Publ. Mat. **68** (2024), 593–613
DOI: 10.5565/PUBLMAT6822410

UNIFORMLY ERGODIC PROBABILITY MEASURES

Jorge Galindo, Enrique Jordá, and Alberto Rodríguez-Arenas

Abstract: Let G be a locally compact group and μ be a probability measure on G. We consider the convolution operator $\lambda_1(\mu)\colon L_1(G)\to L_1(G)$ given by $\lambda_1(\mu)f=\mu*f$ and its restriction $\lambda_1^0(\mu)$ to the augmentation ideal $L_1^0(G)$. Say that μ is uniformly ergodic if the Cesàro means of the operator $\lambda_1^0(\mu)$ converge uniformly to 0, that is, if $\lambda_1^0(\mu)$ is a uniformly mean ergodic operator with limit 0, and that μ is uniformly completely mixing if the powers of the operator $\lambda_1^0(\mu)$ converge uniformly to 0.

We completely characterize the uniform mean ergodicity of the operator $\lambda_1(\mu)$ and the uniform convergence of its powers, and see that there is no difference between $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$ in these regards. We prove in particular that μ is uniformly ergodic if and only if G is compact, μ is adapted (its support is not contained in a proper closed subgroup of G), and 1 is an isolated point of the spectrum of μ. The last of these three conditions can actually be replaced by μ being spread out (some convolution power of μ is not singular). The measure μ is uniformly completely mixing if and only if G is compact, μ is spread out, and the only unimodular value in the spectrum of μ is 1.

2020 Mathematics Subject Classification: Primary: 43A05; Secondary: 43A20, 43A30, 46H99, 47A35.

Key words: ergodic measure, uniformly ergodic measure, random walk, mean ergodic operator, uniformly mean ergodic operator, convolution operator, locally compact group, measure algebra.

1. Introduction

A probability measure μ on a locally compact group G defines the transition probabilities of a random walk on G. It is common knowledge in ergodic theory that the ergodicity of a random walk can be characterized in terms of the fixed points of its Markov operator. In the present case, the Markov operator is identified with the *convolution operator* $\lambda_1(\mu)\colon L_1(G)\to L_1(G)$, $\lambda_1(\mu)f=\mu*f$. This is reflected in the following theorem stated in [**34**], which can be regarded as the starting point of this work.

Theorem 1.1 ([**34**, Proposition 1.2])**.** *Let G be a locally compact group and let μ be a probability measure on G. Consider the operator $\lambda_1^0(\mu)$ that results from restricting $\lambda_1(\mu)$ to the augmentation ideal*

$$L_1^0(G)=\left\{f\in L_1(G):\int f(x)\,\mathrm{d}m_G(x)=0\right\}.$$

The following are equivalent:

(i) *The random walk defined by μ is ergodic.*
(ii) *The Cesàro means of the operator $\lambda_1^0(\mu)$ converge to 0 in the strong operator topology.*
(iii) *If $f\in L_\infty(G)$ and $\mu^**f=f$, where $\mu^*(A)=\mu(A^{-1})$, then f is constant almost everywhere.*

A probability measure μ can be ergodic (and even spread out) while μ^* is not; see Remark on page 9 of [**17**], which leans on results of Azencott [**5**]. This shows that we cannot use μ instead of μ^* in condition (iii) of Theorem 1.1.

If the group G is abelian or compact, the Choquet–Deny and Kawada–Itô theorems ([**8, 25**]) show, respectively, that the random walk induced by μ is ergodic (and we then say that μ itself is *ergodic*) if and only if its support is not contained in a proper closed subgroup of G, i.e., if μ is *adapted*. If the support of μ is not even contained in a translate of a proper closed normal subgroup (we say then that μ is *strictly aperiodic*), μ has a stronger property: the strong operator limit of $\lambda_1^0(\mu^n)$ is 0. We say then that μ is *completely mixing*. All these facts are well known and presented in Theorem 2.9 below.

In this paper we address the problem of characterizing ergodicity when the strong operator topology is replaced by the operator norm. Our approach will be operator-theoretic and focuses on the uniform mean ergodicity (operator norm convergence of the Cesàro means) of the operators $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$ and on the uniform ergodicity of μ, where μ is said to be *uniformly ergodic* if $\lim_n \big\| \frac{1}{n} \sum_{k=1}^{n} \lambda_1^0(\mu)^k \big\| = 0$.

After two preliminary sections, we devote Section 3 to the operator $\lambda_1^0(\mu)$ and its relation with uniform ergodicity and Section 4 to presenting a first characterization of uniform ergodicity which shows its several angles and reflects what was previously known on the subject.

In Section 5 spectral conditions are introduced. It is shown in [**16**] that, for an adapted measure μ and an abelian group G, $\lambda_1(\mu)$ is uniformly mean ergodic if and only if the support of μ is contained in a compact subgroup of G and 1 is isolated in the spectrum of μ. One of our main results here, Theorem 5.10, is that this theorem still holds true when G is not abelian, and that the condition of having 1 as an isolated point of the spectrum of μ is in fact equivalent, when G is compact and μ is adapted, to μ being spread out.

The subject of uniform mean ergodicity of convolution operators has been discussed in the book by Revuz, [**33**, Section 6.3], where uniformly ergodic convolution operators on $L_\infty(G)$ that converge to a rank-one operator are characterized, as well as in a brief incursion by Mustafayev and Topal [**32**, Theorem 1.2]. To the best of our knowledge, no spectral characterizations of uniform ergodicity of convolution operators on $L_1(G)$, G any locally compact group, have been obtained before.

Finally, we complete our study of the operator $\lambda_1^0(\mu)$ and characterize quasi-compactness and convergence of the powers of the operators $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$. Convergence in norm of the sequence of iterates $(\lambda_1^0(\mu^n))$ is shown to be equivalent to that of $(\lambda_1(\mu^n))$, providing a spectral extension of results of Anoussis and Gatzouras in [**3**].

2. Preliminaries

We use this section to set some notations and recall a few definitions and basic known facts.

2.1. Mean ergodic operators. For a bounded linear operator $T\colon X \to X$ on a Banach space X, we use the following notations for the iterates and averages:

$$T^n = T \circ \overset{n}{\cdots} \circ T, \quad T_{[n]} = \frac{T + \cdots + T^n}{n}.$$

Definitions 2.1. Let X be a Banach space and let $T\colon X \to X$ be a bounded linear operator. The operator is

(i) *mean ergodic* if the sequence $(T_{[n]}x)_n$ is convergent for every $x \in X$;

(ii) *uniformly mean ergodic* if the sequence $(T_{[n]})_n$ is convergent in the operator norm.

If T is mean ergodic, the limit of $(T_{[n]}x)_n$ is always a linear projection $P\colon X \to X$, satisfying $\ker P = \overline{(I-T)X}$. Moreover, $X = \ker P \oplus \ker(I-T)$.

Uniform mean ergodicity for power-bounded operators is completely characterized by Lin.

Theorem 2.2 ([**28**, Theorem])**.** *Let T be a power-bounded (i.e., so that $(\|T^n\|)_n$ is a bounded sequence) linear operator on a Banach space X. The operator T is uniformly mean ergodic if, and only if, $(I-T)X$ is closed. The Banach space X can then be decomposed as $X = (I-T)X \oplus \ker(I-T)$.*

Lin's theorem completed the following theorem, due to Dunford [**14**], which reveals that the uniform mean ergodicity of an operator can be characterized through its spectral properties. See also [**26**, Theorem 2.2.7] and [**28**].

Theorem 2.3. *Let T be a power-bounded linear operator on a complex Banach space X. Then T is uniformly mean ergodic if, and only if, either $1 \notin \sigma(T)$ or it is a pole of order 1 of the resolvent.*

For the next result, due to Yosida and Kakutani, we recall that an operator T is quasi-compact if there exists a compact operator K such that $\|T^n - K\| < 1$ for some $n \geq 1$. By $\sigma_p(T)$ we will denote the point spectrum of T, the set of its eigenvalues, and $\mathbb{T}$ will denote the set of complex numbers of modulus 1.

Theorem 2.4 ([**37**, Theorem 4 and its Corollary, p. 205])**.** *Let T be a power-bounded and quasi-compact linear operator on a complex Banach space X. The following assertions hold:*

(i) *$(T_{[n]})_n$ converges uniformly to a finite-rank projection P.*

(ii) *$(T^n)_n$ converges in norm to a finite-rank projection P if, and only if, $\sigma_p(T) \cap \mathbb{T} \subseteq \{1\}$.*

(iii) *$(T^n)_n$ converges in norm to 0 if, and only if, $\sigma_p(T) \cap \mathbb{T} = \varnothing$.*

2.2. Convolution operators. We will be working with convolutions on σ-compact locally compact groups. If G is a locally compact, σ-compact group, e will stand for its identity element and m_G for its Haar measure, the (essentially unique) Borel measure that is left-invariant on a locally compact group G. The Banach space of continuous functions on G vanishing at infinity is denoted by $C_0(G)$ and that of compactly supported functions, by $C_{00}(G)$. The Banach space of bounded regular measures is $M(G) = C_0(G)^*$. We denote by $L_1(G,\mu)$ the Banach space of equivalence classes of integrable functions with respect to a measure μ on G. If $\mu = \mathrm{m}_G$ is the Haar measure, we simply write $L_1(G)$. We denote by $L_1^0(G)$ the augmentation ideal, the space of functions with integral 0. We regard $L_1(G)$ as an ideal of $M(G)$ through the embedding $f \mapsto f \cdot d\mathrm{m}_G$.

Definitions 2.5. Let $\mu, \mu_1, \mu_2 \in M(G)$. We consider

(i) the convolution of measures:

$$\langle \mu_1 * \mu_2, h\rangle = \iint h(xy)\, d\mu_1(x)\, d\mu_2(y), \quad \text{for } h \in C_{00}(G);$$

(ii) the left-convolution operators: $\lambda_p(\mu)\colon L_p(G) \to L_p(G)$, $1 \le p \le \infty$, given by

$$\lambda_p(\mu)f(s) = (\mu * f)(s) = \int f(x^{-1}s)\,d\mu(x), \quad \text{for } f \in L_p(G),\ s \in G;$$

(iii) the operator $\lambda_1^0(\mu)\colon L_1^0(G) \to L_1^0(G)$ that arises when $\lambda_1(\mu)$ is restricted to the augmentation ideal $L_1^0(G)$;

(iv) the convolution iterates μ^n and the convolution averages $\mu_{[n]}$, respectively: $\mu^n = \mu * \cdots * \mu$ and $\mu_{[n]} = \frac{\mu+\cdots+\mu^n}{n}$.

2.3. Ergodic measures. A measure $\mu \in M(G)$ on a locally compact group G is said to be ergodic when the random walk it generates is ergodic and it is said to be completely mixing when the random walk is mixing. For an operator-theoretic approach, however, the following, equivalent, definitions are more suitable. They are, in fact, the ones chosen by most authors; see, e.g., [**4, 11, 23, 30, 34**] (note that Rosenblatt ([**34**]) uses the terms *ergodic by convolutions* and *mixing by convolutions*).

