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BILINEAR EMBEDDING FOR SCHRÖDINGER-TYPE OPERATORS WITH COMPLEX COEFFICIENTS

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Abstract: We prove a variant of the so-called bilinear embedding theorem for operators in divergence form with complex coefficients and with nonnegative locally integrable potentials, subject to mixed boundary conditions, and acting on arbitrary open subsets of $\mathbb{R}^d$.

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1. Introduction and statement of the main result

Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set. Denote by $\mathcal{A}(\Omega)$ the family of all complex *uniformly strictly accretive* (also called *elliptic*) $n \times n$ matrix functions on Ω with L^∞ coefficients. That is, the set of all measurable $A\colon \Omega \to \mathbb{C}^{d\times d}$ for which there exist $\lambda, \Lambda > 0$ such that for almost all $x \in \Omega$ we have

$$\operatorname{Re}\langle A(x)\xi, \xi\rangle \geqslant \lambda|\xi|^2, \qquad \forall \xi \in \mathbb{C}^d; \tag{1.1}$$

$$|\langle A(x)\xi, \eta\rangle| \leqslant \Lambda|\xi||\eta|, \quad \forall \xi, \eta \in \mathbb{C}^d. \tag{1.2}$$

Elements of $\mathcal{A}(\Omega)$ will also more simply be referred to as *accretive* or *elliptic matrices*. For any $A \in \mathcal{A}(\Omega)$ denote by $\lambda(A)$ the largest admissible λ in (1.1) and by $\Lambda(A)$ the smallest Λ in (1.2).

Denote by $H^1_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$. Let $\mathscr{V}$ be a closed subspace of $H^1(\Omega)$ containing $H^1_0(\Omega)$, that is,

$$H^1_0(\Omega) \subseteq \mathscr{V} \subseteq H^1(\Omega). \tag{1.3}$$

Recall that $H^1_0(\mathbb{R}^d) = H^1(\mathbb{R}^d)$; see [**1**, Corollary 3.19] for a reference.

Furthermore, let $V \in L^1_{\mathrm{loc}}(\Omega)$ be a nonnegative function. We define the operator, formally denoted by $Lu = -\operatorname{div}(A\nabla u) + Vu$, in the standard manner via sesquilinear forms; see e.g. [**35**, Sections 4.1 and 4.7]. Before proceeding, we state that all the integrals in this paper will be taken with respect to the Lebesgue measure. As the ambient space we will always take $\mathcal{H} = L^2(\Omega)$.

Let the form $\mathfrak{a} = \mathfrak{a}_{A,V} = \mathfrak{a}_{A,V,\mathscr{V}}$ be given by its domain

$$\mathcal{D}(\mathfrak{a}) = \left\{ u \in \mathscr{V};\ \int_\Omega V|u|^2 < \infty \right\} \tag{1.4}$$

and, for $u, v \in \mathcal{D}(\mathfrak{a})$, the formula

$$\mathfrak{a}(u, v) := \int_\Omega (\langle A\nabla u, \nabla v\rangle_{\mathbb{C}^n} + Vu\bar{v}). \tag{1.5}$$

We define $L = L_{A,V} = L_{A,V,\mathscr{V}}$ to be the unbounded, densely defined, closed operator on $L^2(\Omega)$, associated with $\mathfrak{a}_{A,V}$. See [**35**, Section 1.2.3] for information about this construction. So we have

$$\int_\Omega \langle Lu, v\rangle_{\mathbb{C}^n} = \int_\Omega (\langle A\nabla u, \nabla v\rangle_{\mathbb{C}^n} + Vu\bar{v}), \quad \forall u \in \mathcal{D}(L),\, v \in \mathcal{D}(\mathfrak{a}).$$

In accordance with Davies [**11**, 1.8] we call L a *generalized Schrödinger operator* and V its *potential.*

Given $\vartheta \in (0, \pi)$, define the *(open) sector of angle* ϑ by

$$\mathbf{S}_\vartheta = \{z \in \mathbb{C} \setminus \{0\};\ |\arg z| < \vartheta\}.$$

Also set $\mathbf{S}_0 = (0, \infty)$. The basic properties of $\mathfrak{a}$ are recalled in the following result. Let $\nu(A)$ be the opening angle of the smallest sector whose closure contains the *numerical range* of A; see [**8**, (2.9)] for the definition of $\nu(A)$. Surely, the latter was based on the classical definition of the numerical range [**30**, p. 267]. The definition of accretivity implies $0 \leqslant \nu(A) \leqslant \arccos(\lambda/\Lambda) < \pi/2$ for $A \in \mathcal{A}(\Omega)$. Other notions appearing in the next statement can be found in [**35**].

Theorem 1.1. *For every* $\phi \in \mathbb{R}$ *such that* $|\phi| < \pi/2 - \nu(A)$*, the form* $e^{i\phi}\mathfrak{a}$ *is densely defined, sectorial, and closed.*

Sectorial forms are automatically accretive and continuous; see e.g. [**35**, Proposition 1.8]. Therefore, by [**35**, Proposition 1.27 and Theorem 1.54], the operator $-L$ generates on $L^2(\Omega)$ a strongly continuous semigroup of operators

$$T_t = T_t^{A,V} = T_t^{A,V,\mathscr{V}} = \exp(-tL), \quad t > 0,$$

which is holomorphic and contractive in the sector $\mathbf{S}_{\pi/2-\nu(A)}$. Hence T_t maps $L^2(\Omega)$ into $\mathcal{D}(L) \subseteq \mathscr{V}$ [**23**, Theorem II.4.6], and since $\mathscr{V} \subseteq H^1(\Omega)$, the spatial gradient $\nabla T_t f$ is well defined. By [**39**, p. 72], given $f \in L^2(\Omega)$, we can redefine each $T_t f$ on a set of measure zero, in such a manner that for almost every $x \in \Omega$ the function $t \mapsto T_t f(x)$ is real-analytic on $(0, \infty)$.

1.1. Special classes of boundary conditions. Here we describe certain classes of domains $\mathscr{V}$ that, on top of (1.3), satisfy additional conditions which will be assumed throughout the rest of this paper.

We say that the space $\mathscr{V} \subset H^1(\Omega)$ is *invariant* under:

- the function $p\colon \mathbb{C} \to \mathbb{C}$, if $u \in \mathscr{V}$ implies $p(u) := p \circ u \in \mathscr{V}$;
- the family $\mathscr{P}$ of functions $\mathbb{C} \to \mathbb{C}$, if it is invariant under all $p \in \mathscr{P}$.

Define a function $P\colon \mathbb{C} \to \mathbb{C}$ by

$$P(\zeta) = \begin{cases} \zeta; & |\zeta| \leqslant 1, \\ \zeta/|\zeta|; & |\zeta| \geqslant 1. \end{cases} \tag{1.6}$$

Thus $P(\zeta) = \min\{1, |\zeta|\} \operatorname{sign} \zeta$, where sign is defined as in [**35**, (2.2)]:

$$\operatorname{sign} \zeta := \begin{cases} \zeta/|\zeta|; & \zeta \neq 0, \\ 0; & \zeta = 0. \end{cases}$$

Let $\mathscr{V}$ be a closed subspace of $H^1(\Omega)$ containing $H^1_0(\Omega)$ and such that

$$\mathscr{V} \text{ is invariant under the function } P. \tag{1.7}$$

It is well known (see [**35**, Proposition 4.11]) that (1.7) is satisfied in these notable cases which will feature in our *bilinear embedding* (Theorem 1.4):

(a) $\mathscr{V} = H^1_0(\Omega)$,
(b) $\mathscr{V} = H^1(\Omega)$,
(c) $\mathscr{V}$ is the closure in $H^1(\Omega)$ of $\{u|_\Omega;\ u \in C_c^\infty(\mathbb{R}^d\backslash\Gamma)\}$, where Γ is a (possibly empty) closed subset of $\partial\Omega$.

When $\mathscr{V}$ falls into any of the special cases (a)–(c) from Subsection 1.1, we say that $L = L_{A,V,\mathscr{V}}$ is subject to (a) *Dirichlet*, (b) *Neumann*, or (c) *mixed boundary conditions.*

1.2. The *p*-ellipticity condition. The concept of p-ellipticity was introduced by the present authors in [**8**] as follows.

Given $A \in \mathcal{A}(\Omega)$ and $p \in (1,\infty)$, we say that A is *p-elliptic* if $\Delta_p(A) > 0$, where

$$\Delta_p(A) := \operatorname*{ess\,inf}_{x\in\Omega} \min_{\substack{\xi\in\mathbb{C}^d\\|\xi|=1}} \operatorname{Re}\langle A(x)\xi, \xi + |1-2/p|\bar{\xi}\rangle_{\mathbb{C}^d}. \tag{1.8}$$

Equivalently, A is p-elliptic if there exists $C = C(A,p) > 0$ such that for a.e. $x \in \Omega$,

$$\operatorname{Re}\langle A(x)\xi, \xi + |1-2/p|\bar{\xi}\rangle_{\mathbb{C}^d} \geqslant C|\xi|^2, \quad \forall \xi \in \mathbb{C}^d. \tag{1.9}$$

It follows straight from (1.8) that Δ_p is invariant under conjugation of p, meaning that $\Delta_p(A) = \Delta_q(A)$ when $1/p + 1/q = 1$.

The concept of p-ellipticity generalizes the notion of ellipticity [**15**]. Indeed, the number $\Delta_p(A)$ describes the interplay between the matrix A and the Lebesgue exponent p, which is "hidden" in the classical notion of ellipticity, as the latter corresponds to taking $p = 2$. Namely, $\Delta_2(A) = \lambda(A)$. On the other hand, for any fixed $p > 2$, the class of complex p-elliptic matrices is strictly smaller than the class of all elliptic matrices. Thus when $p > 2$, the condition of p-ellipticity at the same time strengthens the classical ellipticity condition.

Denote by $\mathcal{A}_p(\Omega)$ the class of all p-elliptic matrix functions on Ω. It is known, see [**8**], that $\{\mathcal{A}_p(\Omega);\ p \in [2,\infty)\}$ is a decreasing chain of matrix classes such that

$$\begin{aligned}\{\text{elliptic matrices on } \Omega\} &= \mathcal{A}_2(\Omega),\\ \{\text{real elliptic matrices on } \Omega\} &= \bigcap_{p\in[2,\infty)} \mathcal{A}_p(\Omega).\end{aligned}$$

Since we will be dealing with pairs of matrices, it is useful to introduce further notation, as in [**8, 7**]:

$$\begin{aligned}\Delta_p(A,B) &= \min\{\Delta_p(A), \Delta_p(B)\},\\ \lambda(A,B) &= \min\{\lambda(A), \lambda(B)\},\\ \Lambda(A,B) &= \max\{\Lambda(A), \Lambda(B)\}.\end{aligned}$$

While the present authors were preparing [**8**], M. Dindoš and J. Pipher were working on their own article [**13**]. They found a sharp condition, see [**13**, (1.3)], which implies reverse Hölder inequalities for weak solutions of elliptic operators in divergence form with complex coefficients. It turned out that their condition, devised independently of [**8**], namely, as a strengthening of [**10**, (2.25)], was exactly equivalent to (1.9). The same authors have since then been successfully continuing their line of exploration of p-ellipticity in PDEs; see their recent papers [**14, 12**].

The notion of p-ellipticity emerged in [**8**] after several years of gradually distilling the *Bellman-function-heat-flow method* (see Subsection 3.2), initiated in [**36, 41**], through [**18, 19, 17, 5, 6, 4**]. More information about the genesis of p-ellipticity can be found in [**8, 9**].

1.3. Semigroup properties on L^p. Our first result is Theorem 1.2. It generalizes the implication $(a) \Rightarrow (b)$ of [**8**, Theorem 1.3]. See also [**7**, Proposition 1], where it was proven in special cases (a)–(c) from Subsection 1.1, and $\phi = 0$, $V = 0$. The proof of Theorem 1.2 is a modification of the one from [**8**], the main difference being that instead of [**35**, Theorem 4.7] we now use a more general result [**35**, Theorem 4.31]. In all of those cases, we build on a criterion by Nittka (Theorem 2.2). Assuming again that $\phi = 0$, $V = 0$, and $\mathscr{V}$ is one of the special cases (a)–(c) from Subsection 1.1, a proof of Theorem 1.2 different from the one above, yet still resting on Nittka's theorem, was recently found by Egert [**20**, Proposition 13]. Compare also with theorems by ter Elst et al. [**22, 21**].

