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ON AN ALMOST SHARP LIOUVILLE-TYPE THEOREM FOR FRACTIONAL NAVIER–STOKES EQUATIONS

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Abstract: We investigate existence, Liouville-type theorems, and regularity results for the 3D stationary and incompressible fractional Navier–Stokes equations: in this setting the usual Laplacian is replaced by its fractional power $(-\Delta)^{\frac{\alpha}{2}}$ with $0 < \alpha < 2$. By applying a fixed-point argument, weak solutions can be obtained in the Sobolev space $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ and if we add an extra integrability condition, stated in terms of Lebesgue spaces, then we can prove for some values of α that the zero function is the unique smooth solution. The additional integrability condition is almost sharp for $3/5 < \alpha < 5/3$. Moreover, in the case $1 < \alpha < 2$ a gain of regularity is established under some conditions, although the study of regularity in the regime $0 < \alpha \leq 1$ seems for the moment to be an open problem.

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1. Introduction and presentation of the results

In this article we study existence, regularity, and uniqueness properties of the 3D fractional Navier–Stokes equations which are given by the following system:

$$(1.1)\qquad \begin{cases} (-\Delta)^{\frac{\alpha}{2}}\vec{u}(x) + (\vec{u}\cdot\vec{\nabla})\vec{u}(x) + \vec{\nabla}p(x) - \vec{f}(x) = 0, & \text{with } 0 < \alpha < 2, \\ \operatorname{div}(\vec{u})(x) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Here, the fractional operator $(-\Delta)^{\frac{\alpha}{2}}$ is defined at the Fourier level by the symbol $|\xi|^{\alpha}$. Using the traditional notation, the vector field $\vec{u}\colon \mathbb{R}^3 \to \mathbb{R}^3$ represents the velocity of the fluid, $p\colon \mathbb{R}^3 \to \mathbb{R}$ is the internal pressure of the fluid, and $\vec{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a given external force.

Before presenting our results related to system (1.1), it is worthwhile recalling some facts about the usual stationary Navier–Stokes equations. Indeed, note that when $\alpha = 2$, (1.1) is exactly the problem given by the classical incompressible Navier–Stokes equations

$$(1.2)\qquad \begin{cases} -\Delta\vec{u}(x) + (\vec{u}\cdot\vec{\nabla})\vec{u}(x) + \vec{\nabla}p(x) - \vec{f}(x) = 0, & \\ \operatorname{div}(\vec{u})(x) = 0, & x \in \mathbb{R}^3. \end{cases}$$

The problem (1.2) can be studied from different points of view; we first observe that the pressure p can be easily deduced from the velocity field $\vec{u}$ and the external force $\vec{f}$ since, due to the divergence-free property of $\vec{u}$, we have that

$$p = \frac{1}{(-\Delta)}\operatorname{div}((\vec{u}\cdot\vec{\nabla})\vec{u} - \vec{f}),$$

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and this fact allows us to focus our study on the velocity field $\vec{u}$ (note that the same identity can be easily deduced from system (1.1) since in both cases we have $\operatorname{div}(\vec{u}) = 0$). Now, concerning existence problems for the Navier–Stokes equations (1.2), if we assume that $\vec{f} \in \dot{H}^{-1}(\mathbb{R}^3)$ and that $\operatorname{div}(\vec{f}) = 0$, then it is an easy exercise to construct solutions $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$ (see, for instance, [**9**, Theorem 16.2]) and moreover it is not hard to prove that these solutions are regular. However, a priori it is not known whether these solutions are unique, and an interesting open problem (initially mentioned in [**4**] and also stated in [**12**]) is the following: show that any solution $\vec{u}$ of the problem

$$-\Delta\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = 0, \tag{1.3}$$

which satisfies the conditions

$$\vec{u} \in \dot{H}^1(\mathbb{R}^3) \quad \text{and} \quad \vec{u}(x) \to 0 \text{ as } |x| \to +\infty, \tag{1.4}$$

is identically equal to zero.

Note that, by the classical Sobolev embeddings, we have $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, but this does not seem enough to conclude that a solution $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$ of equation (1.3) is null. Nevertheless, if we assume some additional hypotheses, for example $\vec{u} \in E(\mathbb{R}^3)$, where E is a nice functional space, then statements of the following form have been shown:

if $\vec{u} \in \dot{H}^1(\mathbb{R}^3) \cap E(\mathbb{R}^3)$ is a solution of equation (1.3) in $\mathbb{R}^3$, then we have $\vec{u} \equiv 0$,

and this sort of result is known in the literature as a *Liouville theorem* for the Navier–Stokes equations. In [**4**] the case $E = L^{\frac{9}{2}}(\mathbb{R}^3)$ was studied. The space $E = BMO^{-1}(\mathbb{R}^3)$ was considered in [**7**] and other functional spaces can also be taken into account; for example, Morrey spaces were considered in [**6**], Lorentz spaces in [**8**], and Besov spaces in [**11**].

We observe that if we want to consider only one "simple" additional hypothesis, then a general Liouville-type theorem was proved in [**3**] with $E = L^q(\mathbb{R}^3)$ for some

$$3 \le q \le \frac{9}{2}. \tag{1.5}$$

It is very interesting to note here that there is a *gap* between this set of values and the integrability condition given in (1.4)—which is $\vec{u} \in L^6(\mathbb{R}^3)$ due to the Sobolev embedding—as at present we do not know how to fill the distance between $\frac{9}{2}$ and 6. Thus the following problem: "*show that any solution $\vec{u}$ of* (1.3)*, with $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$ and $\vec{u} \in L^q(\mathbb{R}^3)$ for some $\frac{9}{2} < q < 6$, is identically equal to zero*" remains, to the best of our knowledge, an open problem.

Let us come back now to the fractional Navier–Stokes equations (1.1). In particular, we are interested in understanding how the previous uniqueness results vary if we replace the Laplacian Δ with the operator $(-\Delta)^{\frac{\alpha}{2}}$, with $0 < \alpha < 2$. In [**14**], the authors use the Caffarelli–Silvestre extension [**2**] to show that, for $0 < \alpha < 2$, a smooth weak solution $u \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ to (1.1) is trivial if $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$. On the other hand, since $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ embeds into $L^{\frac{6}{3-\alpha}}(\mathbb{R}^3)$, then for $\alpha < 5/3$ it is reasonable to expect that the assumption $\vec{u} \in L^{\frac{9}{2}}(\mathbb{R}^3)$ may be replaced by a more natural Lebesgue space whose exponent depends on the value of α. Thus the main purpose of our work is twofold: to attempt to improve the understanding of the gap of integrability in Liouville-type theorems for the Navier–Stokes equation by studying the hypodissipative case, and to see whether the gap of integrability in the hypodissipative case differs qualitatively from that of the classical Navier–Stokes system.

We begin by proving that, under some mild assumptions over the external force $\vec{f}$, there exists at least one solution $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$. Indeed, we have:

Theorem 1 (Existence). *Fix $0 < \alpha < 2$ and consider $\vec{f} \in \dot{H}^{-1}(\mathbb{R}^3) \cap \dot{H}^{-\frac{\alpha}{2}}(\mathbb{R}^3)$ an external force such that $\operatorname{div}(\vec{f}) = 0$. There exists a divergence-free vector field $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ and a pressure $p \in \dot{H}^{\alpha-\frac{3}{2}}(\mathbb{R}^3)$, such that $(\vec{u}, p)$ is a solution of the stationary fractional Navier–Stokes equations* (1.1).

The existence of certain weak solutions to the fractional Navier–Stokes equations (1.1) has already been studied in [**13**] via the Caffarelli–Silvestre extension [**2**]; our approach is quite different. We use the Schaefer fixed-point theorem, which is a useful tool when dealing with the existence of solutions for partial differential equations. In order to apply this general fixed-point theorem, we will regularize equation (1.1), and to recover the initial equation we will need to study a limit by considering subsequences. This will give us a solution but we will lose uniqueness.

The study of the potential uniqueness of such solutions is in general a completely different open problem (besides the case $\alpha = 5/3$, which was studied in [**14**]). However, if we add some extra conditions, we can obtain interesting conclusions and in this regard we have our next result:

Theorem 2 (Liouville-type). *Consider the stationary fractional Navier–Stokes equations*

$$(-\Delta)^{\frac{\alpha}{2}}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = 0, \quad \operatorname{div}(\vec{u}) = 0, \quad 0 < \alpha < 2. \tag{1.6}$$

Assume that $\vec{u}$, p are smooth functions that satisfy (1.6) *and consider a positive parameter $0 < \epsilon < 2\alpha$.*

(i) *Let $\alpha = 1$. If $\vec{u} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap L^{\frac{6-\epsilon}{2}}(\mathbb{R}^3)$, then we have that $\vec{u} = 0$.*

(ii) *Let $1 < \alpha < 2$ and fix the parameter $0 < \epsilon < 2\alpha$ such that*

$$1 + \frac{\epsilon}{3} \leq \alpha \leq \frac{5}{3} + \frac{2}{9}\epsilon. \tag{1.7}$$

If $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3) \cap L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$, then we have $\vec{u} = 0$.