Definitions 2.6. A probability measure $\mu \in M(G)$ is:

(i) *ergodic* if $\lim_n \|\mu_{[n]} * f\|_1 = 0$, for all $f \in L_1^0(G)$. In other words, if $\lambda_1^0(\mu)$ is mean ergodic and the strong operator limit of the means is the null operator;

(ii) *completely mixing* if the sequence of iterates $(\lambda_1^0(\mu^n))_{n\in\mathbb{N}}$ converges to 0 in the strong operator topology.

Azencott ([**5**, p. 43, Corollaire]) proved that the existence of an ergodic probability implies amenability (see also [**34**, Proposition 1.9]). J. Rosenblatt ([**34**, Theorem 1.10]) proved that in any amenable group there is an ergodic probability.

If in Definitions 2.6 the strong operator topology is replaced by the operator norm, two new concepts arise.

Definitions 2.7. We say that probability measure $\mu \in M(G)$ is:

(i) *uniformly ergodic* if $\lambda_1^0(\mu)$ is uniformly mean ergodic and the operator norm limit of the means is the null operator;

(ii) *uniformly completely mixing* if the sequence of iterates $(\lambda_1^0(\mu^n))_{n\in\mathbb{N}}$ converges to 0 in the operator norm.

Note that uniform ergodicity of $\mu \in M(G)$ comprises convergence to 0 of the means of the operator $\lambda_1^0(\mu)$. The example $\mu = \delta_e$ shows that this property is stronger than uniform mean ergodicity of $\lambda_1^0(\mu)$.

The analysis of the (uniform) ergodicity or the (uniform) complete mixing of a measure μ is closely related to the algebraic properties of its support subgroup H_μ. We denote by H_μ the smallest closed subgroup of G that contains S_μ, the support of μ. To give a hint on this relation we need to set some definitions that have already been informally introduced.

Definitions 2.8. A probability measure $\mu \in M(G)$ is:

(i) *adapted* if $H_\mu = G$;

(ii) *strictly aperiodic* if it is adapted and the only normal closed subgroup N satisfying $S_\mu \subseteq xN$ for some $x \in G$ is actually $N = G$;

(iii) *spread out* if μ^n is not singular for some $n \in \mathbb{N}$, i.e., if there is $n \in \mathbb{N}$ such that $\mu^n(A) = 1$ implies $\mathrm{m}_G(A) > 0$.

We now present some well-known facts that will be needed later on. The first two statements are proved by Lin and Wittmann in Corollary 2.7 of [**30**]. The third statement is the Kawada–Itô theorem [**25**]. For the last sentence one has to apply Proposition 3.1 of [**16**] to the previous one. See also Corollary 3.2 of [**13**].

Theorem 2.9. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure. The following assertions hold:*

(i) *If μ is ergodic, then $G = \overline{\bigcup_j \bigcup_k S_\mu^{-j} S_\mu^k}$. Hence μ ergodic implies that μ is adapted.*

(ii) *If μ is completely mixing, then $G = \overline{\bigcup_j S_\mu^{-j} S_\mu^j}$. Hence, μ completely mixing implies that μ is strictly aperiodic.*

(iii) *If G is compact, $(\mu_{[n]})_{n\in\mathbb{N}}$ converges, in the weak* topology of $M(G)$, to the Haar measure of H_μ. If μ is strictly aperiodic, $(\mu^n)_{n\in\mathbb{N}}$ converges, in the weak* topology of $M(G)$, to the Haar measure of G. The measure μ is then completely mixing.*

Remark 2.10. Complete mixing and ergodicity are known to be equivalent for strictly aperiodic measures under some restrictions on the measure and/or the group. This is the case, for instance, when the group is in [SIN] (i.e., has a neighborhood basis consisting of sets U such that $Ux = xU$ for every $x \in G$); see [**23**, Theorem 5.2]. Looking from the measure side, Glasner ([**17**]), leaning on Foguel ([**15**]), showed that complete mixing and ergodicity are equivalent for any spread out strictly aperiodic measure. Some other cases have been sorted out in [**4**] and [**11**] but the problem of the equivalence, *the complete mixing problem*, remains open.

2.4. Fourier–Stieltjes transforms. To study the spectrum of μ when G is compact, we will rely on the Fourier–Stieltjes transform, which associates every continuous irreducible unitary representation π of G on a Hilbert space $\mathbb{H}_\pi$ with a homomorphism between $M(G)$ and $\mathcal{L}(\mathbb{H}_\pi)$ the bounded operators on $\mathbb{H}_\pi$. By a unitary representation of G, we understand a continuous homomorphism of G into the group of unitary operators on a finite-dimensional Hilbert space $\mathbb{H}_\pi$, where the latter is assumed to carry the weak operator topology. A representation π of G on a Hilbert space $\mathbb{H}$ is *irreducible* if there is no nontrivial closed subspace V of $\mathbb{H}$ such that $\pi(G)V \subseteq V$.

Definition 2.11 (Fourier–Stieltjes transform)**.** If $\mu \in M(G)$ is a bounded measure on a compact group G and $\pi\colon G \to \mathcal{U}(\mathbb{H}_\pi)$ is a unitary representation of G on a finite-dimensional Hilbert space $\mathbb{H}_\pi$, the *Fourier–Stieltjes transform* of μ at π is the operator $\widehat{\mu}(\pi) \in \mathcal{L}(\mathbb{H}_\pi)$ defined by

$$\langle \widehat{\mu}(\pi)\xi, \eta\rangle = \int_G \overline{\langle \pi(t)\xi, \eta\rangle}\, d\mu(t), \quad \text{for every } \xi, \eta \in \mathbb{H}_\pi.$$

The basic properties of Fourier–Stieltjes transforms are given below. Statements (i), (ii), (iii), and (iv) are proved in Section 28 of [**20**]. Statement (v) is Theorem 27.17 of [**20**].

Theorem 2.12 (Some properties of the Fourier–Stieltjes transform)**.** *Let G be a compact group.*

(i) *If $\mu_i \in M(G)$, $i = 1, 2$, are bounded measures and $\pi\colon G \to \mathcal{U}(\mathbb{H}_\pi)$ is a unitary representation of G on a finite-dimensional Hilbert space $\mathbb{H}_\pi$, then $\widehat{\mu_1 * \mu_2}(\pi) = \widehat{\mu_1}(\pi)\widehat{\mu_2}(\pi)$.*

(ii) *If $\mu \neq 0$, there is an irreducible unitary representation of G on a finite-dimensional Hilbert space, $\pi\colon G \to \mathcal{U}(\mathbb{H}_\pi)$ such that*

$$\widehat{\mu}(\pi) \neq 0.$$

(iii) *If $\mu \in M(G)$ and π is an irreducible unitary representation of G on a finite-dimensional Hilbert space, then*

$$\sigma(\widehat{\mu}(\pi)) \subseteq \sigma(\mu).$$

(iv) *Let $\mathbf{1}$ denote the trivial one-dimensional representation. If $\mathbf{1} \neq \pi$ is an irreducible representation of G, then $\widehat{\mathrm{m}_G}(\pi) = 0$.*

(v) *Let $\pi\colon G \to \mathbb{H}_\pi$ be an irreducible unitary representation. If $\{\xi_1, \ldots, \xi_n\}$ is an orthonormal basis of $\mathbb{H}_\pi$, then*

$$\int |\langle \pi(t)\xi_i, \xi_j\rangle|^2 \,\mathrm{dm}_G(t) = \begin{cases} \dfrac{1}{n}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We next add two simple known facts that will be needed later on. We provide proofs for the sake of completeness.

Corollary 2.13. *Let G be a compact group and $\mu \in M(G)$ a bounded measure.*

(i) *For every irreducible representation π and every $\xi \in \mathbb{H}_\pi$, $\|\xi\| = 1$, there is $f_{\pi,\xi} \in L_1(G)$ such that $\widehat{f_{\pi,\xi}}(\pi)\xi = \xi$.*

(ii) *If μ is adapted and π is an irreducible unitary representation of G with $1 \in \sigma(\widehat{\mu}(\pi))$, then $\pi = \mathbf{1}$, the one-dimensional trivial representation.*

Proof: We first prove (i). Define $f_{\pi,\xi}$ by

$$f_{\pi,\xi}(t) = n\langle \pi(t)\xi, \xi\rangle,$$

where $n = \dim \mathbb{H}_\pi$. Since π is continuous for the weak operator topology, $f_{\pi,\xi} \in L_1(G)$. If $\{\xi, \xi_2, \ldots, \xi_n\}$ is an orthonormal basis of $\mathbb{H}_\pi$ containing ξ, statement (v) of Theorem 2.12 shows that $\langle \widehat{f_{\pi,\xi}}(\pi)\xi, \xi\rangle = 1$ and that $\langle \widehat{f_{\pi,\xi}}(\pi)\xi, \xi_j\rangle = 0$, for every $j = 2, \ldots, n$. It follows that $\widehat{f_{\pi,\xi}}(\pi)\xi = \xi$.

The proof of (ii) is standard. Assume that μ is an adapted measure and that π is an irreducible representation of G with $1 \in \sigma(\widehat{\mu}(\pi))$. There is therefore $\xi \in \mathbb{H}_\pi$, $\|\xi\| = 1$, with

$$1 = \int \overline{\langle \pi(t)\xi, \xi\rangle}\, d\mu(t).$$

Since $|\langle \pi(t)\xi, \xi\rangle| \leq 1$, for every $t \in G$, it follows that $\langle \pi(t)\xi, \xi\rangle = 1$ for every $t \in S_\mu$, and so that $\pi(t)\xi = \xi$ for every $t \in S_\mu$. We then have that

$$S_\mu \subseteq N_\xi := \{t \in G : \pi(t)\xi = \xi\}.$$

Since N_ξ is a closed subgroup of G, adaptedness of μ implies that $N_\xi = G$. As $\mathbb{H}_\pi$ cannot contain proper subspaces invariant under π, we conclude that $\pi = \mathbf{1}$. $\square$

3. The norm of $\boldsymbol{\lambda_1^0(\mu)}$

The ergodicity properties of the measure μ derive from those of the operator $\lambda_1^0(\mu)$. While the norm of $\lambda_1(\mu)$ is always $\|\mu\|$, see [**18**, p. 47], that is not necessarily the case for $\lambda_1^0(\mu)$. It suffices to observe that, when G is a compact group, $\lambda_1^0(\mathrm{m}_G) = 0$. We see in this section that invariance of μ (and, *a fortiori*, compactness of G) is the only obstacle to the equality $\|\lambda_1^0(\mu)\| = \|\mu\|$.