Theorem 1.2. *Choose $p > 1$, $A \in \mathcal{A}(\Omega)$, $\phi \in \mathbb{R}$, such that $|\phi| < \pi/2 - \nu(A)$ and $\Delta_p(e^{i\phi}A) \geqslant 0$, and a nonnegative $V \in L^1_{\rm loc}(\Omega)$. Then, for every $\mathscr{V}$ satisfying* (1.7),

$$\left(e^{-te^{i\phi}L_{A,V,\mathscr{V}}}\right)_{t>0}$$

extends to a contractive semigroup on $L^p(\Omega)$.

The next corollary extends [**7**, Lemma 17].

Corollary 1.3. *Choose $p > 1$, $A \in \mathcal{A}_p(\Omega)$, and a nonnegative $V \in L^1_{\rm loc}(\Omega)$. Let $\mathscr{V}$ satisfy* (1.7). *Then there exists $\vartheta = \vartheta(p, A) > 0$ such that if $|1-2/r| \leqslant |1-2/p|$, then $\{T_z^{A,V};\, z \in \mathbf{S}_\vartheta\}$ is holomorphic and contractive in $L^r(\Omega)$.*

The proofs of these results will be given in Section 2.

1.4. Main result: the bilinear embedding theorem for pairs of complex p-elliptic operators with mixed boundary conditions. In this section we assume boundary conditions that are less general than those from our contractivity result (Theorem 1.2). Namely, we take pairs $\mathscr{V}$, $\mathscr{W}$ which are of the form (a)–(c) from Subsection 1.1. We needed this restriction in order to tackle technical issues which arise in the proof of this note's main result, the *dimension-free bilinear embedding theorem* which we formulate next.

Theorem 1.4. *Choose $p > 1$. Let q be its conjugate exponent, i.e., $1/p + 1/q = 1$. Suppose that $A, B \in \mathcal{A}_p(\Omega) > 0$. Let $V, W \in L^1_{\rm loc}(\Omega)$ be nonnegative. Assume that the operators $L_{A,V,\mathscr{V}}$ and $L_{B,W,\mathscr{W}}$ are subject to Dirichlet, Neumann, or mixed boundary conditions, cf. (a)–(c) from Subsection 1.1.*

There exists $C > 0$ such that for any $f, g \in (L^p \cap L^q)(\Omega)$ we have

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_t^{A,V,\mathscr{V}} f|^2 + V|T_t^{A,V,\mathscr{V}} f|^2}\,\sqrt{|\nabla T_t^{B,W,\mathscr{W}} g|^2 + W|T_t^{B,W,\mathscr{W}} g|^2} \leqslant C\|f\|_p\|g\|_q.$$

We may choose $C > 0$ which depends on p, A, B, but not on the dimension d nor on the potentials V, W.

This result incorporates several earlier theorems as special cases, including:

- $V = W$, $\Omega = \mathbb{R}^d$, $A = B$ equal and real [**18**, Theorem 1],
- $V = W = 0$, $\Omega = \mathbb{R}^d$ [**8**, Theorem 1.1],
- $V = W = 0$ [**7**, Theorem 2].

See also [**19**, Theorem 1] for a variant with $A = B = I$, $V = W$, $\Omega = \mathbb{R}^d$, and involving the semigroup generated by the *square root* of the operator L. This variant also bore consequences in the shape of dimension-free L^p estimates of Riesz transforms associated with the harmonic oscillator (Hermite operator), which are optimal with respect to p; see [**19**, Corollary 1]. The difficulty that we encounter in this paper is the generality of the setting in terms of domains (Ω), matrices (A, B), and potentials (V, W), which requires significantly more effort in order to complete the proof.

Various types of bilinear embeddings have been proven in the last 20 years, often admitting important consequences, such as Riesz transform estimates and optimal holomorphic functional calculus. The present authors' efforts aimed at proving bilinear embedding as in [**8**, Theorem 1.1] eventually gave rise to the concept of p-ellipticity summarized in Subsection 1.2. See [**9**] and the above references for more historical background and motivation. We also remark that p-ellipticity is the sharp condition for *dimension-free* bilinear embeddings; see [**8**, Section 1.4] for a precise statement.

Finally we remark that one of the natural ways to extend the present work is to investigate bilinear embeddings for elliptic operators L with *first-order terms*. This was recently done by A. Poggio [**37**]. In order to run the argument in that context, one needs to adequately extend the p-ellipticity condition so as to take into account the presence of first-order perturbations.

2. Proof of Theorem 1.2

We first recall the notion of L^p*-dissipativity* of sesquilinear forms. It was introduced by Cialdea and Maz'ya in [**10**, Definition 1] for the case of forms defined on $C_c^1(\Omega)$ and associated with complex matrices. In [**8**, Definition 7.1] the present authors extended their definition as follows.

Definition 2.1. Let X be a measure space, $\mathfrak{b}$ a sesquilinear form defined on the domain $\mathcal{D}(\mathfrak{b}) \subset L^2(X)$, and $1 < p < \infty$. Denote

$$\mathcal{D}_p(\mathfrak{b}) := \{u \in \mathcal{D}(\mathfrak{b});\ |u|^{p-2}u \in \mathcal{D}(\mathfrak{b})\}.$$

We say that $\mathfrak{b}$ is L^p*-dissipative* if

$$\operatorname{Re} \mathfrak{b}(u, |u|^{p-2}u) \geqslant 0, \quad \forall u \in \mathcal{D}_p(\mathfrak{b}).$$

The following theorem is due to Nittka [**33**, Theorem 4.1]. We remark that Nittka formulated his result for sectorial forms, but it seems that sectoriality is not needed for our version of his result, since it is not needed for Ouhabaz's criterion [**35**, Theorem 2.2], on which Nittka's own criterion is based. Of course, the forms we are dealing with in this paper are all sectorial anyway.

Theorem 2.2. *Let (Ω, μ) be a measure space. Suppose that the sesquilinear form $\mathfrak{a}$ on $L^2 = L^2(\Omega, \mu)$ is densely defined, accretive, continuous, and closed. Let $\mathcal{L}$ be the operator associated with $\mathfrak{a}$ in the sense of* [**35**, Section 1.2.3].

Take $p \in (1, \infty)$ and define $B^p := \{u \in L^2 \cap L^p;\ \|u\|_p \leqslant 1\}$. Let $\mathbf{P}_{B^p}$ be the orthogonal projection $L^2 \to B^p$. Then the following assertions are equivalent:

(i) $\|\exp(-t\mathcal{L})f\|_p \leqslant \|f\|_p$ *for all $f \in L^2 \cap L^p$ and all $t \geqslant 0$;*

(ii) $\mathcal{D}(\mathfrak{a})$ *is invariant under $\mathbf{P}_{B^p}$ and $\mathfrak{a}$ is L^p-dissipative.*

Define for $p > 1$ the operator $\mathcal{I}_p \colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{I}_p \xi = \xi + (1 - 2/p)\bar{\xi}. \tag{2.1}$$

Observe that $\mathcal{I}_p$ appears in (1.8) and (1.9).

The formula below already appeared in [**8**, (7.5)], where it was considered under the assumptions $f, |f|^{p-2}f \in H_0^1(\Omega)$. Here we need a stronger version that we state next. For the reader's convenience, we prove it in Appendix B; see Corollary B.4 and Remark B.5.

Lemma 2.3. *Suppose that $p>1$ and that f and $|f|^{p-2}f$ belong to $H^1_{\rm loc}(\Omega)$. Then*

$$\nabla(|f|^{p-2}f)=\frac{p}{2}|f|^{p-2}\operatorname{sign} f\cdot \mathcal{I}_p(\operatorname{sign}\bar{f}\cdot\nabla f). \tag{2.2}$$

Consequently,

$$|\nabla(|f|^{p-2}f)|\sim|f|^{p-2}|\nabla f|. \tag{2.3}$$

Hence, for f, p as in Lemma 2.3 and any $B\in\mathcal{A}(\Omega)$ we have

$$\langle B\nabla f,\nabla(|f|^{p-2}f)\rangle_{\mathbb{C}^d}=\frac{p}{2}|f|^{p-2}\langle B(\operatorname{sign}\bar{f}\cdot\nabla f),\mathcal{I}_p(\operatorname{sign}\bar{f}\cdot\nabla f)\rangle_{\mathbb{C}^d}. \tag{2.4}$$

Note the symmetric structure of the inner product above, which is expressed in the appearance of $(\operatorname{sign}\bar{f})\nabla f$ in both factors.

Taking real parts and recalling (1.8) and (2.1) we conclude

$$\operatorname{Re}\langle B\nabla f,\nabla(|f|^{p-2}f)\rangle_{\mathbb{C}^d}\geqslant\frac{p}{2}\Delta_p(B)|f|^{p-2}|\nabla f|^2. \tag{2.5}$$

This inequality will be eventually used in the proof of Theorem 1.2.

Remark 2.4. In view of Lemma 2.3 it does not come as a surprise that the auxiliary operator $\mathcal{I}_p$ is also a part of the Hessian of $|\zeta|^p$. More precisely, we have the following formula, valid for $\zeta\in\mathbb{C}\backslash\{0\}$, $\xi\in\mathbb{C}^d$, which is a reformulation of [**8**, (5.5)]:

$$d^2F_p(\zeta)\xi=\frac{p^2}{2}|\zeta|^{p-2}\operatorname{sign}\zeta\cdot\mathcal{I}_p(\operatorname{sign}\bar{\zeta}\cdot\xi).$$

Here $F_p(\zeta)=|\zeta|^p$ and $d^2F_p(\zeta)$ is the Hessian of F_p, calculated at $\zeta\in\mathbb{C}\equiv\mathbb{R}^2$ and applied to $\xi\in\mathbb{C}^d\equiv\mathbb{R}^d\times\mathbb{R}^d$. Let us reveal the underlying connections further.

An expression akin to (2.4) appears when one differentiates with respect to t the integral of $|\exp(-tL_B)\varphi|^p$ and then integrates by parts. Indeed, this gives

$$-\int_\Omega\frac{d}{dt}|e^{-tL_B}\varphi|^p=\operatorname{Re}\int_\Omega\langle B\nabla f,2\nabla[(\partial_{\bar{z}}F_p)(f)]\rangle_{\mathbb{C}^d}\quad\text{with } f=e^{-tL_B}\varphi.$$

Furthermore, one checks that

$$2\nabla[(\partial_{\bar{z}}F_p)(f)]=[d^2F_p(f)\otimes I_{\mathbb{R}^d}]\nabla f. \tag{2.6}$$

At this point observe that $2(\partial_{\bar{z}}F_p)(f)=p|f|^{p-2}f$ and recall Lemma 2.3.

Proof of Theorem 1.2: We will use Nittka's invariance criterion (Theorem 2.2). Under our assumptions on ϕ, the form $\mathfrak{b}:=e^{i\phi}\mathfrak{a}$ falls into the framework of Nittka's criterion, by Theorem 1.1. The operator associated with $\mathfrak{b}$ is $e^{i\phi}L_{A,V}$.

In order to apply Theorem 2.2, we must check the following:

(a) $\mathcal{D}(\mathfrak{b})=\mathcal{D}(\mathfrak{a})$ is invariant under $\mathbf{P}_{B^p}$;
(b) $\mathfrak{b}$ is L^p-dissipative.

Let us start with (a). Let $-\Delta+V$ denote the operator associated with the form $\mathfrak{a}_{I,V,\mathscr{V}}$. By the basic assumption (1.7) and [**35**, Theorem 4.31*2)*], the semigroup

$$(e^{-t(-\Delta+V)})_{t>0}$$

is contractive on $L^\infty(\Omega)$, and thus, by interpolation with the L^2-estimates, on $L^p(\Omega)$ for all $p\in[1,\infty]$. Hence Nittka's Theorem 2.2 gives that $\mathcal{D}(\mathfrak{a}_{I,V,\mathscr{V}})$ is invariant under $\mathbf{P}_{B^p}$. Now use that $\mathcal{D}(\mathfrak{a}_{I,V,\mathscr{V}})=\mathcal{D}(\mathfrak{a})=\mathcal{D}(\mathfrak{b})$.