(iii) *Let $\frac{3}{5} < \alpha < 1$ and consider a parameter $0 < \epsilon < 2\alpha$ such that*

$$1 - \frac{\epsilon}{3} \leq \alpha \leq \frac{5}{3} - \frac{2}{9}\epsilon. \tag{1.8}$$

If $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3) \cap L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3) \cap L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3)$, then we have $\vec{u} = 0$.

Some remarks are in order. Indeed, we first note that the general condition $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$, stated in all the items above, is rather natural since from Theorem 1 we know how to construct solutions in this functional space. Second, by the classical Sobolev embeddings we have $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3) \subset L^{\frac{6}{3-\alpha}}(\mathbb{R}^3)$ but we have neither $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ nor $\vec{u} \in L^{\frac{6+\varepsilon}{3-\alpha}}(\mathbb{R}^3)$ for $\epsilon > 0$, and we can thus see that the conditions stated in the theorem are actual *additional* hypotheses which help us to obtain this Liouville-type result. Next we note that if $\alpha \to 2$, then by the condition (1.7) we have $\frac{3}{2} \leq \epsilon \leq 3$ and this leads us to the Lebesgue spaces $L^q(\mathbb{R}^3)$ with $3 \leq q \leq \frac{9}{2}$, which is exactly the condition (1.5) stated above, and we recover the known results for the classical stationary Navier–Stokes equations regarding additional Lebesgue space hypotheses. We observe also that in the range $1 \leq \alpha \leq \frac{5}{3}$, then, following the relationship (1.7) (or (1.8)), we can consider very small values for the parameter $\epsilon > 0$ and thus the additional information $L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ (or $L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3)$) comes closer and closer to the critical space $L^{\frac{6}{3-\alpha}}(\mathbb{R}^3)$: in the case of the stationary fractional Navier–Stokes equation we can almost fill the gap between the space $L^{\frac{6}{3-\alpha}}(\mathbb{R}^3)$ and the additional information required to deduce Liouville-type theorems. However, we cannot simply take $\epsilon \to 0$, as the information conveyed by the hypotheses (with $\epsilon > 0$) is needed to obtain our results. Note finally that the lower limit $\frac{3}{5}$ stated in the third item is related to some technical issues. To finish, let us mention that we do not claim any optimality on the different relationships stated here.

To continue, we observe now that smoothness was taken for granted in the previous theorem, but this condition is redundant in some cases. Indeed, if we study the regularity of the solutions obtained in Theorem 1, we have the following result.

Theorem 3 (Regularity)**.** *Consider the stationary fractional Navier–Stokes equations* (1.6)*.*

(i) *If* $\frac{5}{3} < \alpha < 2$*, then the solutions* $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ *obtained via Theorem 1 above are smooth.*
(ii) *Let* $1 < \alpha \leq \frac{5}{3}$ *and consider a solution* $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$*. If we assume that* $\vec{u} \in L^\infty(\mathbb{R}^3)$*, then these solutions are smooth.*

We can see that the smoothness hypothesis in Theorem 2 is actually not necessary in the case $\frac{5}{3} < \alpha < 2$. Nevertheless, if $1 < \alpha \leq \frac{5}{3}$, the regularizing effect of the operator $(-\Delta)^{\frac{\alpha}{2}}$ seems to be too weak to obtain a gain of regularity and an additional hypothesis is thus warranted. For the sake of simplicity we assumed here a very strong condition, namely $\vec{u} \in L^\infty(\mathbb{R}^3)$, but we believe that other more general conditions can be considered. The study of the regularity in the case $0 < \alpha \leq 1$ is considerably more difficult and technical to handle and, to the best of our knowledge, it constitutes an open problem that will not be addressed here.

Note that when proving these results the main difference between the local operator Δ and the non-local operator $(-\Delta)^{\frac{\alpha}{2}}$, with $0 < \alpha < 2$, lies in the study of the commutators. Indeed, in the first case (see [**3**] or [**6**]) the classical Leibniz formula is easy to apply as it is a pointwise relationship, while in the non-local case studied here we will use Lemma 2.3 below, which is more difficult to apply since it is an integral relationship.

Finally, let us point out a few open research directions. First, other functional spaces (such as Besov, Triebel–Lizorkin, Lorentz, Morrey spaces, etc.) can possibly be used to develop all the previous theorems: indeed, most of the tools used here possess a similar behavior in these spaces. However, the L^2-based Sobolev spaces are enough to highlight the behavior of the fractional Navier–Stokes equations considered here. Secondly, although Liouville-type problems comprise a major open area related to the Navier–Stokes regularity problem, we do not study in this article how to adapt our work to the time-dependent problem. In the case of the fractional Navier–Stokes equations, the study of the ancient solutions would require a completely new article.

The plan of the article is the following: in Section 2 we recall some notation and useful results. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2. The last section is devoted to proving Theorem 3.

2. Preliminaries

For $1 < p < +\infty$ and for $s > 0$ we define the homogeneous Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^3)$ by the condition

$$\|f\|_{\dot{W}^{s,p}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < +\infty.$$

In the special case when $p = 2$ we simply write $\dot{W}^{s,2}(\mathbb{R}^3) = \dot{H}^s(\mathbb{R}^3)$. The non-homogeneous Sobolev spaces $W^{s,p}(\mathbb{R}^3)$ are defined by the condition

$$\|f\|_{W^{s,p}} = \|f\|_{L^p} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < +\infty,$$

from which we easily deduce the embedding $W^{s,p}(\mathbb{R}^3) \subset \dot{W}^{s,p}(\mathbb{R}^3)$. Note also that, if $s_1 > s_0 > 0$, then we have the space inclusion $W^{s_1,p}(\mathbb{R}^3) \subset W^{s_0,p}(\mathbb{R}^3)$. As the Sobolev spaces will constitute our main framework, we recall in the following lemmas some classical and useful results.

Lemma 2.1 (Sobolev embeddings)**.**

(i) *For $0 < s < \frac{3}{p}$ and $1 < p, q < +\infty$, if we have the relationship $-\frac{3}{q} = s - \frac{3}{p}$, then we have the classical Sobolev inequality*

$$\|f\|_{L^q} \leq C\|f\|_{\dot{W}^{s,p}}, \quad \textit{for each } f \in C_c^\infty(\mathbb{R}^n).$$

(ii) *If $0 < s_0 < s_1$ and $1 < p_0, p_1 < +\infty$ are such that $s_0 - \frac{3}{p_0} = s_1 - \frac{3}{p_1}$, then we have the following Sobolev space inclusion:*

$$\dot{W}^{s_1,p_1}(\mathbb{R}^3) \subset \dot{W}^{s_0,p_0}(\mathbb{R}^3).$$

Lemma 2.2 (Rellich–Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. If $0 < s < \frac{3}{p}$, then for all $1 \leq q < \frac{3p}{3-sp}$ we have the following compact inclusion:*

$$\dot{W}^{s,p}(\Omega) \Subset L^q(\Omega).$$

A useful consequence of this lemma is that any uniformly bounded sequence in $\dot{W}^{s,p}(\Omega)$ has a subsequence that converges in $L^q(\Omega)$.

Lemma 2.3 (Fractional Leibniz rule)**.**

(i) *Consider f, g two smooth functions. Then we have the estimate*

$$\|(-\Delta)^{\frac{s}{2}}(fg)\|_{L^p} \leq C\|(-\Delta)^{\frac{s}{2}} f\|_{L^{p_0}}\|g\|_{L^{p_1}} + C\|f\|_{L^{q_0}}\|(-\Delta)^{\frac{s}{2}} g\|_{L^{q_1}},$$

where $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{q_0} + \frac{1}{q_1}$, with $0 < s$, $1 < p < +\infty$, and $1 < p_0, p_1, q_0, q_1 \leq +\infty$.

(ii) *For $0 < s, s_1, s_2 < 1$ with $s = s_1 + s_2$ and $1 < p, p_1, p_2 < +\infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have*

$$\|(-\Delta)^{\frac{s}{2}}(fg) - (-\Delta)^{\frac{s}{2}}(f)g - (-\Delta)^{\frac{s}{2}}(g)f\|_{L^p} \leq C\|(-\Delta)^{\frac{s_1}{2}} f\|_{L^{p_1}}\|(-\Delta)^{\frac{s_2}{2}} g\|_{L^{p_2}}.$$

See [**10**] and [**5**] for a proof of these estimates. In the case of the L^2-based Sobolev spaces we also have the following useful estimate:

Lemma 2.4 (Product rule in Sobolev spaces)**.** *For $0 \leq s < +\infty$ and $0 < \delta < \frac{3}{2}$,*

$$\|fg\|_{\dot{H}^{s+\delta-\frac{3}{2}}} \leq C(\|f\|_{\dot{H}^\delta}\|g\|_{\dot{H}^s} + \|g\|_{\dot{H}^\delta}\|f\|_{\dot{H}^s}).$$

See [**9**, Lemma 7.3] for a proof of this inequality.