Lemma 3.1. *Let G be a locally compact group and let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$. Then*

$$\|\lambda_1^0(\mu)\| \geq \frac{1}{2} \sup_{x \in G} \|\mu - \mu * \delta_x\|.$$

Proof: Consider a two-sided approximate identity $\{e_\alpha : \alpha \in \Lambda\}$ in $L_1(G)$, with $\|e_\alpha\| = 1$ for every $\alpha \in \Lambda$; see [**12**, Proposition 9.1.8]. It is then easy to prove that $\{e_\alpha : \alpha \in \Lambda\}$ converges to δ_e in the $\sigma(M(G), C_0(G))$-topology. Since convolution is separately $\sigma(M(G), C_0(G))$-continuous, we have that $(\mu - \mu * \delta_x) * e_\alpha$ converges to $\mu - \mu * \delta_x$ in the $\sigma(M(G), C_0(G))$-topology.

The proposition cited above is actually based on the simple equality

$$\mu * (e_\alpha - \delta_x * e_\alpha) = (\mu - \mu * \delta_x) * e_\alpha \quad (x \in G \text{ and } \alpha \in \Lambda). \tag{1}$$

One can first use functions $f_n \in C_0(G)$ so that $\|\mu - \mu * \delta_x\| \le |\langle \mu - \mu * \delta_x, f_n\rangle| + 1/(2n)$ to show that, for each $n \in \mathbb{N}$, there is $\alpha_n \in \Lambda$ with

$$\|\mu - \mu * \delta_x\| \le \|(\mu - \mu * \delta_x) * e_{\alpha_n}\| + \frac{1}{n}.$$

From this, using (1) and taking into account that $e_\alpha - \delta_x * e_\alpha \in L_1^0(G)$, we have

$$\|\mu - \mu * \delta_x\| \le \|\lambda_1^0(\mu)\| \|e_{\alpha_n} - \delta_x * e_{\alpha_n}\| + \frac{1}{n}, \text{ for each } n \in \mathbb{N}$$

and hence

$$\|\mu - \mu * \delta_x\| \le 2\|\lambda_1^0(\mu)\|,$$

yielding the result. □

Proposition 3.2. *Let G be a locally compact group that is* not compact *and let $\mu \in M(G)$ with $\|\mu\| = 1$. Then $\|\lambda_1^0(\mu)\| = 1$.*

Proof: We first assume that S_μ is compact. Then there is $x \in G$ such that $S_\mu \cap S_\mu x = \varnothing$, which implies that $\|\mu - \delta_x * \mu\| = 2$. We can then apply Lemma 3.1 and obtain that $\|\lambda_1^0(\mu)\| = 1$.

In general, we can always find a sequence μ_n of measures with compact support such that μ_n converges in norm to μ. Since $\|\lambda_1^0(\mu_n)\| = 1$ for every n, $\|\lambda_1^0(\mu)\|$ must be 1 as well. □

Corollary 3.3. *Let G be a locally compact group and let μ be a probability measure. If μ is uniformly ergodic, then G is compact.*

Proof: The measure $\mu_{[n]}$ is again a probability measure. If G is not compact, Proposition 3.2 shows that $\|\lambda_1^0(\mu_{[n]})\| = 1$ and, hence, μ cannot be uniformly ergodic. □

The previous corollary could have also been deduced from the following proposition, which should be compared to Theorem 2.9(i).

Proposition 3.4. *Let $\mu \in M(G)$ be a probability measure with support S_μ. If μ is uniformly ergodic, then there exists $n \in \mathbb{N}$ such that*

$$G = \bigcup_{1 \le j,k \le n} S_\mu^{-j} S_\mu^k.$$

Proof: We proceed by contradiction. Assume $G \neq \bigcup_{1 \le j,k \le n} S_\mu^{-j} S_\mu^k$, for any $n \in \mathbb{N}$.

Since $S_{\mu_{[n]}} = \bigcup_{j=1}^n S_\mu^j$, the assumption implies that $G \neq S_{\mu_{[n]}}^{-1} S_{\mu_{[n]}}$ and therefore that, for each $n \in \mathbb{N}$, there exists $x_n \in G$ such that $S_{\mu_{[n]}} \cdot x_n \cap S_{\mu_{[n]}} = \varnothing$. From this, we have that $\|\mu_{[n]} - \mu_{[n]} * \delta_{x_n}\| = 2$, for every $n \in \mathbb{N}$, and we conclude, upon applying Lemma 3.1, that μ is not uniformly ergodic. □

4. Uniform mean ergodicity of convolution operators

In this section we approach a characterization (Theorem 4.7) of uniform mean ergodicity of the operators $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$ in terms of how close μ is to m_{H_μ}.

Our attention is attracted right away to measures with H_μ compact. This is because of the following result, proved in [**16**, Theorem 5.4].

Theorem 4.1. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability. Then $\lambda_1(\mu)$ is mean ergodic if and only if H_μ is compact.*

We first see that we can restrict our work to adapted measures (and compact groups).

4.1. Reduction to the adapted case. Let G be a locally compact group and let $\mu \in M(G)$ be a bounded measure. We will use the symbol $\underline{\mu}$ to denote the measure μ when seen as a measure in $M(H_\mu)$. We see here that $\lambda_1(\mu)$ is uniformly mean ergodic if and only if $\lambda_1(\underline{\mu})$ is.

We first note the relation between uniform convergence of powers and Cesàro means of the convolution operator $\lambda_1(\mu)$ with the weak*-convergence of convolution powers and Cesàro means of the measure itself.

Proposition 4.2. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure.*

(i) *The operator $\lambda_1(\mu)$ is uniformly mean ergodic if and only if H_μ is compact and*

$$\lim_n \|\lambda_1(\mu_{[n]} - \mathrm{m}_{H_\mu})\| = 0.$$

(ii) *The sequence $(\lambda_1(\mu^n))$ is norm convergent if and only if H_μ is compact and*

$$\lim_n \|\lambda_1(\mu^n - \mathrm{m}_{H_\mu})\| = 0.$$

Proof: By Theorem 4.1, H_μ must be compact whenever $\lambda_1(\mu)$ is uniformly mean ergodic.

Theorem 2.9(iii) shows that $\mu_{[n]}$ converges to m_{H_μ} in the $\sigma(M(H_\mu), C_0(H_\mu))$-topology. This implies right away convergence in the $\sigma(M(G), C_0(G))$-topology. The weak*-SOT sequential continuity of λ_1, see [**16**, Proposition 3.1], then shows that once we assume that either $\lambda_1(\mu_{[n]})$ or $\lambda_1(\mu^n)$ is convergent, the limit must be to $\lambda_1(\mathrm{m}_{H_\mu})$. $\square$

The desired equivalence is now an immediate consequence of the last proposition, after recalling that $\|\lambda_1(\nu)\| = \|\nu\|$, for every $\nu \in M(G)$; see [**18**, p. 47].

Corollary 4.3. *Let G be a locally compact group and let $\mu \in M(G)$. The operator $\lambda_1(\mu)$ is uniformly mean ergodic if and only if $\lambda_1(\underline{\mu})$ is. Furthermore, $(\lambda_1(\mu^n))$ is norm convergent in $\mathcal{L}(L_1(G))$ if and only if $(\lambda_1(\underline{\mu}^n))$ is norm convergent in $\mathcal{L}(L_1(H_\mu))$.*

4.2. Uniform mean ergodicity of $\boldsymbol{\lambda_1(\mu)}$ vs. uniform mean ergodicity of $\boldsymbol{\lambda_1^0(\mu)}$. Theorem 4.1 shows that the operators $\lambda_1^0(\mu)$ and $\lambda_1(\mu)$ are very different from the point of view of mean ergodicity. If G is abelian and $\mu \in M(G)$ is adapted, the operator $\lambda_1^0(\mu)$ is always mean ergodic (this is the Choquet–Deny theorem, [**8**]) but $\lambda_1(\mu)$ will only be mean ergodic when G is compact; the case $\mu = \delta_1 \in M(\mathbb{Z})$ is a particularly simple case. The situation is completely different in the uniform case.

Proposition 4.4. *Let $T \in \mathcal{L}(X)$ be a bounded operator on a Banach space X and let X_0 be a hyperplane of X which is invariant under T, i.e., with $T(X_0) \subseteq X_0$. Then T is uniformly mean ergodic if and only if $T|_{X_0}$ is.*

Proof: It is clear that T uniformly mean ergodic implies that $T|_{X_0}$ is uniformly mean ergodic.

Assume now that $T|_{X_0}$ is uniformly mean ergodic. The ergodic decomposition of Theorem 2.2 then yields

$$X_0 = (I - T)(X_0) \oplus \ker(I - T|_{X_0}).$$

Since X_0 is a hyperplane, there is $y \in X$ such that $X = X_0 \oplus \langle y \rangle$. This shows that $X = (I - T)(X_0) \oplus \ker(I - T|_{X_0}) \oplus \langle y \rangle$.

Now let $a, b \in X_0$ and $\alpha \in \mathbb{C}$ be such that $b = Tb$ and

$$(I - T)y = (I - T)a + b + \alpha y,$$

and define $\tilde{y} = b + \alpha y$. Then $\tilde{y} \in (I - T)(X)$. We now check that $(I - T)(X) = (I - T)(X_0) \oplus \langle \tilde{y} \rangle$. Pick to that end an arbitrary $x = x_0 + \beta y \in X$, $x_0 \in X_0$, $\beta \in \mathbb{C}$. Then

$$\begin{aligned} (I - T)x &= (I - T)x_0 + \beta(I - T)y \\ &= (I - T)x_0 + \beta(I - T)a + \beta(b + \alpha y) \in (I - T)X_0 + \langle \tilde{y} \rangle. \end{aligned}$$

Having proved that $(I-T)(X) = (I-T)(X_0) \oplus \langle \tilde{y} \rangle$, we see that $(I-T)(X)$ is closed and, hence, that T is uniformly mean ergodic by Theorem 2.2. $\square$

After noting that $L_1^0(G)$, being the kernel of the integral functional, is a hyperplane of $L_1(G)$ that is invariant under $\lambda_1(\mu)$, the following corollary is immediately obtained.

Corollary 4.5. *Let G be a locally compact group and $\mu \in M(G)$ be a probability measure. The operator $\lambda_1(\mu)$ is uniformly mean ergodic if and only if $\lambda_1^0(\mu)$ is uniformly mean ergodic.*

Remark 4.6. (a) The proof of Proposition 4.4 shows that the equivalence between the uniform mean ergodicity of $\lambda_1(\mu)$ and that of $\lambda_1^0(\mu)$ holds for any measure $\mu \in M(G)$, with $\|\mu^n\| \leq M$, $n \in \mathbb{N}$.

(b) For the particular case of probability measures, a more direct way to prove Corollary 4.5 was pointed out to us by one of the referees. It consists in observing that, as a consequence of Lin's theorem (Theorem 2.2), a power-bounded operator $T \in L(X)$ on a Banach space X is uniformly mean ergodic if and only if its restriction to a closed subspace X_0 (not necessarily a hyperplane) containing the T-invariant subspace $\overline{(I - T)(X)}$ is uniformly mean ergodic. Indeed, if one assumes that T is uniformly mean ergodic on $L := \overline{(I - T)(X)}$, the ergodic decomposition of $T|_L$ yields that $(I-T)(X) \subseteq L = (I-T)(L) \subseteq (I-T)(X)$ so that $(I-T)(X)$ is closed in X. Theorem 2.2 then applies to show that T is uniformly mean ergodic. If μ is a probability measure, then $(I - \lambda_1^0(\mu))(L_1(G)) \subseteq L_1^0(G)$ and this argument applies to $\lambda_1(\mu)$.