The statement (b) follows from the (weak) p-ellipticity of $e^{i\phi}A$ virtually without changing the argument from [**8**]. Indeed, if $u \in \mathcal{D}_p(\mathfrak{b})$, we get from (2.5), applied with $B = e^{i\phi}A$, that

$$\operatorname{Re}\mathfrak{b}(u, |u|^{p-2}u) = \operatorname{Re}\langle e^{i\phi}A\nabla u, \nabla(|u|^{p-2}u)\rangle_{L^2(\Omega)} + \cos\phi \int_\Omega V|u|^p$$

is a sum of two nonnegative terms. □

Proof of Corollary 1.3: By the continuity of $\varepsilon \mapsto \Delta_r(e^{i\varepsilon}A)$, see [**8**, Section 5.4], and monotonicity and symmetry properties of $r \mapsto \Delta_r(e^{i\varepsilon}A)$, see [**8**, Corollary 5.16 and Proposition 5.8], there exists $\vartheta = \vartheta(p, A) > 0$ such that $\Delta_r(e^{i\varepsilon}A) > 0$ for all $\varepsilon \in [-\vartheta, \vartheta]$ and all $r > 1$ satisfying $|1-2/r| \leqslant |1-2/p|$. The contractivity part now follows from Theorem 1.2 and the relation $T^{A,V}_{te^{i\varepsilon}} = \exp(-te^{i\varepsilon}L_{A,V})$, whereupon analyticity is a consequence of a standard argument [**23**, Theorem II.4.6]. □

3. Proof of Theorem 1.4

In proving Theorem 1.4 we will combine and enhance the following tools:

- contractivity and analyticity properties of the semigroups T_t on L^p [**8**],
- convexity properties of the appropriate *Bellman function* [**19, 18, 5, 8**],
- analysis of the *heat flow* associated with the regularized Bellman function [**5, 8, 7**].

The first item was already settled in Theorem 1.2. We treat the remaining two main steps in separate sections as follows.

3.1. Bellman function. Unless specified otherwise, we assume everywhere in this section that $p \geqslant 2$ and $q = p/(p-1)$. Let $\delta > 0$. The Bellman function we use is the function $\mathcal{Q} = \mathcal{Q}_{p,\delta} \colon \mathbb{C} \times \mathbb{C} \to \mathbb{R}_+$ defined by

$$\mathcal{Q}(\zeta, \eta) := |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2|\eta|^{2-q}; & |\zeta|^p \leqslant |\eta|^q, \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q; & |\zeta|^p \geqslant |\eta|^q. \end{cases} \tag{3.1}$$

This function is due to Nazarov and Treil. See [**8**] or [**7**] for an up-to-date account on previous appearances of $\mathcal{Q}$ in the literature.

It is a direct consequence of the above definition that the function $\mathcal{Q}$ belongs to $C^1(\mathbb{C}^2)$ and is of order C^2 everywhere *except* on the set

$$\Upsilon = \{(\zeta, \eta) \in \mathbb{C} \times \mathbb{C};\ (\eta = 0) \vee (|\zeta|^p = |\eta|^q)\}.$$

We shall use the notation from [**8**, Section 2.2] to denote the *generalized Hessians* $H^{(A,B)}_{\mathcal{Q}}[v; \omega]$ and $H^A_F[\zeta; \xi]$. Vaguely speaking, if $N \in \mathbb{N}$, σ is a N-tuple of complex numbers, X a N-tuple of vectors from $\mathbb{C}^d$, $\mathbf{A}$ a N-tuple of matrices from $\mathbb{C}^{d\times d}$, and $\Phi \colon \mathbb{C}^N \to \mathbb{R}$ is of class C^2, then one has

$$H^{\mathbf{A}}_\Phi[\sigma; X] = \langle d^2\Phi(\sigma)X, \mathbf{A}X\rangle_{(\mathbb{C}^d)^N}.$$

In comparison with the exact definition [**8**] we have omitted here the tensorization of the Hessian $d^2\Phi(\omega)$ with the $d\times d$ identity matrix, as well as appropriate identifications of complex objects (vectors, matrices) with their real counterparts.

This is a stronger version of [**8**, Theorem 5.2] suited for potentials. It evokes [**19**, Theorem 3], where similar properties of $\mathcal{Q}$ were proven, also for the purpose of treating Schrödinger operators. The main distinction is that here we treat arbitrary complex matrix functions A, B, while in [**19**, Theorem 3] we only addressed the case $A = B \equiv I$. The reader is also referred to [**16**].

Theorem 3.1. *Choose $p \geqslant 2$ and $A, B \in \mathcal{A}_p(\Omega)$. Then there exists a continuous function $\tau\colon \mathbb{C}^2 \to [0,+\infty)$ such that $\tau^{-1} = 1/\tau$ is locally integrable in $\mathbb{C}^2$ and continuous on $\mathbb{C}^2 \setminus \{(0,0)\}$, and $\delta \in (0,1)$ such that $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ as in (3.1) admits the following properties:*

(i) *for any $v = (\zeta,\eta) \in \mathbb{C}^2 \setminus \Upsilon$, $\omega = (\omega_1,\omega_2) \in \mathbb{C}^d \times \mathbb{C}^d$, and a.e. $x \in \Omega$, we have*

$$H_{\mathcal{Q}}^{(A(x),B(x))}[v;\omega] \gtrsim \tau|\omega_1|^2 + \tau^{-1}|\omega_2|^2;$$

(ii) *for any $(\zeta,\eta) \in \mathbb{C}^2$, we have*

$$(\partial_\zeta \mathcal{Q})(\zeta,\eta)\cdot\zeta \gtrsim \tau|\zeta|^2 \quad \textit{and} \quad (\partial_\eta \mathcal{Q})(\zeta,\eta)\cdot\eta \gtrsim \tau^{-1}|\eta|^2.$$

The implied constants depend on A, B, and p, but not on the dimension d.

We may take $\tau(\zeta,\eta) = \max\{|\zeta|^{p-2}, |\eta|^{2-q}\}$.

Remark 3.2. It seems that, for arbitrary functions, the first property in general does *not* imply the second one *with the same* τ, even in the case $A = B \equiv I$. See [**19**, Section 2.2].

In order to prove Theorem 3.1 we start with a pair of elementary equivalences which are variants of [**8**, Lemma 5.24].

Lemma 3.3. *Suppose that $\alpha,\beta,\gamma \in \mathbb{R}$. The following statements are equivalent:*

(i) *There exists $\tau > 0$ such that $\alpha x^2 - 2\beta xy + \gamma y^2 \geqslant \tau x^2 + \tau^{-1}y^2$ for all $x,y \in \mathbb{R}$;*

(ii) *$\alpha,\gamma > 0$ and $\sqrt{\alpha\gamma} - |\beta| \geqslant 1$.*

Proof: (i) $\Rightarrow$ (ii): By taking $x \neq 0 = y$ and $x = 0 \neq y$ we get

$$\alpha \geqslant \tau \geqslant \frac{1}{\gamma} > 0. \tag{3.2}$$

The assumption (i) implies that $\alpha x^2 - 2\beta xy + \gamma y^2 \geqslant 2|xy|$ and hence

$$\alpha x^2 + \gamma y^2 \geqslant 2(|\beta|+1)|xy|, \quad \forall x,y \in \mathbb{R}.$$

Dividing by y^2 and writing $t = |x/y|$ we get

$$\alpha t^2 - 2(|\beta|+1)t + \gamma \geqslant 0, \quad \forall t > 0.$$

Clearly the above inequality is then also valid for $t \leqslant 0$, and thus for all $t \in \mathbb{R}$. Of course, this is possible if and only if $(|\beta|+1)^2 - \alpha\gamma \leqslant 0$.

(ii) $\Rightarrow$ (i): Define $\tau := \sqrt{\alpha/\gamma}$. Our assumption implies (3.2). Then for any $x,y \in \mathbb{R}$ we have

$$\begin{aligned}
\alpha x^2 - 2\beta xy + \gamma y^2 - \tau x^2 - \tau^{-1}y^2 &= (\alpha-\tau)x^2 + (\gamma-\tau^{-1})y^2 - 2\beta xy \\
&\geqslant 2\sqrt{(\alpha-\tau)(\gamma-\tau^{-1})}|xy| - 2|\beta||xy| \\
&= \frac{2|xy|}{\sqrt{(\alpha-\tau)(\gamma-\tau^{-1})} + |\beta|}((\alpha-\tau)(\gamma-\tau^{-1}) - \beta^2).
\end{aligned}$$

Since

$$(\alpha-\tau)(\gamma-\tau^{-1}) - \beta^2 = (\sqrt{\alpha\gamma}-1)^2 - \beta^2 \geqslant 0,$$

our proof is complete. $\square$

Corollary 3.4. *Suppose $a, b, c \in \mathbb{R}$. The following conditions are equivalent:*

(i) *there exist $C, \tau > 0$ such that $ax^2 - 2bxy + cy^2 \geqslant C(\tau x^2 + \tau^{-1} y^2)$ for all $x, y \in \mathbb{R}$;*
(ii) *$a, c > 0$ and $ac - b^2 > 0$.*

In this case the largest admissible choice for C is $C = \sqrt{ac} - |b|$. Moreover, we may take $\tau = \sqrt{a/c}$.

Proof: Divide the inequality from the first statement by C and use Lemma 3.3. □

Proof of Theorem 3.1: We follow, and adequately modify, the proof of [**8**, Theorem 5.2], which was in turn modeled after the proofs of [**19**, Theorem 3], [**5**, Theorem 15], and [**6**, Theorem 5.2]. We shall be using the notation $F_p(\zeta) = |\zeta|^p$ for $\zeta \in \mathbb{C}$, introduced on page 88.

Let us start with the case $p = 2$. In this special case the Bellman function reads

$$\mathcal{Q}(\zeta, \eta) = (1 + \delta) F_2(\zeta) + F_2(\eta) \quad \text{for all } \zeta, \eta \in \mathbb{C}. \tag{3.3}$$

Therefore, by [**8**, Lemma 5.6],

$$\begin{aligned} H_{\mathcal{Q}}^{(A,B)}[v; \omega] &= (1 + \delta) H_{F_2}^{A}[\zeta; \omega_1] + H_{F_2}^{B}[\eta; \omega_2] \\ &= 2(1 + \delta) \operatorname{Re}\langle A\omega_1, \omega_1 \rangle + 2 \operatorname{Re}\langle B\omega_2, \omega_2 \rangle \\ &\geqslant 2\lambda(A, B)(|\omega_1|^2 + |\omega_2|^2). \end{aligned} \tag{3.4}$$

On the other hand, trivial calculations show that

$$\text{if } \Phi = F_r \otimes F_s \text{ with } r, s \geqslant 0, \text{ then } \begin{cases} (\partial_\zeta \Phi)(\zeta, \eta) \cdot \zeta = (r/2)\Phi(\zeta, \eta), \\ (\partial_\eta \Phi)(\zeta, \eta) \cdot \eta = (s/2)\Phi(\zeta, \eta). \end{cases} \tag{3.5}$$

Thus by keeping in mind (3.3) and applying (3.5) separately with $(r, s) = (2, 0)$ and $(r, s) = (0, 2)$, we arrive at

$$\begin{aligned} (\partial_\zeta \mathcal{Q})(\zeta, \eta) \cdot \zeta &= (1 + \delta)|\zeta|^2, \\ (\partial_\eta \mathcal{Q})(\zeta, \eta) \cdot \eta &= |\eta|^2. \end{aligned} \tag{3.6}$$

By combining (3.4) and (3.6) we prove the theorem in the case $p = 2$ with $\tau = 1$.

Now consider the case $p > 2$. Denote $\mathsf{u} = |\zeta|$, $\mathsf{v} = |\eta|$, $\mathsf{A} = |\omega_1|$, $\mathsf{B} = |\omega_2|$. Recall the notation (1.8). We divide $\mathbb{C}^2 \setminus \Upsilon$ into two natural subdomains in which we analyze the gradients and Hessians of $\mathcal{Q}$ separately.

First assume that $\boxed{\mathsf{u}^p > \mathsf{v}^q > 0}$. Then, by [**8**, proof of Theorem 5.2, p. 3205 top],

$$H_{\mathcal{Q}}^{(A,B)}[v; \omega] \geqslant \frac{pq\Delta_p(A, B)}{2} [(p-1)\mathsf{u}^{p-2}\mathsf{A}^2 + (q-1)\mathsf{v}^{q-2}\mathsf{B}^2].$$

So in this case we may take $\tau = (p - 1)\mathsf{u}^{p-2}$, as in [**19**, proof of Theorem 3].

