedding, and Young's inequality imply that for any $\sigma > 0$,

$$\begin{aligned}
\int_B |fu\psi^q|\, dx &\le \|f\psi^{\frac{q}{p'}}\|_{L^{p_*+\mathfrak{A}}(B)} \times \|u\psi^{\frac{q}{p}}\|_{L^{(p_*+\mathfrak{A})'}(B)} \\
&\le c(\mathsf{data})\|f\psi^{\frac{q}{p'}}\|_{L^{p_*+\mathfrak{A}}(B)} \times ([u\psi^{\frac{q}{p}}]_{W^{s,p}(B)} + R^{-s}\|u\psi^{\frac{q}{p}}\|_{L^p(B)}) \\
&\le \frac{c}{\sigma^{p-1}}\|f\|^{p'}_{L^{p_*+\mathfrak{A}}(B)} + \sigma([u\psi^{\frac{q}{p}}]^p_{W^{s,p}(B)} + R^{-sp}\|u\psi^{\frac{q}{p}}\|^p_{L^p(B)}).
\end{aligned} \tag{3.2}$$

In addition, since the kernel coefficient $a(\cdot,\cdot,\cdot,\cdot)$ satisfies the uniform ellipticity condition (1.1) in (A2), we have the result of the lemma by following the proof as presented in [**37**, Theorem 3.1] with (3.2). □

A dual pair (μ, U). We now introduce the notion of dual pair (μ, U), which is an essential tool to obtain the self-improving property of a weak solution u to $(\mathcal{P})$. Assume that ϵ is a sufficiently small positive number such that

$$\epsilon \in \left(0, \min\left\{\frac{s}{p}, 1-s\right\}\right),$$

which will be determined later, in Lemma 3.6. Let us define a measure μ in $\mathbb{R}^{2N}$ by

$$\mu(\mathcal{A}) = \int_{\mathcal{A}} \frac{dx\,dy}{|x-y|^{N-\epsilon p}}, \quad \mathcal{A} \subset \mathbb{R}^{2N} \text{ is a measurable subset.}$$

We write $\mathcal{B}(x_0, R) := B_R(x_0) \times B_R(x_0)$ with $x_0 \in \mathbb{R}^N$ and $R > 0$. Then we observe some properties of the measure μ as below.

Lemma 3.2 (see [**36**, Theorem 3.1])**.** *Let us write $k\mathcal{B}(x_0, R) = \mathcal{B}(x_0, kR)$ for $k > 0$. Then we have*

$$\frac{\mu(k\mathcal{B}(x_0,R))}{\mu(\mathcal{B}(x_0,R))} = k^{N+p\epsilon} \quad \text{and} \quad \mu(\mathcal{B}(x_0,R)) = \frac{cR^{N+p\epsilon}}{\epsilon},$$

where $c = c(N,p,\epsilon)$ satisfies $\frac{1}{C(N,p)} \le c \le C(N,p)$ for some constant $C(N,p) \ge 1$.

For $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, we define some functions as below:

$$\begin{cases} B(x,y) = b(x,y)|x-y|^{(s-t)q+\epsilon(q-p)}, \quad U(x,y) = \frac{|u(x)-u(y)|}{|x-y|^{s+\epsilon}}, \\ \mathbf{H}(x,y,U) = U^p + BU^q, \quad G(x,y,U) = \mathbf{H}(x,y,U)^{\frac{1}{p'}}, \\ F(x,y) = \begin{cases} |f(x)| & (x,y) \in \Omega \times \Omega, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{3.3}$$

Then we notice that

$$G(x,y,U) \in L^{p'}_{\mathrm{loc}}(\Omega \times \Omega; d\mu) \quad \text{and} \quad F \in L^{p_*+\delta_0}_{\mathrm{loc}}(\Omega \times \Omega; d\mu).$$

For convenience of notations, we set

$$m = \frac{Np + \epsilon p^2}{N + sp + \epsilon p}, \quad \tau = s + \epsilon - \frac{\epsilon p}{m}, \quad \alpha = \frac{m}{p} < 1, \quad \theta = \frac{s - \epsilon(p-1)}{N + \epsilon p},$$

and

$$\beta_i = 2^{i(-\frac{sp}{p-1} + (s+\epsilon))}, \quad \text{for nonnegative integers } i.$$

Then we check directly the following:

$$m \in (1,p), \quad p = \frac{Nm}{N - \tau m}, \quad \text{and} \quad \sum_{i=0}^{\infty} \beta_i < \infty. \tag{3.4}$$

We now state the following fractional Sobolev inequality.

Lemma 3.3 (see [**36**, Lemma 4.2])**.** *Let $B_R(x_0) \subset \Omega$ and let $u \in W^{s,p}(B_R(x_0))$. Then for all $\eta \in [1,p]$, we have*

$$\left(\fint_{B_R(x_0)} |u(x) - (u)_{R,x_0}|^\eta \, dx\right)^{\frac{1}{\eta}} \le \frac{c}{\epsilon^{\frac{1}{m}}} R^{s+\epsilon} \left(\fint_{\mathcal{B}(x_0,R)} U^m \, d\mu\right)^{\frac{1}{m}},$$

for some constant $c = c(\mathsf{data})$.

With the aid of the Caccioppoli-type inequality (see Lemma 3.1) and Lemma 3.3, we prove the following diagonal reverse Hölder-type inequality with a nonlocal tail.

Lemma 3.4. *Let u be a local weak solution to $(\mathcal{P})$. Let $0 < R \leq \frac{1}{8}$ and choose a positive integer l such that*

$$x_0 \in \widehat{\Omega}, \quad 2^l R \leq 2, \quad \text{and} \quad B_{2^l R}(x_0) \subset \widetilde{\Omega}. \tag{3.5}$$

Then, for $B \equiv B_R(x_0)$, the following holds:

$$\begin{aligned}
\left(\fint_{\frac{1}{2}\mathcal{B}} G(x,y,U)^{p'}\, d\mu \right)^{\frac{1}{p'}} &\leq c \frac{\sigma^{-(p-1)}}{\epsilon^{\frac{1}{p'\alpha} - \frac{1}{p'}}} \left(\fint_{\mathcal{B}} G(x,y,U)^{p'\alpha}\, d\mu \right)^{\frac{1}{p'\alpha}} \\
&\quad + c \frac{\sigma}{\epsilon^{\frac{1}{p'\alpha} - \frac{1}{p'}}} \sum_{j=0}^{l} \beta_j^{p-1} \left(\fint_{2^j \mathcal{B}} G(x,y,U)^{p'\alpha}\, d\mu \right)^{\frac{1}{p'\alpha}} \\
&\quad + c\sigma \epsilon^{\frac{1}{p'}} [\epsilon \mu(\mathcal{B})]^{\theta} T(u - (u)_{2^l R, x_0}; x_0, 2^l R) \\
&\quad + \frac{c[\epsilon\mu(\mathcal{B})]^{\theta}}{\epsilon^{\frac{1}{p_* + \mathfrak{A}} - \frac{1}{p'}}} \left(\fint_{\mathcal{B}} F^{p_* + \mathfrak{A}}\, d\mu \right)^{\frac{1}{p_* + \mathfrak{A}}},
\end{aligned} \tag{3.6}$$

for some constant $c = c(\mathsf{data}_1)$ which is independent of l and $\sigma \in (0,1)$.

Proof: Let l be a fixed positive number satisfying (3.5). Using Lemma 3.1 with $r = \frac{R}{2}$ and $\|u\|_{L^\infty(B_R)} \leq c$, we deduce

$$\begin{aligned}
&\underbrace{\int_{\frac{1}{2}B} \fint_{\frac{1}{2}B} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy + \int_{\frac{1}{2}B} \fint_{\frac{1}{2}B} b(x,y) \frac{|u(x)-u(y)|^q}{|x-y|^{N+tq}}\, dx\, dy}_{I_1} \\
&\qquad \leq \underbrace{\frac{c}{R^{sp}} \fint_B |u(x) - (u)_{R,x_0}|^p\, dx}_{I_2} + \underbrace{cT(u - (u)_{R,x_0}; x_0, R) \fint_B \psi^q(x) |u(x) - (u)_{R,x_0}|\, dx}_{I_3} \\
&\qquad\quad + \underbrace{cR^{sp'} \left(\int_B |f(x)|^{p_* + \mathfrak{A}}\, dx \right)^{\frac{p'}{p_* + \mathfrak{A}}}}_{I_4}.
\end{aligned}$$

We estimate each I_i, for $i = 1, 2, 3$, and 4, to discover the reverse Hölder inequality (3.6).

Estimate of I_1. By (3.3) and Lemma 3.2, we observe that

$$\fint_{\frac{1}{2}\mathcal{B}} G(x,y,U)^{p'}\, d\mu \leq c \frac{\epsilon}{R^{\epsilon p}} I_1.$$

Estimate of I_2. In light of Lemma 3.3, we have

$$I_2 \leq c \frac{R^{\epsilon p}}{\epsilon^{\frac{p}{m}}} \left(\fint_{\mathcal{B}} G(x,y,U)^{p'\alpha}\, d\mu \right)^{\frac{1}{\alpha}}.$$

Estimate of I_3. A simple calculation with (3.5) gives us

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_R} \frac{|u(y)-(u)_{R,x_0}|^{q-1}}{|x_0-y|^{N+tq}}\,dy &\le \sum_{i=0}^{l-1}\int_{B_{2^{i+1}R}\setminus B_{2^iR}} \frac{|u(y)-(u)_{R,x_0}|^{q-1}}{|x_0-y|^{N+tq}}\,dy \\
&\quad + \int_{\mathbb{R}^N\setminus B_{2^lR}} \frac{|u(y)-(u)_{R,x_0}|^{q-1}}{|x_0-y|^{N+tq}}\,dy \\
&=: \sum_{i=0}^{l-1} I_{3,i} + I_{3,l}.
\end{aligned} \tag{3.7}
$$

We first note from the facts $\|u\|_{L^\infty(B_2)} \le c$ and $p \le q$ that

$$
\begin{aligned}
I_{3,i}^{\frac{1}{p-1}} &\le c\left((2^iR)^{-qt} \fint_{B_{2^{i+1}R}} |u-(u)_{B_R}|^{p-1}\,dy\right)^{\frac{1}{p-1}} \\
&\le c(2^iR)^{-\frac{qt}{p-1}}\left[\left(\fint_{B_{2^{i+1}R}} |u-(u)_{B_{2^{i+1}R}}|^{p-1}\,dy\right)^{\frac{1}{p-1}} + \sum_{j=0}^{k} |(u)_{B_{2^{j+1}R}}-(u)_{B_{2^jR}}|\right] \\
&\le c(2^iR)^{-\frac{qt}{p-1}} \sum_{j=1}^{i+1}\left(\fint_{B_{2^{j+1}R}} |u-(u)_{B_{2^jR}}|^{p-1}\,dy\right)^{\frac{1}{p-1}} \\
&\le c(2^iR)^{-\frac{sp}{p-1}} \sum_{j=1}^{i+1}\left(\fint_{B_{2^{j+1}R}} |u-(u)_{B_{2^jR}}|^{p-1}\,dy\right)^{\frac{1}{p-1}},
\end{aligned}
$$

where we have used the relations $2^lR \le 2$ and $tq \le sp$ in the last inequality. Then, by Lemma 3.3, we obtain

$$
I_{3,i}^{\frac{1}{p-1}} \le c(2^iR)^{-\frac{sp}{p-1}} \sum_{j=1}^{i+1} \frac{2^{j(s+\epsilon)}R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}}\left(\fint_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}.
$$

We now employ the following Minkowski's inequality

$$
\left(\sum_{j=0}^{i} (I_{3,i}^{\frac{1}{p-1}})^{p-1}\right)^{\frac{1}{p-1}} \le \sum_{j=0}^{i} I_{3,i}^{\frac{1}{p-1}}
$$

and Fubini's theorem to deduce that

$$
\begin{aligned}
\left(\sum_{i=0}^{l-1} I_{3,i}\right)^{\frac{1}{p-1}} &\le c\sum_{i=0}^{l-1}(2^iR)^{-\frac{sp}{p-1}} \sum_{j=1}^{i+1} \frac{2^{j(s+\epsilon)}R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}}\left(\fint_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}} \\
&\le c\sum_{j=1}^{l}\sum_{i=j-1}^{l-1}(2^iR)^{-\frac{sp}{p-1}} \frac{2^{j(s+\epsilon)}R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}}\left(\fint_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}} \\
&\le c\sum_{j=1}^{l} R^{-\frac{sp}{p-1}+s+\epsilon}\frac{\beta_j}{\epsilon^{\frac{1}{m}}}\left(\fint_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}.
\end{aligned}
$$

By using Minkowski's inequality once again, we next note that

$$
\begin{aligned}
I_{3,l}^{\frac{1}{q-1}} \leq & \left(\int_{\mathbb{R}^n \setminus B_{2^l R}} \frac{|u-(u)_{B_{2^l R}}|^{q-1}}{|y|^{n+tq}} \, dy \right)^{\frac{1}{q-1}} \\
& + \sum_{j=0}^{l-1} \left(\int_{\mathbb{R}^n \setminus B_{2^i R}} \frac{|(u)_{B_{2^{j+1}R}} - (u)_{B_{2^j R}}|^{q-1}}{|y|^{n+tq}} \, dy \right)^{\frac{1}{q-1}} .
\end{aligned}
\tag{3.8}
$$

We further estimate the second term on the right-hand side of (3.8) by means of Lemma 3.3 as below:

$$
\begin{aligned}
& \sum_{j=0}^{l-1} \left(\int_{\mathbb{R}^n \setminus B_{2^l R}} \frac{|(u)_{B_{2^{j+1}R}} - (u)_{B_{2^j R}}|^{q-1}}{|y|^{n+tq}} \, dy \right)^{\frac{1}{q-1}} \\
& \quad \leq c \sum_{j=0}^{l-1} \left(\int_{\mathbb{R}^n \setminus B_{2^l R}} \frac{|(u)_{B_{2^{j+1}R}} - (u)_{B_{2^j R}}|^{p-1}}{|y|^{n+tq}} \, dy \right)^{\frac{1}{q-1}} \\
& \quad \leq c \sum_{j=0}^{l-1} (2^l R)^{-\frac{qt}{q-1}} |(u)_{B_{2^{j+1}R}} - (u)_{B_{2^j R}}|^{\frac{p-1}{q-1}} \\
& \quad \leq c \sum_{j=1}^{l-1} \left((2^l R)^{-\frac{sp}{p-1}} \frac{2^{j(s+\epsilon)} R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}} \left(\fint_{2^j \mathcal{B}} U^m \, d\mu \right)^{\frac{1}{m}} \right)^{\frac{p-1}{q-1}} .
\end{aligned}
$$

We next claim that

$$
I_{3,l}^{\frac{1}{p-1}} \leq cT(u - (u)_{2^l R, x_0}; x_0, 2^l R)^{\frac{1}{p-1}} + c \sum_{j=1}^{l-1} (2^j R)^{-\frac{sp}{p-1}} \frac{2^{j(s+\epsilon)} R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}} \left(\fint_{2^j \mathcal{B}} U^m \, d\mu \right)^{\frac{1}{m}} . \tag{3.9}
$$