4.3. A first characterization. The arguments in the previous sections lead us to the classic characterization of uniform mean ergodicity, summarized in Theorem 4.7. Part of that theorem, the equivalence of (ii) and (iv) for instance, hold for general Markov operators, see [**21**], while those involving spread out measures are specific to convolution operators. A proof of Theorem 4.7 can be found in [**33**, Section 6.3]; see also [**6**]. When the group is connected, more can be said; we refer to Theorem 5.19 and the subsequent discussion. Since, given the tools already introduced, the proof of the theorem is rather straightforward, we have decided to put it together using the notations of the present paper, which will be handy later on. Note that some of the references mentioned in this paragraph deal with convolution operators on $L_\infty(G)$ instead of $L_1(G)$. Since convolution operators on $L_\infty(G)$ are adjoint to convolution operators on $L_1(G)$, both approaches yield the same classification (cf. Corollary 5.12 below). This characterization will be completed in Section 5 by adding a spectral condition.

Theorem 4.7. *Let G be a compact group and $\mu \in M(G)$ be an adapted probability measure. The following are equivalent:*

(i) *μ is uniformly ergodic.*
(ii) *$\lambda_1(\mu)$ is uniformly mean ergodic.*
(iii) *μ is spread out.*
(iv) *$\lambda_1(\mu)$ is quasi-compact.*
(v) *$\lambda_1^0(\mu)$ is quasi-compact.*

Proof: Proposition 4.2 and Corollary 4.5 prove that (i) and (ii) are equivalent.

If μ is not spread out, then $\mu_{[n]}$ is a singular measure for every $n \in \mathbb{N}$. It follows that

$$\|\lambda_1(\mu_{[n]} - \mathrm{m}_G)\| = \|\mu_{[n]} - \mathrm{m}_G\| \geq 1, \quad \text{for every } n \in \mathbb{N}.$$

We deduce from Proposition 4.2 that (ii) implies (iii).

Assume that condition (iii) holds. Then there exist $n \in \mathbb{N}$ and $f \in L_1(G)$ such that $\mu^n = f\,\mathrm{dm}_G + \mu_s$, with μ_s singular with respect to m_G and $\|\mu_s\| < 1$. This yields $\|\lambda_1(\mu^n) - \lambda_1(f\mathrm{m}_G)\| = \|\mu^n - f\mathrm{m}_G\| = \|\mu_s\| < 1$. Since $\lambda_1(f\mathrm{m}_G)$ is a compact operator by Theorem 4 of [**1**], we get that $\lambda_1(\mu)$ is quasi-compact.

If $\lambda_1(\mu)$ is quasi-compact, so will be its restriction $\lambda_1^0(\mu)$ to $L_1^0(G)$. So (iv) implies (v).

Finally, (v) implies (ii) by Theorem 2.4 and Corollary 4.5. □

The implication (iii) ⇒ (ii) is specific to convolutions. The example in [**34**, pp. 213–214] shows that a stationary Markov operator P may satisfy (iii) without being uniformly ergodic on L_1.

Remark 4.8. By Corollaries 4.3 and 3.3, Theorem 4.7 actually characterizes uniform mean ergodicity of $\lambda_1(\mu)$ for any probability measure and every locally compact group.

5. Spectral characterization of uniformly ergodic probability measures

The deep connection between uniform ergodicity and spectral properties of operators is set forth in Theorem 2.3. It is immediately clear that a uniformly mean ergodic operator cannot have 1 as an accumulation point of its spectrum. If dealing with a normal operator defined on a Hilbert space, then the converse is true under mild conditions. This is the departure point of the following result, proved as Theorem 4.10 in our previous paper [**16**].

Theorem 5.1. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure with $\mu^* * \mu = \mu * \mu^*$. Then $\lambda_1(\mu)$ is uniformly mean ergodic if and only if H_μ is compact and 1 is an isolated point of $\sigma(\lambda_1(\mu)) = \sigma(\mu)$.*

We have used here that $\sigma(\mu) = \sigma(\lambda_1(\mu))$. This is an easy consequence of Wendel's theorem [**36**] to the effect that a bounded operator T on $L_1(G)$ that commutes with right translations is necessarily of the form $T = \lambda_1(\nu)$ for some $\nu \in M(G)$. In particular, when $s \notin \sigma(\lambda_1(\mu))$, the operator $T = (\lambda_1(\mu) - sI)^{-1}$ will correspond to the convolution operator associated to the measure $(\mu - s\delta_e)^{-1}$.

The main objective of this section is to remove the normality condition $\mu^* * \mu = \mu * \mu^*$ in Theorem 5.1, and, therefore, to obtain a full spectral characterization of uniformly ergodic measures.

5.1. When 1 is an accumulation of $\sigma(\mu)$, μ is adapted. Our analysis will rely on the well-known Riesz idempotent operators that spectral sets induce. We describe them below for the convenience of the reader.

Theorem 5.2 (Riesz idempotent operator, [**27**, Theorem 4.20])**.** *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that Δ is a closed and open subset of $\sigma(T)$. Consider a function $\psi\colon \Lambda \to \mathbb{C}$ that is analytic on an open set $\Lambda \subseteq \mathbb{C}$ that contains $\sigma(T)$ and satisfies*

$$\psi(\Delta) = \{1\} \quad \text{and} \quad \psi(\sigma(T) \setminus \Delta) = \{0\}.$$

The function ψ then defines, through Riesz functional calculus (see Theorem 4.7 from [**10**]*), an operator $E_\Delta \in \mathcal{L}(X)$, with the following properties:*

(i) *E_Δ is a projection.*
(ii) *If $S \in \mathcal{L}(X)$ and $ST = TS$, then $E_\Delta S = SE_\Delta$.*
(iii) *$\sigma(TE_\Delta) = \Delta \cup \{0\}$.*
(iv) *$\sigma(T|_{E_\Delta(X)}) = \Delta$.*

We now need a few lemmas. The first one can be stated in terms of multipliers of Banach algebras. Recall that a right multiplier M of a Banach algebra $\mathcal{A}$ is a bounded linear operator $M\colon \mathcal{A} \to \mathcal{A}$ such that $M(ab) = aM(b)$.

Lemma 5.3. *Let $\mathcal{A}$ be a Banach algebra and let $M_1, M_2\colon \mathcal{A} \to \mathcal{A}$ be two commuting right multipliers of $\mathcal{A}$. Assume that M_2 is a projection and set $J := M_2(\mathcal{A})$. If $M_1|_J$ is invertible, then there is a right multiplier $M'\colon \mathcal{A} \to \mathcal{A}$ such that $M'|_J = (M_1|_J)^{-1}$.*

Proof: Let $C := (M_1|_J)^{-1}$ and define $\widetilde{C}\colon \mathcal{A} \to \mathcal{A}$ by

$$\widetilde{C} = C \circ M_2.$$

Since $\widetilde{C}|_J = C$, it will be enough to check that $\widetilde{C}$ is a right multiplier.

Let $a, b \in \mathcal{A}$. Then

$$aCM_2(b) = CM_1(aCM_2(b)) = C(aM_1CM_2(b)) = C(aM_2(b)).$$

It follows that

$$a\widetilde{C}(b) = C(aM_2(b)) = C(M_2(ab)) = \widetilde{C}(ab). \qquad \square$$

The second lemma is a well-known linear algebra fact.

Lemma 5.4. *Let T_1 and T_2 be commuting bounded operators on a complex Banach space X. If $z \in \mathbb{C}$ is an eigenvalue of T_1 and its eigenspace $V_z = \{v \in X : T_1 v = zv\}$ is finite-dimensional, then there exists $0 \neq u \in V_z$, which is an eigenvector for T_2.*

Proof: Since T_1 and T_2 commute, $T_2|_{V_z}\colon V_z \to V_z$ is a linear operator on the finite-dimensional space V_z and therefore has at least one eigenvalue. All eigenvectors of $T_2|_{V_z}$ are then eigenvectors of T_2. $\square$

Theorem 5.5. *Let G be a compact group and $\mu \in M(G)$ be an adapted probability measure. The following assertions are equivalent:*

(i) *1 is an isolated point of $\sigma(\mu)$.*
(ii) *μ is spread out.*
(iii) *$\lambda_1(\mu)$ is uniformly mean ergodic.*

Proof: After Theorem 4.7, which shows that (ii) implies (iii), and Theorem 2.3, giving that (iii) implies (i), it only remains to prove that (i) implies (ii).

Since $\{1\}$ is open and closed in $\sigma(\lambda_1(\mu))$, there is $0 < r < 1$ small enough such that $B(1, r) \cap \sigma(\mu) = \{1\}$, where $B(z, r)$ denotes the ball of radius r centered in $z \in \mathbb{C}$.

Now we consider $K := \overline{B\big(0,1+\frac{r}{2}\big)} \setminus B(1,r) \cup \overline{B\big(1,\frac{r}{2}\big)}$. Clearly, $\sigma(\mu) \subseteq K$. Define ψ on K as

$$\psi(z) = \begin{cases} 0, & \text{when } z \in \overline{B}\big(0,1+\frac{r}{2}\big) \setminus B(1,r), \\ 1, & \text{when } z \in \overline{B}\big(1,\frac{r}{2}\big). \end{cases}$$

Since the function ψ can be analytically extended to an open set Λ containing K, the Riesz functional calculus defines an operator $E_1 = \psi(\lambda_1(\mu)) \in \mathcal{L}(L_1(G))$ with the properties stated in Theorem 5.2. Properties (i) and (ii) of that theorem show that E_1 is a projection that commutes with right translations. Wendel's classical theorem (see [**36**]) shows then that $E_1 = \lambda_1(\nu)$ for some idempotent $\nu \in M(G)$.

Let $\pi\colon G \to \mathcal{U}(\mathbb{H}_\pi)$ be an irreducible representation of G, different from $\mathbf{1}$, the trivial one-dimensional representation. Assume that $\widehat{\nu}(\pi) \neq 0$.