Regarding the last pair of estimates, since here $\mathcal{Q}_{p,\delta}$ is a linear combination of functions $F_p \otimes F_0$ and $F_0 \otimes F_q$, it follows from (3.5) that, similarly to (3.6),

$$\begin{aligned} (\partial_\zeta \mathcal{Q})(\zeta, \eta) \cdot \zeta &= C_1(p, \delta)|\zeta|^p, \\ (\partial_\eta \mathcal{Q})(\zeta, \eta) \cdot \eta &= C_2(p, \delta)|\eta|^q. \end{aligned}$$

Therefore, since $\mathsf{v}^{q-2} > \mathsf{u}^{2-p} > 0$, with $\tau = (p - 1)\mathsf{u}^{p-2}$ we get

$$\begin{aligned} (\partial_\zeta \mathcal{Q})(\zeta, \eta) \cdot \zeta &= \widetilde{C_1}(p, \delta)\tau|\zeta|^2, \\ (\partial_\eta \mathcal{Q})(\zeta, \eta) \cdot \eta &\geqslant \widetilde{C_2}(p, \delta)\tau^{-1}|\eta|^2. \end{aligned}$$

Suppose now that $\boxed{\mathsf{u}^p < \mathsf{v}^q}$. Extend the definition (1.8) to all $p \in (0, \infty]$. Then, as in [**8**, proof of Theorem 5.2],

$$H_{\mathcal{Q}}^{(A,B)}[v;\omega] \geqslant 2\delta\left(\lambda(A)\mathsf{v}^{2-q}\mathsf{A}^2 - 2(2-q)\Lambda(A,B)\mathsf{A}\mathsf{B} + \frac{\Gamma}{4}\mathsf{v}^{q-2}\mathsf{B}^2\right),$$

where

$$\Gamma = \frac{q^2\Delta_q(B)}{\delta} + (2-q)^2\Delta_{2-q}(B).$$

Since $\Delta_p(B) > 0$, we have that Γ grows to infinity as $\delta \searrow 0$. Since we also have $\lambda(A) > 0$, there exists $\delta = \delta(p, A, B) > 0$ such that

$$\frac{\lambda(A)\Gamma}{4} > [(2-q)\Lambda(A,B)]^2,$$

which through Corollary 3.4 implies the existence of $\tau > 0$ that accommodates the first requirement of Theorem 3.1. Moreover, we may take $\tau = D\mathsf{v}^{2-q}$, where $D = 2\sqrt{\lambda(A)/\Gamma}$.

Now consider the gradient estimates. In the domain $\{\mathsf{u}^p < \mathsf{v}^q\}$ we have

$$\mathcal{Q} = F_p \otimes F_0 + F_0 \otimes F_q + \delta F_2 \otimes F_{2-q}.$$

Again (3.5) implies that with (the above chosen) $\tau = D\mathsf{v}^{2-q}$ we get

$$(\partial_\zeta \mathcal{Q})(\zeta,\eta)\cdot\zeta = \frac{p}{2}|\zeta|^p + \delta|\zeta|^2|\eta|^{2-q} \geqslant \delta|\eta|^{2-q}|\zeta|^2 \geqslant \frac{\delta}{D}\tau|\zeta|^2,$$

$$(\partial_\eta \mathcal{Q})(\zeta,\eta)\cdot\eta = \frac{q}{2}|\eta|^q + \delta\cdot\frac{2-q}{2}|\zeta|^2|\eta|^{2-q} \geqslant \frac{q}{2}|\eta|^{q-2}|\eta|^2 \geqslant \frac{Dq}{2}\tau^{-1}|\eta|^2.$$

This finishes the proof of the theorem. □

Identification operators. We will explicitly identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$. For each $d \in \mathbb{N}$ consider the operator $\mathcal{V}_d\colon \mathbb{C}^d \to \mathbb{R}^d \times \mathbb{R}^d$, defined by

$$\mathcal{V}_d(\alpha + i\beta) = (\alpha, \beta).$$

One has, for all $z, w \in \mathbb{C}^d$,

$$\operatorname{Re}\langle z, w\rangle_{\mathbb{C}^d} = \langle \mathcal{V}_d(z), \mathcal{V}_d(w)\rangle_{\mathbb{R}^{2d}}.$$

If $(\omega_1,\omega_2) \in \mathbb{C}^d \times \mathbb{C}^d$, then $\mathcal{V}_{2d}(\omega_1,\omega_2) = (\operatorname{Re}\omega_1, \operatorname{Re}\omega_2, \operatorname{Im}\omega_1, \operatorname{Im}\omega_2) \in (\mathbb{R}^d)^4$. On $\mathbb{C}^d \times \mathbb{C}^d$ define another identification operator, $\mathcal{W}_{2d}\colon \mathbb{C}^d \times \mathbb{C}^d \to (\mathbb{R}^d)^4$, by

$$\mathcal{W}_{2d}(\omega_1,\omega_2) = (\mathcal{V}_d(\omega_1), \mathcal{V}_d(\omega_2)) = (\operatorname{Re}\omega_1, \operatorname{Im}\omega_1, \operatorname{Re}\omega_2, \operatorname{Im}\omega_2).$$

When the dimensions of the spaces on which the identification operators act are clear, we will sometimes omit the indices and instead of $\mathcal{V}_n$, $\mathcal{W}_m$ only write $\mathcal{V}$, $\mathcal{W}$. For example,

$$\mathcal{W}(\zeta,\eta) = (\mathcal{V}(\zeta), \mathcal{V}(\eta)), \quad \forall \zeta,\eta \in \mathbb{C}.$$

For functions Φ on spaces $\mathbb{C}^k$ we will sometimes use their "pullbacks" defined on $\mathbb{R}^{2k}$, namely

$$\Phi_{\mathcal{V}} = \Phi \circ \mathcal{V}^{-1} \quad \text{or} \quad \Phi_{\mathcal{W}} = \Phi \circ \mathcal{W}^{-1}.$$

Regularization of $\mathcal{Q}$. Denote by $*$ the convolution in $\mathbb{R}^4$ and let $(\varphi_\kappa)_{\kappa>0}$ be a nonnegative, smooth, and compactly supported approximation of the identity on $\mathbb{R}^4$. Explicitly, $\varphi_\kappa(y) = \kappa^{-4}\varphi(y/\kappa)$, where φ is smooth, nonnegative, radial, of integral 1, and supported in the closed unit ball in $\mathbb{R}^4$. If $\Phi\colon \mathbb{C}^2 \to \mathbb{R}$, define $\Phi * \varphi_\kappa = (\Phi_{\mathcal{W}} * \varphi_\kappa) \circ \mathcal{W}\colon \mathbb{C}^2 \to \mathbb{R}$. Explicitly, for $v \in \mathbb{C}^2$,

$$\begin{aligned}(\Phi * \varphi_\kappa)(v) &= \int_{\mathbb{R}^4} \Phi_{\mathcal{W}}(\mathcal{W}(v) - y)\varphi_\kappa(y)\, dy \\ &= \int_{\mathbb{R}^4} \Phi(v - \mathcal{W}^{-1}(y))\varphi_\kappa(y)\, dy.\end{aligned} \tag{3.7}$$

Now we can formulate a version of Theorem 3.1 for the mollifications $\mathcal{Q} * \varphi_\kappa$, in the fashion of [**19**, Theorem 4]. It also strengthens [**8**, Corollary 5.5].

Theorem 3.5. *Choose $p \geqslant 2$ and $A, B \in \mathcal{A}_p(\Omega)$. Let $\delta \in (0,1)$ and $\tau\colon \mathbb{C}^2 \to [0,\infty)$ be as in Theorem 3.1. Then for $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ as in* (3.1) *and any $v = (\zeta, \eta) \in \mathbb{C}^2$ we have, for a.e. $x \in \Omega$ and every $\omega = (\omega_1, \omega_2) \in \mathbb{C}^d \times \mathbb{C}^d$,*

$$H^{(A(x),B(x))}_{\mathcal{Q}*\varphi_\kappa}[v;\omega] \gtrsim (\tau * \varphi_\kappa)(v) \cdot |\omega_1|^2 + (\tau^{-1} * \varphi_\kappa)(v) \cdot |\omega_2|^2, \tag{3.8}$$

with the implied constant depending on A, B, and p, but not on the dimension d.

Proof: As in [**8**, proof of Corollary 5.5] we obtain, for $v \in \mathbb{C}^2$, $\omega \in \mathbb{C}^d \times \mathbb{C}^d$, and $\kappa > 0$,

$$H^{(A,B)}_{\mathcal{Q}*\varphi_\kappa}[v;\omega] = \int_{\mathbb{R}^4} H^{(A,B)}_{\mathcal{Q}}[v - \mathcal{W}^{-1}(y);\omega]\varphi_\kappa(y)\, dy.$$

The first estimate of Theorem 3.1 now gives

$$H^{(A,B)}_{\mathcal{Q}*\varphi_\kappa}[v;\omega] \gtrsim \int_{\mathbb{R}^4} (\tau(v - \mathcal{W}^{-1}(y))|\omega_1|^2 + \tau^{-1}(v - \mathcal{W}^{-1}(y))|\omega_2|^2)\varphi_\kappa(y)\, dy.$$

By recalling the convention (3.7), we see that we just obtained (3.8). □

3.2. Heat flow. As announced before, we prove the bilinear embedding by means of a *heat flow* technique applied to the Nazarov–Treil function $\mathcal{Q}$. We follow the outline of the method in [**8, 7**], where we proved the theorem for $V = W = 0$. The presence of nonzero potentials, considered in this paper, calls for settling a couple of technical problems which do not appear in the homogeneous case. As a historical note we mention that the early versions of the heat flow method associated with Bellman functions go back to the papers by Petermichl–Volberg [**36**] and Volberg–Nazarov [**41**].

Proving bilinear embedding on arbitrary open sets Ω [**7**], as opposed to proving it for $\Omega = \mathbb{R}^d$ [**8**], requires a major modification of the heat flow argument. See [**7**, Section 1.4] for an explanation. The gist of the problem is to justify *integration by parts*, which was overcome in [**7**] by approximating $\mathcal{Q}$ by a specifically constructed sequence of functions; see [**7**, Theorem 16].

For $f, g \in (L^p \cap L^q)(\Omega)$ and $A, B \in \mathcal{A}(\Omega)$ define

$$\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T^{A,V,\mathscr{V}}_t f, T^{B,W,\mathscr{W}}_t g), \quad t > 0.$$

Known estimates of $\mathcal{Q}$ and its gradient [**4**, Theorem 4] and the analyticity of $(T^A_t)_{t>0}$ and $(T^B_t)_{t>0}$ (see Theorem 1.2) imply that $\mathcal{E}$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ with a continuous derivative. As in our previous works involving heat flow, our aim is to prove two-sided estimates of

$$-\int_0^\infty \mathcal{E}'(t)\, dt, \tag{3.9}$$

which will then, in a by now familiar Bellman-heat fashion (see e.g. [**8, 7**] and the references there) merge into bilinear embedding.

Regarding the *upper estimates* of (3.9), we use upper pointwise estimates on $\mathcal{Q}$ (see, for example, [**8**, Proposition 5.1]) to get

$$-\int_0^\infty \mathcal{E}'(t)\,dt \leqslant \mathcal{E}(0) \lesssim \|f\|_p + \|g\|_q. \tag{3.10}$$

Now we turn to *lower estimates.* For

$$(v, w) = (T_t^{A,V,\mathscr{V}} f, T_t^{B,W,\mathscr{W}} g) \tag{3.11}$$

we have, by Corollary 1.3, that $v, w \in L^p \cap L^q$ and

$$-\mathcal{E}'(t) = 2\operatorname{Re}\int_\Omega (\langle(\partial_{\bar\zeta}\mathcal{Q})(v, w), L_{A,V}v\rangle_{\mathbb{C}} + \langle(\partial_{\bar\eta}\mathcal{Q})(v, w), L_{B,W}w\rangle_{\mathbb{C}}). \tag{3.12}$$

See [**7**, Section 6.1] for more details on how to justify (3.12).