3. Proof of Theorem 1

We apply the Leray projector $\mathbb{P}(\vec{\psi}) = \vec{\psi} + \vec{\nabla}\frac{1}{(-\Delta)}(\vec{\nabla}\cdot\vec{\psi})$ to obtain on the one hand the following equation of the velocity (recall that $\operatorname{div}(\vec{u}) = \operatorname{div}(\vec{f}) = 0$):

$$(-\Delta)^{\frac{\alpha}{2}}\vec{u} + \mathbb{P}((\vec{u}\cdot\vec{\nabla})\vec{u}) - \vec{f} = 0, \tag{3.1}$$

and on the other hand, using the divergence-free condition of $\vec{u}$ and $\vec{f}$, we have the equation for the pressure:

$$p = \frac{1}{(-\Delta)}(\operatorname{div}((\vec{u}\cdot\vec{\nabla})\vec{u})). \tag{3.2}$$

We can thus focus our study on the velocity field $\vec{u}$ and then we will deduce the properties needed for the pressure p. In order to solve equation (3.1) we will first consider a function $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ such that $0 \leq \theta(x) \leq 1$ with $\theta(x) = 1$ if $|x| \leq 1$ and $\theta(x) = 0$ if $|x| > 2$; then for $R > 1$ we set $\theta_R(x) = \theta\big(\frac{x}{R}\big)$. With this auxiliary function and for some $0 < \epsilon < 1$ we study the following equation:

$$-\epsilon\Delta\vec{u} + (-\Delta)^{\frac{\alpha}{2}}\vec{u} + \mathbb{P}([(\theta_R\vec{u}) \cdot \vec{\nabla}](\theta_R\vec{u})) - \vec{f} = 0, \quad \operatorname{div}(\vec{u}) = 0. \tag{3.3}$$

Observe that, at least formally, if we make $\epsilon \to 0$ and $R \to +\infty$, we recover equation (3.1).

The previous equation (3.3) can be seen as a perturbation of the stationary Navier–Stokes system (1.3) and we will study the existence of solutions for this modified problem using the structure of the usual stationary Navier–Stokes. Indeed, we note that this equation can be rewritten as

$$\vec{u} = T_{R,\epsilon}(\vec{u}), \tag{3.4}$$

where

$$T_{R,\epsilon}(\vec{u}) = \frac{-1}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}(\mathbb{P}([(\theta_R\vec{u}) \cdot \vec{\nabla}](\theta_R\vec{u})) - \vec{f}). \tag{3.5}$$

Thus, in order to obtain a solution for the problem $\vec{u} = T_{R,\epsilon}(\vec{u})$ we will apply the Schaefer fixed-point theorem (see [**9**, Theorem 16.1]):

Theorem 4 (Schaefer)**.** *Consider the following functional space:*

$$E = \{\vec{v}\colon \mathbb{R}^3 \to \mathbb{R}^3 : \vec{v} \in \dot{H}^1(\mathbb{R}^3) \text{ and } \operatorname{div}(\vec{v}) = 0\}. \tag{3.6}$$

If

(i) *the operator $T_{R,\epsilon}$ defined in* (3.5) *is continuous and compact in the space E, and*

(ii) *if $\vec{u} = \lambda T_{R,\epsilon}(\vec{u})$ for any $\lambda \in [0,1]$, then we have $\|\vec{u}\|_{\dot{H}^1} \leq M$,*

then equation (3.4) *admits at least one solution $\vec{u} \in E$.*

As we can see, in order to obtain a solution of the modified problem (3.3), it is enough to verify the two points of the previous theorem. We decompose our study into some propositions and corollaries that will be helpful in the sequel.

Proposition 3.1. *The application $T_{R,\epsilon}$ is continuous and compact in the space E.*

Proof: We start writing

$$\begin{aligned} \|T_{R,\epsilon}(\vec{u})\|_{\dot{H}^1} &= \left\| \frac{-\Delta}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]} \frac{-1}{(-\Delta)} \mathbb{P}([(\theta_R\vec{u}) \cdot \vec{\nabla}](\theta_R\vec{u}) - \vec{f}) \right\|_{\dot{H}^1} \\ &= \left\| \frac{-\Delta}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]} \mathcal{T}_R(\vec{u}) \right\|_{\dot{H}^1}, \end{aligned}$$

where the operator $\mathcal{T}_R(\vec{u})$ is given by

$$\mathcal{T}_R(\vec{u}) = \frac{-1}{(-\Delta)}(\mathbb{P}([(\theta_R\vec{u}) \cdot \vec{\nabla}](\theta_R\vec{u})) - \vec{f}). \tag{3.7}$$

Observe now that the symbol σ_ϵ associated to the operator $\frac{-\Delta}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}$ is $\sigma_\epsilon(\xi) = \frac{|\xi|^2}{\epsilon|\xi|^2 + |\xi|^\alpha}$, which is a bounded Fourier multiplier, i.e., we have the uniform estimate $\sigma_\epsilon(\xi) \leq \frac{C}{\epsilon}$, so we can write

$$\|T_{R,\epsilon}(\vec{u})\|_{\dot{H}^1} = \left\| \frac{-\Delta}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]} \mathcal{T}_R(\vec{u}) \right\|_{\dot{H}^1} \leq \frac{C}{\epsilon} \|\mathcal{T}_R(\vec{u})\|_{\dot{H}^1},$$

but from the proof of Theorem 16.2 in [**9**] we know that the operator $\mathcal{T}_R(\vec{u})$ is a continuous and compact operator in the space E (recall that we have the hypothesis $\vec{f} \in \dot{H}^{-1}(\mathbb{R}^3)$) and we can deduce from this fact that the operator $T_{R,\epsilon}$ is itself continuous and compact in the space E. □

We now need to establish some additional estimates.

Proposition 3.2. *If $\vec{u}$ belongs to the functional space E given in* (3.6) *and if $\vec{u}$ satisfies*

$$\vec{u} = \frac{-1}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}(\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})) - \vec{f}),$$

then we have $\vec{u} \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$, with $0 < \alpha < 2$.

Proof: By the previous proposition we already know that if $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$, then the quantity

$$\frac{-1}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}(\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})) - \vec{f})$$

also belongs to $\dot{H}^1(\mathbb{R}^3)$: we only need to study if $\vec{u} \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$. We thus write

$$\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} = \|(-\Delta)^{\frac{\alpha}{4}}\vec{u}\|_{L^2} = \left\|\frac{(-\Delta)^{\frac{1}{2}+\frac{\alpha}{4}}}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}\frac{(-\Delta)^{\frac{1}{2}}}{(-\Delta)}(-\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})) + \vec{f})\right\|_{L^2};$$

note that the symbol $\widetilde{\sigma}_\epsilon(\xi) = \frac{|\xi|^{1+\frac{\alpha}{2}}}{\epsilon|\xi|^2+|\xi|^\alpha}$ is a bounded Fourier multiplier as we have $\widetilde{\sigma}_\epsilon(\xi) \leq \frac{C}{\epsilon}$ and we write

$$\begin{aligned}
\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} &\leq \frac{C}{\epsilon}\left\|\frac{(-\Delta)^{\frac{1}{2}}}{(-\Delta)}(-\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})) + \vec{f})\right\|_{L^2} \\
&= \frac{C}{\epsilon}\left\|\frac{-1}{(-\Delta)}(\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})) - \vec{f})\right\|_{\dot{H}^1} \\
&\leq \frac{C}{\epsilon}\|\mathcal{T}_R(\vec{u})\|_{\dot{H}^1},
\end{aligned}$$

where we used the definition of the operator $\mathcal{T}_R$ given in (3.7) above. We recall now that the operator $\mathcal{T}_R$ is bounded in the space $\dot{H}^1(\mathbb{R}^3)$ (recall that we have $\vec{f} \in \dot{H}^{-1}(\mathbb{R}^3)$). Then, as we are assuming that $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$, we have:

$$\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} \leq \frac{C_R}{\epsilon}\|\vec{u}\|_{\dot{H}^1}\|\vec{u}\|_{\dot{H}^1} < +\infty.$$

We have thus proved that $\vec{u} \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$. □

This proposition shows us that, although the operator $T_{R,\epsilon}$ defined in (3.5) is bounded in the space $\dot{H}^1(\mathbb{R}^3)$, we have some additional boundedness properties in the space $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ with $0 < \alpha < 2$.