If the only eigenvalue of $\widehat{\nu}(\pi)$ was 0, the characteristic polynomial of $\widehat{\nu}(\pi)$ would be $P(z) = z^n$ and the Cayley–Hamilton theorem would imply that $(\widehat{\nu}(\pi))^n = 0$. But ν is idempotent and $(\widehat{\nu}(\pi))^n = \widehat{\nu^n}(\pi) = \widehat{\nu}(\pi)$. With this and Lemma 5.4, we conclude that there are $z_0, z \in \mathbb{C}$, $z_0 \neq 0$, and $\xi \in \mathbb{H}_\pi$, $\xi \neq 0$, such that

$$\begin{aligned} \widehat{\nu}(\pi)\xi &= z_0\xi \quad \text{and} \\ \widehat{\mu}(\pi)\xi &= z\,\xi. \end{aligned}$$

The measure ν being idempotent, we have that, actually, $z_0 = 1$. Hence, $\widehat{(\mu * \nu)}(\pi)\xi = z\xi$. As we know that $\sigma(\mu * \nu) = \{0, 1\}$ (statement (iii) of Theorem 5.2) and that 1 cannot be an eigenvalue of $\widehat{\mu}(\pi)$ (we are using here Corollary 2.13(ii)), it follows from (iii) of Theorem 2.12 that

$$\begin{aligned} \widehat{\nu}(\pi)\xi &= \xi \quad \text{and} \\ \widehat{\mu}(\pi)\xi &= 0. \end{aligned}$$

On the other hand, statement (iv) of Theorem 5.2 shows that $0 \notin \sigma(\lambda_1(\mu)|_{\nu * L_1(G)})$. There is therefore an operator $C\colon \nu * L_1(G) \to \nu * L_1(G)$ inverse to $\lambda_1(\mu)|_{\nu * L_1(G)}$. We now apply Lemma 5.3 to $M_1 = \lambda_1(\mu)$ and $M_2 = \lambda_1(\nu)$. The conclusion of the lemma together with Wendel's theorem provides us with a measure $\sigma \in M(G)$ such that

$$\sigma * \mu * \nu * f = \nu * f, \quad \text{for every } f \in L_1(G). \tag{2}$$

Let $f_{\pi,\xi} \in L_1(G)$ be as defined in Corollary 2.13(i). If we take Fourier–Stieltjes transforms in equation (2) with $f = f_{\pi,\xi}$ and apply them to the representation π and then to the vector ξ (equation (2) is actually used in the third equality), we get:

$$\begin{aligned} \xi = \widehat{\nu}(\pi)\xi &= \widehat{\nu}(\pi)\widehat{f_{\pi,\xi}}(\pi)\xi \\ &= \widehat{\sigma}(\pi)\widehat{\mu}(\pi)\widehat{\nu}(\pi)\widehat{f_{\pi,\xi}}(\pi)\xi \\ &= \widehat{\sigma}(\pi)\widehat{\mu}(\pi)\xi = 0. \end{aligned}$$

This is in contradiction with our choice of ξ. It follows that $\widehat{\nu}(\pi) = 0$ for every π different from the trivial one-dimensional representation $\mathbf{1}$.

Since $\widehat{\nu}(\mathbf{1}) = 1$ (it has to be a nonzero idempotent complex number), we conclude that $\widehat{\nu}(\pi) = \widehat{\mathrm{m}_G}(\pi)$ for every irreducible representation π of G and, hence (Theorem 2.12(ii)), that $\nu = \mathrm{m}_G$.

We have thus that $E_1 = \psi(\lambda_1(\mu)) = \lambda_1(\mathrm{m}_G)$. Finally, we observe that, since $\mathbb{C}\setminus K$ is connected and ψ can be holomorphically extended to an open set containing K, there is, by Runge's theorem (see [**10**, Corollary III.8.5]), a sequence $(\psi_n)_{n\in\mathbb{N}}$ of polynomials such that $\lim \psi_n = \psi$ uniformly on K. Having identified $E_1 = \psi(\lambda_1(\mu))$ with $\lambda_1(\mathrm{m}_G)$, we see that $\lambda_1(\mathrm{m}_G) = \lim \psi_n(\lambda_1(\mu))$. The fact that $\lambda_1 \colon M(G) \to \mathcal{L}(L_1(G))$ is a linear isometry yields that for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ with

$$\|\mathrm{m}_G - \psi_n(\mu)\| < \varepsilon.$$

Now, ψ_n being a polynomial, functional calculus shows that, for each $n \in \mathbb{N}$, there are $k_n \in \mathbb{N}$, and $(\alpha_{j,n})_{j=0}^{k_n} \subset \mathbb{C}$ such that $\psi_n(\mu) = \sum_{j=0}^{k_n} \alpha_{j,n}\mu^j$, which implies that not all μ^k can be singular, since singular measures constitute a closed subspace of $M(G)$. □

Remark 5.6. The existence of Rajchman measures in $\mathbb{T}$ (measures whose Fourier–Stieltjes transforms vanish at infinity in $\mathbb{Z}$) that are not spread out shows that one cannot replace "1 isolated in $\sigma(\mu)$" with "1 isolated in $\sigma(\lambda_2(\mu))$" (which, in the case of abelian groups, coincides with the *natural* spectrum $\overline{\mu(\widehat{G})}$). Examples of such measures can be deduced from the proof of Theorem 3.9 of [**38**]; see also Remark 6.4 in [**16**].

5.2. Connecting the spectrum of $\boldsymbol{\lambda_1(\mu)}$ with that of $\boldsymbol{\lambda_1^0(\mu)}$ and $\boldsymbol{\lambda_1(\underline{\mu})}$. We see in this subsection that once 1 is an accumulation point in the spectrum of one of the convolution operators $\lambda_1(\mu)$, $\lambda_1^0(\mu)$, or $\lambda_1(\underline{\mu})$, it is an accumulation point in the other two spectra. This leads to clean characterizations of uniform ergodicity where any of them can be used.

The following notations concerning a bounded operator $T \colon X \to X$ on a Banach space X will prove useful in this section. The symbol $\sigma_{\mathrm{ap}}(T)$ will denote the approximate spectrum of T. We recall that $z \in \sigma_{\mathrm{ap}}(T)$ whenever there exists a sequence (x_n) in the unit sphere of X such that $\lim \|T(x_n) - zx_n\| = 0$. If $A \subset \mathbb{C}$, ∂A will denote the boundary of A and $\mathrm{Acc}(A)$ will stand for the set of cluster points of a set A.

Proposition 5.7. *Let $T \colon X \to X$ be a continuous and linear operator on a Banach space X with $\|T\| = 1$. Then, $1 \in \mathrm{Acc}(\sigma(T))$ if, and only if, $1 \in \mathrm{Acc}(\sigma_{\mathrm{ap}}(T))$.*

Proof: One direction is obvious; we prove the other one. Assume that $1 \in \mathrm{Acc}(\sigma(T))$. Then, for every $n \in \mathbb{N}$, there is $z_n \in \sigma(T) \cap \left\{w \in \mathbb{C} : d(w,1) \leq \frac{1}{n}\right\} \setminus \{1\}$ where d here stands for the Euclidean distance in $\mathbb{C}$.

Let $t_n = \sup\{t \geq 0 : [z_n - ti, z_n + ti] \subseteq \sigma(T)\}$, where $[z_n - ti, z_n + t_i]$ denotes the obvious vertical segment in $\mathbb{C}$. Since $\sigma(T)$ is a compact set, one can choose correctly the sign so that $w_n = z_n \pm t_n i \in \partial\sigma(T)$.

Now define x_n as the intersection of the line $1 - \frac{1}{n} + ti$ with $\mathbb{T}$, the boundary of $\mathbb{D}$, the unit disk, so $x_n = 1 - \frac{1}{n} \pm \frac{\sqrt{2n-1}}{n} i$. Since $\sigma(T) \subseteq \overline{\mathbb{D}}$, we have that, by construction, $0 \leq t_n \leq \frac{\sqrt{2n-1}}{n}$. This concludes the proof, for $w_n \in \partial\sigma(T) \subseteq \sigma_{\mathrm{ap}}(T)$ [**10**, Chapter VII, Proposition 6.7], and we have just seen that $\lim_n w_n = \lim_n z_n = 1$. □

Proposition 5.8. *Let $\mu \in M(G)$ be a probability measure on a noncompact group G. Then $\sigma_{\mathrm{ap}}(\lambda_1^0(\mu)) = \sigma_{\mathrm{ap}}(\lambda_1(\mu))$.*

Proof: We only have to show that $\sigma_{\mathrm{ap}}(\lambda_1(\mu)) \subseteq \sigma_{\mathrm{ap}}(\lambda_1^0(\mu))$. To that end, let $z \in \sigma_{\mathrm{ap}}(\lambda_1(\mu))$, then there is a sequence $(f_n)_n \subseteq L_1(G)$, with $\|f_n\|_1 = 1$ and satisfying $\lim_n \|\mu * f_n - zf_n\| = 0$. Since, for each $n \in \mathbb{N}$, $\|\lambda_1^0(f_n \cdot m_G)\| = 1$, by Proposition 3.2 there exists $g_n \in L_1^0(G)$, with $\|g_n\|_1 = 1$ such that $\frac{1}{2} < \|f_n * g_n\|_1 \leq 1$.

We conclude by observing that $(f_n * g_n)/\|f_n * g_n\|_1 \in L_1^0(G)$ and that

$$\lim_{n\to\infty} \left\| \mu * \frac{f_n * g_n}{\|f_n * g_n\|_1} - z \cdot \frac{f_n * g_n}{\|f_n * g_n\|_1} \right\|_1 \leq 2 \lim_{n\to\infty} \|\mu * f_n * g_n - z \cdot f_n * g_n\|_1$$

$$\leq 2 \lim_{n\to\infty} \|\mu * f_n - z \cdot f_n\|_1 = 0. \qquad \square$$

Corollary 5.9. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure.*

(i) 1 *is isolated in* $\sigma(\lambda_1(\mu))$ *if and only if* 1 *is not an accumulation point of* $\sigma(\lambda_1^0(\mu))$.
(ii) 1 *is isolated in* $\sigma(\lambda_1(\mu))$ *if and only if* 1 *is isolated in* $\sigma(\lambda_1(\underline{\mu}))$.

Proof: We first prove (i). Our argument will be different depending on whether G is compact or not.

If G is compact, the map $f \mapsto (f - \int f \, \mathrm{d}m_G) \oplus \int f \, \mathrm{d}m_G$ establishes a linear isometry between $L_1(G)$ and $L_1^0(G) \oplus \mathbb{C}$. Moreover, since both the closed hyperplane $L_1^0(G)$ and its complement are $\lambda_1(\mu)$-invariant, we can decompose $\lambda_1(\mu)$ as $\lambda_1(\mu) = \lambda_1^0(\mu) \oplus \mathrm{Id}$. It follows that $\sigma(\lambda_1^0)(\mu) \cup \{1\} = \sigma(\lambda_1(\mu))$, which certainly implies the result.

If G is not compact, Propositions 5.7 and 5.8 together yield the result directly.