3.3. Special case: bounded potentials. First we prove the bilinear embedding under the additional assumption that V, W are (nonnegative and) *essentially bounded.* In that case, $\mathcal{D}(L_{A,V}) = \mathcal{D}(L_{A,0})$ and for $u \in \mathcal{D}(L_{A,V})$ we have

$$L_{A,V}u = L_{A,0}u + Vu$$

and the same for B and W. Consequently, (3.12) gives

$$-\mathcal{E}'(t) = I_1 + I_2, \tag{3.13}$$

where

$$I_1 = 2\operatorname{Re}\int_\Omega (\langle(\partial_{\bar\zeta}\mathcal{Q})(v, w), L_{A,0}v\rangle_{\mathbb{C}} + \langle(\partial_{\bar\eta}\mathcal{Q})(v, w), L_{B,0}w\rangle_{\mathbb{C}}),$$

$$I_2 = 2\int_\Omega [V(\partial_\zeta\mathcal{Q})(v, w)\cdot v + W(\partial_\eta\mathcal{Q})(v, w)\cdot w].$$

The integral I_2 is absolutely convergent by the upper pointwise estimates of the gradient of $\mathcal{Q}$ (see, for example, [**8**, Proposition 5.1]), Hölder's inequality, the contractivity of the semigroups from (3.11) on L^p and L^q, respectively (cf. Theorem 1.2), and the assumption that $V, W \in L^\infty(\Omega)$.

We will estimate the terms I_1, I_2 separately.

3.3.1. Estimate of I_1. As in [**7**, Section 6.1, p. 23] we get

$$I_1 \geqslant \liminf_{\kappa\searrow 0}\int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\kappa}[(v, w); (\nabla v, \nabla w)].$$

(It is here that we needed to restrict the choice of $\mathscr{V}$, $\mathscr{W}$ to (a)–(c) from Subsection 1.1. Besides, let us remark that this step entails integration by parts referred to on page 93; see also [**7**, Section 2.4] for a clarification of the notion of integration by parts as perceived in this paper.)

Next we apply Theorem 3.5 for

$$I_1 \gtrsim \liminf_{\kappa\searrow 0}\int_\Omega [(\tau * \varphi_\kappa)(v, w)\cdot|\nabla v|^2 + (\tau^{-1} * \varphi_\kappa)(v, w)\cdot|\nabla w|^2].$$

Now, since both τ and τ^{-1} are continuous on $\mathbb{C}^2 \setminus \{(0, 0)\}$ (see Theorem 3.1), Fatou's lemma implies

$$I_1 \gtrsim \int_{\Omega\setminus\{v=0,\, w=0\}} [\tau(v, w)\cdot|\nabla v|^2 + \tau(v, w)^{-1}\cdot|\nabla w|^2].$$

3.3.2. Estimate of I_2. Using that V, W are nonnegative, we get from Theorem 3.1

$$I_2 \gtrsim \int_{\Omega\setminus\{v=0,\, w=0\}} (\tau(v,w)\cdot V|v|^2 + \tau(v,w)^{-1}\cdot W|w|^2).$$

The last two estimates, along with (3.13) and the fact that for $u \in H^1(\Omega)$ we have $\nabla u = 0$ almost everywhere on $\{u = 0\}$, yield

$$-\,\mathcal{E}'(t) \gtrsim \int_\Omega \sqrt{|\nabla v|^2 + V|v|^2}\sqrt{|\nabla w|^2 + W|w|^2}. \tag{3.14}$$

3.3.3. Summary. By merging (3.10) and (3.14) we get

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_t^{A,V,\mathscr{V}} f|^2 + V|T_t^{A,V,\mathscr{V}} f|^2}\sqrt{|\nabla T_t^{B,W,\mathscr{W}} g|^2 + W|T_t^{B,W,\mathscr{W}} g|^2}\, dx\, dt \lesssim \|f\|_p + \|g\|_q.$$

Now use the standard trick and replace f by μf and g by $\mu^{-1} g$, with $\mu > 0$, and minimize in μ. This finishes the proof of Theorem 1.4 in the case of *bounded* V, W.

3.4. General case: unbounded potentials. Theorem 1.4 will follow from the special case of bounded potentials already proven in Subsection 3.3, once we prove the following approximation result.

Let $U \in L^1_{\rm loc}(\Omega)$ be a nonnegative function. For each $n \in \mathbb{N}$ define

$$U_n := \min\{U, n\}.$$

We also set $U_\infty = U$.

Theorem 3.6. *For all $f \in L^2(\Omega)$, $A \in \mathcal{A}(\Omega)$, $U \in L^1_{\rm loc}(\Omega)$, and all $t > 0$ we have*

$$\begin{aligned} \nabla T_t^{A,U_n} f &\to \nabla T_t^{A,U} f \qquad \text{in } L^2(\Omega;\mathbb{C}^d),\\ U_n^{1/2} T_t^{A,U_n} f &\to U^{1/2} T_t^{A,U} f \quad \text{in } L^2(\Omega), \end{aligned}$$

as $n \to \infty$.

The proof will be given in page 99. First we need a few technical results.

Notation 3.7. Until the end of this chapter we will work with a single matrix function A. Therefore, in order to make the text more readable, we will from now on omit A in the notation for the operators and semigroups. For example, we will write T_t^U instead of $T_t^{A,U}$ and L_U instead of $L_{A,U}$.

Recall that $\nu(A)$ was defined on page 84. It then follows from the positivity of U and the estimate [**35**, (1.26)] that the operators L_{U_n}, $n \in \mathbb{N} \cup \{\infty\}$, are uniformly sectorial of angle $\nu = \nu(A)$ in the sense that

$$\|(\zeta - L_{U_n})^{-1}\|_2 \leqslant \frac{1}{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\nu)}, \quad \forall \zeta \in \mathbb{C}\setminus\overline{\mathbf{S}}_\nu. \tag{3.15}$$

We will use the next lemma, whose proof is based on an idea of Ouhabaz [**34**] that we learnt from [**3**].

Lemma 3.8. *For all $f \in L^2(\Omega)$ and all $s > 0$ we have*

$$(s + L_{U_n})^{-1} f \to (s + L_U)^{-1} f \quad \text{in } L^2(\Omega), \text{ as } n \to \infty. \tag{3.16}$$

Sketch of the proof: The proof is based on the argument presented in [**3**, pp. 19–20]. Let us outline the main steps.

Recall the definitions (1.4) and (1.5) and consider the sesquilinear forms $\mathfrak{a} = \mathfrak{a}_U$ and $\mathfrak{a}_n := \mathfrak{a}_{U_n}$. We define operations on forms as in [**30**, Chapter VI, §1.1]. For $z \in \mathbb{C}$ and $n \in \mathbb{N}$ denote

$$\mathfrak{a}_z := \operatorname{Re}\mathfrak{a} + z\operatorname{Im}\mathfrak{a},$$
$$\mathfrak{a}_{n,z} := \operatorname{Re}\mathfrak{a}_n + z\operatorname{Im}\mathfrak{a}_n.$$

Note that $\mathfrak{a} = \mathfrak{a}_i$. Set $\delta = \delta(A) = \cot\nu(A)$ and $\mathcal{O} := \{z \in \mathbb{C};\ |\operatorname{Re} z| < \delta\}$. It can be shown that if $z \in \mathcal{O}$, then $\mathfrak{a}_z$ and $\mathfrak{a}_{n,z}$ are closed sectorial forms.

Let L_z and $L_{n,z}$ be the operators associated with $\mathfrak{a}_z$ and $\mathfrak{a}_{n,z}$, respectively. For $z \in \mathcal{O}$ and $s > 0$, the operator $s+L_z$ is invertible and $\|(s+L_z)^{-1}f\|_2 \leqslant \|f\|_2/s$, cf. (3.15). A theorem by Kato [**30**, p. 395], see also [**40**], shows that $z \mapsto (s+L_z)^{-1}$ is holomorphic as a map from $\mathcal{O}$ to the space of bounded linear operators on $L^2(\Omega)$. The same holds for the map $z \mapsto (s + L_{n,z})^{-1}$.

A monotone convergence theorem for sequences of symmetric sesquilinear forms (see [**30**, Theorem 3.13a, p. 461] and [**38**, Theorem 3.1]) gives that for every $s > 0$, $z \in (-\delta, \delta)$, and $f \in L^2(\Omega)$ we have

$$(s + L_{n,z})^{-1}f \to (s + L_z)^{-1}f \quad \text{in } L^2(\Omega), \text{ as } n \to \infty.$$

A vector-valued version of Vitali's theorem ([**2**, Theorem A.5]) implies that $(s + L_{n,z})^{-1}f \to (s + L_z)^{-1}f$ for all $z \in \mathcal{O}$, and (3.16) follows by taking $z = i$. □

Proposition 3.9. *For all $f \in L^2(\Omega)$ and all $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\nu$ we have*

$$(\zeta - L_{U_n})^{-1}f \to (\zeta - L_U)^{-1}f \qquad \text{in } L^2(\Omega), \tag{3.17}$$

$$\nabla(\zeta - L_{U_n})^{-1}f \to \nabla(\zeta - L_U)^{-1}f \qquad \text{in } L^2(\Omega;\mathbb{C}^d), \tag{3.18}$$

$$U_n^{1/2}(\zeta - L_{U_n})^{-1}f \to U^{1/2}(\zeta - L_U)^{-1}f \quad \text{in } L^2(\Omega), \tag{3.19}$$

as $n \to \infty$.

Proof: Recall the notation $U_\infty = U$. Fix $f \in L^2(\Omega)$. For $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\nu$ set

$$u_n(\zeta) := (L_{U_n} - \zeta)^{-1}f \in \mathcal{D}(L_{U_n}) \subseteq \mathscr{V} \subseteq H^1(\Omega).$$

By ellipticity of A, for every $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\nu$ we have

$$\begin{aligned} \lambda\|\nabla u_n\|_2^2 + \|U_n^{1/2}u_n\|_2^2 &\leqslant \operatorname{Re}\int_\Omega [\langle A\nabla u_n, \nabla u_n\rangle + U_n u_n \overline{u}_n] \\ &= \operatorname{Re}\int_\Omega (L_{U_n}u_n)\overline{u_n} \\ &= \operatorname{Re}\int_\Omega f\overline{u_n} + (\operatorname{Re}\zeta)\int_\Omega |u_n|^2 \\ &\leqslant \|f\|_2\|u_n\|_2 + |\operatorname{Re}\zeta| \cdot \|u_n\|_2^2. \end{aligned}$$

Therefore, the uniform sectoriality estimate (3.15) gives, for all $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\nu$,

$$\|u_n(\zeta)\|_2 + \|\nabla u_n(\zeta)\|_2 + \|U_n^{1/2}u_n(\zeta)\|_2 \leqslant C_{\lambda,\nu}(\zeta)\|f\|_2, \tag{3.20}$$

where $C_{\lambda,\nu}(\zeta) > 0$ is continuous in ζ.

Now temporarily fix $s > 0$ and set

$$u_n = u_n(-s),$$
$$u = u_\infty(-s).$$

By (3.20), the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, hence it admits a weakly convergent subsequence. That is, there exists a subsequence of indices $(n_j)_{j\in\mathbb{N}}$ and a function $w \in H^1(\Omega)$ such that $u_{n_j} \rightharpoonup w$ in $H^1(\Omega)$. Here the symbol $\rightharpoonup$ denotes weak convergence (in other words, convergence in the weak topology; see e.g. [**24**, Appendix D.4] for an explicit description in L^p). Lemma 3.8 reads

$$\lim_{n\to\infty} u_n = u \quad \text{in } L^2(\Omega), \quad \forall s > 0, \tag{3.21}$$

which implies that $w = u$. Thus $u_{n_j} \rightharpoonup u$ in $H^1(\Omega)$.

Again by (3.20), the sequence $(U_n^{1/2}u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. From (3.21) and a standard theorem we derive a subsequence $(u_{n_l})_{l\in\mathbb{N}}$ such that $u_{n_l} \to u$ almost everywhere on Ω. Recall that $U_n^{1/2} \to U^{1/2}$ pointwise on Ω, just by the construction of U_n. Hence $U_{n_l}^{1/2}u_{n_l} \to U^{1/2}u$ almost everywhere on Ω. Now a well-known theorem [**28**, Theorem 13.44] gives $U_{n_l}^{1/2}u_{n_l} \rightharpoonup U^{1/2}u$ in $L^2(\Omega)$.

We proved that in L^2 we have

$$u_n \to u, \quad \nabla u_{n_k} \rightharpoonup \nabla u, \quad \text{and} \quad U_{n_k}^{1/2}u_{n_k} \rightharpoonup U^{1/2}u. \tag{3.22}$$

We now show that the last two convergences in (3.22) are also in the normed topology of $L^2(\Omega)$.