Indeed, if $p = q$, it is a direct computation. We now assume that $p < q$. Then, by Hölder's inequality, we have

$$
\begin{aligned}
I_{3,l}^{\frac{1}{p-1}} & \leq (I_{3,l}^{\frac{1}{q-1}})^{\frac{q-1}{p-1}} \\
& \leq cT(u - (u)_{2^l R, x_0}; x_0, 2^l R)^{\frac{1}{p-1}} \\
& \quad + c \left(\sum_{j=1}^{l-1} (2^{l-j})^{\frac{-sp}{q-1}} \left((2^j R)^{-\frac{sp}{p-1}} \frac{2^{j(s+\epsilon)} R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}} \left(\fint_{2^j \mathcal{B}} U^m \, d\mu \right)^{\frac{1}{m}} \right)^{\frac{p-1}{q-1}} \right)^{\frac{q-1}{p-1}} \\
& \leq cT(u - (u)_{2^l R, x_0}; x_0, 2^l R)^{\frac{1}{p-1}} \\
& \quad + c \sum_{j=1}^{l-1} (2^j R)^{-\frac{sp}{p-1}} \frac{2^{j(s+\epsilon)} R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}} \left(\fint_{2^j \mathcal{B}} U^m \, d\mu \right)^{\frac{1}{m}} ,
\end{aligned}
$$

which proves (3.9). Consequently,

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_R} \frac{|u(y)-(u)_{R,x_0}|^{q-1}}{|x_0-y|^{N+tq}}\,dy &\le \left(\sum_{i=0}^{l-1} I_{3,i}^{\frac{1}{p-1}} + I_{3,l}^{\frac{1}{p-1}}\right)^{p-1}\\
&\le cT(u-(u)_{2^lR,x_0};x_0,2^lR)\\
&\quad + c\frac{R^{\epsilon p-(s+\epsilon)}}{\epsilon^{\frac{p-1}{m}}}\sum_{j=0}^{l}\left(\beta_j\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}\right)^{p-1},
\end{aligned}
\tag{3.10}
$$

where we have used the following algebraic inequality:

$$
\begin{aligned}
&\left(\sum_{j=1}^{l} R^{-\frac{sp}{p-1}+s+\epsilon}\frac{\beta_j}{\epsilon^{\frac{1}{m}}}\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}\right)^{p-1}\\
&\quad\le \left(\sum_{j=1}^{l} R^{-\frac{sp}{+}(s+\epsilon)(p-1)}\frac{\beta_j}{\epsilon^{\frac{p-1}{m}}}\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{p-1}{m}}\right)\left(\sum_{j=1}^{l}\beta_j\right)^{p-2}\\
&\quad\le c\frac{R^{\epsilon p-(s+\epsilon)}}{\epsilon^{\frac{p-1}{m}}}\sum_{j=0}^{l}\left(\beta_j\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}\right)^{p-1}.
\end{aligned}
$$

Similarly, we estimate

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_R} \frac{|u(y)-(u)_{R,x_0}|^{p-1}}{|x_0-y|^{N+sp}}\,dy &\le cT(u-(u)_{2^lR,x_0};x_0,2^lR)\\
&\quad + c\frac{R^{\epsilon p-(s+\epsilon)}}{\epsilon^{\frac{p-1}{m}}}\sum_{j=0}^{l}\left(\beta_j\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}\right)^{p-1}.
\end{aligned}
\tag{3.11}
$$

Coupling (3.10) and (3.11), we get

$$
\begin{aligned}
T(u-(u)_{R,x_0};x_0,R) &\le c\frac{R^{\epsilon p-(s+\epsilon)}}{\epsilon^{\frac{p-1}{m}}}\sum_{j=0}^{l}\left(\beta_j\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}\right)^{p-1}\\
&\quad + cT(u-(u)_{2^lR,x_0};x_0,2^lR).
\end{aligned}
\tag{3.12}
$$

On the other hand, we have

$$
⨍_B \psi^q(x)|u(x)-(u)_{R,x_0}|\,dx \le c\frac{R^{s+\epsilon}}{\epsilon^{\frac{1}{m}}}\left(⨍_{\mathcal{B}} U^m\,d\mu\right)^{\frac{1}{m}}.
\tag{3.13}
$$

Consequently, using Young's inequality with (3.12) and (3.13), we obtain

$$
\begin{aligned}
I_3 &\le c\frac{\sigma^{p'}R^{\epsilon p}}{\epsilon^{\frac{p}{m}}}\left(\sum_{j=0}^{l}\beta_j\left(⨍_{2^j\mathcal{B}} U^m\,d\mu\right)^{\frac{p-1}{m}}\right)^{p'}\\
&\quad + c\sigma^{p'}R^{sp'}T(u-(u)_{2^lR,x_0};x_0,2^lR)^{p'} + c\frac{\sigma^{-p}R^{\epsilon p}}{\epsilon^{\frac{p}{m}}}\left(⨍_{\mathcal{B}} U^m\,d\mu\right)^{\frac{p}{m}}\\
&\le c\frac{\sigma^{p'}R^{\epsilon p}}{\epsilon^{\frac{p}{m}}}\left(\sum_{j=0}^{l}\beta_j\left(⨍_{2^j\mathcal{B}} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}}\right)^{p'}\\
&\quad + c\sigma^{p'}R^{sp'}T(u-(u)_{2^lR,x_0};x_0,2^lR)^{p'} + c\frac{\sigma^{-p}R^{\epsilon p}}{\epsilon^{\frac{p}{m}}}\left(⨍_{\mathcal{B}} G^{p'\alpha}\,d\mu\right)^{\frac{p}{m}}.
\end{aligned}
$$

Estimate of I_4. Recalling the definition of F from (3.3), we get

$$I_4 \leq c\frac{R^{sp'}}{\epsilon^{\frac{p'}{p_*+\mathfrak{A}}}}\left(\fint_{\mathcal{B}} F^{p_*+\mathfrak{A}}\,d\mu\right)^{\frac{p'}{p_*+\mathfrak{A}}}.$$

Eventually, we combine the estimates of I_1, I_2, I_3, and I_4 to discover that

$$\begin{aligned}\fint_{\frac{1}{2}\mathcal{B}} G(x,y,U)^{p'}\,d\mu \leq{}& \frac{c\sigma^{-p}}{\epsilon^{\frac{p}{m}-1}}\left(\fint_{\mathcal{B}} G(x,y,U)^{p'\alpha}\,d\mu\right)^{\frac{1}{\alpha}}\\ &+\frac{c\sigma^{p'}}{\epsilon^{\frac{p}{m}-1}}\left(\sum_{j=0}^{l}\beta_j^{p-1}\left(\fint_{2^j\mathcal{B}} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}}\right)^{p'}\\ &+c\epsilon\sigma^{p'}R^{sp'-\epsilon p}T(u-(u)_{2^lR,x_0};x_0,2^lR)^{p'}\\ &+c\frac{R^{sp'-\epsilon p}}{\epsilon^{\frac{p'}{p_*+\mathfrak{A}}-1}}\left(\fint_{\mathcal{B}} F^{p_*+\mathfrak{A}}\,d\mu\right)^{\frac{p'}{p_*+\mathfrak{A}}},\end{aligned}$$

which implies (3.6). □

Now we are ready to give and prove a level set estimate for G in $\mathcal{B}(x_0,2\rho_0)\subset\widetilde{\Omega}\times\widetilde{\Omega}$ with $\rho_0\leq 1$ and $x_0\in\widehat{\Omega}$. First, we introduce a few more functionals. For every $\mathcal{B}(x,R)\subset\mathcal{B}(x_0,2\rho_0)$, we define

$$\Upsilon(x,R):=\left(\fint_{\mathcal{B}(x,R)} F^{p_*+\mathfrak{A}+\delta_f}\,d\mu\right)^{\frac{1}{p_*+\mathfrak{A}+\delta_f}},$$

where $\delta_f\in\left(0,\frac{\delta_0}{2}\right]$ will be determined later, in Lemma 3.6, and

$$\begin{aligned}\mathrm{Tail}(x,R):={}&\sum_{i=0}^{l}\beta_i^{p-1}\left(\fint_{2^i\mathcal{B}(x,R)} G(x,y,U)^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}}\\ &+\epsilon^{\frac{1}{p'}}[\epsilon\mu(\mathcal{B}(x,R))]^{\theta}T(u-(u)_{2^lR,x};x,2^lR),\end{aligned}\tag{3.14}$$

for some positive integer l such that

$$\frac{\rho_0}{2}\leq 2^lR<\rho_0.$$

We also define

$$\Psi_M(x,R):=\left(\fint_{\mathcal{B}(x,R)} G^{p'}\,d\mu\right)^{\frac{1}{p'}}+M\frac{[\mu(\mathcal{B}(x,R))]^{\theta}}{\epsilon^{\frac{1}{p_*+\mathfrak{A}}-\frac{1}{p'}}}\left(\fint_{\mathcal{B}(x,R)} F^{p_*+\mathfrak{A}}\,d\mu\right)^{\frac{1}{p_*+\mathfrak{A}}},$$

where $M\geq 1$ will be chosen later, in (3.32). Now we set

$$\Xi(x,R):=\Upsilon(x,R)+\mathrm{Tail}(x,R)+\Psi_M(x,R).$$

In particular, we denote

$$\Xi_0:=\Upsilon(x_0,2\rho_0)+\Psi_1(x_0,2\rho_0)+T(u-(u)_{2\rho_0,x_0};x_0,2\rho_0).\tag{3.15}$$

For convenience, we write

$$\theta_u=\frac{(p+1)(1-\alpha)}{\alpha},\quad \theta_f=(p_*+\mathfrak{A}+\delta_f)\left(\frac{(p_*+\mathfrak{A})\theta}{1-(p_*+\mathfrak{A})\theta}\right),\quad\text{and}\quad \tilde{\theta}_f=\frac{(p_*+\mathfrak{A})(1+\theta\delta_f)}{1-(p_*+\mathfrak{A})\theta}.$$

In this setting, we now describe an integral estimate of G on superlevel sets.

Lemma 3.5. *Suppose that u is a local weak solution to $(\mathcal{P})$. Take $B \equiv B_{2\rho_0}(x_0) \subset \widetilde{\Omega}$ with $0 < \rho_0 \leq 1$ and $x_0 \in \widehat{\Omega}$. Let $\frac{\rho_0}{2} \leq r < \rho \leq \rho_0$. Then there exist constants $c_\alpha = c_\alpha(\mathsf{data}_1) \geq 1$, $c_f = c_f(\mathsf{data}_1, \epsilon) \geq 1$, and $\kappa_f = \kappa_f(\mathsf{data}_1, \epsilon) \in (0,1)$ such that the inequality*

$$\begin{aligned} \frac{1}{\lambda^{p'}} \int_{\mathcal{B}(x_0,r)\cap\{G>\lambda\}} G^{p'}\, d\mu &\leq \frac{c_\alpha}{\epsilon^{\theta_u}\lambda^{p'\alpha}} \int_{\mathcal{B}(x_0,\rho)\cap\{G>\lambda\}} G^{p'\alpha}\, d\mu \\ &\quad + \frac{c_f \lambda_0^{\theta_f}}{\lambda^{\tilde{\theta}_f}} \int_{\mathcal{B}(x_0,\rho)\cap\{F>\kappa_f\lambda\}} F^{p_*+\mathfrak{A}}\, d\mu \end{aligned} \tag{3.16}$$

holds whenever $\lambda \geq \lambda_0$, where

$$\lambda_0 := \frac{c}{\epsilon^{\frac{1}{p'\alpha}}} \left(\frac{\rho_0}{\rho - r}\right)^{2N+p} \Xi_0, \tag{3.17}$$

for some constant $c = c(\mathsf{data}_1)$.

Proof: Let $\kappa \in (0,1)$ be a parameter which will be determined later, in (3.24). Define

$$\lambda_1 := \frac{1}{\kappa} \sup_{\frac{\rho-r}{40^N} \leq R \leq \frac{\rho_0}{2}} \ \sup_{x \in B_r(x_0)} \{\Psi_M(x,R) + \Upsilon(x,R) + \mathrm{Tail}(x,R)\}. \tag{3.18}$$

Now, we prove the lemma in five steps.

Step 1: Upper bound on λ_1. We estimate the upper bound of λ_1 as follows. For any $x \in B_r(x_0)$ and $\frac{\rho-r}{40^N} \leq R \leq \frac{\rho_0}{2}$, using the doubling property of μ, we have

$$\Upsilon(x,R) \leq c\left(\frac{2\rho_0}{\rho - r}\right)^{N+p} \Upsilon(x_0, 2\rho_0) \quad \text{and } \Psi_M(x,R) \leq c\left(\frac{2\rho_0}{\rho - r}\right)^{N+p} \Psi_M(x_0, 2\rho_0).$$

On the other hand, using Hölder's inequality and similar tail estimates as in (3.7), we see that

$$\begin{aligned} \mathrm{Tail}(x,R) &\leq c\sum_{i=0}^{l} \left(\frac{2\rho_0}{2^i R}\right)^{N+\epsilon p} \beta_i^{p-1} \left(\fint_{\mathcal{B}(x_0,2\rho_0)} G^{p'\alpha}\, d\mu\right)^{\frac{1}{p'\alpha}} + c\left(\fint_{\mathcal{B}(x_0,2\rho_0)} G^{p'\alpha}\, d\mu\right)^{\frac{1}{p'\alpha}} \\ &\quad + cT(u - (u)_{2\rho_0,x_0}; x, 2\rho_0) \\ &\leq c\left(\frac{\rho_0}{\rho - r}\right)^{N+\epsilon p} \left(\fint_{\mathcal{B}(x_0,2\rho_0)} G^{p'\alpha}\, d\mu\right)^{\frac{1}{p'\alpha}} \\ &\quad + c\left(\frac{\rho_0}{\rho - r}\right)^{N+sp} T(u - (u)_{2\rho_0,x_0}; x_0, 2\rho_0), \end{aligned}$$

where in the last line, we have used the relation

$$|y - x| \geq |y - x_0| - |x - x_0| \geq |y - x_0| \frac{\rho - r}{\rho_0}, \quad \text{for } y \in B_{2\rho_0}(x)^c.$$

Thus, we get

$$\lambda_1 \leq \frac{c(\mathsf{data}_1, M)}{\kappa} \left(\frac{\rho_0}{\rho - r}\right)^{N+p} \Xi_0. \tag{3.19}$$

Step 2: Vitali covering. We start with an exit-time argument as in [**28**] and [**37**] to cover the diagonal level set of G. We focus on handling the tail term which is different from the previous works. Define the diagonal level set of the functional Ψ_M by

$$D_{\kappa\lambda} := \left\{ (x,x) \in \mathcal{B}(x_0,r) : \sup_{0 \le R \le \frac{\rho-r}{40^N}} \Psi_M(x,R) > \kappa\lambda \right\}, \tag{3.20}$$

for some $\lambda \ge \lambda_1$ which will be specified in (3.24). Note that for each $(x,x) \in \mathcal{B}(x_0,r)$ and $R \in \left[\frac{\rho-r}{40^N}, \frac{\rho_0}{2}\right]$, we have $\Psi_M(x,R) \le \kappa\lambda_1 \le \kappa\lambda$. Therefore, for each $(x,x) \in D_{\kappa\lambda}$, there exists a constant $0 < R(x) \le \frac{\rho-r}{40^N}$ such that

$$\Psi_M(x,R(x)) \ge \kappa\lambda \quad \text{and} \quad \Psi_M(x,R) \le \kappa\lambda, \quad \text{for any } R \in \left(R(x), \frac{\rho-r}{40^N} \right]. \tag{3.21}$$