Proposition 3.3. *Let $0 \leq \lambda \leq 1$. If $\vec{u}$ belongs to the functional space E given in* (3.6) *and if $\vec{u}$ satisfies*

$$\vec{u} = \lambda\left[\frac{-1}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}(\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})) - \vec{f})\right], \tag{3.8}$$

for $0 < \alpha < 2$, then we have the inequality

$$\epsilon\|\vec{u}\|^2_{\dot{H}^1} + \|\vec{u}\|^2_{\dot{H}^{\frac{\alpha}{2}}} \leq \lambda\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}}\|\vec{f}\|_{\dot{H}^{-\frac{\alpha}{2}}}. \tag{3.9}$$

Proof: Let us first observe that since we are working in the space E we have enough regularity to show that $\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\in \dot{H}^{-1}(\mathbb{R}^3)$. Indeed, we write by the properties of the Leray projector and by the Sobolev embedding $\dot{H}^{-1}(\mathbb{R}^3)\subset L^{\frac{6}{5}}(\mathbb{R}^3)$:

$$\|\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\|_{\dot{H}^{-1}}\leq C\|[(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})\|_{\dot{H}^{-1}}\leq C\|[(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u})\|_{L^{\frac{6}{5}}}.$$

Now, by the Hölder inequalities we obtain

$$\leq C\sum_{j=1}^{3}\|(\theta_R u_j)\partial_j(\theta_R\vec{u})\|_{L^{\frac{6}{5}}}\leq C\sum_{j=1}^{3}\|\theta_R u_j\|_{L^3}\|\partial_j(\theta_R\vec{u})\|_{L^2}$$

$$\leq C\sum_{j=1}^{3}\|\theta_R\|_{L^6}\|u_j\|_{L^6}(\|(\partial_j\theta_R)\vec{u}\|_{L^2}+\|\theta_R(\partial_j\vec{u})\|_{L^2})$$

$$\leq C\sum_{j=1}^{3}\|\theta_R\|_{L^6}\|u_j\|_{L^6}(\|\partial_j\theta_R\|_{L^3}\|\vec{u}\|_{L^6}+\|\theta_R\|_{L^\infty}\|\partial_j\vec{u}\|_{L^2})\leq C_R\|\vec{u}\|_{L^6}(\|\vec{u}\|_{L^6}+\|\vec{u}\|_{\dot{H}^1})$$

$$\leq C_R\|\vec{u}\|_{\dot{H}^1}\|\vec{u}\|_{\dot{H}^1}<+\infty,$$

where we used the Sobolev embedding $\dot{H}^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$ in the last estimate above. With this information at hand and since $\operatorname{div}(\vec{u})=0$ we can write, by the properties of the Leray projector:

$$\int_{\mathbb{R}^3}\vec{u}\cdot\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\,dx=\int_{\mathbb{R}^3}\vec{u}\cdot([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\,dx,$$

but since by an integration by parts we have

$$\int_{\mathbb{R}^3}\vec{u}\cdot([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\,dx=-\int_{\mathbb{R}^3}\vec{u}\cdot([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\,dx$$

we deduce that

$$\int_{\mathbb{R}^3}\vec{u}\cdot([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\,dx=0. \tag{3.10}$$

With this information, we now rewrite equation (3.8) in the following form:

$$[-\epsilon\Delta+(-\Delta)^{\frac{\alpha}{2}}]\vec{u}=-\lambda[\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))-\vec{f}]=-\lambda\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))+\lambda\vec{f},$$

from which we deduce

$$-\epsilon\int_{\mathbb{R}^3}\Delta\vec{u}\cdot\vec{u}\,dx+\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}\vec{u}\cdot\vec{u}\,dx=-\lambda\int_{\mathbb{R}^3}\mathbb{P}([(\theta_R\vec{u})\cdot\vec{\nabla}](\theta_R\vec{u}))\cdot\vec{u}\,dx+\lambda\int_{\mathbb{R}^3}\mathbb{P}(\vec{f})\cdot\vec{u}\,dx.$$

Using identity (3.10), by the properties of the Leray projector, since $\operatorname{div}(\vec{u})=0$, and using the properties of the operators Δ and $(-\Delta)^{\frac{\alpha}{2}}$, we obtain

$$\epsilon\|\vec{u}\|^2_{\dot{H}^1}+\|\vec{u}\|^2_{\dot{H}^{\frac{\alpha}{2}}}=\lambda\int_{\mathbb{R}^3}(-\Delta)^{-\frac{\alpha}{4}}\vec{f}\cdot(-\Delta)^{\frac{\alpha}{4}}\vec{u}\,dx\leq\lambda\|\vec{f}\|_{\dot{H}^{-\frac{\alpha}{2}}}\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}},$$

and we have proved estimate (3.9). □

This estimate has several consequences and we gather them in the following corollary:

Corollary 3.1. *In the general framework of Proposition 3.3,* i.e., *if $\vec{u}$ belongs to the functional space E given in* (3.6) *and if $\vec{u}$ satisfies equation* (3.8)*, then we have the following points:*

(i) *For $\epsilon > 0$ one has the inequality*

$$\|\vec{u}\|_{\dot{H}^1} \leq \frac{1}{\sqrt{2\epsilon}} \|\vec{f}\|_{\dot{H}^{-\frac{\alpha}{2}}}. \tag{3.11}$$

(ii) *We also have the uniform estimate*

$$\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} \leq \|\vec{f}\|_{\dot{H}^{-\frac{\alpha}{2}}}. \tag{3.12}$$

Proof: From estimate (3.9) we write, by the Young inequalities for the product:

$$\epsilon\|\vec{u}\|^2_{\dot{H}^1} + \|\vec{u}\|^2_{\dot{H}^{\frac{\alpha}{2}}} \leq \frac{\lambda}{2}(\|\vec{f}\|^2_{\dot{H}^{-\frac{\alpha}{2}}} + \|\vec{u}\|^2_{\dot{H}^{\frac{\alpha}{2}}}),$$

from which we easily obtain $\epsilon\|\vec{u}\|^2_{\dot{H}^1} + \left(1 - \frac{\lambda}{2}\right)\|\vec{u}\|^2_{\dot{H}^{\frac{\alpha}{2}}} \leq \frac{\lambda}{2}\|\vec{f}\|^2_{\dot{H}^{-\frac{\alpha}{2}}}$ and since $0 \leq \lambda \leq 1$ we easily deduce the two desired estimates. □

End of the proof of Theorem 1: By the Schaefer fixed-point theorem, in order to obtain the existence of a solution of the problem (3.4)–(3.5) we only need to prove the two points given in Theorem 4. Thus, for some fixed parameters $R > 1$, $\epsilon > 0$, we know by Proposition 3.1 that the application $T_{R,\epsilon}$ is continuous and compact in the space E given in (3.6). The second point of Theorem 4 is given by estimate (3.11) stated in Corollary 3.1. We thus have the existence of a solution of the problem

$$\vec{u} = \frac{-1}{[-\epsilon\Delta + (-\Delta)^{\frac{\alpha}{2}}]}(\mathbb{P}([(\theta_R\vec{u}) \cdot \vec{\nabla}](\theta_R\vec{u})) - \vec{f}),$$

where $\vec{u} \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$.

Remark 3.1. Note that the solution obtained above depends on the parameters $R > 1$ and $\epsilon > 0$ and they will be denoted by $\vec{u}_{R,\epsilon}$.

Now we need to recover the initial problem by making $R \to +\infty$ and $\epsilon \to 0$ in the solutions $\vec{u}_{R,\epsilon}$. To do so, we will first fix $\epsilon > 0$ and then we will take the limit when $R \to +\infty$. Indeed, we observe that for a fixed $\epsilon > 0$ we have the uniform (in R) estimate (3.12), and thus there exists a sequence $R_k \to +\infty$ such that $\vec{u}_{R_k,\epsilon}$ converges weakly in $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ to some limit $\vec{u}_\epsilon$. Moreover, by Lemma 2.2, we have for all $0 < \alpha < 2$ the strong convergence of $\vec{u}_{R_k,\epsilon}$ to some limit $\vec{u}_\epsilon$ in the space $L^2_{\text{loc}}(\mathbb{R}^3)$. These two facts allow us to obtain a weak convergence (in $\mathcal{D}'$) of the nonlinear term $([(\theta_R\vec{u}_{R,\epsilon}) \cdot \vec{\nabla}](\theta_R\vec{u}_{R,\epsilon}))$ to $([\vec{u}_\epsilon \cdot \vec{\nabla}]\vec{u}_\epsilon)$ when $R \to +\infty$. We thus obtain a function $\vec{u}_\epsilon$ which is a solution of the problem

$$-\epsilon\Delta\vec{u}_\epsilon + (-\Delta)^{\frac{\alpha}{2}}\vec{u}_\epsilon + \mathbb{P}([\vec{u}_\epsilon \cdot \vec{\nabla}]\vec{u}_\epsilon) - \vec{f} = 0.$$