By ellipticity,

$$\begin{aligned}
J_{n_k} &:= s\|u_{n_k} - u\|_2^2 + \lambda\|\nabla u_{n_k} - \nabla u\|_2^2 + \|U_{n_k}^{1/2}u_{n_k} - U^{1/2}u\|_2^2 \\
&\leqslant s\|u_{n_k}\|_2^2 + s\|u\|_2^2 - 2s\,\mathrm{Re}\int_\Omega u_{n_k}\bar{u} + \|U_{n_k}^{1/2}u_{n_k}\|_2^2 + \|U^{1/2}u\|_2^2 \\
&\quad - 2\,\mathrm{Re}\int_\Omega U_{n_k}^{1/2}U^{1/2}u_{n_k}\bar{u} + \mathrm{Re}\int_\Omega \langle A(\nabla u_{n_k} - \nabla u), (\nabla u_{n_k} - \nabla u)\rangle \\
&= I^0 + I^1_{n_k} + I^2_{n_k} + I^3_{n_k},
\end{aligned}$$

where

$$\begin{aligned}
I^0 &= s\|u\|_2^2 + \mathrm{Re}\int_\Omega \langle A\nabla u, \nabla u\rangle + \|U^{1/2}u\|_2^2, \\
I^1_{n_k} &= s\|u_{n_k}\|_2^2 + \mathrm{Re}\int_\Omega \langle A\nabla u_{n_k}, \nabla u_{n_k}\rangle + \|U_{n_k}^{1/2}u_{n_k}\|_2^2, \\
I^2_{n_k} &= -2s\,\mathrm{Re}\int_\Omega u_{n_k}\bar{u} - 2\,\mathrm{Re}\int_\Omega U_{n_k}^{1/2}U^{1/2}u_{n_k}\bar{u}, \\
I^3_{n_k} &= -\,\mathrm{Re}\bigg(\int_\Omega \langle A\nabla u_{n_k}, \nabla u\rangle + \int_\Omega \langle A\nabla u, \nabla u_{n_k}\rangle\bigg).
\end{aligned}$$

Sending $k \to \infty$, we obtain

$$I^0 = \operatorname{Re} \int_\Omega ((s + L_U)u)\bar{u} = \operatorname{Re} \int_\Omega f\bar{u},$$

because $u \in \mathcal{D}(L_U)$;

$$I^1_{n_k} = \operatorname{Re} \int_\Omega ((s + L_{U_{n_k}})u_{n_k})\bar{u}_{n_k} = \operatorname{Re} \int_\Omega f\bar{u}_{n_k} \to \operatorname{Re} \int_\Omega f\bar{u},$$

because $u_{n_k} \to u$ in $L^2(\Omega)$;

$$I^2_{n_k} = -2 \operatorname{Re}\biggl(s \int_\Omega u_{n_k}\bar{u} + \int_\Omega (U^{1/2}_{n_k} u_{n_k})(U^{1/2}\bar{u}) \biggr) \to -2s\|u\|_2^2 - 2\|U^{1/2}u\|_2^2,$$

by (3.22), since $u \in \mathcal{D}(\mathfrak{a}_U)$ implies $U^{1/2}\bar{u} \in L^2(\Omega)$; and finally

$$I^3_{n_k} \to -2 \operatorname{Re} \int_\Omega \langle A\nabla u, \nabla u\rangle$$

by (3.22) again, since $A \in \mathcal{A}(\Omega)$ implies $|A\nabla u|, |A^*\nabla u| \lesssim |\nabla u| \in L^2(\Omega)$.

Therefore, using that $u \in \mathcal{D}(L_U)$, we obtain, as $k \to \infty$,

$$I^0 + I^1_{n_k} \to 2 \operatorname{Re} \int_\Omega f\bar{u},$$

$$I^2_{n_k} + I^3_{n_k} \to -2 \operatorname{Re} \int_\Omega ((s + L_U)u)\bar{u} = -2 \operatorname{Re} \int_\Omega f\bar{u}.$$

It follows that $J_{n_k} \to 0$ as $k \to \infty$, so

$$\nabla u_{n_k} \to \nabla u \quad \text{and} \quad U^{1/2}_{n_k} u_{n_k} \to U^{1/2}u \quad \text{in } L^2(\Omega), \tag{3.23}$$

as desired.

By repeating verbatim the argument following (3.22), we may prove that every *subsequence* of $(u_n)_n$ has its own subsequence for which (3.23) holds. Therefore, by a standard convergence argument involving subsequences, (3.18) and (3.19) hold for all $\zeta = -s$, $s > 0$. (Recall that for (3.17) we already know that, by virtue of(3.21).)

It remains to prove (3.17), (3.18), and (3.19) for all $\zeta \in (\mathbb{C} \setminus \overline{\mathbf{S}}_\nu) \setminus (-\infty, 0)$. Fix $f \in L^2(\Omega)$ and for each $n \in \mathbb{N} \cup \{\infty\}$ consider the function

$$G_n : \mathbb{C} \setminus \overline{\mathbf{S}}_\nu \to L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega; \mathbb{C}^d) =: H$$

given by

$$G_n(\zeta) = (u_n(\zeta), U_n^{1/2} u_n(\zeta), \nabla u_n(\zeta)).$$

By (3.20), the family $\{G_n : n \in \mathbb{N}_+\}$ is locally uniformly bounded in $\mathbb{C} \setminus \overline{\mathbf{S}}_\nu$. Hence [**2**, Proposition A.3] implies that the functions G_n are holomorphic, because the complex-valued function $\langle G_n(\cdot), \mathbf{g}\rangle_H$ is holomorphic for all $\mathbf{g}$ from the norming subspace $L^2(\Omega) \times C_c^\infty(\Omega; \mathbb{C}^d) \times C_c^\infty(\Omega)$ of H. Moreover, since we have already proven that $G_n(-s) \to G_\infty(-s)$ in H for all $s > 0$, it follows from Vitali's theorem [**2**, Theorem A.5] that the convergence holds true for all $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\nu$. □

Proof of Theorem 3.6: We use the standard representation of the analytic semigroup $T_t^{U_n}$, $n \in \mathbb{N} \cup \{\infty\}$, by means of a Cauchy integral (see, for example, [**23**, Chapter II] or [**27**, Lemma 2.3.2]). We used it earlier in the proof of [**8**, Lemma A.4]. Fix $\delta > 0$, $\vartheta > \nu(A)$, and denote by γ the positively oriented boundary of $\mathbf{S}_\vartheta \cup \{\zeta \in \mathbb{C};\ |\zeta| < \delta\}$. Then

$$\|\nabla T_t^{U_n} f - \nabla T_t^U f\|_2 \lesssim \int_\gamma e^{-t\,\mathrm{Re}\,\zeta} \|\nabla(\zeta - L_{U_n})^{-1} f - \nabla(\zeta - L_U)^{-1} f\|_2\,|d\zeta|,$$

$$\|U_n^{1/2} T_t^{U_n} f - U^{1/2} T_t^U f\|_2 \lesssim \int_\gamma e^{-t\,\mathrm{Re}\,\zeta} \|U_n^{1/2}(\zeta - L_{U_n})^{-1} f - U^{1/2}(\zeta - L_U)^{-1} f\|_2\,|d\zeta|.$$

By Proposition 3.9, the integrands converge to zero, as $n \to \infty$. An examination of the constant $C_{\lambda,\nu}(\zeta)$ from (3.20) shows that for ζ along the curve γ we have $C_{\lambda,\nu}(\zeta) \lesssim 1$ uniformly in $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \gamma$. This means that we can apply the dominated convergence theorem in the two integrals above and complete the proof. □

Appendix A. Invariance of form domains under normal contractions

Following [**35**, Section 2.4], we say that a function $p\colon \mathbb{C} \to \mathbb{C}$ is a *normal contraction* if it is Lipschitz on $\mathbb{C}$ with constant 1 and $p(0) = 0$. Denote by $\mathscr{N}$ the set of all normal contractions. Define $T\colon \mathbb{C} \to \mathbb{C}$ by $T(\zeta) = (\mathrm{Re}\,\zeta)^+$, where $x^+ = \max\{x, 0\}$. Recall that we defined $P\colon \mathbb{C} \to \mathbb{C}$ in (1.6). Functions P, T belong to the class $\mathscr{N}$. Moreover, they are in a particular sense fundamental representatives of this class, as we show next.

Proposition A.1. *Let $\mathscr{V}$ be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Then $\mathscr{V}$ is invariant under P and T if and only if it is invariant under the (whole) class $\mathscr{N}$.*

Proof: Suppose that $\mathscr{V}$ is invariant under P and T. Let Δ be the Euclidean Laplacian on Ω subject to the boundary conditions embodied in $\mathscr{V}$. That is, $-\Delta$ is the operator arising from the form $\mathfrak{b}$, defined by $\mathcal{D}(\mathfrak{b}) = \mathscr{V}$ and

$$\mathfrak{b}(u, v) = \int_\Omega \langle \nabla u, \nabla v \rangle_{\mathbb{C}^d}, \quad \forall u, v \in \mathscr{V}. \tag{A.1}$$

Parts *1)* and *2)* of [**35**, Theorem 4.31] imply that $(e^{-t(-\Delta)})_{t>0}$ is sub-Markovian. Now [**35**, Theorem 2.25] implies that $\mathscr{V}$ is invariant under $\mathscr{N}$.

The implication in the opposite direction is obvious, as $P, T \in \mathscr{N}$. □

Proposition A.2. *When $\mathscr{V}$ is any of the special cases (a)–(c) from Subsection 1.1, then $\mathscr{V}$ is invariant under $\mathscr{N}$.*

Proof: By [**35**, Theorem 2.25], it suffices to find a sesquilinear form $\mathfrak{b}$ such that:

- $\mathcal{D}(\mathfrak{b}) = \mathscr{V}$;
- $\mathfrak{b}$ is symmetric, accretive, and closed on $L^2(\Omega)$;
- if $\mathscr{L}$ is the operator associated with $\mathfrak{b}$, then the semigroup $\exp(-t\mathscr{L})$ is sub-Markovian.

We define $\mathfrak{b}$ on $\mathscr{V}$ by (A.1). Thus $\mathscr{L} = -\Delta$, with boundary conditions embodied in $\mathscr{V}$. Then $\mathcal{D}(\mathfrak{b}) = \mathscr{V}$ by construction, and the form is clearly symmetric and accretive. It is closed by Theorem 1.1. In order to check that the semigroup is sub-Markovian, we have to check (cf. [**35**, Definition 2.12]) that it is positive and contractive on L^∞. Now, these properties are proven in [**35**, Corollary 4.3] and [**35**, Corollary 4.10], respectively. □

Appendix B. Chain rule formula

In this section we give *two* proofs of the formulæ (2.2) and (2.3). The first follows the description of Sobolev spaces involving the notion of weak differentiability, while the second one is based on their description by means of absolute continuity along lines.

B.1. The distributional proof. Here we prove several versions of the chain rule in $H^1(\Omega)$, aiming for those that will lead to the formula (2.2). The first one is modeled after [**29**, Lemma 10.2.3]. Other useful references for this section are [**42**, Theorem 2.1.11], [**25**, Theorem 4.4], and [**26**, Section 7.4].

Recall the standard notion of a weak derivative [**24**, p. 256]. If all weak partial derivatives $\partial_{x_j} u$ exist, we denote by ∇u the weak gradient $(\partial_{x_1} u, \dots, \partial_{x_n} u)$.

Theorem B.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $u, v \in W^{1,1}_{\rm loc}(\Omega)$ real functions. Suppose that $F \in C^1(\mathbb{R}^2)$ is real and $|\nabla F| \in L^\infty(\mathbb{R}^2)$. Then $F \circ (u,v)$ is weakly differentiable and*

$$\nabla[F \circ (u,v)] = F_x(u,v) \cdot \nabla u + F_y(u,v) \cdot \nabla v. \tag{B.1}$$

Proof: Set $M := \| |\nabla F| \|_{L^\infty(\mathbb{R}^2)}$. By Lagrange's theorem, for any $a, b \in \mathbb{R}^2$ there exists some $\xi \in \mathbb{R}^2$ on the line segment between a and b such that

$$F(a) - F(b) = \langle \nabla F(\xi), a - b \rangle. \tag{B.2}$$

In particular, $|F(a)| \leqslant |F(0)| + M|a|$. This implies that $F(u,v) \in L^1_{\rm loc}(\Omega)$.