Using Vitali's covering lemma, we find that there is a collection $\{\mathcal{B}(x_j, 2R(x_j))\}_{j\in\mathbb{N}}$ of disjoint open sets with center $(x_j,x_j) \in D_{\kappa\lambda}$ such that

$$D_{\kappa\lambda} \subset \bigcup_j \mathcal{B}(x_j, 10R(x_j)). \tag{3.22}$$

Let us write $R_j \equiv R(x_j)$ and $\mathcal{B}_j \equiv \mathcal{B}(x_j, R(x_j))$ for each positive integer j. From (3.21) and the doubling property of the measure μ (see Lemma 3.2), we have

$$\sum_j \int_{10\mathcal{B}_j} G^{p'}\, d\mu \le \sum_j \mu(10\mathcal{B}_j)[\Psi_M(x_j,10R_j)]^{p'} \le 10^{N+\epsilon p}(\kappa\lambda)^{p'} \sum_j \mu(\mathcal{B}_j).$$

Step 3: Off-diagonal estimate of G. For this, we follow the method described in [**36**, Subsection 4.3]. Since we know that $u \in W^{s,p}(B_{2\rho_0}(x_0)) \cap L^\infty(B_{2\rho_0}(x_0))$ with (3.22) and functions G and H which are described in [**36**] are the same, an inspection of Subsection 4.3 in [**36**] shows that it remains valid for our case, too. Therefore, we have a desired result similar to [**36**, Lemma 4.10] as follows. There is a constant

$$\kappa = \frac{\epsilon^{\frac{1}{p'\alpha}}}{c_\kappa} \quad \text{with } c_\kappa = c_\kappa(\mathsf{data}_1) \ge 1 \tag{3.23}$$

such that

$$\int_{\mathcal{B}(x_0,r)\cap\{G>\lambda\}} G^{p'}\, d\mu \le 10^{N+p}\kappa^{p'}\lambda^{p'} \sum_j \mu(\mathcal{B}_j) + c\lambda^{p'-p'\alpha} \int_{\mathcal{B}(x_0,\rho)\cap\{G>\kappa\lambda\}} G^{p'\alpha}\, d\mu, \tag{3.24}$$

for some constant $c_0 = c_0(\mathsf{data}_1)$, whenever

$$\lambda \ge \max\left\{ \lambda_1, \frac{c_1}{\epsilon^{\frac{1}{p'}}} \left(\frac{\rho_0}{\rho - r} \right)^{2N+p} \Xi_0 \right\} =: \lambda_2.$$

Step 4: Estimate of $\mu(\mathcal{B}_j)$. This step is to establish the existence of constants $c_4 = c_4(\mathsf{data}_1)$ and $c_5 = c_5(\mathsf{data}_1, \epsilon)$ such that

$$\begin{aligned} \sum_j \mu(\mathcal{B}_j) \le{}& \frac{c_4}{\epsilon^{p-p\alpha}\kappa^{p'\alpha}\lambda^{p'\alpha}} \int_{\mathcal{B}(x_0,\rho)\cap\{G>\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\, d\mu \\ &+ \frac{c_5\lambda_1^{\theta_f}}{(\hat{\kappa}\kappa\lambda)^{\tilde{\theta}_f}} \int_{\mathcal{B}(x_0,\rho)\cap\{F>\hat{\kappa}\kappa\lambda\}} F^{p_*+\mathfrak{A}}\, d\mu. \end{aligned} \tag{3.25}$$

Indeed, by (3.20), it follows that at least one of the following inequalities hold:

$$\left(\fint_{\mathcal{B}_j} G^{p'}\,d\mu\right)^{\frac{1}{p'}} \geq \frac{\kappa\lambda}{2} \quad \text{or} \tag{3.26}$$

$$\frac{M[\mu(\mathcal{B}_j)]^\theta}{\epsilon^{\frac{1}{p_*+\mathfrak{A}}-\frac{1}{p'}}}\left(\fint_{\mathcal{B}_j} F^{p_*+\mathfrak{A}}\,d\mu\right)^{\frac{1}{p_*+\mathfrak{A}}} \geq \frac{\kappa\lambda}{2}. \tag{3.27}$$

Case 1. We assume that (3.26) holds. Then, from (3.4), we observe that

$$\begin{aligned}\kappa\lambda \leq{}& \frac{c\sigma^{-(p-1)}}{\epsilon^{\frac{1}{p'\alpha}-\frac{1}{p'}}}\left(\fint_{2\mathcal{B}_j} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}} + \frac{c\sigma}{\epsilon^{\frac{1}{p'\alpha}-\frac{1}{p'}}}\sum_{i=0}^{l_j-1}\beta_i^{p-1}\left(\fint_{2^{i+1}\mathcal{B}_j} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}}\\ &+ c\sigma\epsilon^{\frac{1}{p'}}[\epsilon\mu(2\mathcal{B}_j)]^\theta T(u-(u)_{2^{l_j}R_j,x_0};x_j,2^{l_j}R_j)\\ &+ \frac{c[\epsilon\mu(2\mathcal{B}_j)]^\theta}{\epsilon^{\frac{1}{p_*+\mathfrak{A}}-\frac{1}{p'}}}\left(\fint_{2\mathcal{B}_j} F^{p_*+\mathfrak{A}}\,d\mu\right)^{\frac{1}{p_*+\mathfrak{A}}},\end{aligned} \tag{3.28}$$

for some positive integer l_j such that $\frac{\rho_0}{2} \leq 2^{l_j}R_j < \rho_0$. Note that since $R_j \leq \frac{\rho-r}{40^N} \leq \frac{\rho_0}{40^N}$, we have $l_j \geq 3$. Therefore, by (3.21), we see that

$$\left(\fint_{2^i\mathcal{B}_j} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}} \leq \kappa\lambda, \tag{3.29}$$

for $i = 0, 1, \ldots, l_j - 1$. With (3.18), we have

$$\begin{aligned}\kappa\lambda &\geq \mathrm{Tail}(x_j, 2^{l_j-1}R_j)\\ &\geq \sum_{k=0}^{1}\beta_k^{p-1}\left(\fint_{2^{l_j-1+k}\mathcal{B}_j} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}} + \epsilon^{\frac{1}{p'}}[\epsilon\mu(2\mathcal{B}_j)]^\theta T(u-(u)_{2^{l_j}R_j,x_0};x_j,2^{l_j}R_j)\end{aligned} \tag{3.30}$$

and

$$\kappa\lambda \geq \frac{M[\mu(2\mathcal{B}_j)]^\theta}{\epsilon^{\frac{1}{p_*+\mathfrak{A}}-\frac{1}{p'}}}\left(\fint_{2\mathcal{B}_j} F^{p_*+\mathfrak{A}}\,d\mu\right)^{\frac{1}{p_*+\mathfrak{A}}}, \tag{3.31}$$

where we have used the fact that

$$\frac{\rho-r}{40^N} \leq 2^{l_j-1}R_j < \frac{\rho_0}{2}.$$

Applying (3.29), (3.30), and (3.31) to (3.28), we then discover that there are constants $c_1 = c_1(\mathsf{data}_1)$ and $c_2 = c_2(\mathsf{data}_1)$ such that

$$\kappa\lambda \leq \frac{c\sigma^{-(p-1)}}{\epsilon^{\frac{1}{p'\alpha}-\frac{1}{p'}}}\left(\fint_{2\mathcal{B}_j} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}} + \frac{c_1\sigma}{\epsilon^{\frac{1}{p'\alpha}-\frac{1}{p'}}}\kappa\lambda + \frac{c_2}{M}\kappa\lambda.$$

By taking

$$\sigma = \frac{\epsilon^{\frac{1}{p'\alpha}-\frac{1}{p'}}}{4c_1} \quad \text{and} \quad M = 4c_2, \tag{3.32}$$

we see that

$$\kappa\lambda \leq \frac{c}{\epsilon^{\frac{p}{p'\alpha}-\frac{p}{p'}}}\left(\fint_{2\mathcal{B}_j} G^{p'\alpha}\,d\mu\right)^{\frac{1}{p'\alpha}},$$

which yields

$$\mu(\mathcal{B}_j)(\kappa\lambda)^{p'\alpha} \leq \frac{c}{\epsilon^{p-p\alpha}}\int_{2\mathcal{B}_j} G^{p'\alpha}\,d\mu.$$

Since

$$\begin{aligned}\frac{c}{\epsilon^{p-p\alpha}}\int_{2\mathcal{B}_j} G^{p'\alpha}\,d\mu &\le \frac{c}{\epsilon^{p-p\alpha}}\left(\int_{2\mathcal{B}_j\cap\{G\le\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu+\int_{2\mathcal{B}_j\cap\{G\ge\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu\right)\\ &\le \frac{c_3}{\epsilon^{p-p\alpha}}\left((\tilde{\kappa}\kappa\lambda)^{p'\alpha}\mu(\mathcal{B}_j)+\int_{2\mathcal{B}_j\cap\{G\ge\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu\right)\end{aligned}$$

(thanks to the doubling property in Lemma 3.2), by choosing $\tilde{\kappa}=\frac{\epsilon^{\frac{p}{p'\alpha}-\frac{p}{p'}}}{(2c_3)^{\frac{1}{p'\alpha}}}$ we obtain

$$\mu(\mathcal{B}_j)\le\frac{c_4}{\epsilon^{p-p\alpha}(\kappa\lambda)^{p'\alpha}}\int_{2\mathcal{B}_j\cap\{G\ge\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu, \tag{3.33}$$

for some constant $c_4=c_4(\mathsf{data}_1)$.

Case 2. If (3.27) occurs, we follow the proof exactly as in [**36**, Subsection 4.2] so that there is a constant $c_5=2\big(\frac{4M(L+1)}{\epsilon^{\frac{1}{p_*+\mathfrak{A}}-\frac{1}{p'}}}\big)^{\frac{p_*+\mathfrak{A}}{1-(p_*+\mathfrak{A})\theta}}$ with $L=\mu(\mathcal{B}_2)=c(N,p,\epsilon)$ such that

$$\begin{aligned}\mu(\mathcal{B}_j)&\le\frac{c_5\lambda_1^{((p_*+\mathfrak{A})+\delta_f)\theta(p_*+\mathfrak{A})/(1-(p_*+\mathfrak{A})\theta)}}{(\hat{\kappa}\kappa\lambda)^{(1+\theta\delta_f)(p_*+\mathfrak{A})/(1-(p_*+\mathfrak{A})\theta)}}\int_{\mathcal{B}_j\cap\{F>\hat{\kappa}\kappa\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu\\ &=\frac{c_5\lambda_1^{\theta_f}}{(\hat{\kappa}\kappa\lambda)^{\tilde{\theta}_f}}\int_{\mathcal{B}_j\cap\{F>\hat{\kappa}\kappa\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu,\end{aligned} \tag{3.34}$$

provided

$$\hat{\kappa}\le\left(\frac{1}{4}\right)^{\frac{1-(p_*+\mathfrak{A})\theta}{p_*+\mathfrak{A}}}\frac{\epsilon^{\frac{1}{p_*+\mathfrak{A}}-\frac{1}{p'}}}{4M(L+1)}. \tag{3.35}$$

Since $\{2\mathcal{B}_j\}$ is a collection of disjoint open sets contained in $\mathcal{B}(x_0,\rho)$, the two estimates in (3.33) and (3.34) imply (3.25).

Step 5: Conclusion. We are now ready to complete the proof. An elementary calculation gives

$$\underbrace{\int_{\mathcal{B}(x_0,r)\cap\{G>\tilde{\kappa}\kappa\lambda\}} G^{p'}\,d\mu}_{I_1}\le\underbrace{\lambda^{p'-p'\alpha}\int_{\mathcal{B}(x_0,r)\cap\{G>\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu}_{I_2}+\underbrace{\int_{\mathcal{B}(x_0,r)\cap\{G>\lambda\}} G^{p'}\,d\mu}_{I_3}.$$

By (3.24), we have

$$I_3\le\underbrace{10^{N+p}\kappa^{p'}\lambda^{p'}\sum_j\mu(\mathcal{B}_j)}_{I_{3,1}}+\underbrace{c\lambda^{p'-p'\alpha}\int_{\mathcal{B}(x_0,\rho)\cap\{G>\kappa\lambda\}} G^{p'\alpha}\,d\mu}_{I_{3,2}},$$

for any $\lambda\ge\lambda_2$. Using (3.25), we have

$$\begin{aligned}I_{3,1}\le&\frac{10^{N+p}c_4(\kappa\lambda)^{p'}}{\epsilon^{p-p\alpha}\kappa^{p'\alpha}\lambda^{p'\alpha}}\int_{\mathcal{B}(x_0,\rho)\cap\{G>\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu\\ &+\frac{10^{N+p}c_5(\kappa\lambda)^{p'}\lambda_1^{\theta_f}}{(\hat{\kappa}\kappa\lambda)^{\tilde{\theta}_f}}\int_{\mathcal{B}(x_0,\rho)\cap\{F>\hat{\kappa}\kappa\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu,\end{aligned}$$

where $c_4=c_4(\mathsf{data}_1)$ and $c_5=c_5(\mathsf{data}_1,\epsilon)$. Note that

$$\frac{\kappa^{p'-p'\alpha}}{\epsilon^{p-p\alpha}}=c\frac{\epsilon^{\frac{1-\alpha}{\alpha}}}{\epsilon^{p(1-\alpha)}}=c\epsilon^{-p+\frac{p}{m}}\ge c, \tag{3.36}$$

where we have used (3.23), (3.4), and $\epsilon \in (0,1)$. Therefore, combining the above estimate with I_3 and using (3.36), we obtain

$$I_1 \leq \frac{c(\kappa\lambda)^{p'}}{\epsilon^{p-p\alpha}\kappa^{p'\alpha}\lambda^{p'\alpha}} \int_{\mathcal{B}(x_0,\rho)\cap\{G>\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu + \frac{10^{N+p}c_5(\kappa\lambda)^{p'}\lambda_1^{\theta_f}}{(\hat{\kappa}\kappa\lambda)^{\tilde{\theta}_f}} \int_{\mathcal{B}(x_0,\rho)\cap\{F>\hat{\kappa}\kappa\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu.$$