Similarly, since we have the uniform (in ϵ) control $\|\vec{u}_\epsilon\|_{\dot{H}^{\frac{\alpha}{2}}} \leq \|\vec{f}\|_{\dot{H}^{-\frac{\alpha}{2}}}$ given in estimate (3.12), there exists a subsequence $\epsilon_k \to 0$ such that $\vec{u}_{\epsilon_k}$ converges weakly to a limit $\vec{u}$ in the space $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$. Again, by Lemma 2.2, we obtain the strong convergence of $\vec{u}_{\epsilon_k}$ to $\vec{u}$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ and from these facts we obtain the weak convergence (in $\mathcal{D}'$) of the quantity $[\vec{u}_\epsilon \cdot \vec{\nabla}]\vec{u}_\epsilon$ to $(\vec{u} \cdot \vec{\nabla})\vec{u}$ when $\epsilon \to 0$. We have thus obtained a solution $\vec{u}$ of the equation

$$(-\Delta)^{\frac{\alpha}{2}}\vec{u} + \mathbb{P}((\vec{u} \cdot \vec{\nabla})\vec{u}) - \vec{f} = 0,$$

which belongs to the space $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$ and satisfies $\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} \leq \|\vec{f}\|_{\dot{H}^{-\frac{\alpha}{2}}}$.

To end the proof, we need to study the pressure p. By equation (3.2) and by the divergence-free property of $\vec{u}$ we can write

$$\|p\|_{\dot{H}^{\alpha-\frac{3}{2}}} = \|(-\Delta)^{\frac{\alpha-\frac{3}{2}}{2}} p\|_{L^2} = \left\| \frac{(-\Delta)^{\frac{\alpha-\frac{3}{2}}{2}}}{(-\Delta)} \operatorname{div}(\operatorname{div}(\vec{u}\otimes\vec{u})) \right\|_{L^2}$$

$$= \|(-\Delta)^{\frac{\alpha-\frac{3}{2}}{2}} (\vec{u}\otimes\vec{u})\|_{L^2} = \|\vec{u}\otimes\vec{u}\|_{\dot{H}^{\alpha-\frac{3}{2}}}.$$

Now by the product rule in Sobolev spaces given in Lemma 2.4 we have

$$\|\vec{u}\otimes\vec{u}\|_{\dot{H}^{\alpha-\frac{3}{2}}} \leq C\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}}\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} < +\infty,$$

from which we easily deduce that $\|p\|_{\dot{H}^{\alpha-\frac{3}{2}}} < +\infty$ and this ends the proof of Theorem 1. □

4. Proof of Theorem 2

We start the proof of this theorem with the following:

Lemma 4.1. *Let $(\vec{u}, p)$ be a solution of the fractional Navier–Stokes equation*

$$(-\Delta)^{\frac{\alpha}{2}}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = 0, \quad \operatorname{div}(\vec{u}) = 0.$$

If we have $\vec{u} \in L^q(\mathbb{R}^3)$ for some $2 < q < +\infty$, then the pressure p belongs to the space $L^{\frac{q}{2}}(\mathbb{R}^3)$.

Proof: Applying the divergence operator to the equation above and using the divergence-free property of $\vec{u}$ we obtain $\operatorname{div}((\vec{u}\cdot\vec{\nabla})\vec{u}) + \operatorname{div}(\vec{\nabla}p) = 0$, which leads us to the equation $\Delta p = -\operatorname{div}(\operatorname{div}(\vec{u}\otimes\vec{u}))$, from which we deduce the expression $p = \frac{1}{(-\Delta)}\operatorname{div}(\operatorname{div}(\vec{u}\otimes\vec{u}))$. Thus, taking the $L^{\frac{q}{2}}(\mathbb{R}^3)$ norm, we have

$$\|p\|_{L^{\frac{q}{2}}} = \left\| \frac{1}{(-\Delta)} \operatorname{div}(\operatorname{div}(\vec{u}\otimes\vec{u})) \right\|_{L^{\frac{q}{2}}}.$$

Since $1 < \frac{q}{2} < +\infty$, the Riesz transforms are bounded in the space $L^{\frac{q}{2}}(\mathbb{R}^3)$ and we can write

$$\left\| \frac{1}{(-\Delta)} \operatorname{div}(\operatorname{div}(\vec{u}\otimes\vec{u})) \right\|_{L^{\frac{q}{2}}} \leq C\|\vec{u}\otimes\vec{u}\|_{L^{\frac{q}{2}}} \leq C\|\vec{u}\|_{L^q}\|\vec{u}\|_{L^q} < +\infty,$$

and we obtain that $p \in L^{\frac{q}{2}}(\mathbb{R}^3)$. □

This simple remark allows us to deduce some integrability results for the pressure from the information available on the velocity field $\vec{u}$.

We will now prove that in the framework of Theorem 2 the unique solution of equation (1.6) is the trivial solution. For this we consider $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ a smooth cut-off function given by $0 \leq \theta \leq 1$, $\theta(x) = 1$ if $|x| < \frac{1}{2}$, and $\theta(x) = 0$ if $|x| \geq 1$. For $R > 1$ a real parameter, we define the function

$$\theta_R(x) = \theta\left(\frac{x}{R}\right).$$

In particular, we have $\theta_R(x) = 1$ if $|x| < \frac{R}{2}$ and $\theta_R(x) = 0$ if $|x| \geq R$ and thus $\operatorname{supp}(\theta_R) \subset B_R$, where B_R denotes the ball $B(0, R)$. With this auxiliary function, we multiply equation (1.6) by $(\theta_R\vec{u})$ and we integrate:

$$\int_{\mathbb{R}^3} \underbrace{(-\Delta)^{\frac{\alpha}{2}}\vec{u}\cdot(\theta_R\vec{u})}_{(1)} + \underbrace{(\vec{u}\cdot\vec{\nabla})\vec{u}\cdot(\theta_R\vec{u})}_{(2)} + \underbrace{\vec{\nabla}p\cdot(\theta_R\vec{u})}_{(3)}\, dx = 0, \tag{4.1}$$

and we study each one of these terms separately. For the first term in (4.1) we write, using the properties of the operator $(-\Delta)^{\frac{\alpha}{2}}$:

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}\vec{u}\cdot(\theta_R\vec{u})\,dx=\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot(-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})\,dx$$
$$=\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot[((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R+(-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})-((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R]\,dx,$$

and we have

$$\begin{aligned}\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}\vec{u}\cdot(\theta_R\vec{u})\,dx &= \int_{B_R}|(-\Delta)^{\frac{\alpha}{4}}\vec{u}|^2\theta_R\,dx\\ &\quad+\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot[(-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})-((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R]\,dx,\end{aligned}\tag{4.2}$$

where we used the fact that $\mathrm{supp}(\theta_R)\subset B_R$ in the second integral above.

For the second term of (4.1) we have:

$$\int_{\mathbb{R}^3}(\vec{u}\cdot\vec{\nabla})\vec{u}\cdot(\theta_R\vec{u})\,dx=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}u_j(\partial_{x_j}u_i)(\theta_R u_i)\,dx=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\theta_R u_j(\partial_{x_j}u_i)u_i\,dx$$
$$=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\theta_R u_j(\partial_{x_j}\left(\frac{u_i^2}{2}\right))\,dx,$$

and by an integration by parts we obtain

$$\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\theta_R u_j\left(\partial_{x_j}\left(\frac{u_i^2}{2}\right)\right)dx=-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\theta_R(\partial_{x_j}u_j)\frac{u_i^2}{2}\,dx-\int_{\mathbb{R}^3}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{u}|^2}{2}\vec{u}\right)dx.$$

Now, using the fact that $\mathrm{div}(\vec{u})=0$, we have that the second integral above is null and we can write

$$\int_{\mathbb{R}^3}(\vec{u}\cdot\vec{\nabla})\vec{u}\cdot(\theta_R\vec{u})\,dx=-\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{u}|^2}{2}\vec{u}\right)dx,\tag{4.3}$$

where we used the support property of the auxiliary function θ_R.