Clearly, $\Phi := F_x(u,v) \cdot \nabla u + F_y(u,v) \cdot \nabla v \in L^1_{\rm loc}(\Omega)$. Take $\phi \in C_c^\infty(\Omega)$ and write $K := \operatorname{supp} \phi$. Let η be the *standard mollifier* on $\mathbb{R}^2$ [**24**, Appendix C.5]. If $0 < \varepsilon < d(K, \partial\Omega)/2$, define $u_\varepsilon = u * \eta_\varepsilon$ on $\Omega_\varepsilon := \{x \in \Omega;\, d(x, \partial\Omega) > \varepsilon\}$ and the same for v. Let also $K^\varepsilon := \{x \in \Omega;\, d(x, K) < \varepsilon\}$. Observe that $K \subset K^\varepsilon \subset \Omega_\varepsilon$ for any ε as specified before. From (B.2) we get

$$|F(u,v) - F(u_\varepsilon, v_\varepsilon)| \leqslant M |(u,v) - (u_\varepsilon, v_\varepsilon)|. \tag{B.3}$$

Then for such $\varepsilon > 0$ we get

$$\begin{aligned}
\int_\Omega F(u,v) \nabla \phi &= \int_{K^\varepsilon} F(u,v) \nabla \phi \\
&= \lim_{\varepsilon \to 0} \int_{K^\varepsilon} F(u_\varepsilon, v_\varepsilon) \nabla \phi \\
&= -\lim_{\varepsilon \to 0} \int_{K^\varepsilon} (F_x(u_\varepsilon, v_\varepsilon) \nabla u_\varepsilon + F_y(u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) \cdot \phi \\
&= -\int_{K^\varepsilon} \Phi \cdot \phi \\
&= -\int_\Omega \Phi \cdot \phi.
\end{aligned}$$

The second identity above follows from (B.3) and the fact that $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1_{\rm loc}(\Omega)$ [**25**, Theorem 4.1(iii)].

The third identity follows from the usual integration by parts [**24**, Appendix C.2]. The fact that we make no assumptions on the boundary of K^ε is not an obstacle because of the presence of the function ϕ, which is smooth and compactly supported in K^ε. Thus, instead of with K^ε, we may work with its subset whose boundary is piecewise smooth; namely, a finite union of open balls containing the support of ϕ.

The fourth identity follows from the estimate

$$\begin{aligned}\left|\int_K F_x(u_\varepsilon,v_\varepsilon)\nabla u_\varepsilon\cdot\phi-\int_K F_x(u,v)\nabla u\cdot\phi\right|&\\ \leqslant\left|\int_K F_x(u_\varepsilon,v_\varepsilon)(\nabla u_\varepsilon-\nabla u)\cdot\phi\right|&+\left|\int_K[F_x(u_\varepsilon,v_\varepsilon)-F_x(u,v)]\nabla u\cdot\phi\right|\\ \leqslant M\|\nabla u_\varepsilon-\nabla u\|_{L^1(K)}\|\phi\|_{L^\infty(K)}&+\int_K|F_x(u_\varepsilon,v_\varepsilon)-F_x(u,v)||\nabla u|\cdot|\phi|.\end{aligned}$$

The first term in the last row tends to zero because $u_\varepsilon\to u$ in $W^{1,1}_{\rm loc}(\Omega)$; see [**25**, Theorem 4.1(vi)]. The second term in the last row tends to zero because of the estimate $|F_x(u_\varepsilon,v_\varepsilon)-F_x(u,v)||\nabla u|\cdot|\phi|\leqslant 2M\|\phi\|_{L^\infty(K)}|\nabla u|\in L^1(K)$, the fact that $(u_\varepsilon,v_\varepsilon)\to(u,v)$ a.e. K, cf. [**25**, Theorem 4.1(iv)], and the dominated convergence theorem. □

Theorem B.2. *Let Ω be an open subset of $\mathbb{R}^n$ and $u,v\in H^1_{\rm loc}(\Omega)$ real functions. Suppose that $F=F(x,y)\colon\mathbb{R}^2\to\mathbb{R}$ is of class C^1 and such that*

$$F(u,v)\ \textit{and}\ |\nabla F|(u,v)\ \textit{belong to}\ L^2_{\rm loc}(\Omega).$$

Then $F(u,v)$ is weakly differentiable and (B.1) *holds.*

Proof: We reduce the proof to Theorem B.1. To this end we apply an approximation argument as follows.

Let $\eta\in C_c^\infty(\mathbb{R}^2)$ be such that $\eta\in[0,1]$ everywhere, $\eta\equiv1$ on $B(0,1)$, and $\eta\equiv0$ on $B(0,2)^c$. For $R>0$ and $\xi\in\mathbb{R}^2$ define

$$\eta_R(\xi):=\eta\left(\frac{\xi}{R}\right)\quad\text{and}\quad F_R(\xi):=F(\xi)\eta_R(\xi).$$

Take $j\in\{1,\dots,n\}$, $\varphi\in C_c^\infty(\Omega)$, and let $K:=\operatorname{supp}\varphi$.

Write $a:=(u,v)$. Since $F(a)\in L^2_{\rm loc}(\Omega)\subset L^2(K)\subset L^1(K)$, $\partial_{x_j}\varphi\in L^\infty(K)$, and $|\eta|\leqslant1$, we may apply the dominated convergence theorem and conclude that

$$\int_\Omega F(a)\cdot\partial_{x_j}\varphi=\int_K F(a)\cdot\partial_{x_j}\varphi=\lim_{R\to\infty}\int_K F_R(a)\cdot\partial_{x_j}\varphi=\lim_{R\to\infty}\int_\Omega F_R(a)\cdot\partial_{x_j}\varphi.\tag{B.4}$$

Because $F_R\in C_c^1(\mathbb{R}^2)$, we may apply Theorem B.1 and obtain

$$\lim_{R\to\infty}\int_\Omega F_R(a)\cdot\partial_{x_j}\varphi=-\lim_{R\to\infty}\int_\Omega[(\partial_xF_R)(a)\partial_{x_j}u+(\partial_yF_R)(a)\partial_{x_j}v]\varphi.\tag{B.5}$$

We calculate

$$\partial_xF_R=\partial_xF\cdot\eta_R+F\cdot(\partial_x\eta)\left(\frac{\cdot}{R}\right)\frac{1}{R}\tag{B.6}$$

and similarly for ∂_yF_R. Thus it suffices to show that

$$\begin{aligned}&\lim_{R\to\infty}\int_\Omega(\partial_xF)(a)\cdot\eta\left(\frac{a}{R}\right)\cdot\partial_{x_j}u\cdot\varphi=\int_\Omega(\partial_xF)(a)\cdot\partial_{x_j}u\cdot\varphi,\\ &\lim_{R\to\infty}\frac{1}{R}\int_\Omega F(a)\cdot(\partial_x\eta)\left(\frac{a}{R}\right)\cdot\partial_{x_j}u\cdot\varphi=0,\end{aligned}\tag{B.7}$$

and similarly for ∂_yF_R.

Regarding the *first integral*, since $|\eta| \leqslant 1$ and φ is bounded,

$$\left|(\partial_x F)(a) \cdot \eta\left(\frac{a}{R}\right) \cdot \partial_{x_j} u \cdot \varphi\right| \leqslant \|\varphi\|_\infty \cdot |(\nabla F)(a)| \cdot |\partial_x u| \cdot \chi_K.$$

Due to our assumptions on F and a, this function belongs to $L^1(\Omega)$. Now we may apply the dominated convergence theorem and obtain the first identity in (B.7).

Regarding the *second integral*, we have

$$\left|F(a) \cdot (\partial_x \eta)\left(\frac{a}{R}\right) \cdot \partial_{x_j} u \cdot \varphi\right| \leqslant \|\varphi\|_\infty \|\partial_x \eta\|_\infty |F(a)\chi_K||\nabla u|.$$

This implies, by Hölder's inequality, that

$$\frac{1}{R}\left|\int_\Omega F(a) \cdot (\partial_x \eta)\left(\frac{a}{R}\right) \cdot \partial_{x_j} u \cdot \varphi\right| \leqslant \frac{1}{R}\|\varphi\|_\infty \cdot \|\partial_x \eta\|_\infty \cdot \|F(a)\|_{L^2(K)} \cdot \|\partial_x u\|_{L^2(K)}.$$

Clearly, this vanishes as $R \to \infty$, which proves the second identity in (B.7).

We point out that the assumption $F(u,v) \in L^2_{\rm loc}(\Omega)$ was mainly used to justify applying the Lebesgue dominated convergence theorem.

Going back to (B.4), i.e., combining it with (B.5), (B.6), and (B.7), we obtain

$$\int_\Omega F(a) \cdot \partial_{x_j} \varphi = -\int_\Omega [(\partial_x F)(a) \cdot \partial_{x_j} u + (\partial_y F)(a) \cdot \partial_{x_j} v]\varphi,$$

which is precisely what had to be proven. □

Chain rule for complex functions. Now we also consider *complex* functions in $H^1_{\rm loc}(\Omega)$. More precisely, we formulate a "complex version" of Theorem B.2. Recall the operators $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar z} = (\partial_x + i\partial_y)/2$.

Theorem B.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Take functions $f = u + iv \in H^1_{\rm loc}(\Omega)$ and $\Phi\colon \mathbb{C} \to \mathbb{C}$. Define $\widetilde\Phi\colon \mathbb{R}^2 \to \mathbb{C}$ by $\widetilde\Phi(x,y) := \Phi(x+iy)$. Assume that $\widetilde\Phi \in C^1(\mathbb{R}^2)$ and that*

$$\Phi \circ f = \widetilde\Phi(u,v) \text{ and } |\nabla\widetilde\Phi|(u,v) \text{ belong to } L^2_{\rm loc}(\Omega).$$

Then $\Phi \circ f$ is weakly differentiable and we have

$$\nabla(\Phi \circ f) = (\partial_z \widetilde\Phi)(u,v) \cdot \nabla f + (\partial_{\bar z} \widetilde\Phi)(u,v) \cdot \nabla \bar f. \tag{B.8}$$

Proof: Introduce $U, V\colon \mathbb{R}^2 \to \mathbb{R}$ by $U = \operatorname{Re}\widetilde\Phi$ and $V = \operatorname{Im}\widetilde\Phi$. That is,

$$U(x,y) = \operatorname{Re}\Phi(x+iy),$$

$$V(x,y) = \operatorname{Im}\Phi(x+iy).$$

Another way of writing the theorem's conclusion (B.8) is

$$\nabla(\Phi \circ f) = (\partial_x U)(u,v)\nabla u + (\partial_y U)(u,v)\nabla v + i[(\partial_x V)(u,v)\nabla u + (\partial_y V)(u,v)\nabla v].$$

In order to prove it, apply Theorem B.2 separately to $U(u,v) = \operatorname{Re}(\Phi \circ f)$ and $V(u,v) = \operatorname{Im}(\Phi \circ f)$. □

By taking Φ of the form $\widetilde\Phi = \partial_{\bar z} F$ for some $F \in C^2(\mathbb{R}^2)$, one eventually recovers (2.6).

Proof of Lemma 2.3. Our foremost special case is discussed in the corollary below. We introduce the notation $p^* = \max\{p,q\}$, where $p>1$ and $1/p+1/q=1$.

Define, for $f \in \mathbb{C}$ and $F \in \mathbb{C}^d$,

$$\tau_p(f) = |f|^{p-2} f,$$

$$\Lambda_p(f,F) = \frac{p}{2}|f|^{p-2}\operatorname{sign} f \cdot \mathcal{I}_p(\operatorname{sign}\bar{f}\cdot F).$$

In this notation, the formula (2.2) reads $\nabla(\tau_p(f)) = \Lambda_p(f,\nabla f)$.