After some elementary algebraic manipulations, we observe that

$$I_1 \leq \frac{c(\tilde{\kappa}\kappa\lambda)^{p'-p'\alpha}}{\epsilon^{p-p\alpha}(\tilde{\kappa}\kappa)^{p'-p'\alpha}} \int_{\mathcal{B}(x_0,\rho)\cap\{G>\tilde{\kappa}\kappa\lambda\}} G^{p'\alpha}\,d\mu + \frac{10^{N+p}c_5(\kappa\lambda)^{p'}\lambda_1^{\theta_f}}{(\hat{\kappa}\kappa\lambda)^{\tilde{\theta}_f}} \int_{\mathcal{B}(x_0,\rho)\cap\{F>\hat{\kappa}\kappa\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu, \tag{3.37}$$

whenever $\lambda \geq \lambda_2$. We reformulate estimate (3.37) as follows:

$$\int_{\mathcal{B}(x_0,r)\cap\{G>\lambda\}} G^{p'}\,d\mu \leq \frac{c}{\epsilon^{p-p\alpha}(\tilde{\kappa}\kappa)^{p'-p'\alpha}}\lambda^{p'-p'\alpha} \int_{\mathcal{B}(x_0,\rho)\cap\{G>\lambda\}} G^{p'\alpha}\,d\mu + \frac{c_6(\mathsf{data}_1,\epsilon)\lambda_1^{\theta_f}}{\lambda^{\tilde{\theta}_f-p'}} \int_{\mathcal{B}(x_0,\rho)\cap\{F>\frac{\hat{\kappa}}{\tilde{\kappa}}\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu,$$

provided $\lambda \geq \tilde{\kappa}\kappa\lambda_2$. Now we take a number $\hat{\kappa} > 0$ sufficiently small so that (3.35) and $\kappa_f := \frac{\hat{\kappa}}{\tilde{\kappa}} \leq 1$ hold. Since $\tilde{\kappa}\kappa = \frac{\epsilon^{\frac{1+p}{p'\alpha}-\frac{p}{p'}}}{c}$ for some constant $c = c(\mathsf{data}_1) > 1$ and $\lambda_0 \geq \tilde{\kappa}\kappa\lambda_2$ by (3.19), we conclude that (3.16) holds whenever $\lambda \geq \lambda_0$. ☐

Lemma 3.6. *Let u be a local weak solution to $(\mathcal{P})$. Take $\mathcal{B}(x_0, 2\rho_0) \subset \widetilde{\Omega} \times \widetilde{\Omega}$ with $0 < \rho_0 \leq 1$ and $x_0 \in \widehat{\Omega}$ and write $\mathcal{B} \equiv \mathcal{B}(x_0, \rho_0)$. Then there exist positive constants ϵ, $\delta \in (0,1)$, $\delta_f \in \left(0, \frac{\delta_0}{2}\right)$, and c depending on data_1 and δ_0 such that*

$$\left(\fint_{\frac{1}{2}\mathcal{B}} G(x,y,U)^{p'+\delta}\,d\mu\right)^{\frac{1}{p'+\delta}} \leq c\left(\fint_{2\mathcal{B}} G(x,y,U)^{p'}\,d\mu\right)^{\frac{1}{p'}} + cT(u-(u)_{2\rho_0,x_0}; x_0, 2\rho_0) + c\left(\fint_{2\mathcal{B}} F^{p_*+\mathfrak{A}+\delta_f}\,d\mu\right)^{\frac{1}{p_*+\mathfrak{A}+\delta_f}}.$$

Proof: Let $\frac{\rho_0}{2} < r < \rho < \rho_0$. We now set the parameters δ, δ_f, $\tilde{\delta}$, and ϵ depending only on data_1 and δ_0 such that

$$\frac{c_\alpha\delta}{\epsilon^{\theta_u}(p'-p'\alpha+\delta)} \leq \frac{1}{16} \quad \text{and} \quad \delta < p' - \tilde{\theta}_f + \delta + \tilde{\delta} < \delta_f. \tag{3.38}$$

To this end, we consider the two cases depending on the relationship between N and sp.

Case 1. Assume that $sp \geq N$. Set $\delta_f = \min\{\frac{\delta_0}{2}, \frac{1}{2(p-1)}\}$ and choose

$$\epsilon < \min\left\{\frac{s}{p}, (1-s)\right\} \tag{3.39}$$

such that

$$\frac{1}{p} - \frac{p-1}{2p}\delta_f < \theta, \tag{3.40}$$

which is possible because

$$\theta = \frac{s - \epsilon(p-1)}{N + \epsilon p} < \frac{s}{N}$$

is a decreasing function with respect to ϵ and $sp \geq N$. Since

$$\frac{1}{p-1} - \mathfrak{A} - \frac{\delta_f}{2}(1 + (1+\mathfrak{A})\theta) < \frac{1}{p-1} - \frac{\delta_f}{2} \quad \text{and} \quad p'\theta < p'(1+\mathfrak{A})\theta,$$

by (3.40), we see that

$$\frac{1}{p-1} - \mathfrak{A} - \frac{\delta_f}{2}(1 + (1+\mathfrak{A})\theta) < p'(1+\mathfrak{A})\theta,$$

which can be rewritten as

$$p'(1 - (1+\mathfrak{A})\theta) - (1+\mathfrak{A}) < \frac{\delta_f}{2}(1 + (1+\mathfrak{A})\theta),$$

where we have used the following fact:

$$\frac{1}{p-1} - \mathfrak{A} = p' - (1+\mathfrak{A}).$$

Dividing each side by $(1 - (1+\mathfrak{A})\theta)$, we find that

$$p' - \frac{(1+\mathfrak{A})}{(1-(1+\mathfrak{A})\theta} < \frac{\delta_f}{2}\frac{(1+(1+\mathfrak{A})\theta)}{(1-(1+\mathfrak{A})\theta)} = \frac{\delta_f}{2} + \frac{((1+\mathfrak{A})\theta)}{(1-(1+\mathfrak{A})\theta)}\delta_f,$$

which is equivalent to

$$p' - \tilde{\theta}_f < \frac{\delta_f}{2}. \tag{3.41}$$

Let δ be any nonnegative number satisfying

$$\delta < \min\left\{\frac{spp'\epsilon^{(sp^2+sp+N)/N}}{16c_\alpha(N+sp)}, \frac{1}{8}\delta_f\right\}. \tag{3.42}$$

In light of (3.39) and (3.42), we see that

$$\frac{spp'\epsilon^{(sp^2+sp+N)/N}}{16c_\alpha(N+sp+p)} \leq \frac{s^2p}{(p-1)16c_\alpha(N+sp+p)} \leq \frac{1}{(p-1)16c_\alpha}$$

and

$$2\delta < \delta_f \quad \text{with} \quad \frac{c_\alpha\delta}{\epsilon^{\theta_u}(p' - p'\alpha + \delta)} \leq \frac{1}{16}.$$

Moreover, we note by (3.40) and (3.1) that

$$p' - \tilde{\theta}_f + \theta_f = p' - (p_* + \mathfrak{A}) \geq \frac{1}{2(p-1)} > \delta. \tag{3.43}$$

Combine (3.41) and (3.43) to find a constant $\tilde{\delta} \in [0, \theta_f]$ such that

$$\delta < p' - \tilde{\theta}_f + \delta + \tilde{\delta} < \delta_f.$$

Case 2. Assume that $sp < N$. With an elementary algebraic manipulation as in [**37**, Theorem 5.1], we find that there are constants $\epsilon \in (0,1)$ and $\delta_f \in \big(0, \min\{\frac{1}{2(p-1)}, \frac{\delta_0}{2}\}\big)$ depending on data_1 and δ_0 satisfying the following:

$$\frac{\epsilon p(p')^2}{N+\epsilon p} < \delta_f \le \frac{\epsilon p(N+sp')}{N(s-\epsilon(p-1))}. \tag{3.44}$$

Let δ be any nonnegative number such that

$$\delta \le \min\left\{ \frac{spp'\epsilon^{(sp^2+sp)/N}}{16c_\alpha(N+sp+p)}, \frac{1}{p-1}\frac{\epsilon pp'(N+sp')}{N^2+2N\epsilon p+\epsilon spp'} \right\}, \tag{3.45}$$

where $c_\alpha = c_\alpha(\mathsf{data}_1)$ is determined as in (3.46). By proceeding exactly as in [**37**, Theorem 5.1], we find that the conditions (3.44) and (3.45) imply (3.38) by taking $\tilde{\delta} = 0$.

From the choice of δ, δ_f, $\tilde{\delta}$, and ϵ satisfying (3.38), we now prove the higher integrability of G. We first apply Lemma 3.5 with ϵ, δ_f, and δ which satisfy (3.38). Then we see that there are some constants $c_\alpha = c_\alpha(\mathsf{data}_1) \ge 1$, $c_f = c_f(\mathsf{data}_1, \delta_0) \ge 1$, and $\kappa_f = \kappa_f(\mathsf{data}_1, \delta_0) \in (0,1)$ such that

$$\frac{1}{\lambda^{p'}}\int_{\mathcal{B}_r\cap\{G>\lambda\}} G^{p'}\,d\mu \le \frac{c_\alpha}{\epsilon^{\theta_u}\lambda^{p'\alpha}}\int_{\mathcal{B}_\rho\cap\{G>\lambda\}} G^{p'\alpha}\,d\mu + \frac{c_f\lambda_0^{\theta_f}}{\lambda^{\tilde{\theta_f}}}\int_{\mathcal{B}_\rho\cap\{F>\kappa_f\lambda\}} F^{p_*+\mathfrak{A}}\,d\mu, \tag{3.46}$$

whenever $\lambda \ge \lambda_0$ with

$$\lambda_0 := c_0(\mathsf{data}_1, \delta_0)\left(\frac{\rho_0}{\rho - r}\right)^{2N+p}\Xi_0.$$

Let us define a truncated function $G_m(x,y) = \min\{G(x,y), m\}$ for $(x,y) \in \mathcal{B}_{2\rho_0}$ with $m > \lambda_0$ and a measure $d\nu = G^{p'}\,d\mu$ in $\mathcal{B}_{2\rho_0}$. We then observe that

$$\begin{aligned}
\int_{\mathcal{B}_r} G_m^\delta G^{p'}\,d\mu &= \int_{\mathcal{B}_r} G_m^\delta\,d\nu \\
&= \delta\int_0^\infty \lambda^{\delta-1}\nu(\mathcal{B}_r\cap\{G_m>\lambda\})\,d\lambda \\
&= \delta\int_0^{\lambda_0} \lambda^{\delta-1}\nu(\mathcal{B}_r\cap\{G_m>\lambda\})\,d\lambda + \delta\int_{\lambda_0}^\infty \lambda^{\delta-1}\nu(\mathcal{B}_r\cap\{G_m>\lambda\})\,d\lambda \\
&\le \lambda_0^\delta\int_{\mathcal{B}_r} G^{p'}\,d\mu + \delta\int_{\lambda_0}^\infty \lambda^{\delta-1}\nu(\mathcal{B}_r\cap\{G_m>\lambda\})\,d\lambda \\
&= \underbrace{\lambda_0^\delta\int_{\mathcal{B}_r} G^{p'}\,d\mu}_{I_1} + \underbrace{\delta\int_{\lambda_0}^m \lambda^{\delta-1}\int_{\mathcal{B}_r\cap\{G>\lambda\}} G^{p'}\,d\mu d\lambda}_{I_2},
\end{aligned}$$

where we have used an integral formula of a distribution function of G. We next estimate I_1 and I_2 as follows.

Estimate of I_1. By the definition of λ_0, we find

$$I_1 \le \lambda_0^\delta\mu(\mathcal{B}_{2\rho_0})\fint_{\mathcal{B}_{2\rho_0}} G^{p'}\,d\mu \le c\lambda_0^{p'+\delta}\mu(\mathcal{B}_{2\rho_0}).$$

Estimate of I_2. Using (3.46), we discover that

$$I_2 \le \underbrace{\delta \int_{\lambda_0}^{m} \lambda^{\delta-1} \frac{c_\alpha \lambda^{p'}}{\epsilon^{\theta_u} \lambda^{p'\alpha}} \int_{\mathcal{B}_\rho \cap \{G>\lambda\}} G^{p'\alpha}\, d\mu d\lambda}_{I_{2,1}}$$

$$+ \underbrace{\delta \int_{\lambda_0}^{m} \lambda^{\delta-1} \frac{c_f \lambda_0^{\theta_f} \lambda^{p'}}{\lambda^{\tilde{\theta}_f}} \int_{\mathcal{B}_\rho \cap \{F>\kappa_f \lambda\}} F^{p_*+\mathfrak{A}}\, d\mu\, d\lambda}_{I_{2,2}}.$$

By (3.38), we see that

$$\begin{aligned} I_{2,1} &\le \frac{c_\alpha \delta}{\epsilon^{\theta_u}} \int_0^\infty \lambda^{p'-p'\alpha+\delta-1} \int_{\mathcal{B}_\rho \cap \{G_m>\lambda\}} G^{p'\alpha}\, d\mu\, d\lambda \\ &= \frac{c_\alpha \delta}{\epsilon^{\theta_u}(p'-p'\alpha+\delta)} \int_{\mathcal{B}_\rho} G_m^{\delta+p'-p'\alpha} G^{p'\alpha}\, d\mu \\ &\le \frac{1}{16} \int_{\mathcal{B}_\rho} G_m^{\delta+p'-p'\alpha} G^{p'\alpha}\, d\mu \le \frac{1}{16} \int_{\mathcal{B}_\rho} G_m^{\delta} G^{p'}\, d\mu. \end{aligned}$$

We next estimate $I_{2,2}$ as follows:

$$\begin{aligned} I_{2,2} &\le \lambda_0^{\theta_f-\tilde{\delta}} \delta \int_{\lambda_0}^{m} c_f \lambda^{p'-\tilde{\theta}_f+\delta+\tilde{\delta}-1} \int_{\mathcal{B}_\rho \cap \{F>\kappa_f \lambda\}} F^{p_*+\mathfrak{A}}\, d\mu\, d\lambda \\ &\le c\lambda_0^{\theta_f-\tilde{\delta}} \mu(\mathcal{B}_{2\rho_0}) \fint_{\mathcal{B}_{2\rho_0}} F^{p_*+\mathfrak{A}+\delta+p'-\tilde{\theta}_f+\tilde{\delta}}\, d\mu \\ &\le c\mu(\mathcal{B}_{2\rho_0}) \lambda_0^{\theta_f-\tilde{\delta}} \left(\fint_{\mathcal{B}_{2\rho_0}} F^{p_*+\mathfrak{A}+\delta_f}\, d\mu \right)^{\frac{p_*+\mathfrak{A}+\delta+\tilde{\delta}+p'-\tilde{\theta}_f}{p_*+\mathfrak{A}+\delta_f}} \\ &\le c\mu(\mathcal{B}_{2\rho_0}) \lambda_0^{\theta_f-\tilde{\delta}} \Upsilon_0(x_0, 2\rho_0)^{p_*+\mathfrak{A}+\delta+\tilde{\delta}+p'-\tilde{\theta}_f} \le c(\mathsf{data}_1, \delta_0) \mu(\mathcal{B}_{2\rho_0}) \lambda_0^{p'+\delta}, \end{aligned}$$

where we have used (3.38) with Hölder's inequality, (3.15), and (3.17). We combine estimates I_1 and I_2 to obtain

$$\int_{\mathcal{B}_r} G_m^{\delta} G^{p'}\, d\mu \le \frac{1}{16} \int_{\mathcal{B}_\rho} G_m^{\delta} G^{p'}\, d\mu + c\mu(\mathcal{B}_{2\rho_0}) \lambda_0^{p'+\delta}.$$

Due to the doubling property and (3.15), we discover that

$$\left(\frac{\mu(\mathcal{B}_r)}{\mu(\mathcal{B}_\rho)} \fint_{\mathcal{B}_r} G_m^{\delta} G^{p'}\, d\mu \right)^{\frac{1}{p'+\delta}} \le \left(\frac{1}{16} \fint_{\mathcal{B}_\rho} G_m^{\delta} G^{p'}\, d\mu \right)^{\frac{1}{p'+\delta}} + c\left(\frac{\mu(\mathcal{B}_{2\rho_0})}{\mu(\mathcal{B}_\rho)} \right)^{\frac{1}{p'+\delta}} \lambda_0,$$

and a few elementary manipulations with (3.17) gives

$$\left(\fint_{\mathcal{B}_r} G_m^{\delta} G^{p'}\, d\mu \right)^{\frac{1}{p'+\delta}} \le \frac{1}{2} \left(\fint_{\mathcal{B}_\rho} G_m^{\delta} G^{p'}\, d\mu \right)^{\frac{1}{p'+\delta}} + c\left(\frac{\rho_0}{\rho-r} \right)^{2N+p} \Xi_0.$$

Therefore, we rewrite the above inequality as

$$\varphi(r) \le \frac{1}{2}\varphi(\rho) + c\left(\frac{\rho_0}{\rho-r} \right)^{2N+p} \Xi_0,$$

where we have defined $\varphi(\tau) := \left(\fint_{\mathcal{B}_\tau} G_m^\delta G^{p'}\, d\mu\right)^{\frac{1}{p'+\delta}}$ for $\tau \in \left[\frac{\rho_0}{2}, \rho_0\right]$. By Lemma 2.1, we obtain

$$\left(\fint_{\mathcal{B}_{\frac{\rho_0}{2}}} G_m^\delta G^{p'}\, d\mu\right)^{\frac{1}{p'+\delta}} \le c\Xi_0,$$

where $c = c(\mathsf{data}_1, \delta_0)$ is independent of m. Thus, by taking $m \to \infty$, we conclude that

$$\left(\fint_{\mathcal{B}_{\frac{\rho_0}{2}}} G^{p'+\delta}\, d\mu\right)^{\frac{1}{p'+\delta}} \le c\Xi_0.$$

Recalling the definition of Ξ_0 as stated in (3.15), we complete the proof of the lemma. □

Since we have obtained a higher integrability result for G, we now prove our first main result.