Finally, for the last term of (4.1), by an integration by parts, using again the fact $\mathrm{div}(\vec{u})=0$ and the support property of θ_R, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\vec{\nabla}p\cdot(\theta_R\vec{u})\,dx&=\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\partial_{x_i}p)\theta_R u_i\,dx=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}p\,\partial_{x_i}(\theta_R u_i)\,dx\\ &=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}p(\partial_{x_i}\theta_R)(u_i)\,dx=-\int_{B_R}\vec{\nabla}\theta_R\cdot(p\vec{u})\,dx.\end{aligned}\tag{4.4}$$

Thus, with the expressions (4.2), (4.3), and (4.4), we can rewrite equation (4.1) in the following manner:

$$\int_{B_R} |(-\Delta)^{\frac{\alpha}{4}}\vec{u}|^2\theta_R\,dx + \int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot[(-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u}) - ((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R]\,dx$$
$$- \int_{B_R}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{u}|^2}{2}\vec{u}\right)dx - \int_{B_R}\vec{\nabla}\theta_R\cdot(p\vec{u})\,dx = 0,$$

from which we obtain the equation

$$\int_{B_R} |(-\Delta)^{\frac{\alpha}{4}}\vec{u}|^2\theta_R\,dx = \int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot[((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R - (-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})]\,dx$$
$$+ \int_{B_R}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{u}|^2}{2}\vec{u}\right)dx$$
$$+ \int_{B_R}\vec{\nabla}\theta_R\cdot(p\vec{u})\,dx.$$

We recall now that since $0 \leq \theta_R(x) \leq 1$ and $\theta_R(x) = 1$, if $|x| < \frac{R}{2}$, we have the estimate

$$\int_{B_{\frac{R}{2}}} |(-\Delta)^{\frac{\alpha}{4}}\vec{u}|^2\,dx \leq \int_{B_R} |(-\Delta)^{\frac{\alpha}{4}}\vec{u}|^2\theta_R\,dx,$$

and we can write

$$\int_{B_{\frac{R}{2}}} |(-\Delta)^{\frac{\alpha}{4}}\vec{u}|^2\,dx \leq \underbrace{\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot[((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R - (-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})]\,dx}_{(I_a)}$$
$$+ \underbrace{\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{u}|^2}{2}\vec{u}\right)dx}_{(I_b)} \tag{4.5}$$
$$+ \underbrace{\int_{B_R}\vec{\nabla}\theta_R\cdot(p\vec{u})\,dx}_{(I_c)}.$$

We will now show that (a) $\lim_{R\to+\infty} I_a = 0$, (b) $\lim_{R\to+\infty} I_b = 0$, and (c) $\lim_{R\to+\infty} I_c = 0$. Indeed:

(a) For the first term above, we write by the Cauchy–Schwarz inequality

$$\begin{aligned} I_a &= \int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{4}}\vec{u}\cdot[((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R - (-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})]\,dx \\ &\leq \|(-\Delta)^{\frac{\alpha}{4}}\vec{u}\|_{L^2}\|((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R - (-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u})\|_{L^2} \\ &\leq \|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}}(\|(-\Delta)^{\frac{\alpha}{4}}(\theta_R\vec{u}) - ((-\Delta)^{\frac{\alpha}{4}}\vec{u})\theta_R - ((-\Delta)^{\frac{\alpha}{4}}\theta_R)\vec{u}\|_{L^2} + \|((-\Delta)^{\frac{\alpha}{4}}\theta_R)\vec{u}\|_{L^2}). \end{aligned}$$

We now apply the second point of Lemma 2.3 to obtain the estimate

$$I_a \leq \|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}}(\|(-\Delta)^{\frac{\alpha_1}{4}}\theta_R\|_{L^{p_1}}\|(-\Delta)^{\frac{\alpha_2}{4}}\vec{u}\|_{L^{p_2}} + \|((-\Delta)^{\frac{\alpha}{4}}\theta_R)\vec{u}\|_{L^2}),$$

where $\alpha = \alpha_1 + \alpha_2$, $0 < \alpha, \alpha_1, \alpha_2 < 2$, and $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$.

Recall now that we have $0 < \alpha < 2$ and that we are assuming in all the cases stated in Theorem 2 the condition $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ with $0 < \epsilon < 2\alpha$, thus by the Hölder inequality with $\frac{1}{2} = \frac{2\alpha-\epsilon}{12-2\epsilon} + \frac{3-\alpha}{6-\epsilon}$ we have

$$\begin{aligned} I_a &\leq \|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} (\|(-\Delta)^{\frac{\alpha_1}{4}}\theta_R\|_{L^{p_1}} \|(-\Delta)^{\frac{\alpha_2}{4}}\vec{u}\|_{L^{p_2}} + \|(-\Delta)^{\frac{\alpha}{4}}\theta_R\|_{L^{\frac{12-2\epsilon}{2\alpha-\epsilon}}} \|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}}) \\ &\leq \|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} (C R^{-\frac{\alpha_1}{2}+\frac{3}{p_1}} \|(-\Delta)^{\frac{\alpha_2}{4}}\vec{u}\|_{L^{p_2}} + C R^{-\frac{\alpha}{2}+3\frac{2\alpha-\epsilon}{12-2\epsilon}} \|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}}), \end{aligned}$$

where we used the properties of the function θ_R in the last estimate above. Let us also note that, due to the complex interpolation theory (see [**1**, Theorem 6.4.5]), we have

$$[\dot{H}^{\frac{\alpha}{2}}, L^{\frac{6-\epsilon}{3-\alpha}}]_\nu = \dot{W}^{\frac{\alpha_2}{2},p_2} \quad \text{and} \quad \|(-\Delta)^{\frac{\alpha_2}{4}}\vec{u}\|_{L^{p_2}} = \|\vec{u}\|_{\dot{W}^{\frac{\alpha_2}{2},p_2}} \leq C\|\vec{u}\|^{\nu}_{\dot{H}^{\frac{\alpha}{2}}} \|\vec{u}\|^{1-\nu}_{L^{\frac{6-\epsilon}{3-\alpha}}},$$

with the relationships

$$\alpha_2 = \nu\alpha, \quad \frac{1}{p_2} = \frac{\nu}{2} + (1-\nu)\frac{3-\alpha}{6-\epsilon} \quad \text{for some} \quad 0 < \nu < 1, \tag{4.6}$$

and then we can write

$$I_a \leq \|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} (C R^{-\frac{\alpha_1}{2}+\frac{3}{p_1}} \|\vec{u}\|^{\nu}_{\dot{H}^{\frac{\alpha}{2}}} \|\vec{u}\|^{1-\nu}_{L^{\frac{6-\epsilon}{3-\alpha}}} + C R^{-\frac{\alpha}{2}+3\frac{2\alpha-\epsilon}{12-2\epsilon}} \|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}}).$$

But since we have $\alpha = \alpha_1 + \alpha_2$ and $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$, following the conditions (4.6) above, we obtain that $\alpha_1 = (1-\nu)\alpha$ and $\frac{1}{p_1} = (1-\nu)\frac{2\alpha-\epsilon}{12-2\epsilon}$ and we can write

$$I_a \leq C\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} (R^{(1-\nu)[-\frac{\alpha}{2}+\frac{6\alpha-3\epsilon}{12-2\epsilon}]} \|\vec{u}\|^{\nu}_{\dot{H}^{\frac{\alpha}{2}}} \|\vec{u}\|^{1-\nu}_{L^{\frac{6-\epsilon}{3-\alpha}}} + R^{-\frac{\alpha}{2}+\frac{6\alpha-3\epsilon}{12-2\epsilon}} \|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}}).$$

Observe that since $0 < \alpha < 2$ and $0 < \epsilon < 2\alpha$ we always have $\frac{6\alpha-3\epsilon}{12-2\epsilon} < \frac{\alpha}{2}$ and thus all the powers of the parameter R in the right-hand side above are negative. Moreover, we have $\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} < +\infty$ and $\|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}} < +\infty$, so we obtain

$$\lim_{R\to+\infty} I_a = 0. \tag{4.7}$$

Remark 4.1. Note that in all the cases $\alpha = 1$, $1 < \alpha < 2$, and $\frac{3}{5} < \alpha < 1$ stated in Theorem 2, in order to obtain the previous limit (4.7) we only require the information $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ for some $0 < \epsilon < 2\alpha$ and no further conditions are needed for the parameter ϵ. The conditions (1.7) and (1.8) will appear in the study of the limits $\lim\limits_{R\to+\infty} I_b$ and $\lim\limits_{R\to+\infty} I_c$.