We now prove statement (ii). Since $M(H_\mu) \subseteq M(G)$ as unitary Banach algebras, it follows that

$$\sigma(\lambda_1(\mu)) = \sigma(\mu) \subseteq \sigma(\underline{\mu}) = \sigma(\lambda_1(\underline{\mu})).$$

Again, we just have to show that if 1 is an accumulation point of $\sigma(\lambda_1(\underline{\mu}))$, then 1 also accumulates in $\sigma(\lambda_1(\mu))$. We can deduce from Proposition 5.7 that it suffices to show that $\sigma_{\mathrm{ap}}(\lambda_1(\underline{\mu})) \subseteq \sigma_{\mathrm{ap}}(\lambda_1(\mu))$. Let $\lambda \in \sigma_{\mathrm{ap}}(\lambda_1(\underline{\mu}))$. There is $f_n \in L_1(H_\mu)$ such that $\|f_n\| = 1$ and $\lim_n \|\mu * f_n - \lambda f_n\| = 0$. Now, we observe that, for every n, $f_n \, \mathrm{d}m_H \in M(G)$ and $1 = \|f_n\| = \|\lambda_1(f_n \, \mathrm{d}m_H)\|$, therefore there will be $g_n \in L_1(G)$, $\|g_n\| = 1$, such that $\|\lambda_1(f_n)(g_n)\| = \|f_n * g_n\| > \frac{1}{2}$. Choosing $h_n := (1/\|f_n * g_n\|) f_n * g_n \in L_1(G)$ we get

$$\lim_n \|\mu * h_n - \lambda h_n\| = 0,$$

and $\lambda \in \sigma_{\mathrm{ap}}(\lambda_1(\mu))$. $\square$

5.3. The spectral characterizations. The results of the previous subsections can now be put to work.

Theorem 5.10. *Let G be a compact group and $\mu \in M(G)$ be an adapted probability measure. The following are equivalent:*

(i) μ *is uniformly ergodic.*
(ii) $\lambda_1(\mu)$ *is uniformly mean ergodic.*
(iii) μ *is spread out.*
(iv) $\lambda_1(\mu)$ *is quasi-compact.*
(v) $\lambda_1^0(\mu)$ *is quasi-compact.*
(vi) 1 *is an isolated point of* $\sigma(\mu)$.
(vii) 1 *is not an accumulation point of* $\sigma(\lambda_1^0(\mu))$.

Proof: Statements (i)–(v) have already been shown to be equivalent, by Theorem 4.7. Statement (ii) implies statement (vi) by Theorem 2.3 together with the equality $\sigma(\lambda_1(\mu)) = \sigma(\mu)$. Statement (vi) implies statement (iii), by Theorem 5.5. Statements (vi) and (vii) are equivalent by Corollary 5.9(i). $\square$

We can also characterize uniform mean ergodicity of $\lambda_1(\mu)$ for any $\mu \in M(G)$, in terms of μ itself. Recall that we denote by $\underline{\mu}$ the measure μ seen as a measure in $M(H_\mu)$.

Theorem 5.11. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure. The following statements are equivalent:*

(i) $\lambda_1(\mu)$ *is a uniformly mean ergodic operator.*

(ii) $\lambda_1^0(\mu)$ *is a uniformly mean ergodic operator.*

(iii) *The measure $\underline{\mu}$ is uniformly ergodic.*

(iv) H_μ *is compact and* 1 *is an isolated point of* $\sigma(\mu)$.

Proof: The equivalence of the first three statements follows from Theorem 5.10 and Corollary 4.5. If Corollary 3.3 is taken into account, Theorem 5.10 also shows that these three statements are equivalent to H_μ being compact and 1 being isolated in $\sigma(\lambda_1(\underline{\mu}))$. We conclude with Corollary 5.9(ii). $\square$

As already remarked after Theorem 1.1, the measure μ^* need not be ergodic when μ is. Furthermore, M. Rosenblatt ([**35**]) produces an example of a stationary (irreversible) Markov chain P that defines an operator T which is uniformly mean ergodic on L_1 while T is not uniformly mean ergodic on L_∞, so T^* is not uniformly mean ergodic on L_1. Corollary 5.12 shows that adjoints behave better in our setting.

Corollary 5.12. *Let G be a locally compact group and $\mu \in M(G)$ be a probability measure. The following statements are equivalent to statements (i)–(iv) of Theorem 5.11.*

(v) $\lambda_\infty(\mu)$ *is mean ergodic.*

(vi) $\lambda_\infty(\mu^*)$ *is mean ergodic.*

(vii) $\lambda_1(\mu^*)$ *is uniformly mean ergodic.*

Proof: We need to recall for this proof that the adjoint operator of $\lambda_1(\mu)$ is the operator $\lambda_\infty(\mu^*)\colon L_\infty(G) \to L_\infty(G)$ [**19**, Theorem 20.23]. Since inverse sets of Haar-null sets are Haar-null, and $(\mu^*)^n = (\mu^n)^*$, [**19**, Theorem 20.22], $\underline{\mu}$ will be spread out if and only if $\underline{\mu}^*$ is. Theorem 5.11 then shows that $\lambda_1(\mu)$ is uniformly mean ergodic if and only if $\lambda_1(\mu^*)$ is. Since the adjoint of a uniformly mean ergodic operator is uniformly mean ergodic as well and $\lambda_\infty(\mu)$ is the adjoint of $\lambda_1(\mu^*)$, the equivalence of statement (v) here and statement (i) of Theorem 5.11 follows. To conclude the proof we only need Lotz's theorem [**31**, Theorem 5], which ensures that a power-bounded operator on $L_\infty(G)$ is uniformly mean ergodic whenever it is mean ergodic. $\square$

We also note the following *abelian* consequence of Theorem 5.5.

Corollary 5.13. *Let G be a locally compact* abelian *group. The following are equivalent.*

(i) $\lambda_1(\mu)$ *is uniformly mean ergodic.*

(ii) 1 *is isolated in* $\sigma(\mu)$.

Proof: Theorem 5.11 shows that (i) implies (ii). By the same characterization, the converse statement will be proved if we show that 1 isolated in $\sigma(\mu)$ implies that H_μ is compact.

If H_μ is not compact, then $\widehat{H_\mu}$ is not discrete and one can find a net $\{\chi_\alpha\}_\alpha \subseteq \widehat{H_\mu}\setminus\{\mathbf{1}\}$ that converges to the trivial character $\mathbf{1}$. The net $\{\widehat{\underline{\mu}}(\chi_\alpha)\}_\alpha$ is then contained in $\mathbb{T}\setminus\{1\}$, by Corollary 2.13, and converges to 1. It follows that 1 is not isolated in $\sigma(\lambda_1(\underline{\mu}))$. Corollary 5.9(ii) then shows that 1 is not isolated in $\sigma(\mu)$. $\square$

Remark 5.14. As we have observed, uniform ergodicity of $\lambda_1(\mu)$ is a strong property. Indeed, uniformly ergodic stationary Markov operators on L_1 are uniformly mean ergodic on each L_p, $1 < p < \infty$, [**35**, Theorem 1, p. 211]. The converse does not hold; the adjoint T^* of the operator T constructed in [**34**, pp. 213–214] is uniformly mean ergodic in L_2, since T is, but is not uniformly mean ergodic in L_1. Careful constructions involving Rajchman measures provide examples of convolution operators that are uniformly mean ergodic on L_2 but are not uniformly ergodic in L_1. Constructions of that sort can be found in Theorem 5.6 of [**7**] (where Rajchman measures whose spectrum is the whole disk are constructed) and in Theorem 6.2 of [**9**] (where Rajchman measures that are not spread out are constructed).

It may be worth remarking that, although the operator $\lambda_2(\mu)$ can be uniformly mean ergodic without $\lambda_1(\mu)$ being so, the proof of the preceding corollary shows that, still, for $\lambda_2(\mu)$ to be uniformly mean ergodic it is necessary that H_μ is compact.

Remark 5.15. The authors of [**32**] observe that Host and Parreau's characterization of the measures $\mu \in M(G)$ such that $\mu * L_1(G)$ is closed in $M(G)$ ([**22**]) implies that $\lambda_1(\mu)$ is uniformly mean ergodic if and only if μ is of the form $\mu = \delta_e + \lambda * \theta$, with λ an invertible measure and θ an idempotent measure. Hence, Theorems 5.10 and 5.11 together show that this description for probability measures is equivalent to H_μ being compact and $\underline{\mu}$ being spread out.

5.4. Uniform convergence of powers of convolution operators. Our results on uniform ergodicity can also be translated to uniform complete mixing and, more generally, to the convergence of the powers of $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$.

We first characterize, see Theorem 5.16 below, uniformly completely mixing measures. The equivalence between statements (i) and (iii) of this characterization follows from Theorem 4.1 of [**3**]. Since condition (ii) is trivially equivalent to (i) under the considered hypothesis, only the equivalence between (iv) and (ii) needs to be proved. For the sake of completeness, we also offer here a short alternative proof of the equivalence between the first three statements based on Theorem 5.10 and classic ergodic arguments.

Theorem 5.16. *Let G be a compact group and let $\mu \in M(G)$ be an adapted probability measure. The following assertions are equivalent.*

(i) *The sequence $(\lambda_1(\mu^n))$ is norm convergent.*
(ii) *The measure μ is uniformly completely mixing.*
(iii) *μ is spread out and strictly aperiodic.*
(iv) *1 is isolated in $\sigma(\mu)$ and $\sigma(\mu) \cap \mathbb{T} = \{1\}$.*
(v) *$\lambda_1(\mu)$ is uniformly mean ergodic and $\|\lambda_1(\mu^{n+1}) - \lambda_1(\mu^n)\| \to 0$.*

Proof: Since G is compact, $\lambda_1(\mu) = \lambda_1^0(\mu) \oplus \mathrm{Id}$, hence Proposition 4.2 shows that (i) and (ii) are equivalent. Condition (ii) implies that $\lambda_1^0(\mu)$ is uniformly mean ergodic and μ is completely mixing. Uniform ergodicity implies that μ is spread out (Theorem 5.10). Completely mixing and Lin and Wittmann's Theorem 2.9(ii) yield that μ is strictly aperiodic. Hence (ii) implies (iii).

If we assume (iii), then $\lambda_1^0(\mu)$ is quasi-compact by Theorem 5.10, and Kawada and Ito's Theorem 2.9(iii) yields that $(\lambda_1^0(\mu^n))$ is SOT convergent to 0, hence $\sigma_p(\lambda_1^0(\mu)) \cap \mathbb{T} = \varnothing$. Yosida and Kakutani's Theorem 2.4 implies then that $(\|\lambda_1^0(\mu^n)\|)$ is convergent to 0, hence we have (ii). Statements (i)–(iii) are thus shown to be equivalent.

Assume that the first three equivalent conditions hold. In a Banach algebra the elements whose sequence of powers is convergent to 0 are those whose spectral radius is smaller than 1, hence (ii) implies $r(\lambda_1^0(\mu)) < 1$. Since $\sigma(\mu) = \sigma(\lambda_1(\mu)) = \sigma(\lambda_1^0(\mu)) \cup \{1\}$, (iv) necessarily holds.