Proof of Theorem 1.1: For any $x_0 \in \widehat{\Omega}$, there is a $\rho_0 \in (0,1]$ such that

$$B_{2\rho_0}(x_0) \subset \widetilde{\Omega}.$$

We now need to normalize the solution u. Define

$$\begin{aligned} &\tilde{u}(x) = u(\rho_0 x + x_0), \quad \tilde{f}(x) = \rho_0^{sp} f(\rho_0 x + x_0), \quad \text{for } x \in B_2, \\ &\tilde{a}(x,y,z,w) = a(\rho_0 x + x_0, \rho_0 y + x_0, z, w), \\ &\tilde{b}(x,y) = \rho_0^{sp-tq} b(\rho_0 x + x_0, \rho_0 y + x_0), \qquad \text{for } x,y \in \mathbb{R}^N \times \mathbb{R}^N. \end{aligned}$$

Then we have

$$\mathcal{L}_{\tilde{a},\tilde{b}} \tilde{u} = \tilde{f} \quad \text{in } B_2,$$

with

$$\begin{aligned} &0 < \Lambda^{-1} \le \tilde{a}(x,y,z,w) \le \Lambda \quad \text{and} \\ &0 \le \tilde{b}(x,y) \le \rho_0^{sp-tq}\Lambda. \end{aligned}$$

For $\mathcal{B} := B_1 \times B_1$, by Lemma 3.6, there are sufficiently small positive numbers δ_1, $\epsilon \in (0,1)$, $\delta_f \in (0, \delta_0)$, and c depending on data_1 and δ_0 such that

$$\begin{aligned} \left(\fint_{\frac{1}{2}\mathcal{B}} G(x,y,\tilde{U})^{p'(1+\delta_1)}\, d\mu\right)^{\frac{1}{p'(1+\delta_1)}} &\le c\left(\fint_{2\mathcal{B}} G(x,y,\tilde{U})^{p'}\, d\mu\right)^{\frac{1}{p'}} \\ &\quad + c\tilde{T}(\tilde{u} - (\tilde{u})_{2,0}; 0, 2\rho_0) \\ &\quad + c\left(\fint_{2\mathcal{B}} \tilde{F}^{p_*+\mathfrak{A}+\delta_f}\, d\mu\right)^{\frac{1}{p_*+\mathfrak{A}+\delta_f}}, \end{aligned} \tag{3.47}$$

and

$$s + \frac{p\epsilon\delta_1}{p(1+\delta_1)} < 1, \tag{3.48}$$

where

$$\tilde{T}(\tilde{u} - (\tilde{u})_{2,0}; 0, 2\rho_0) := \int_{\mathbb{R}^N \setminus B_2} \left(\frac{|\tilde{u}(y) - (\tilde{u})_{2,0}|^{p-1}}{|y|^{N+sp}} + \|\tilde{b}\|_{L^\infty} \frac{|\tilde{u}(y) - (\tilde{u})_{2,0}|^{q-1}}{|y|^{N+tq}}\right) dy.$$

Since $\tilde{u} \in L^\infty(B_2)$ and $tq \le sp$, we see that

$$\int_{2\mathcal{B}} G(x,y,\tilde{U})^{p'}\, d\mu \le c[\tilde{u}]^p_{W^{s,p}(B_2)}. \tag{3.49}$$

From [**33**, Proposition 2.5] with (3.48), we discover that

$$\begin{aligned}[\tilde{u}]^{p-1}_{W^{s+\frac{N\delta}{p(1+\delta)},p(1+\delta)}(B_{1/2})} &\le c[\tilde{u}]^{p-1}_{W^{s+\frac{p\epsilon\delta_1}{p(1+\delta_1)},p(1+\delta_1)}(B_{1/2})} \\ &\le c\left(\int_{\frac{1}{2}\mathcal{B}} G(x,y,\tilde{U})^{p'(1+\delta_1)}\, d\mu\right)^{\frac{1}{p'(1+\delta_1)}},\end{aligned} \tag{3.50}$$

where $\delta = \delta(\mathsf{data}_1, \delta_0)$ is a sufficiently small positive number such that

$$\frac{N\delta}{p(1+\delta)} < \frac{p\epsilon\delta_1}{p(1+\delta_1)} \quad \text{and} \quad p\delta \le p\delta_1.$$

We combine the estimates (3.47), (3.49), and (3.50) to obtain

$$\begin{aligned}&\left(\int_{B_{\frac{1}{2}}}\int_{B_{\frac{1}{2}}} \left(\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}}\right)^{(1+\delta)} dx\, dy\right)^{\frac{1}{p'(1+\delta)}} \\ &\quad \le c\left[[\tilde{u}]^{p-1}_{W^{s,p}(B_2)} + \tilde{T}(\tilde{u}-(\tilde{u})_{2,0};0,2) + \left(\int_{B_2} |\tilde{f}(x)|^{p_*+\mathfrak{A}+\delta_f}\, dx\right)^{\frac{1}{p_*+\mathfrak{A}+\delta_f}}\right].\end{aligned}$$

By scaling back, noting $u \in L^\infty(B_{2\rho_0}(x_0))$ and using Hölder's inequality with $\mathfrak{A}+\delta_f < \delta_0$, we conclude the estimate (1.5). Finally, the standard covering argument gives that $u \in W^{s+\frac{N\delta}{p(1+\delta)},p(1+\delta)}_{\mathrm{loc}}(\widetilde{\Omega})$. □

4. The Hölder continuity

We first focus on a local weak solution

$$u \in \mathcal{W}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt}(\mathbb{R}^N)$$

to

$$\mathcal{L}_{a,b}\, u = f \quad \text{in } \Omega, \tag{$\mathcal{PA}$}$$

where the coefficient function a is a VMO function and is independent of the solution u, and $f \in L^\gamma_{\mathrm{loc}}(\Omega)$ with $\gamma > \max\{1, \frac{N}{ps}\}$. Then, from [**22**, Theorem 4.5] and using the Caccioppoli-type estimate of Lemma 3.1 (to control the quantity $[u]_{W^{s,p}}$, appearing there), we can get the following continuity result.

Lemma 4.1. *Suppose that $2 \le p \le q \le ps/t$ and that the functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are locally translation invariant in $\Omega \times \Omega$. Let u be a local weak solution to the problem ($\mathcal{PA}$) with $f \equiv 0$. Then $u \in C^\alpha_{\mathrm{loc}}(\Omega)$ for all $\alpha \in (0,\Theta_0)$, where $\Theta_0 := \min\{\frac{ps}{p-1}, 1\}$.*

More precisely, for $B_{2\rho_0} \equiv B_{2\rho_0}(x_0) \Subset \Omega$ with $\rho_0 \in (0,1]$ and for all $\alpha \in (0,\Theta_0)$, there exists a positive constant c depending only on data and α such that

$$[u]_{C^\alpha(B_{\rho_0/4})} \le \frac{c}{\rho_0^\alpha}[\|u\|_{L^\infty(B_{\rho_0/2})} + 1 + T_{ps}(u;x_0,\rho_0/2) + T_{qt}(u;x_0,\rho_0/2)]^{\beta(q-p)+1},$$

where $\beta \in \mathbb{N}$ depends only on N, p, s, and α.

Concerning the case when the coefficients need not be locally translation invariant, we have the following approximation lemma.

Lemma 4.2. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\mathsf{data}, \epsilon) > 0$ such that for any weak solution u to $(\mathcal{PA})$ in $B_4 \equiv B_4(0)$ with*

$$\sup_{B_4} |u| \le 1, \quad T_{ps}(u; 0, 4) + T_{qt}(u; 0, 4) \le 1$$

and

$$\left(\fint_{B_4} |f|^\gamma \, dx \right)^{1/\gamma} + \fint_{B_4} \fint_{B_4} (|a(x,y) - (a)_{4,0}| + |b(x,y) - (b)_{4,0}|) \, dx \, dy \le \delta,$$

there exists a weak solution v to

$$\begin{cases} \mathcal{L}_{\tilde{a},\tilde{b}} v = 0 & \text{in } B_2, \\ v = u & \text{in } \mathbb{R}^N \setminus B_2, \end{cases} \tag{4.1}$$

such that

$$\|u - v\|_{L^\infty(B_1)} \le \epsilon,$$

where

$$\tilde{a}(x,y) = \begin{cases} (a)_{4,0} & \text{if } (x,y) \in B_4 \times B_4, \\ a(x,y) & \text{otherwise}, \end{cases} \quad \text{and} \quad \tilde{b}(x,y) = \begin{cases} (b)_{4,0} & \text{if } (x,y) \in B_4 \times B_4, \\ b(x,y) & \text{otherwise}. \end{cases} \tag{4.2}$$

Proof: The existence of a weak solution v to (4.1) is given by Theorem 5.1 below. To prove the claim, we proceed by the method of contradiction. Suppose there exist $\epsilon_0 > 0$ and sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, $\{f_k\}_{k\in\mathbb{N}}$, and $\{u_k\}_{k\in\mathbb{N}}$ such that

$$\mathcal{L}_{a_k,b_k} u_k = f_k \quad \text{in } B_4 \tag{4.3}$$

with

$$\sup_{B_4} |u_k| \le 1, \quad T_{ps}(u_k; 0, 4) + T_{qt}(u_k; 0, 4) \le 1 \tag{4.4}$$

and

$$\left(\fint_{B_4} |f_k|^\gamma \, dx \right)^{1/\gamma} + \fint_{B_4} \fint_{B_4} (|a_k(x,y) - (a_k)_{4,0}| + |b_k(x,y) - (b_k)_{4,0}|) \, dx \, dy \le \frac{1}{k},$$

but for any weak solution v_k to

$$\begin{cases} \mathcal{L}_{\tilde{a}_k,\tilde{b}_k} v_k = 0 & \text{in } B_2, \\ v_k = u_k & \text{in } \mathbb{R}^N \setminus B_2, \end{cases} \tag{4.5}$$

there holds

$$\|u_k - v_k\|_{L^\infty(B_1)} > \epsilon_0. \tag{4.6}$$

Set $w_k := u_k - v_k$. Then, from Lemmas A.1 and A.2, we see that $v_k \in L^\infty(B_2)$ and hence $w_k \in L^\infty(B_4)$. On account of [**22**, Lemma 5.1], we check that w_k is a well-defined test function to the weak formulation of problems (4.3) and (4.5). We next claim that

$$\int_{B_{3/2}} |w_k(x)|^{p_s^*} \, dx \to 0 \quad \text{as } k \to \infty. \tag{4.7}$$

Testing w_k to (4.3) and (4.5), we see that

$$I_0 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{a}_k(x,y)([u_k(x)-u_k(y)]^{p-1}-[v_k(x)-v_k(y)]^{p-1})(w_k(x)-w_k(y))\,d\mu_1$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{b}_k(x,y)([u_k(x)-u_k(y)]^{q-1}-[v_k(x)-v_k(y)]^{q-1})(w_k(x)-w_k(y))\,d\mu_2$$
$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\tilde{a}_k(x,y)-a_k(x,y))[u_k(x)-u_k(y)]^{p-1}(w_k(x)-w_k(y))\,d\mu_1$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\tilde{b}_k(x,y)-b_k(x,y))[u_k(x)-u_k(y)]^{q-1}(w_k(x)-w_k(y))\,d\mu_2$$
$$+\int_{B_4} f_k w_k\,dx =: I_1+I_2.$$

Now we estimate each I_i for $i = 0$, 1, 2, and 3.