(b) For the second term of (4.5) we recall that $\theta_R(x) = 1$ if $|x| < \frac{R}{2}$ and $\theta_R(x) = 0$ if $|x| \geq R$ and thus we have

$$\operatorname{supp}(\vec{\nabla}\theta_R) \subset \{x \in \mathbb{R}^3 : \tfrac{R}{2} < |x| < R\} = \mathcal{C}(\tfrac{R}{2}, R), \tag{4.8}$$

and with this remark we can write

$$I_b = \int_{B_R} \vec{\nabla}\theta_R \cdot \left(\frac{|\vec{u}|^2}{2}\vec{u}\right) dx = \int_{\mathcal{C}(\frac{R}{2},R)} \vec{\nabla}\theta_R \cdot \left(\frac{|\vec{u}|^2}{2}\vec{u}\right) dx.$$

In order to study the limit when $R \to +\infty$, we decompose our study following the values of α and the information available. Indeed:

- If $\alpha = 1$, we have $\vec{u} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and thus by the Sobolev embeddings we also have $\vec{u} \in L^3(\mathbb{R}^3)$, so we can write:

$$I_b \leq C\|\vec{\nabla}\theta_R\|_{L^\infty(\mathcal{C}(\frac{R}{2},R))}\|\vec{u}\|^3_{L^3(\mathcal{C}(\frac{R}{2},R))} \leq CR^{-1}\|\vec{u}\|^3_{L^3(\mathcal{C}(\frac{R}{2},R))}, \tag{4.9}$$

from which we easily deduce that

$$\lim_{R\to+\infty} I_b = 0.$$

- If $1 < \alpha < 2$, we know by hypothesis that $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ and recall that we have in this case the condition (1.7), i.e., $1 + \frac{\epsilon}{3} \leq \alpha \leq \frac{5}{3} + \frac{2}{9}\epsilon$. Thus, if $1 + \frac{\epsilon}{3} < \alpha \leq \frac{5}{3} + \frac{2}{9}\epsilon$, by the Hölder inequality with $\frac{3\alpha-3-\epsilon}{6-\epsilon} + 3\big(\frac{3-\alpha}{6-\epsilon}\big) = 1$, we can write

$$\begin{aligned} I_b &\leq C\|\vec{\nabla}\theta_R\|_{L^{\frac{6-\epsilon}{3\alpha-3-\epsilon}}(\mathcal{C}(\frac{R}{2},R))}\|\vec{u}\|^3_{L^{\frac{6-\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))} \\ &\leq CR^{-1+3\frac{3\alpha-3-\epsilon}{6-\epsilon}}\|\vec{u}\|^3_{L^{\frac{6-\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))}. \end{aligned}$$

Then if $1 + \frac{\epsilon}{3} < \alpha < \frac{5}{3} + \frac{2}{9}\epsilon$, the power of the parameter R above is negative and then the quantity above will tend to 0 if $R \to +\infty$. But if $\alpha = \frac{5}{3} + \frac{2}{9}\epsilon$, we have $-1 + 3\frac{3\alpha-3-\epsilon}{6-\epsilon} = 0$; then since $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$, we will have $\|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))} \xrightarrow[R\to+\infty]{} 0$. Finally, if $1 + \frac{\epsilon}{3} = \alpha$, then we have $L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3) = L^3(\mathbb{R}^3)$; moreover, the power of R is negative and equal to -1 and we can proceed as in (4.9). Thus, in any case we obtain

$$\lim_{R\to+\infty} I_b = 0.$$

Note that in the case $1 < \alpha < 2$ besides the condition $0 < \epsilon < 2\alpha$ we need the relationship (1.7) between α and ϵ.

- If $\frac{3}{5} < \alpha < 1$, in this case we have the additional condition $\vec{u} \in L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3)$ with the relationship $1 - \frac{\epsilon}{3} \leq \alpha \leq \frac{5}{3} - \frac{2}{9}\epsilon$ (recall the condition (1.8)). As above, if $1 - \frac{\epsilon}{3} < \alpha \leq \frac{5}{3} - \frac{2}{9}\epsilon$, by the Hölder inequality with $\frac{3\alpha-3+\epsilon}{6+\epsilon} + 3\big(\frac{3-\alpha}{6+\epsilon}\big) = 1$, we obtain

$$I_b \leq CR^{-1+3\frac{3\alpha-3+\epsilon}{6+\epsilon}}\|\vec{u}\|^3_{L^{\frac{6+\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))}.$$

Note that if $1-\frac{\epsilon}{3} < \alpha < \frac{5}{3}-\frac{2}{9}\epsilon$, the power of the parameter R is negative, while if $\alpha = \frac{5}{3} - \frac{2}{9}\epsilon$, we have $-1+3\frac{3\alpha-3+\epsilon}{6+\epsilon} = 0$, but we have $\|\vec{u}\|_{L^{\frac{6+\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))} \xrightarrow[R\to+\infty]{} 0$. Observe also that if $\alpha = 1 - \frac{\epsilon}{3}$, then $L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3) = L^3(\mathbb{R}^3)$, the power of R is equal -1 and we can proceed as in (4.9). In any case we have

$$\lim_{R\to+\infty} I_b = 0.$$

Remark 4.2. Note that when $\frac{3}{5} < \alpha < 1$ we need the information $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ with the condition $0 < \epsilon < 2\alpha$ in order to obtain the limit (4.7) for the term I_a, but we also need the information $\vec{u} \in L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3)$ with the constraint (1.8) to obtain that $\lim\limits_{R\to+\infty} I_b = 0$.

Note also that the lower limit $\frac{3}{5} < \alpha$ is a consequence of the conditions $1 - \frac{\epsilon}{3} \leq \alpha$ and $0 < \epsilon < 2\alpha$. We recall that these conditions are technical and we do not claim any optimality on them.

(c) For the last term of (4.5) we write, using the support property (4.8):

$$I_c = \int_{B_R} \vec{\nabla}\theta_R \cdot (p\vec{u})\, dx = \int_{\mathcal{C}(\frac{R}{2},R)} \vec{\nabla}\theta_R \cdot (p\vec{u})\, dx.$$

The study of this term is very similar to that of the previous one since by Lemma 4.1 we also have some information on the pressure p. Indeed:

- If $\alpha = 1$, we have $\vec{u} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and thus by the Sobolev embeddings we have $\vec{u} \in L^3(\mathbb{R}^3)$ but we also have $p \in L^{\frac{3}{2}}(\mathbb{R}^3)$ by Lemma 4.1, and we write

$$\begin{aligned} I_c &\leq C\|\vec{\nabla}\theta_R\|_{L^\infty(\mathcal{C}(\frac{R}{2},R))}\|p\|_{L^{\frac{3}{2}}(\mathcal{C}(\frac{R}{2},R))}\|\vec{u}\|_{L^3(\mathcal{C}(\frac{R}{2},R))} \\ &\leq CR^{-1}\|p\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\|\vec{u}\|_{L^3(\mathbb{R}^3)}, \end{aligned} \tag{4.10}$$

  from which we easily deduce that

$$\lim_{R\to+\infty} I_c = 0.$$

- If $1 < \alpha < 2$, we have $\vec{u} \in L^{\frac{6-\epsilon}{3-\alpha}}(\mathbb{R}^3)$ and by Lemma 4.1 we have $p \in L^{\frac{6-\epsilon}{6-2\alpha}}(\mathbb{R}^3)$. If $1 + \frac{\epsilon}{3} < \alpha \leq \frac{5}{3} + \frac{2}{9}\epsilon$, by the Hölder inequality with $\frac{3\alpha-3-\epsilon}{6-\epsilon} + \frac{6-2\alpha}{6-\epsilon} + \frac{3-\alpha}{6-\epsilon} = 1$ we obtain

$$\begin{aligned} I_c &\leq \|\vec{\nabla}\theta_R\|_{L^{\frac{6-\epsilon}{3\alpha-3-\epsilon}}(\mathcal{C}(\frac{R}{2},R))}\|p\|_{L^{\frac{6-\epsilon}{6-2\alpha}}(\mathcal{C}(\frac{R}{2},R))}\|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))} \\ &\leq CR^{-1+3\frac{3\alpha-3-\epsilon}{6-\epsilon}}\|p\|_{L^{\frac{6-\epsilon}{6-2\alpha}}(\mathcal{C}(\frac{R}{2},R))}\|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))}. \end{aligned}$$

  If $1 + \frac{\epsilon}{3} < \alpha < \frac{5}{3} + \frac{2}{9}\epsilon$, the power of the parameter R is then negative and we have $\lim\limits_{R\to+\infty} I_c = 0$, while if $\alpha = \frac{5}{3} + \frac{2}{9}\epsilon$, we use the fact that $\|p\|_{L^{\frac{6-\epsilon}{6-2\alpha}}(\mathcal{C}(\frac{R}{2},R))}$, $\|\vec{u}\|_{L^{\frac{6+\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))} \xrightarrow[R\to+\infty]{} 0$. Now in the case $\alpha = 1 + \frac{\epsilon}{3}$, we have $\vec{u} \in L^3(\mathbb{R}^3)$, $p \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and we can proceed as in (4.10). We thus have $\lim\limits_{R\to+\infty} I_c = 0$.