Suppose conversely that (iv) holds. Then μ is uniformly ergodic (Theorem 5.10) and $1 \notin \sigma(\lambda_1^0(\mu))$, for otherwise also $1 \in \sigma(\lambda_1^0(\mu_{[n]}))$ for each $n \in \mathbb{N}$, against uniform ergodicity. Using again that $\sigma(\mu) = \sigma(\lambda_1(\mu)) = \sigma(\lambda_1^0(\mu)) \cup \{1\}$ we conclude that $r(\lambda_1^0(\mu)) < 1$, which is equivalent to (ii).

Condition (v) being equivalent to condition (iv) is the theorem of Katznelson and Tzafriri [**24**]. □

Remark 5.17. The previous theorem, together with Proposition 3.2, provides a solution to the uniform version of the complete mixing: a uniformly ergodic and strictly aperiodic measure is necessarily uniformly completely mixing.

If we do not assume μ to be adapted, Theorem 5.16 can be stated as follows.

Theorem 5.18. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure. The following statements are equivalent:*

(i) $(\lambda_1(\mu^n))$ *is convergent in* $(\mathcal{L}(L_1(G)), \|\cdot\|)$.
(ii) $(\lambda_1^0(\mu^n))$ *is convergent in* $(\mathcal{L}(L_1^0(G)), \|\cdot\|)$.
(iii) $\lambda_1^0(\mu)$ *is uniformly mean ergodic and* $(\lambda_1^0(\mu^n))$ *is convergent in* $\mathcal{L}(L_1^0(G))$ *endowed with the strong operator topology.*
(iv) H_μ *is compact,* $\underline{\mu}$ *is strictly aperiodic, and* 1 *is isolated in* $\sigma(\mu)$.

Proof: (i) implies (ii) and (ii) implies (iii) are trivial. Assertion (iii) implies that $\underline{\mu}$ is uniformly ergodic by Theorem 5.11. This implies that $\sigma_p(\lambda_1^0(\underline{\mu})) \cap \mathbb{T} \subseteq \{1\}$ and, by Theorem 5.10, that $\lambda_1^0(\underline{\mu})$ must be quasi-compact. Theorem 2.4 proves then that $\underline{\mu}$ is uniformly completely mixing. Statement (iv) then follows from Theorem 5.16, via Corollary 3.3, and Corollary 5.9. This latter corollary shows that we can get statement (i) from statement (iv) by applying Theorem 5.16 to $\underline{\mu}$ and then Corollary 4.3. □

If G is connected, the condition of strict aperiodicity can be dropped from Theorem 5.16.

Theorem 5.19. *Let G be a locally compact* connected *group and $\mu \in M(G)$ be a probability measure. Then the following statements are equivalent:*

(i) μ *is uniformly completely mixing.*
(ii) μ *is uniformly ergodic.*
(iii) G *is compact and* μ *is spread out.*

Proof: It is trivial that (i) implies (ii). Conditions (ii) and (iii) are equivalent by Corollary 3.3 and Theorem 4.7. We see that the conjunction of (iii) and (ii) implies (i). Let μ be a uniformly ergodic measure on a compact group G. By Theorem 5.16 we only need to show that μ is strictly aperiodic. We proceed by contradiction and assume that there are a proper normal closed subgroup H and a point $x \in G$ such that $S_\mu \subseteq xH$. Note that $\mathrm{m}_G(H) = 0$, for otherwise it would be open and, bearing in mind that G is connected, we would have $H = G$. Now we have

$$\bigcup_{1 \le j,k \le n} S_\mu^{-j} S_\mu^k \subseteq \bigcup_{j=-n}^{n} x^j H,$$

for each $n \in \mathbb{N}$. Since $\mathrm{m}_G(H) = 0$ we can apply Proposition 3.4, to conclude that μ is not uniformly ergodic. □

Remark 5.20. If we add G compact to the hypothesis of the previous theorem (a condition that, as it turns out, is necessary), the resulting statement is completely equivalent to Theorem 3 of [**6**], where Bhattacharya proves that the sequence (μ^n) converges to m_G for every nonsingular probability measure μ on a connected compact group G. This statement is formally weaker than Theorem 5.19. If one only assumes that μ is spread out as is done in Theorem 5.19, [**6**, Theorem 3] would just show that

$\lim_n \|\mu^{kn} - \mathrm{m}_G\| = 0$ for some $k \in \mathbb{N}$. But, as a matter of fact, from here one readily deduces that μ is uniformly completely mixing. It suffices to factorize, for any $n \in \mathbb{N}$, $\mu^n = \mu^{km_n} * \mu^{j_n}$, for some $0 \leq j_n < n$ and $m_n \in \mathbb{N}$ and recall that $\mathrm{m}_G * \nu = \mathrm{m}_G$ for any probability measure ν.

Other proofs of this can be found in the literature; see for instance [**33**, Chapter 6, Exercise 3.19] or [**3**, Corollary 4.2, Remark (5)].

Example 5.21. Let A be an arc 0 in $\mathbb{T}$ of length less than 1/2, centered at 1. Then $\mu = \frac{1}{\mathrm{m}_{\mathbb{T}}(A)} \mathbf{1}_A \mathrm{m}_{\mathbb{T}}$ is uniformly completely mixing, even if $\|\lambda_1^0(\mu)\| = 1$, by Lemma 3.1.

Remark 5.22. The above example shows that for an adapted probability measure μ on a compact group, contrary to what happens with $\lambda_1(\mu)$, the inequality $r(\lambda_1^0(\mu)) < \|\lambda_1^0(\mu)\|$ is possible. The inequality $r(\lambda_1^0(\mu)) < 1$ for a probability $\mu \in M(G)$ is actually equivalent, for any locally compact group G, to μ being uniformly completely mixing by the very definition and basic theory of Banach algebras. Hence, the inequality $r(\lambda_1^0(\mu)) = \|\lambda_1^0(\mu)\| = 1$ is fulfilled by any probability $\mu \in M(G)$ which is not uniformly completely mixing.

If G is abelian but not connected, measures failing to satisfy Theorem 5.19 can always be constructed. We thus have the following characterization of connected groups in the category of compact abelian groups.

Theorem 5.23. *Let G be a compact abelian group. The following assertions are equivalent:*

(i) *G is connected.*
(ii) *Every uniformly ergodic probability measure on G is uniformly completely mixing.*

Proof: After Theorem 5.19 we only have to show that statement (ii) implies statement (i). Suppose to that end that G is not connected. Then G has a maximal proper normal subgroup H which is open (since $\widehat{G}$ is not torsion-free [**19**, 24.25], there must exist $\chi \in \widehat{G}$ such that $\langle \chi \rangle$ has a prime number of elements, then $H = \langle \chi \rangle^{\perp}$).

Let then $\mu = \mathbf{1}_{xH} \mathrm{m}_G$, for some $x \notin H$. It is easy to see that $H \subsetneq \langle xH \rangle$, while the maximality of H shows that $\overline{H_\mu} = \overline{\langle xH \rangle} = G$. The measure μ is hence adapted and spread out, but it is not strictly aperiodic by construction. Theorems 4.7 and 2.9 then suffice to show that μ is a uniformly ergodic probability measure but it is not uniformly completely mixing. □

Commutativity is essential in the above example. If G is a simple group or, more generally, if G admits no nontrivial continuous characters (see [**30**, Lemma 3.4]), then every adapted measure is strictly aperiodic. The theorem is no longer true either, if the condition *uniform* is dropped. If $x \in \mathbb{T}$ is not a root of unity, then $\delta_x \in M(\mathbb{T})$ is ergodic but not strictly aperiodic.

5.5. A characterization of quasi-compactness of convolution operators. The next theorem should be compared to Theorem 2 from [**29**] and Theorem 3.4 of [**33**], where the equivalence between (i) and (ii) below is proved for a more general class of operators.

Theorem 5.24. *Let G be a locally compact group and let μ be a probability measure. Let T stand for $\lambda_1(\mu)$, $\lambda_\infty(\mu)$, or $\lambda_1^0(\mu)$. The following assertions are equivalent:*

(i) *The operator T is quasi-compact.*
(ii) *$(T_{[n]})_n$ is norm convergent to a finite-dimensional projection.*
(iii) *G is compact, H_μ is open in G, and $\underline{\mu}$ is spread out.*

Proof: That statement (i) implies statement (ii) is the Yosida–Kakutani theorem, Theorem 2.4.

Let us see that (ii) implies (iii). If (ii) holds, then H_μ is compact, by Corollary 5.12. We now proceed by contradiction and suppose that G is either noncompact or that G is compact and H_μ is not open in G. In both cases, we can choose a sequence $(x_k) \subseteq G \setminus H_\mu$ such that $(H_\mu x_k)$ is a disjoint sequence of compact subsets of G. We fix $n \in \mathbb{N}$. One can get a compact neighborhood of the identity U such that $H_\mu x_i U \cap H_\mu x_j U = \varnothing$ whenever $1 \leq i < j \leq n$. To find U, first we choose inductively compact neighborhoods of the identity U_i, $1 \leq i \leq n$, such that $H_\mu x_i \cap H_\mu x_l U_l = \varnothing$ when $1 \leq l < i \leq n$ and $H_\mu x_i U_i \cap H_\mu x_j = \varnothing$ for $1 \leq i < j \leq n$. The set $U := \bigcap_{i=1}^n U_i$ satisfies the required condition. We observe that if f_i is the characteristic function of $x_i U$, then $\operatorname{supp}(\mathrm{m}_{H_\mu} * f_i) = H_\mu x_i U$ for each $i \neq j \leq n$. Hence the dimension of $\lambda_1(\mathrm{m}_{H_\mu})(L_1(G))$ and the dimension of $\lambda_\infty(\mathrm{m}_{H_\mu})(L_\infty(G))$ are at least n. By observing how the convolution operator acts on $f_{ij} = f_i - f_j$, $i \neq j$, we get also that the dimension of $\lambda_1^0(\mathrm{m}_{H_\mu})(L_1^0(G))$ is at least n. Since n is arbitrary, we get that, in all the considered cases, the projection limit of $T_{[n]}$ is not compact.

Finally, we see that statement (iii) implies statement (i). If G is compact and μ is spread out, then there exists n such that μ^n and m_{H_μ} are not singular. If we assume in addition H_μ to be open in G, then μ^n and m_G are not singular either. Hence, $\lambda_1(\mu)$ is quasi-compact. □

Example 5.25. Let $G = \mathbb{T} \oplus K$, K compact. Let $f \in L_1(\mathbb{T})$, $f \geq 0$, and $\int_{\mathbb{T}} f \, \mathrm{dm}_{\mathbb{T}} = 1$, and consider $\mu := f \mathrm{m}_{\mathbb{T}} \oplus \delta_e$. The operator $\lambda_1^0(\mu)$ is then quasi-compact if K is finite, but it is not if K is infinite. In both cases, $\lambda_1^0(\mu)$ is uniformly mean ergodic; even $(\lambda_1^0(\mu^n))$ is convergent in the norm topology.