Estimate of I_0. Using (2.3), we see that

$$I_0 \geq \frac{1}{\Lambda}[w_k]^p_{W^{s,p}(\mathbb{R}^N)}.$$

Estimate of I_2. We first note that there exists a constant$c = c(q)$ such that

$$I_2 = \int_{B_4}\int_{B_4} (\tilde{b}_k(x,y)-b_k(x,y))[u_k(x)-u_k(y)]^{q-1}(w_k(x)-w_k(y))\,d\mu_2$$
$$\leq c\int_{B_4}\int_{B_4} |\tilde{b}_k(x,y)-b_k(x,y)||u_k(x)-u_k(y)|^{p-1}|w_k(x)-w_k(y)|\,d\mu_1,$$

where we have used the fact that $tq \leq ps$ and (4.4). In addition, using Hölder's inequality, (1.2), Theorem 1.1, and Young's inequality, we find that there is a constant $c = c(\mathsf{data})$ which is independent of k such that

$$I_2 \leq c\left(\fint_{B_4}\fint_{B_4} |\tilde{b}_k(x,y)-b_k(x,y)||u_k(x)-u_k(y)|^p\,d\mu_1\right)^{(p-1)/p}[w_k]_{W^{s,p}(B_4)}$$
$$\leq c\left(\fint_{B_4}\fint_{B_4} |\tilde{b}_k(x,y)-b_k(x,y)|\,dx\,dy\right)^{\frac{\sigma(p-1)}{p(1+\sigma)}}$$
$$\times\left(\fint_{B_4}\fint_{B_4}\left(\frac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+ps}}\right)^{(1+\sigma)}dx\,dy\right)^{\frac{(p-1)}{p(1+\sigma)}}[w_k]_{W^{s,p}(B_4)}$$
$$\leq c\left(\frac{1}{k}\right)^{\frac{\sigma}{1+\sigma}}+\frac{I_0}{16},$$

where we have chosen a sufficiently small $\sigma > 0$ so that Theorem 1.1 holds. Likewise, we have

$$I_1 \leq c\left(\frac{1}{k}\right)^{\frac{\sigma}{1+\sigma}}+\frac{I_0}{16}.$$

Estimate of I_3. We use Hölder's inequality, Young's inequality, and the Sobolev–Poincaré inequality to discover that

$$I_3 \le \|f_k\|_{L^\gamma(B_4)}\|w_k\|_{L^{\gamma'}(B_4)} \le c\|f_k\|^{p'}_{L^\gamma(B_4)} + \frac{I_0}{16}.$$

Combining all estimates I_0, I_1, and I_2, we have

$$[w_k]^p_{W^{s,p}(\mathbb{R}^N)} \le c\left(\frac{1}{k}\right)^{\frac{\sigma}{1+\sigma}}, \tag{4.8}$$

where c is independent of k. Therefore the claim (4.7) follows by the Sobolev–Poincaré inequality and (4.8). Moreover, u_k and v_k are Hölder continuous in B_2 with uniform bound independent of k as in [**22**, Lemma 5.1]. By the Arzelà–Ascoli theorem, there is a function w such that $w_k \to w$ in $C^\beta(B_{3/2})$, up to a subsequence, for some $\beta \in (0,1)$. By the uniqueness of the limit together with (4.7), we have that

$$\lim_{k\to\infty} \|u_k - v_k\|_{L^\infty(B_{3/2})} = 0,$$

which is a contradiction to (4.6). □

Lemma 4.3. *Let u be a weak solution to $(\mathcal{PA})$ in $B_4 \equiv B_4(0)$ with*

$$\sup_{B_4} |u| \le 1 \quad \textit{and} \quad T_{ps}(u;0,4) + T_{qt}(u;0,4) \le 1. \tag{4.9}$$

Given $\alpha \in (0,\Theta)$, where Θ is given by (1.6), there exists a small constant $\delta = \delta(\mathsf{data},\alpha) > 0$ such that if kernel coefficients a and b are $(\delta,4)$-vanishing in $B_4 \times B_4$ and

$$\left(\fint_{B_4} |f|^\gamma \, dx\right)^{1/\gamma} \le \delta,$$

then $u \in C^\alpha(B_1)$ with the estimate

$$[u]_{C^\alpha(B_1)} \le c$$

for some constant $c = c(\mathsf{data},\alpha)$.

Proof: Let $\alpha \in (0,\Theta)$ be fixed. We now show that for any $x \in B_1$ there is a constant $A^x \in \mathbb{R}$ such that

$$\sup_{y\in B_r(x)} |u(y) - A^x| \le cr^\alpha,$$

for any $r \in (0,1]$ and for some constant $c = c(\mathsf{data},\alpha)$. Using a translation argument as in [**31**, Proposition 4.2], it suffices to prove the case for $x = 0$. To this end, we show the following claim.

Claim. *There exist $\rho = \rho(\mathsf{data},\alpha) \in (0,1/4)$ and a sequence $\{A_k\}_{k=-1}^\infty$ with $A_{-1} = 0$ such that for all $k \ge 0$,*

$$|A_k - A_{k-1}| \le 2\rho^{(k-1)\alpha}, \quad \sup_{B_4} |u(\rho^k x) - A_k| \le \rho^{k\alpha}, \tag{4.10}$$

and

$$T_{ps}\left(\left(\frac{u(\rho^k x) - A_k}{\rho^{k\alpha}}\right);0,4\right) + T_{qt}\left(\left(\frac{u(\rho^k x) - A_k}{\rho^{k\alpha}}\right);0,4\right) \le 1. \tag{4.11}$$

To prove the claim, we take $\rho > 0$ sufficiently small depending only on data and $\alpha > 0$ such that

$$\rho^{\frac{\Theta-\alpha}{2}} \le \frac{1}{12^{\Theta_0+2}c_1c_2} \min\left\{\left[sp - \left(\frac{\Theta_0+\alpha}{2}\right)\right]^{\frac{1}{p-1}}, 1\right\}, \tag{4.12}$$

where $c_1 = c_1(\mathsf{data}) \geq 1$ and $c_2 = c_2(\mathsf{data}) \geq 1$ are constants which are determined later. For $k = 0$, we take $A_0 = 0$; then (4.10) and (4.11) hold by (4.9). Suppose that (4.10) and (4.11) hold for $k = 0, 1, \ldots, i$. Set

$$u_i(x) = \frac{u(\rho^i x) - A_i}{\rho^{\alpha i}}, \quad f_i(x) = \rho^{(sp-\alpha(p-1))i} f(\rho^i x), \quad x \in \mathbb{R}^N;$$

$$a_i(x,y) = a(\rho^i x, \rho^i y) \quad \text{and} \quad b_i(x,y) = b(\rho^i x, \rho^i y)\rho^{(sp-tq+\alpha(q-p))i}, \quad (x,y) \in \mathbb{R}^{2N}.$$

Then u_i is a weak solution to

$$\mathcal{L}_{a_i,b_i} u_i = f_i, \quad \text{in } B_4.$$

By the inductive assumption, we have

$$\sup_{B_4} |u_i| \leq 1 \quad \text{and} \quad T_{ps}(u_i; 0, 4) + T_{qt}(u_i; 0, 4) \leq 1.$$

Since $\rho < 1$, we notice that

$$\Lambda^{-1} \leq a_i \leq \Lambda \quad \text{and} \quad 0 \leq b_i \leq \Lambda.$$

By Lemma 4.2, we find $\delta_0 = \delta_0(\mathsf{data}, \epsilon)$, corresponding to the given

$$\epsilon = \frac{\rho^\alpha}{16 c_2}. \tag{4.13}$$

Taking $\delta = \frac{\delta_0}{3}$, we see that a_i and b_i are $(\delta, 4)$-vanishing in $B_4 \times B_4$ because a and b are $(\delta, 4)$-vanishing in $B_4 \times B_4$. Therefore, we check that

$$\left(\fint_{B_4} |f_i|^\gamma \, dx \right)^{1/\gamma} + \fint_{B_4} \fint_{B_4} (|a_i(x,y) - (a_i)_{4,0}| + |b_i(x,y) - (b_i)_{4,0}|) \, dx \, dy \leq \delta_0.$$

By Lemma 4.2, there exists a weak solution v_i to the following problem:

$$\begin{cases} \mathcal{L}_{\tilde{a}_i, \tilde{b}_i} v_i = 0 & \text{in } B_2, \\ v_i = u_i & \text{in } \mathbb{R}^N \setminus B_2, \end{cases}$$

such that

$$\|u_i - v_i\|_{L^\infty(B_1)} \leq \epsilon, \tag{4.14}$$

where $\tilde{a}_i$ and $\tilde{b}_i$ are defined as in (4.2), corresponding to a_i and b_i, respectively. Before checking the assumptions (4.10) and (4.11), we specify the constants c_1 and c_2.

1. Constant c_1. We first note that there is a $c = c(\mathsf{data})$ independent of i such that

$$\|v_i\|_{L^{p_s^*}(B_2)} \leq c, \tag{4.15}$$

by following the proof in Lemma 4.2 with (4.9). From (4.15) and (4.14), we see that

$$T_{ps}(v_i; 0, 2\rho) \leq c(\|v_i\|_{L^\infty(B_{3/2})} + \|v_i\|_{L^p(B_2)} T_{ps}(u; 0, 2)) \leq c, \tag{4.16}$$

where $c = c(\mathsf{data})$. In light of Lemma 4.1 and (4.16), there exists a constant $c_1 = c_1(\mathsf{data}) \geq 1$ which is independent of i such that

$$[v_i]_{C^{\tilde{\alpha}}(B_1)} \leq c_1, \tag{4.17}$$

where $\tilde{\alpha} = (\Theta_0 + \alpha)/2 < 1$.

2. Constant c_2. Set

$$c_2 = \max\{1, T_{ps}(1; x_0, R) + T_{qt}(1; x_0, R)\}, \quad \text{for } R > 0 \quad \text{and} \quad x_0 \in \mathbb{R}^N. \tag{4.18}$$

Then we find that $c_2 = c_2(\mathsf{data}) \geq 1$ and it is independent of R and x_0.

Let $A_{i+1} = A_i + \rho^{i\alpha} v_i(0)$. We now check the inductive assumptions (4.10) and (4.11) for $i = k+1$. We first note that (4.14) also implies that

$$|A_{i+1} - A_i| \le \rho^{i\alpha}|v_i(0)| \le 2\rho^{i\alpha}. \tag{4.19}$$

In addition, by (4.12), (4.14), and (4.17), we see that

$$\begin{aligned}\sup_{B_4} |u(\rho^{i+1}x) - A_{i+1}| &= \sup_{B_{4\rho}} |u(\rho^i x) - A_i - \rho^{i\alpha} v_i(0)| \\ &\le \rho^{i\alpha} \sup_{B_{4\rho}} |u_i(x) - v_i(x)| + \rho^{i\alpha} \sup_{B_{4\rho}} |v_i(x) - v_i(0)| \\ &\le \frac{\rho^{(i+1)\alpha}}{16} + c_1 (4\rho)^{\tilde{\alpha}} \rho^{i\alpha} \le \rho^{(i+1)\alpha},\end{aligned}$$

where we have used $\rho \in (0, 1/4)$. Thus, we have shown that (4.10) holds for $k = i+1$. Moreover, we observe that

$$\begin{aligned} J_{s,p} &:= \left((4\rho^{i+1})^{sp} \int_{B_{\rho^i} \setminus B_{4\rho^{i+1}}} \frac{|u(x) - A_{i+1}|^{p-1}}{\rho^{(i+1)\alpha(p-1)}|x|^{N+sp}}\, dx \right)^{\frac{1}{p-1}} \\ &\le \left((4\rho)^{sp} \int_{B_1 \setminus B_{4\rho}} \frac{|u(\rho^i x) - (A_i + v_i(x)\rho^{i\alpha})|^{p-1}}{\rho^{(i+1)\alpha(p-1)}|x|^{N+sp}}\, dx \right)^{\frac{1}{p-1}} \\ &\quad + \left((4\rho)^{sp} \int_{B_1 \setminus B_{4\rho}} \frac{|v_i(x) - v_i(0)|^{p-1}}{\rho^{\alpha(p-1)}|x|^{N+sp}}\, dx \right)^{\frac{1}{p-1}} \\ &\le c_2 \frac{\|u_i - v_i\|_{L^\infty(B_1)}}{\rho^\alpha} + c_1 \left((4\rho)^{sp} \int_{B_1 \setminus B_{4\rho}} \frac{dx}{\rho^{\alpha(p-1)}|x|^{N+sp-\tilde{\alpha}(p-1)}} \right)^{\frac{1}{p-1}} \\ &\le c_2 \frac{\epsilon}{\rho^\alpha} + \frac{c_2 c_1 4^{\tilde{\alpha}}}{(sp - \tilde{\alpha}(p-1))^{\frac{1}{p-1}}} \rho^{\frac{\Theta - \alpha}{2}} \le \frac{1}{8}, \end{aligned} \tag{4.20}$$

where we have used (4.14), (4.17), (4.13), and (4.12). Similarly, we deduce that

$$J_{t,q} \le \frac{1}{8}. \tag{4.21}$$

Consequently, using (4.20), (4.21), and (4.18), we obtain

$$\begin{aligned} \sum_l T_l \left(\left(\frac{u(\rho^{i+1}x) - A_{i+1}}{\rho^{\alpha(i+1)}} \right); 0, 4 \right) &= \sum_l T_l \left(\left(\frac{u(x) - A_{i+1}}{\rho^{\alpha(i+1)}} \right); 0, 4\rho^{i+1} \right) \\ &\le \sum_l (4\rho)^{\Theta_0} T_l \left(\left(\frac{u(x) - A_{i+1}}{\rho^{\alpha(i+1)}} \right); 0, \rho^i \right) + J_{s,p} + J_{t,q} \\ &\le \sum_l (4\rho)^{\Theta_0} T_l \left(\left(\frac{u(x) - A_{i+1}}{\rho^{\alpha(i+1)}} \right); 0, \rho^i \right) + \frac{1}{4} \end{aligned}$$

for $l \in \{ps, qt\}$. With the help of (4.18), (4.19), (4.10), and (4.11) for $k = i$, we further estimate

$$\begin{aligned}
\sum_l (4\rho)^{\Theta_0} T_l\left(\left(\frac{u(x) - A_{i+1}}{\rho^{\alpha(i+1)}}\right); 0, \rho^i\right) \\
&\leq 4^{\Theta_0}\left[\sum_l \rho^{\Theta_0} T_l\left(\left(\frac{u(x) - A_i}{\rho^{\alpha(i+1)}}\right); 0, 4\rho^i\right) + \sum_l \rho^{\Theta_0} T_l\left(\frac{1}{\rho^\alpha}; 0, 4\rho^i\right)\right. \\
&\qquad \left. + \sum_l \rho^{\Theta_0} T_l\left(\left(\frac{A_i - A_{i+1}}{\rho^{\alpha(i+1)}}\right); 0, \rho^i\right)\right] \\
&\leq 4^{\Theta_0}\left[\sum_l \rho^{\Theta_0 - \alpha} T_l\left(\left(\frac{u(x) - A_i}{\rho^{\alpha i}}\right); 0, 4\rho^i\right) + \rho^{\Theta_0 - \alpha} 3c_2\right] \\
&\leq 4^{\Theta_0}[4c_2\rho^{\Theta_0 - \alpha}] \leq \frac{1}{4}.
\end{aligned}$$

It gives that (4.11) holds for $k = i + 1$, hence the claim follows. Thus, from the claim with simple computations (see [**8**]), we see that

$$\lim_{i\to\infty} A_i = A < +\infty.$$

In addition, for any $r \in (0, 1]$, there is a constant $c = c(\mathsf{data}, \alpha)$ such that

$$\begin{aligned}
\|u(x) - A\|_{L^\infty(B_r)} = \|u(x) - A_j\|_{L^\infty(B_r)} + |A - A_j| &\leq \rho^{j\alpha} + \sum_{k=j}^{\infty} 2\rho^{k\alpha} \\
&\leq c\rho^{j\alpha} \leq cr^\alpha,
\end{aligned}$$

where j is the unique nonnegative number satisfying $\rho^{j+1} < r \leq \rho^j$. □

Lemma 4.4. *Let u be a local weak solution to ($\mathcal{PA}$) and let the functions a and b be in VMO. Then for any $\alpha \in (0, \Theta)$, $u \in C^\alpha_{\mathrm{loc}}(\Omega)$.*