- If $\frac{3}{5} < \alpha < 1$, we have $\vec{u} \in L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3)$ and by Lemma 4.1 we also have $p \in L^{\frac{6+\epsilon}{6-2\alpha}}(\mathbb{R}^3)$. As above, if $1 - \frac{\epsilon}{3} < \alpha \leq \frac{5}{3} - \frac{2}{9}\epsilon$, by the Hölder inequality with $\frac{3\alpha-3-\epsilon}{6-\epsilon} + \frac{6-2\alpha}{6-\epsilon} + \frac{3-\alpha}{6-\epsilon} = 1$ we obtain

$$I_c \leq CR^{-1+3\frac{3\alpha-3-\epsilon}{6-\epsilon}}\|p\|_{L^{\frac{6-\epsilon}{6-2\alpha}}(\mathcal{C}(\frac{R}{2},R))}\|\vec{u}\|_{L^{\frac{6-\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))}.$$

  Note that if $1-\frac{\epsilon}{3} < \alpha < \frac{5}{3}-\frac{2}{9}\epsilon$, the power of the parameter R is negative, while if $\alpha = \frac{5}{3} - \frac{2}{9}\epsilon$, we have $-1+3\frac{3\alpha-3+\epsilon}{6+\epsilon} = 0$, but we have $\|\vec{u}\|_{L^{\frac{6+\epsilon}{3-\alpha}}(\mathcal{C}(\frac{R}{2},R))} \xrightarrow[R\to+\infty]{} 0$. Observe also that if $\alpha = 1 - \frac{\epsilon}{3}$, then $\vec{u} \in L^{\frac{6+\epsilon}{3-\alpha}}(\mathbb{R}^3) = L^3(\mathbb{R}^3) < +\infty$ by hypothesis, $p \in L^{\frac{3}{2}}(\mathbb{R}^3) < +\infty$ by Lemma 4.1, and we can proceed as in (4.10). In any case we have $\lim\limits_{R\to+\infty} I_c = 0$.

We have proved that

$$\lim_{R\to+\infty} I_a = 0, \quad \lim_{R\to+\infty} I_b = 0, \quad \text{and} \quad \lim_{R\to+\infty} I_c = 0;$$

thus, by making $R \to +\infty$ in both sides of inequality (4.5) we easily obtain that

$$\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} = 0,$$

from which we deduce by the Sobolev embeddings that $\|\vec{u}\|_{L^{\frac{6}{3-\alpha}}} = 0$ and we finally obtain that $\vec{u} \equiv 0$. Theorem 2 is proved. □

5. Proof of Theorem 3

(i) We start by proving the first point of Theorem 3. Recall that in this case we have $\frac{5}{3} < \alpha < 2$. Thus, applying the Leray projector $\mathbb{P}$ to the fractional Navier–Stokes equation and applying the divergence-free condition, we have the equation $(-\Delta)^{\frac{\alpha}{2}}\vec{u} = -\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))$, which can be rewritten as

$$\vec{u} = -\frac{\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))}{(-\Delta)^{\frac{\alpha}{2}}}.$$

Now, for some index $\sigma > 0$ that will be defined later, we write

$$\|(-\Delta)^{\frac{\sigma}{2}}\vec{u}\|_{L^2} = \left\|(-\Delta)^{\frac{\sigma}{2}}\frac{\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))}{(-\Delta)^{\frac{\alpha}{2}}}\right\|_{L^2} \leq C\|(-\Delta)^{\frac{\sigma-\alpha+1}{2}}(\vec{u}\otimes\vec{u})\|_{L^2},$$

where we used the boundedness properties of the Leray projector in the L^2 space. At this point we apply the product law given in Lemma 2.4 to obtain (since $\vec{u}\in\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$)

$$\|(-\Delta)^{\frac{\sigma-\alpha+1}{2}}(\vec{u}\otimes\vec{u})\|_{L^2} = \|\vec{u}\otimes\vec{u}\|_{\dot{H}^{\sigma-\alpha+1}} \leq C\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}}\|\vec{u}\|_{\dot{H}^{\frac{\alpha}{2}}} < +\infty, \tag{5.1}$$

as long as $\sigma-\alpha+1 = \alpha-\frac{3}{2}$, from which we deduce that $\sigma = 2\alpha-\frac{5}{2}$. Now, since $\alpha > \frac{5}{3}$ we have that $\sigma > \frac{\alpha}{2}$. We have thus proved that

$$\|(-\Delta)^{\frac{\sigma}{2}}\vec{u}\|_{L^2} = \|\vec{u}\|_{\dot{H}^{\sigma}} < +\infty,$$

which is a gain of regularity. By iterating this process we easily obtain that the solutions of equation (1.6) are smooth.

(ii) We now study the second point of Theorem 3, where we have $1 < \alpha \leq \frac{5}{3}$. In this case, we have $\vec{u}\in\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^3)$, which does not seem to be enough to obtain a gain of regularity when applying Lemma 2.4 in estimate (5.1). To circumvent this issue, we will use the additional hypothesis given by $\vec{u}\in L^\infty(\mathbb{R}^3)$, instead of Lemma 2.4 we use the Leibniz fractional inequality given in Lemma 2.3, and in (5.1) we write:

$$\|(-\Delta)^{\frac{\sigma-\alpha+1}{2}}(\vec{u}\otimes\vec{u})\|_{L^2} \leq C\|(-\Delta)^{\frac{\sigma-\alpha+1}{2}}\vec{u}\|_{L^2}\|\vec{u}\|_{L^\infty},$$

which is a finite quantity as long as $\sigma-\alpha+1 = \frac{\alpha}{2}$, which gives $\sigma = \frac{3}{2}\alpha - 1$. Since $1 < \alpha \leq \frac{5}{3}$ we have $\sigma > \frac{\alpha}{2}$ and we have obtained a gain of regularity as we have proved that $\vec{u}\in\dot{H}^{\sigma}(\mathbb{R}^3)$. Again, by iteration we obtain that the solutions of equation (1.6) are smooth.

Theorem 3 is now proved. □

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer-Verlag, Berlin-New York, 1976. DOI: 10.1007/978-3-642-66451-9.

[2] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32(8)** (2007), 1245–1260. DOI: 10.1080/03605300600987306.

[3] D. Chamorro, O. Jarrín, and P.-G. Lemarié-Rieusset, Some Liouville theorems for stationary Navier–Stokes equations in Lebesgue and Morrey spaces, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **38(3)** (2021), 689–710. DOI: 10.1016/j.anihpc.2020.08.006.

[4] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady-State Problems*, Second edition, Springer Monogr. Math., Springer, New York, 2011. DOI: 10.1007/978-0-387-09620-9.

[5] L. Grafakos and S. Oh, The Kato–Ponce inequality, *Comm. Partial Differential Equations* **39(6)** (2014), 1128–1157. DOI: 10.1080/03605302.2013.822885.

[6] O. Jarrín, A remark on the Liouville problem for stationary Navier–Stokes equations in Lorentz and Morrey spaces, *J. Math. Anal. Appl.* **486(1)** (2020), 123871, 16 pp. DOI: 10.1016/j.jmaa.2020.123871.

[7] G. Koch, N. Nadirashvili, G. A. Seregin, and V. Šverák, Liouville theorems for the Navier–Stokes equations and applications, *Acta Math.* **203(1)** (2009), 83–105. DOI: 10.1007/s11511-009-0039-6.

[8] H. Kozono, Y. Terasawa, and Y. Wakasugi, A remark on Liouville-type theorems for the stationary Navier–Stokes equations in three space dimensions, *J. Funct. Anal.* **272(2)** (2017), 804–818. DOI: 10.1016/j.jfa.2016.06.019.

[9] P. G. Lemarié-Rieusset, *The Navier–Stokes Problem in the 21st Century*, CRC Press, Boca Raton, FL, 2016. DOI: 10.1201/b19556.

[10] V. Naibo and A. Thomson, Coifman–Meyer multipliers: Leibniz-type rules and applications to scattering of solutions to PDEs, *Trans. Amer. Math. Soc.* **372(8)** (2019), 5453–5481. DOI: 10.1090/tran/7866.

[11] G. Seregin, Liouville type theorem for stationary Navier–Stokes equations, *Nonlinearity* **29(8)** (2016), 2191–2195. DOI: 10.1088/0951-7715/29/8/2191.

[12] G. Seregin, A Liouville type theorem for steady-state Navier–Stokes equations, Journées *Équations aux dérivées partielles*, Roscoff, 30 mai–3 juin 2016, GDR 2434 (CNRS), Exposé no. IX, 5 p.

[13] L. Tang and Y. Yu, Partial Hölder regularity of the steady fractional Navier–Stokes equations, *Calc. Var. Partial Differential Equations* **55(2)** (2016), Art. 31, 18 pp. DOI: 10.1007/s00526-016-0967-x.

[14] Y. Wang and J. Xiao, A Liouville problem for the stationary fractional Navier–Stokes–Poisson system, *J. Math. Fluid Mech.* **20(2)** (2018), 485–498. DOI: 10.1007/s00021-017-0330-9.

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