Remark 5.26. (a) The equivalence between conditions (ii) and (vi) (or (v) and (vii)) in Theorem 5.10 and the equivalence between conditions (iv) and (v) in Theorem 5.16 do not hold for general operators. The following example was kindly provided by one of the referees. Let V be the Volterra operator on $L_2([0,1])$ and $T = I - V$. It is known that $\sigma(V) = \{0\}$, and hence that $\sigma(T) = \{1\}$ (and then $\|T^{n+1} - T^n\| \to 0$ by Katznelson and Tzafriri [**24**]), and that T is power-bounded; see e.g. [**2**]. It follows that $T^n \to 0$ in the strong operator topology. But, given that $1 \in \sigma(T_{[n]})$ for each $n \in \mathbb{N}$, $\|T_{[n]}\|$ does not converge to 0 (and hence $\|T^n\|$ does not converge to 0, either).

(b) In Theorem 5.18, we have seen that for $T = \lambda_1^0(\mu)$ the sequence (T^n) is norm convergent if and only if T is uniformly mean ergodic and the sequence (T^n) is SOT convergent. Once again, one could wonder if this statement holds for any operator. As in the previous remark, it does not. Let $a = (a_n)$ be a sequence of numbers such that $\lim a_n = -1$ and $|a_n| < 1$ for all $n \in \mathbb{N}$. The multiplication operator $M_a \colon l_1(\mathbb{N}) \to l_1(\mathbb{N})$, $(b_n) \mapsto (a_n b_n)$ satisfies $\|M_a\| = 1$ and $1 \notin \sigma(M_a) = \{a_n : n \in \mathbb{N}\} \cup \{-1\}$, hence M_a is uniformly mean ergodic by Theorem 2.2. Moreover, M_a is easily seen to be SOT-convergent to 0. However, $\|M_a^n\| = 1$ for all $n \in \mathbb{N}$.

When T is quasi-compact, norm convergence of T^n does follow from uniform mean ergodicity and SOT convergence of T^n. This is because $\sigma_p(T) \cap \mathbb{T} \subseteq \{1\}$ whenever (T^n) is a SOT convergent sequence and Theorem 2.4 applies. In Theorem 5.24 and Example 5.25 we see that, in general, uniform mean ergodicity and SOT convergence of the sequence $(\lambda_1^0(\mu^n))$ does not imply quasi-compactness of $\lambda_1^0(\mu)$.

Acknowledgements. The authors would like to express their gratitude to the anonymous referees, who read the paper carefully, and provided us with comments, references, and suggestions which have certainly contributed to improving this work.

The contribution of J. Galindo to this article is part of the grant PID2019-106529 GB-I00 funded by MCIN/AEI/10.13039/501100011033.

The research of E. Jordá was partially supported by the project PID2020-119457 GB-100 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". The research of E. Jordá is also partially supported by GVA-AICO/2021/170.

A. Rodríguez-Arenas acknowledges the support of Acció 3.2 POSDOC/2020/14 of UJI. The research was supported partially by the grant PID2019-106529GB-I00 funded by MCIN/ AEI /10.13039/501100011033.

References

[1] C. A. Akemann, Some mapping properties of the group algebras of a compact group, *Pacific J. Math.* **22(1)** (1967), 1–8. DOI: 10.2140/pjm.1967.22.1.

[2] G. R. Allan, Power-bounded elements and radical Banach algebras, in: *Linear Operators* (Warsaw, 1994), Banach Center Publ. **38**, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1997, pp. 9–16. DOI: 10.4064/-38-1-9-16.

[3] M. Anoussis and D. Gatzouras, A spectral radius formula for the Fourier transform on compact groups and applications to random walks, *Adv. Math.* **188(2)** (2004), 425–443. DOI: 10.1016/j.aim.2003.11.001.

[4] M. Anoussis and D. Gatzouras, On mixing and ergodicity in locally compact motion groups, *J. Reine Angew. Math.* **2008(625)** (2008), 1–28. DOI: 10.1515/CRELLE.2008.088.

[5] R. Azencott, *Espaces de Poisson des groupes localement compacts*, Lecture Notes in Math. **148**, Springer-Verlag, Berlin-New York, 1970.

[6] R. N. Bhattacharya, Speed of convergence of the n-fold convolution of a probability measure on a compact group, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **25** (1972/73), 1–10. DOI: 10.1007/BF00533331.

[7] G. Brown, C. Karanikas, and J. H. Williamson, The asymmetry of $M_0(G)$, *Math. Proc. Cambridge Philos. Soc.* **91(3)** (1982), 407–433. DOI: 10.1017/S0305004100059466.

[8] G. Choquet and J. Deny, Sur l'équation de convolution $\mu = \mu * \sigma$, *C. R. Acad. Sci. Paris* **250** (1960), 799–801.

[9] G. Cohen and M. Lin, L^2-quasi-compact and hyperbounded Markov operators, Preprint (2022). arXiv:2206.08003.

[10] J. B. Conway, *A Course in Functional Analysis*, Second edition, Grad. Texts in Math. **96**, Springer-Verlag, New York, 1990.

[11] C. Cuny, Weak mixing of random walks on groups, *J. Theoret. Probab.* **16(4)** (2003), 923–933. DOI: 10.1023/B:JOTP.0000012000.54810.d2.

[12] H. G. Dales, P. Aiena, J. Eschmeier, K. Laursen, and G. A. Willis, *Introduction to Banach Algebras, Operators, and Harmonic Analysis*, London Math. Soc. Stud. Texts **57**, Cambridge University Press, Cambridge, 2003. DOI: 10.1017/CBO9780511615429.

[13] Y. Derriennic and M. Lin, Convergence of iterates of averages of certain operator representations and of convolution powers, *J. Funct. Anal.* **85(1)** (1989), 86–102. DOI: 10.1016/0022-1236(89)90047-5.

[14] N. Dunford, Spectral theory. I. Convergence to projections, *Trans. Amer. Math. Soc.* **54(2)** (1943), 185–217. DOI: 10.2307/1990329.

[15] S. R. Foguel, Iterates of a convolution on a non abelian group, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **11(2)** (1975), 199–202.

[16] J. Galindo and E. Jordá, Ergodic properties of convolution operators, *J. Operator Theory* **86(2)** (2021), 469–501. DOI: 10.7900/jot.2020jun25.2303.

[17] S. Glasner, On Choquet–Deny measures, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **12(1)** (1976), 1–10.

[18] F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand Mathematical Studies **16**, Van Nostrand Reinhold Co., New York-Toronto-London, 1969.

[19] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I: Structure of Topological Groups. Integration Theory, Group Representations*, Die Grundlehren der mathematischen

Wissenschaften **115**, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. DOI: 10.1007/978-3-662-40409-6.

[20] E. HEWITT AND K. A. ROSS, *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*, Die Grundlehren der mathematischen Wissenschaften **152**, Springer-Verlag, New York-Berlin, 1970. DOI: 10.1007/978-3-662-26755-4.

[21] S. HOROWITZ, Transition probabilities and contractions of L_∞, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** (1972), 263–274. DOI: 10.1007/BF00679131.

[22] B. HOST AND F. PARREAU, Sur un problème de I. Glicksberg: les idéaux fermés de type fini de $M(G)$, *Ann. Inst. Fourier (Grenoble)* **28(3)**, (1978), 143–164. DOI: 10.5802/aif.706.

[23] W. JAWORSKI, Ergodic and mixing probability measures on [SIN] groups, *J. Theoret. Probab.* **17(3)** (2004), 741–759. DOI: 10.1023/B:JOTP.0000040297.84097.57.

[24] Y. KATZNELSON AND L. TZAFRIRI, On power bounded operators, *J. Funct. Anal.* **68(3)** (1986), 313–328. DOI: 10.1016/0022-1236(86)90101-1.

[25] Y. KAWADA AND K. ITÔ, On the probability distribution on a compact group. I., *Proc. Phys.-Math. Soc. Japan (3)* **22(12)** (1940), 977–998. DOI: 10.11429/PPMSJ1919.22.12_977.

[26] U. KRENGEL, *Ergodic Theorems*, With a supplement by Antoine Brunel, De Gruyter Stud. Math. **6**, Walter de Gruyter & Co., Berlin, 1985. DOI: 10.1515/9783110844641.

[27] C. S. KUBRUSLY, *Spectral Theory of Operators on Hilbert Spaces*, Birkhäuser/Springer, New York, 2012. DOI: 10.1007/978-0-8176-8328-3.

[28] M. LIN, On the uniform ergodic theorem, *Proc. Amer. Math. Soc.* **43** (1974), 337–340. DOI: 10.2307/2038891.

[29] M. LIN, Quasi-compactness and uniform ergodicity of Markov operators, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **11(4)** (1975), 345–354.

[30] M. LIN AND R. WITTMANN, Convergence of representation averages and of convolution powers, *Israel J. Math.* **88(1-3)** (1994), 125–157. DOI: 10.1007/BF02937508.

[31] H. P. LOTZ, Uniform convergence of operators on L^∞ and similar spaces, *Math. Z.* **190(2)** (1985), 207–220. DOI: 10.1007/BF01160459.

[32] H. MUSTAFAYEV AND H. TOPAL, Ergodic properties of convolution operators in group algebras, *Colloq. Math.* **165(2)** (2021), 321–340. DOI: 10.4064/cm8214-6-2020.

[33] D. REVUZ, *Markov Chains*, Second edition, North-Holland Math. Library **11**, North-Holland Publishing Co., Amsterdam, 1984.

[34] J. ROSENBLATT, Ergodic and mixing random walks on locally compact groups, *Math. Ann.* **257(1)** (1981), 31–42. DOI: 10.1007/BF01450653.

[35] M. ROSENBLATT, *Markov Processes. Structure and Asymptotic Behavior*, Die Grundlehren der mathematischen Wissenschaften **184**, Springer-Verlag, New York-Heidelberg, 1971. DOI: 10.1007/978-3-642-65238-7.

[36] J. G. WENDEL, Left centralizers and isomorphisms of group algebras, *Pacific J. Math.* **2(2)** (1952), 251–261. DOI: 10.2140/PJM.1952.2.251.

[37] K. YOSIDA AND S. KAKUTANI, Operator-theoretical treatment of Markoff's process and mean ergodic theorem, *Ann. of Math. (2)* **42(1)** (1941), 188–228. DOI: 10.2307/1968993.

[38] M. ZAFRAN, On the spectra of multipliers, *Pacific J. Math.* **47(2)** (1973), 609–626. DOI: 10.2140/pjm.1973.47.609.

Jorge Galindo
Instituto Universitario de Matemáticas y Aplicaciones (IMAC), Universidad Jaume I, E-12071 Castellón, Spain
E-mail address: jgalindo@uji.es

Enrique Jordá
EPS Alcoy, Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Plaza Ferrándiz y Carbonell s/n, E-03801 Alcoy, Spain
E-mail address: ejorda@mat.upv.es

Alberto Rodríguez-Arenas
Departamento de Matemáticas, Universidad Jaume I, E-12071 Castellón, Spain
E-mail address: arenasa@uji.es

Received on April 20, 2023.
Accepted on November 7, 2023.