Proof: Let $\alpha \in (0, \Theta_0)$ be fixed and let $\delta = \delta(\mathsf{data}, \alpha)$ be as obtained in Lemma 4.3. Suppose $B_{\rho_0}(x_0) \Subset \Omega$. It suffices to show that $u \in C^\alpha(\overline{B_{\rho_0}(x_0)})$. Set

$$R := \mathrm{dist}(B_{\rho_0}(x_0), \partial\Omega), \quad R_0 := \rho_0 + R/2, \tag{4.22}$$

and

$$\begin{aligned}
\mathcal{M} := 8c_2\Bigg[\|u\|_{L^\infty(B_{R_0}(x_0))} &+ T_{ps}(u; x_0, R_0) + T_{qt}(u; x_0, R_0) \\
&+ \left(\frac{R_0^{sp - \frac{n}{\gamma}}\|f\|_{L^\gamma(B_{R_0}(x_0))}}{\delta}\right)^{\frac{1}{p-1}} + 1\Bigg] \times \left(\frac{2R_0}{R}\right)^{\frac{N+sp}{p-1}},
\end{aligned}$$

where c_2 is given as in (4.18) of Lemma 4.3. Then we find that there is a constant

$$\rho \in \left(0, \min\left\{\frac{\rho_0}{4}, \frac{R}{4}\right\}\right) \tag{4.23}$$

depending only on data, $\mathcal{M}$, ν_a, and ν_b such that

$$\mathcal{M}^{q-p}\left(\frac{\rho}{4}\right)^{sp-tq} \leq 1$$

(this is possible because of the condition $ps > qt$) and the kernel coefficients a and b are (δ,ρ)-vanishing in $B_{\rho_0}(x_0)\times B_{\rho_0}(x_0)$. We further note that

$$B_\rho(z)\subset B_{R_0}(x_0),\quad \text{for every } z\in B_{\rho_0}(x_0).$$

We define, for any $z\in B_{\rho_0}(x_0)$,

$$u_z(x)=\frac{u\big(\frac{\rho}{4}x+z\big)}{\mathcal{M}},\quad f_z(x)=\left(\frac{\rho}{4}\right)^{sp}\frac{1}{\mathcal{M}^{p-1}}f\left(\frac{\rho}{4}x+z\right),\quad x\in B_4,$$

and

$$a_z(x,y)=a\left(\frac{\rho}{4}x+z,\frac{\rho}{4}y+z\right),\ b_z(x,y)=\mathcal{M}^{q-p}\left(\frac{\rho}{4}\right)^{sp-tq}b\left(\frac{\rho}{4}x+z,\frac{\rho}{4}y+z\right),\ (x,y)\in\mathbb{R}^{2N}.$$

Then we directly see that

$$\mathcal{L}_{a_z,b_z}u_z=f_z,\quad \text{in } B_4(0)$$

with

$$\sup_{B_4}|u_z|\le 1\quad\text{and}\quad\left(\fint_{B_4}|f|^\gamma\,dx\right)^{1/\gamma}\le\delta.$$

On the other hand, for $l\in\{ps,qt\}$, we note that

$$\begin{aligned}\sum_l T_l(u_z;0,4)=\frac{1}{\mathcal{M}}\sum_l T_l(u;z,\rho)&\le\left(\frac{2R_0}{R}\right)^{\frac{N+sp}{p-1}}\frac{1}{\mathcal{M}}\sum_l T_l(u;x_0,R_0)\\&\quad+\frac{1}{\mathcal{M}}\sum_l T_l\left(\left(\frac{R}{2R_0}\right)^{\frac{N+sp}{p-1}}\frac{\mathcal{M}}{8c_2};z,\rho\right)\\&\le\frac{1}{8c_2}+\frac{1}{4}\le 1,\end{aligned}$$

where we have used (4.23), (4.22), and the fact that

$$|y-z|\ge|y-x_0|-|x_0-z|\ge|y-x_0|-\frac{\rho_0}{R_0}|y-x_0|\ge\frac{R}{2R_0}|y-x_0|,\quad y\in B_{R_0}(x_0)^c.$$

Moreover, the kernel coefficients a_z and b_z are $(\delta,4)$-vanishing in $B_4\times B_4$ and the following holds:

$$\Lambda^{-1}\le a_z\le\Lambda\quad\text{and}\quad 0\le b_z\le\Lambda.$$

By Lemma 4.3, $u_z\in C^\alpha(\overline{B_1})$. Scaling it back, we obtain $u\in C^\alpha(\overline{B_\rho(z)})$ for any $z\in\overline{B_{\rho_0}(x_0)}$. Using the standard covering argument as in [**31**, Theorem 4.3], we have the desired result. □

Now we return to our original problem; that is, the coefficient function a has the form $a(x,y,u(x),u(y))$, where u is a solution under consideration.

Lemma 4.5. *For a weak solution $u\in C^\sigma_{\mathrm{loc}}(\Omega)$ to $(\mathcal{P})$, for some $\sigma\in(0,1)$, the coefficient function $a(x,y,u(x),u(y))$ is in VMO on $B_\rho(x_0)\times B_\rho(y_0)$ for any $x_0,y_0\in\mathbb{R}^N$ and $\rho>0$ satisfying $B_\rho(x_0),B_\rho(y_0)\Subset\Omega$.*

Proof: Fix $x_0, y_0 \in \Omega$ and $\rho > 0$ such that $B_\rho(x_0), B_\rho(y_0) \Subset \Omega$. Then, for all $r < \rho$, using the continuity and VMO properties, we have

$$\begin{aligned}
&\fint_{B_r(x_0)} \fint_{B_r(y_0)} \left| a(x,y,u(x),u(y)) - \fint_{B_r(x_0)} \fint_{B_r(y_0)} a(x',y',u(x'),u(y'))\, dx'\, dy' \right| dx\, dy \\
&\quad \le 2 \fint_{B_r(x_0)} \fint_{B_r(y_0)} |a(x,y,u(x),u(y)) - a(x,y,u(x_0),u(y_0))|\, dx\, dy \\
&\qquad + \fint_{B_r(x_0)} \fint_{B_r(y_0)} |a(x,y,u(x_0),u(y_0)) - (a)_{r,x_0,y_0}(u(x_0),u(y_0))|\, dx\, dy \\
&\quad \le 2 \fint_{B_r(x_0)} \fint_{B_r(y_0)} \omega_{a,M}\left(\frac{|u(x)-u(x_0)| + |u(y)-u(y_0)|}{2} \right) dx\, dy + \nu_{a,M}(\rho),
\end{aligned}$$

where $M = 2\max\{\|u\|_{L^\infty(B_\rho(x_0))}, \|u\|_{L^\infty(B_\rho(y_0))}\}$, $\omega_{a,M}$ is given by property (A3), and $\nu_{a,M}$ by (2.2). The right-hand side terms converge to 0 as $\rho \to 0$ due to the assumption (A3) and the VMO condition of Definition 2.2. This proves the lemma. □

Proof of Theorem 1.2: Let $B_{\rho_0}(x_0) \Subset \Omega$ and $\alpha \in (0, \Theta_0)$ be fixed. It suffices to show that $u \in C^\alpha(\overline{B_{\rho_0}(x_0)})$. Set

$$R := \operatorname{dist}(B_{\rho_0}(x_0), \partial\Omega) \quad \text{and} \quad R_0 := \rho_0 + R/2.$$

In light of Lemma 4.5 and simple computations, we see that $A(x,y) := a(x,y,u(x),u(y))$ is in VMO on $B_{R_0}(x_0) \times B_{R_0}(x_0)$, symmetric and satisfies (1.1). Since u solves

$$\mathcal{L}_{A,b}\, u = f \quad \text{in } B_{R_0}(x_0),$$

where $\Lambda^{-1} \le A \le \Lambda$, it gives that $u \in C^\sigma_{\mathrm{loc}}(\Omega)$ for some $\sigma = \sigma(\mathsf{data}) \in (0,1)$. By Lemmas 4.4 and 4.5, the result follows. □

5. The existence result

This section provides the solvability of the following Dirichlet problem:

$$(\mathcal{G}) \qquad \begin{cases} \mathcal{L}_{a(\cdot,u),b}\, u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, and f and g are suitable measurable functions. With $g \in L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt,b}(\mathbb{R}^N)$ and $\Omega \Subset \Omega' \Subset \mathbb{R}^N$, we define

$$X_{g,b}(\Omega,\Omega') := \{v \in \mathcal{W}_b(\Omega') \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt,b}(\mathbb{R}^N) : v = g \quad \text{a.e. in } \mathbb{R}^N \setminus \Omega\},$$

equipped with the norm of $\mathcal{W}_b(\Omega')$. Once again, we will suppress the term b from the above definition whenever it is clear in the context. Now we define the notion of a weak solution to $(\mathcal{G})$ as usual.

Definition 5.1. Let $f \in (\mathcal{W}(\Omega'))^*$ and $g \in \mathcal{W}(\Omega') \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt,b}(\mathbb{R}^N)$, for $\Omega \Subset \Omega' \Subset \mathbb{R}^N$. A function $u \in X_g(\Omega,\Omega')$ is said to be a weak solution of the problem $(\mathcal{G})$, if for all $\phi \in X_0(\Omega,\Omega')$,

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(a(x,y,u(x),u(y)) \frac{[u(x)-u(y)]^{p-1}}{|x-y|^{N+ps}} + b(x,y) \frac{[u(x)-u(y)]^{q-1}}{|x-y|^{N+qt}} \right) (\phi(x)-\phi(y))\, dx\, dy \\
= \langle f, \phi \rangle_{\mathcal{W},\mathcal{W}^*}.
\end{aligned}$$

To prove our existence result, we consider the case when the kernel coefficient $a(\cdot,\cdot,\cdot,\cdot)$ satisfies a global uniform continuity condition (stronger than (A3)), namely

(A3)' the function a is uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$; that is, there is a nondecreasing function $\omega_a\colon [0,\infty) \to [0,\infty)$ with $\omega_a(0) = 0$ and $\lim_{t\downarrow 0} \omega_a(t) = 0$ such that

$$|a(x,y,w,z) - a(x,y,w',z')| \le \omega_a\left(\frac{|w-w'| + |z-z'|}{2}\right) \tag{5.1}$$

for all $z, z', w, w' \in \mathbb{R}$ uniformly in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Theorem 5.1. *Suppose that $2 \le p \le q < \infty$, $s, t \in (0,1)$, and that the coefficients satisfy the assumptions* (A1), (A2), *and* (A3)'. *Let $f \in (\mathcal{W}(\Omega'))^*$ and $g \in \mathcal{W}(\Omega') \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt,b}(\mathbb{R}^N)$, for $\Omega \Subset \Omega' \Subset \mathbb{R}^N$. Then, there exists a weak solution $u \in X_g(\Omega, \Omega')$ to the problem $(\mathcal{G})$. In particular, if $g \in \mathcal{W}(\Omega') \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt}(\mathbb{R}^N)$ and $q \le p^*_s$, then*

$$u \in \mathcal{W}(\Omega') \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt}(\mathbb{R}^N).$$

Proof: We see that, as in the proof of [**4**, Lemma 2.11], the space $X_0(\Omega, \Omega')$ is continuously embedded into $\mathcal{W}(\Omega')$. Moreover, we can directly verify that $X_0(\Omega, \Omega')$ is a separable uniformly convex Banach space. We now define a functional $\mathcal{A}\colon X_0(\Omega,\Omega') \to (\mathcal{W}(\Omega'))^*$ by

$$\mathcal{A} := \mathcal{A}_p + \mathcal{A}_q,$$

where

$$\begin{aligned}
\langle \mathcal{A}_p(v), \phi\rangle &= \int_{\Omega'}\int_{\Omega'} a(x,y,v(x),v(y)) \frac{[v(x)+g(x)-v(y)-g(y)]^{p-1}}{|x-y|^{N+ps}} (\phi(x)-\phi(y))\,dx\,dy \\
&\quad + 2\int_{\mathbb{R}^N\setminus\Omega'}\int_{\Omega} a(x,y,v(x),g(y)) \frac{[v(x)+g(x)-g(y)]^{p-1}}{|x-y|^{N+ps}} \phi(x)\,dx\,dy \\
&=: \langle \mathcal{A}^1_p(v),\phi\rangle + \langle \mathcal{A}^2_p(v),\phi\rangle \quad \text{for all } \phi \in \mathcal{W}(\Omega')
\end{aligned}$$

and $\mathcal{A}_q$ is defined analogously. By virtue of Hölder's inequality and recalling the definition of W (as stated in (2.1)), we obtain

$$\begin{aligned}
|\langle \mathcal{A}_q(v),\phi\rangle| &\le \int_{\Omega'}\int_{\Omega'} b(x,y)\frac{|v(x)+g(x)-v(y)-g(y)|^{q-1}}{|x-y|^{N+qt}}|\phi(x)-\phi(y)|\,dx\,dy \\
&\quad + c(q)\int_{\mathbb{R}^N\setminus\Omega'}\int_{\Omega} b(x,y)\frac{|v(x)+g(x)|^{q-1}+|g(y)|^{q-1}}{|x-y|^{N+qt}}|\phi(x)|\,dx\,dy \\
&\le c([v]^{q-1}_{W^{t,q}_b(\Omega')} + [g]^{q-1}_{W^{t,q}_b(\Omega')})[\phi]_{W^{t,q}_b(\Omega')} \\
&\quad + c\int_{\Omega'} W(x)|v(x)+g(x)|^{q-1}|\phi(x)|\,dx \\
&\quad + c\int_{\Omega}|\phi(x)|\int_{\mathbb{R}^N\setminus\Omega'} b(x,y)\frac{|g(y)|^{q-1}}{|x-y|^{N+qt}}\,dy\,dx \\
&\le c([v]^{q-1}_{W^{t,q}_b(\Omega')} + [g]^{q-1}_{W^{t,q}_b(\Omega')})[\phi]_{W^{t,q}_b(\Omega')} \\
&\quad + c\left(\int_{\Omega'} W(x)|(v+g)(x)|^q\,dx\right)^{\frac{1}{q'}}\left(\int_{\Omega'} W(x)|\phi(x)|^q\,dx\right)^{\frac{1}{q}} \\
&\quad + c\left(\int_{\Omega}|\phi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N} b(x,y)\frac{|g(y)|^{q-1}}{(1+|y|)^{N+tq}}\,dy\right) \\
&\le c(\|v\|^{q-1}_{\mathcal{W}(\Omega')} + \|g\|^{q-1}_{\mathcal{W}(\Omega')} + \|g\|^{q-1}_{L^{q-1}_{qt,b}(\mathbb{R}^N)})\|\phi\|_{\mathcal{W}(\Omega')},
\end{aligned} \tag{5.2}$$

where $c = c(\mathsf{data}, \operatorname{dist}(\Omega, \Omega'))$. A similar result holds for the p-term, too, by simply using the bound (1.1). Consequently, we get that $\mathcal{A}$ is a well-defined operator. Moreover, (5.2) together with its p-counterpart shows that the operator $\mathcal{A}$ is bounded; i.e., it maps bounded sets to bounded sets. We next prove that $\mathcal{A}$ is weakly continuous. For this, let $\{u_k\} \subset X_0(\Omega, \Omega')$ be a sequence such that $u_k \rightharpoonup u$, weakly in $\mathcal{W}(\Omega')$ for some $u \in X_0(\Omega, \Omega')$. Then, we claim that

$$\lim_{k\to\infty} \langle \mathcal{A}(u_k), \phi\rangle = \langle \mathcal{A}(u), \phi\rangle \quad \text{for all } \phi \in X_0(\Omega, \Omega').$$

Using the bound on the function a, we observe that

$$\begin{aligned}
&|\langle \mathcal{A}(u_k) - \mathcal{A}(u), \phi\rangle| \\
&\le \int_{\Omega'}\int_{\Omega'} |a(x,y,u_k(x),u_k(y)) - a(x,y,u(x),u(y))||(u+g)(x) - (u+g)(y)|^{p-1} \\
&\qquad\qquad \times |\phi(x) - \phi(y)|\, d\mu_1 \\
&\quad + 2\int_{\Omega}\int_{\mathbb{R}^N\setminus\Omega'} |a(x,y,u_k(x),g(y)) - a(x,y,u(x),g(y))||u(x)+g(x)-g(y)|^{p-1}|\phi(x)|\, d\mu_1 \\
&\quad + \Lambda\int_{\Omega'}\int_{\Omega'} |[(u_k+g)(x)-(u_k+g)(y)]^{p-1} - [(u+g)(x)-(u+g)(y)]^{p-1}||\phi(x)-\phi(y)|\, d\mu_1 \\
&\quad + 2\Lambda\int_{\Omega}\int_{\mathbb{R}^N\setminus\Omega'} |[(u_k+g)(x)-g(y)]^{p-1} - [(u+g)(x)-g(y)]^{p-1}||\phi(x)|\, d\mu_1 \\
&\quad + |\langle \mathcal{A}_q^1(u_k) - \mathcal{A}_q^1(u), \phi\rangle| + |\langle \mathcal{A}_q^2(u_k) - \mathcal{A}_q^2(u), \phi\rangle|.
\end{aligned} \tag{5.3}$$

By the definition of $X_0(\Omega, \Omega')$ together with the weak convergence and compactness of the Sobolev embedding, we infer that, up to a subsequence, $u_k(x) \to u(x)$ a.e. in Ω'. Hence, using the uniform continuity condition of (5.1), we deduce that the first two terms on the right-hand side of (5.3) converge to 0, as $k \to \infty$. To prove the convergence of the third term, on the contrary, we assume that there exist $\epsilon_0 > 0$ and a subsequence $\{u_k\}$ (up to relabeling) such that

$$\int_{\Omega'}\int_{\Omega'} |[(u_k+g)(x)-(u_k+g)(y)]^{p-1} - [(u+g)(x)-(u+g)(y)]^{p-1}||\phi(x)-\phi(y)|\, d\mu_1 \ge \epsilon_0. \tag{5.4}$$

Since $\{u_k\}$ is bounded in $\mathcal{W}(\Omega')$, using the definition of the norm on $\mathcal{W}(\Omega')$, we observe that the sequence $\left\{\frac{[(u_k+g)(x)-(u_k+g)(y)]^{p-1}}{|x-y|^{\frac{N+ps}{p'}}}\right\}$ is bounded in $L^{p'}(\Omega' \times \Omega')$. Thus, by the reflexivity of the space $L^{p'}$ and the pointwise convergence $u_k \to u$ a.e. in Ω', up to a subsequence (again up to relabeling), we get that

$$\frac{[(u_k+g)(x)-(u_k+g)(y)]^{p-1}}{|x-y|^{\frac{N+ps}{p'}}} \rightharpoonup \frac{[(u+g)(x)-(u+g)(y)]^{p-1}}{|x-y|^{\frac{N+ps}{p'}}} \quad \text{weakly in } L^{p'}(\Omega' \times \Omega'),$$

as $k \to \infty$. Owing to the fact $\frac{|\phi(x)-\phi(y)|}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Omega' \times \Omega')$ (due to $\phi \in \mathcal{W}(\Omega')$), we find a contradiction to (5.4). Consequently, the third term on the right-hand side of (5.3) converges to 0, as $k \to \infty$. Similarly, for the fifth term, we note that the sequence $\Big\{b(x,y)^{\frac{1}{q'}}\frac{[(u_k+g)(x)-(u_k+g)(y)]^{q-1}}{|x-y|^{\frac{N+qt}{q'}}}\Big\}$ is bounded in $L^{q'}(\Omega'\times\Omega')$. Therefore, by the reflexivity of the space $L^{q'}$ and proceeding as above by noting that $b(x,y)^{\frac{1}{q}}\frac{|\phi(x)-\phi(y)|}{|x-y|^{\frac{N+qt}{q}}}\in L^q(\Omega'\times\Omega')$, we get that the fifth term also converges to 0. It remains to prove the convergence of the fourth and the sixth terms on the right-hand side of (5.3). Using (2.4) and Hölder's inequality, we deduce that

$$\begin{aligned}
&\int_\Omega\int_{\mathbb{R}^N\setminus\Omega'} b(x,y)|[(u_k+g)(x)-g(y)]^{q-1}-[(u+g)(x)-g(y)]^{q-1}||\phi(x)|\,d\mu_2\\
&\quad\le c\int_{\Omega'}|u_k(x)-u(x)|^{q-1}|\phi(x)|\int_{\mathbb{R}^N\setminus\Omega'}\frac{b(x,y)}{|x-y|^{N+qt}}\,dy\,dx\\
&\qquad+c\int_\Omega\int_{\mathbb{R}^N\setminus\Omega'} b(x,y)|\phi(x)||u_k(x)-u(x)||u(x)+g(x)-g(y)|^{q-2}\,d\mu_2\\
&\quad\le c\int_{\Omega'}W(x)|u_k(x)-u(x)|^{q-1}|\phi(x)|\,dx+c\Big(\int_{\Omega'}W(x)|u_k(x)-u(x)|^{q-1}|\phi(x)|\,dx\Big)^{\frac{1}{q-1}}\\
&\qquad\times\Big(\int_\Omega\int_{\mathbb{R}^N\setminus\Omega'} b(x,y)|u(x)+g(x)-g(y)|^{q-1}|\phi(x)|\,d\mu_2\Big)^{\frac{q-2}{q-1}},
\end{aligned}$$

where W is as defined in (2.1) with Ω' in place of Ω. From (5.2), we see that the second quantity on the right-hand side of the second term is finite. Then, recalling the definition of the norm on $\mathcal{W}(\Omega')$ and arguing as above (the case of the fifth term), we get that the sixth term on the right-hand side of (5.3) converges to 0. Similarly, we see that the fourth term on the right-hand side of (5.3) tends to 0. Hence, we prove the claim.

Next, to prove coercivity of the operator $\mathcal{A}$, for any $v\in X_0(\Omega,\Omega')$, using Hölder's and Young's inequalities, we first see that

$$\begin{aligned}
\langle\mathcal{A}_p^1(v),v\rangle&=\int_{\Omega'}\int_{\Omega'} a(x,y,v(x),v(y))([(v+g)(x)-(v+g)(y)]^{p-1}-[g(x)-g(y)]^{p-1})\\
&\qquad\qquad\times(v(x)-v(y))\,d\mu_1\\
&\quad+\int_{\Omega'}\int_{\Omega'} a(x,y,v(x),v(y))[g(x)-g(y)]^{p-1}(v(x)-v(y))\,d\mu_1\\
&\ge\frac{1}{c}\int_{\Omega'}\int_{\Omega'}|v(x)-v(y)|^p\,d\mu_1-c\int_{\Omega'}\int_{\Omega'}|g(x)-g(y)|^{p-1}|v(x)-v(y)|\,d\mu_1\\
&\ge\frac{1}{c}\int_{\Omega'}\int_{\Omega'}|v(x)-v(y)|^p\,d\mu_1-c\int_{\Omega'}\int_{\Omega'}|g(x)-g(y)|^p\,d\mu_1,
\end{aligned}\tag{5.5}$$

where we have also used (1.1) and (2.3). Similarly, we discover that

$$\langle\mathcal{A}_q^1(v),v\rangle\ge\frac{1}{c}\int_{\Omega'}\int_{\Omega'} b(x,y)|v(x)-v(y)|^q\,d\mu_2-c\int_{\Omega'}\int_{\Omega'} b(x,y)|g(x)-g(y)|^q\,d\mu_2.\tag{5.6}$$

Furthermore, using the inequality (2.3) once again and the definition of W, we observe that

$$
\begin{aligned}
\langle \mathcal{A}_q^2(v), v\rangle &= \int_\Omega\int_{\mathbb{R}^N\setminus\Omega'} b(x,y)([v(x)+g(x)-g(y)]^{q-1}-[g(x)-g(y)]^{q-1})v(x)\,d\mu_2\\
&\quad+\int_\Omega\int_{\mathbb{R}^N\setminus\Omega'} b(x,y)[g(x)-g(y)]^{q-1}v(x)\,d\mu_2\\
&\geq \frac{1}{c}\int_\Omega\int_{\mathbb{R}^N\setminus\Omega'}|v(x)|^q\frac{b(x,y)}{|x-y|^{N+qt}}\,dx\,dy\\
&\quad-c\int_\Omega\int_{\mathbb{R}^N\setminus\Omega'}(|g(x)|^{q-1}+|g(y)|^{q-1})|v(x)|\frac{b(x,y)}{|x-y|^{N+qt}}\,dx\,dy\\
&\geq \frac{1}{c}\int_{\Omega'}W(x)|v(x)|^q\,dx - c\Big(\int_{\Omega'}W(x)|g(x)|^q\,dx\Big)^{\frac{1}{q'}}\Big(\int_{\Omega'}W(x)|v(x)|^q\,dx\Big)^{\frac{1}{q}}\\
&\quad-c\Big(\int_\Omega|v(x)|\,dx\Big)\|g\|^{q-1}_{L^{q-1}_{qt,b}(\mathbb{R}^N)}\\
&\geq \frac{1}{c}\int_{\Omega'}W(x)|v(x)|^q\,dx - c\int_{\Omega'}W(x)|g(x)|^q\,dx-\epsilon\int_\Omega|v(x)|^p\,dx\\
&\quad-c\epsilon^{\frac{-1}{p-1}}\|g\|^{p'(q-1)}_{L^{q-1}_{qt,b}(\mathbb{R}^N)},
\end{aligned}
\tag{5.7}
$$

where we have also used the fact that $v=0$ in $\Omega'\setminus\Omega$ and Young's inequality on the last line. On a similar note,

$$
\langle \mathcal{A}_p^2(v), v\rangle \geq \frac{1}{2c}\int_{\Omega'}|v(x)|^p\,dx - c\int_{\Omega'}|g(x)|^p\,dx - c\|g\|^p_{L^{p-1}_{ps}(\mathbb{R}^N)}. \tag{5.8}
$$

Finally, combining (5.5), (5.6), (5.7), and (5.8) with $\epsilon=\frac{1}{4c}$, and recalling the definition of the norm on $\mathcal{W}(\Omega')$, we obtain

$$
\begin{aligned}
\langle \mathcal{A}(v), v\rangle \geq \frac{1}{4c}\min\{\|v\|^p_{\mathcal{W}(\Omega')},\|v\|^q_{\mathcal{W}(\Omega')}\} - c\|g\|^p_{W^{s,p}(\Omega')}-c\|g\|^q_{W^{t,q}_b(\Omega')}\\
-c(\|g\|^p_{L^{p-1}_{ps}(\mathbb{R}^N)}+\|g\|^{p'(q-1)}_{L^{q-1}_{qt,b}(\mathbb{R}^N)}),
\end{aligned}
$$

where the constant c depends only on data, Ω, and Ω'. This proves the coercivity of the operator $\mathcal{A}$.

Consequently, by [**38**, Example 2.A, p. 40], it follows that the operator $\mathcal{A}$ is of M-type. Note that $X_0(\Omega,\Omega')$ is a separable reflexive Banach space and $(\mathcal{W}(\Omega'))^*\subset(X_0(\Omega,\Omega'))^*$. Hence, using [**38**, Corollary 2.2, p. 39], we get that the map $\mathcal{A}$ is surjective. Moreover, the last statement is true considering $u\in L^{q-1}(\Omega)$ by $q\leq p_s^*$. This completes the proof of the theorem. □

Appendix A. Boundedness results

We first give a boundedness result for the problem $(\mathcal{P})$ whose proof runs along the same lines of [**23**, Proposition 3.1] by using the Caccioppoli estimate of Lemma 3.1.

Lemma A.1. *Suppose that $q < p_s^*$ and $qt \le ps$. Let u be a local weak solution to the problem $(\mathcal{P})$ in Ω. Then, there exists a constant c depending only on* data *and γ (if $\gamma < \infty$) such that*

$$\|u\|_{L^\infty(B_r(x_0))} \le c\Bigg(\bigg(\fint_{B_{2r}(x_0)} |u(x)|^\vartheta\,dx\bigg)^{q/(p\vartheta)} + \|f\|^{1/(p-1)}_{L^\gamma(B_{2r}(x_0))} + T_{ps}(u;x_0,2r) + T_{qt,b}(u;x_0,2r) + 1\Bigg),$$

provided $B_{2r}(x_0) \Subset \Omega$, where $\vartheta = \max\{q, p\varsigma\}$ with $\varsigma = \frac{p\gamma-(p_s^*)'}{p(\gamma-(p_s^*)')}\big(< \frac{p_s^*}{p}\big)$ if $\gamma < \infty$, while $\varsigma = 1$ if $\gamma = \infty$.

Let $B_{\rho_0} \equiv B_{\rho_0}(0)$. We next consider the following problem:

$$(\mathcal{P}_b) \qquad \begin{cases} \mathcal{L}_{a(\cdot,v),b} v = f & \text{in } B_{3\rho_0/2}, \\ v = g & \text{in } \mathbb{R}^N \setminus B_{3\rho_0/2}, \end{cases}$$

where $g \in \mathcal{W}_b(B_{2\rho_0}) \cap L^\infty(B_{2\rho_0}) \cap L^{p-1}_{ps}(\mathbb{R}^N) \cap L^{q-1}_{qt,b}(\mathbb{R}^N)$ and $f \in L^\gamma_{\text{loc}}(B_{2\rho_0})$ with $\gamma > \max\{1, N/(ps)\}$. Let $v \in X_{g,b}(B_{3\rho_0/2}, B_{2\rho_0})$ be a weak solution to the problem $(\mathcal{P}_b)$. Then, v enjoys the same Caccioppoli-type estimate as in Lemma 3.1 and hence Lemma A.1 holds for v, too. We next see the boundary estimate of the solution v. Precisely, we have the following estimate using [**27**, Theorem 5] and [**23**, Proposition 3.1] with slight modifications.

Lemma A.2. *Suppose that $q < p_s^*$ and $qt \le ps$. Let $v \in X_{g,b}(B_{3\rho_0/2}, B_{2\rho_0})$ be a weak solution to $(\mathcal{P}_b)$ with $f \in L^\gamma(B_{2r}(x_0) \cap B_{3\rho_0/2})$ and $g \in L^\infty(B_{2\rho_0})$, for some $r \in (0, 1/16)$ and $x_0 \in \partial B_{3\rho_0/2}$. Then there is a constant c depending only on* data *and γ (if $\gamma < \infty$) such that*

$$\|v\|_{L^\infty(B_r(x_0))} \le c\Bigg(\bigg(\fint_{B_{2r}(x_0)} |v(x)|^\vartheta\,dx\bigg)^{q/(p\vartheta)} + \|f\|^{1/(p-1)}_{L^\gamma(B_{2r}(x_0)\cap B_{3\rho_0/2})} + T_{ps}(v;x_0,2r) + T_{qt,b}(v;x_0,2r) + \|g\|_{L^\infty(B_{2\rho_0})} + 1\Bigg),$$

where ϑ is the same number as in Lemma A.1.

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Received on November 7, 2022.
Accepted on March 6, 2023.