If $1/p+1/q=1$, then we may easily verify the following pair of *inversion formulæ*:

$$f = \tau_q(\tau_p(f)), \tag{B.9}$$

$$F = \Lambda_q(\tau_p(f),\Lambda_p(f,F)). \tag{B.10}$$

Corollary B.4. *Suppose that $p>1$ and $f\colon \Omega\to\mathbb{C}$ is such that*

$$\tau_{\min\{p,2\}}(f) \in H^1_{\mathrm{loc}}(\Omega), \tag{B.11}$$

$$\tau_{\max\{p,2\}}(f) \in L^2_{\mathrm{loc}}(\Omega). \tag{B.12}$$

Then $\tau_{\max\{p,2\}}(f)$ is also weakly differentiable and (2.2), (2.3) *hold.*

Remark B.5. Observe that we have $\{\tau_{\min\{p,2\}}(f),\tau_{\max\{p,2\}}(f)\} = \{f,|f|^{p-2}f\}$ and that the conditions (B.11), (B.12) can be rewritten as

$$\begin{cases} f\in H^1_{\mathrm{loc}}(\Omega) \text{ and } |f|^{p-2}f\in L^2_{\mathrm{loc}}(\Omega) & \text{if } p\geqslant 2,\\ f\in L^2_{\mathrm{loc}}(\Omega) \text{ and } |f|^{p-2}f\in H^1_{\mathrm{loc}}(\Omega) & \text{if } p\leqslant 2.\end{cases}$$

Since these conditions clearly hold if $f, |f|^{p-2}f \in H^1_{\mathrm{loc}}(\Omega)$, this proves Lemma 2.3.

Proof: We split the proof into two cases (not counting $p=2$, which obviously holds).

First assume that $p>2$. In this case, the function $\Phi(\zeta) = |\zeta|^{p-2}\zeta$ satisfies the assumptions of Theorem B.3. Now notice that we have the identities $\partial_z\widetilde{\Phi} = (p/2)|z|^{p-2}$ and $\partial_{\bar z}\widetilde{\Phi} = (p/2-1)(z/\bar z)|z|^{p-2}$ (and zero at $z=0$) and use (B.8).

Suppose now that $1<p<2$. Write $g=\tau_p(f)$. Because of (B.9), the assumption $f\in L^2_{\mathrm{loc}}(\Omega)$, $\tau_p(f)\in H^1_{\mathrm{loc}}(\Omega)$, now reads $g\in H^1_{\mathrm{loc}}(\Omega)$, $\tau_q(g)\in L^2_{\mathrm{loc}}(\Omega)$. Since $q>2$, we may apply the part proven so far, which gives that $f=\tau_q(g)$ is weakly differentiable and $\nabla f = \nabla(\tau_q(g)) = \Lambda_q(g,\nabla g)$. Now (B.10) returns

$$\nabla(\tau_p(f)) = \nabla g = \Lambda_p(\tau_q(g),\Lambda_q(g,\nabla g)) = \Lambda_p(f,\nabla f),$$

as desired. These reasonings are surely valid on $\{f,g\neq 0\} = \{f=0\} = \{g=0\}$. On the other hand, $\nabla g = 0$ on $\{f,g=0\}$ by a known result. □

B.2. The ACL proof. Here we prove the following version of Lemma 2.3:

Lemma B.6. *Let $f\in H^1(\Omega)$ and $p\in(1,\infty)$. The function $|f|^{p-2}f$ belongs to $H^1(\Omega)$ if and only if $|f|^{p-2}f\in L^2(\Omega)$ and $|f|^{p-2}\nabla f\in L^2(\Omega;\mathbb{C}^n)$. In this case,* (2.2) *and* (2.3) *hold.*

We shall systematically use the ACL characterization of Sobolev spaces [**32**], which, for the sake of simplicity, we enunciate below for the Sobolev space $H^1(\Omega)$ only; see, for example, [**31**, Theorem 11.45].

Proposition B.7. *Let $\Omega\subseteq\mathbb{R}^n$ be an open set. Denote by $\mathcal{L}^k$ the k-dimensional Lebesgue measure. A function $u\in L^2(\Omega)$ belongs to $H^1(\Omega)$ if and only if it has a representative u_* that is absolutely continuous on $\mathcal{L}^{n-1}$-a.e. line segments of Ω that are parallel to the coordinate axes and whose first-order classical partial derivatives belong to $L^2(\Omega)$. Moreover, the classical partial derivatives of u_* agree $\mathcal{L}^n$-a.e. with the weak derivatives of u.*

We shall also use the following results.

Lemma B.8 ([**31**, Lemma 3.60])**.** *Let $I \subseteq \mathbb{R}$ be an interval. Let $h\colon I \to \mathbb{R}^k$, where $k \in \mathbb{N}_+$. Assume that h is differentiable on a set $E \subseteq I$, with $\mathcal{H}^1(h(E)) = 0$ (here $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure on $\mathbb{R}^k$). Then, $h'(t) = 0$ for $\mathcal{L}^1$-a.e. $t \in E$.*

Corollary B.9. *Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $u\colon \Omega \to \mathbb{C}$. Assume that the first-order partial derivatives of u exist in a Lebesgue measurable set $G \subseteq \Omega$, with $\mathcal{H}^1(u(G)) = 0$ (here $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure on $\mathbb{C} = \mathbb{R}^2$). Then, $\nabla u(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in G$.*

Proof: For $E \subseteq \Omega$ and $y \in \mathbb{R}^{n-1}$ let $E^y = \{x \in \mathbb{R} : (x, y) \in E\}$. If $\Omega^y \neq \emptyset$, define $u^y(x) = u(x, y)$ for all $x \in \Omega^y$. Fix $y \in \mathbb{R}^{n-1}$ such that $G^y \neq \emptyset$. Then the function u^y is differentiable at every $x \in G^y$, and $\mathcal{H}^1(u^y(G^y)) \leqslant \mathcal{H}^1(u(G)) = 0$. Hence, by Lemma B.8, we have that $(\partial_x u)(x, y) = (u^y)'(x) = 0$, for $\mathcal{L}^1$-a.e. $x \in G^y$. Let $S = \{(x, y) \in G : (\partial_x u)(x, y) \neq 0\}$. We have proven that $\mathcal{L}^1(S^y) = 0$, for all $y \in \mathbb{R}^{n-1}$. Therefore, by Fubini's theorem, $\mathcal{L}^n(S) = 0$. □

Lemma B.10. *Let $(v_n)_{n\geqslant 1}$ be a bounded sequence in $H^1(\Omega)$ such that $v_n(x) \to v(x)$ for almost every $x \in \Omega$. Then $v \in H^1(\Omega)$ and $v_n \rightharpoonup v$ in $H^1(\Omega)$.*

Suppose further that $\nabla v_n(x) \to F(x)$ for almost every $x \in \Omega$. Then $F = \nabla v$.

Proof: By the Banach–Alaoglu theorem, for any subsequence, there exist another subsequence $(u_{n_k})_{k\geqslant 1}$ and a function $u \in H^1(\Omega)$ such that $u_{n_k} \rightharpoonup u$ in $L^2(\Omega)$ and $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^2(\Omega; \mathbb{C}^n)$. Since v_{n_k} converges almost everywhere to v, it follows from [**28**, Theorem 13.44] that $u = v$ almost everywhere in Ω. This proves that $(v_n)_{n\geqslant 1}$ weakly converges to v in $H^1(\Omega)$. The second part of the lemma follows by using once again [**28**, Theorem 13.44]. □

Remark B.11. We note that in the proof above, instead of relying on the Severini–Egorov theorem as used in [**28**, Theorem 13.44], one can utilize Mazur's lemma.

Proof of Lemma B.6: For every $p \in (1, \infty)$ define $\Phi_p(z) = |z|^{p-2} z$, $z \in \mathbb{C}$. Note that when $p \in (1, 2)$ the function Φ_p is not differentiable at $z = 0$.

Fix $p \in (1, \infty)$ and $f \in H^1(\Omega)$ such that $\Phi_p(f) = |f|^{p-2} f \in H^1(\Omega)$. By Proposition B.7 there exist ACL representatives f_* and g_* of f and $\Phi_p(f)$, respectively. We also set $h_* = \Phi_p(f_*)$. Then, $g_*(x) = h_*(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, and the first-order partial derivatives of both f_* and g_* exist $\mathcal{L}^n$-a.e. in Ω.

Therefore, there exists $\Omega_0 \subset \Omega$ such that $\mathcal{L}^n(\Omega_0) = 0$ and

$$\begin{aligned} &g_*(x) = h_*(x), &&\forall x \in \Omega \setminus \Omega_0; \\ &\exists\, \nabla f_*(x),\, \exists\, \nabla g_*(x), &&\forall x \in \Omega \setminus \Omega_0. \end{aligned} \tag{B.13}$$

Let $E_0 = \{x \in \Omega \setminus \Omega_0 : f_*(x) = 0\}$. Since $\Phi_p \in C^1(\mathbb{C} \setminus \{0\})$, it follows from (B.13) and the classical chain rule that the first-order partial derivatives of h_* exist at every $x \in (\Omega \setminus \Omega_0) \setminus E_0$, and

$$\nabla h_*(x) = \nabla \Phi_p(f_*(x)) = \frac{p}{2} |f_*(x)|^{p-2} \operatorname{sign} f_*(x) \cdot \mathcal{I}_p(\operatorname{sign} \bar{f}_*(x) \cdot \nabla f_*(x)),$$

for all $x \in (\Omega \setminus \Omega_0) \setminus E_0$.

Since $g_*(x) - h_*(x) = 0$ for every $x \in \Omega \setminus \Omega_0$, it follows from Corollary B.9 and the equality above that

$$\nabla g_*(x) = \frac{p}{2} |f_*(x)|^{p-2} \operatorname{sign} f_*(x) \cdot \mathcal{I}_p(\operatorname{sign} \bar{f}_*(x) \cdot \nabla f_*(x)), \quad \mathcal{L}^n\text{-a.e. } x \in (\Omega \setminus \Omega_0) \setminus E_0.$$

On the other hand, the gradient of both f_* and g_* exists at every point of E_0 and

$$g_*(x) = h_*(x) = \Phi_p(f_*(x)) = 0, \quad \forall x \in E_0.$$

Therefore, by Corollary B.9,

$$\nabla g_*(x) = \nabla f_*(x) = 0, \quad \mathcal{L}^n\text{-a.e. } x \in E_0.$$

To prove (2.2), observe that, by Proposition B.7, the weak gradients of $\Phi_p(f)$ and f agree with ∇g_* and ∇f_*, respectively, almost everywhere on Ω with respect to the Lebesgue measure $\mathcal{L}^n$. It now follows from (2.2) and the definition of$\mathcal{I}_p$ that

$$|\nabla(|f|^{p-2}f)| \sim_p |u|^{p-2}|\nabla u|.$$

Given our assumption that $|f|^{p-2}f \in H^1(\Omega)$, we conclude that both $|f|^{p-1}f$ and $|f|^{p-2}\nabla f$ are square-integrable over Ω.

We now prove that if $f \in H^1(\Omega)$, $|f|^{p-1}f \in L^2(\Omega)$, and $|f|^{p-2}\nabla f \in L^2(\Omega;\mathbb{C}^n)$, then $|f|^{p-2}f \in H^1(\Omega)$.

The technical issue here is that when $p<2$, the function $z \mapsto |z|^{p-2}z$ is not locally Lipchitz at $z=0$. To better explain this issue, let us prove by means of Proposition B.7 the well-known fact that the absolute value of a Sobolev function is also a Sobolev function. Let f_* be an ACL representative of f (see Proposition B.7). Since $z \mapsto |z|$ is Lipschitz, the function $|f_*|$ is ACL on Ω, and therefore its gradient exists almost everywhere on Ω. It follows from Lemma B.8 that $\nabla|f_*(x)| = 0 = \nabla f_*(x)$ almost everywhere on the set $\{f_*=0\}$. On the other hand, since $|\cdot| \in C^1(\mathbb{C}\setminus\{0\})$, by the classical chain rule, we have that $\nabla|f_*(x)| = \mathrm{Re}(\mathrm{sign}(\bar{f}_*(x))\nabla f_*(x))$, for all $x \in \Omega$ such that $f_*(x) \neq 0$ and the gradient of f_* exists at x.

For every $\epsilon > 0$ consider the function $\Psi_\epsilon(t,z) = (t+\epsilon)^{p-2}z$, $t > -\epsilon$, $z \in \mathbb{C}$. Let

$$v_\epsilon = \Psi_\epsilon(|f_*|, f_*).$$

Since both f_* and $|f_*|$ are ACL on Ω and $\Psi_\epsilon \in C^1((-\epsilon,+\infty)\times\mathbb{C})$, we have that v_ϵ is ACL on Ω and, by the classical chain rule,

$$\nabla v_\epsilon(x) = (|f_*(x)|+\epsilon)^{p-2}\left(\nabla f_*(x) + (p-2)\,\mathrm{Re}\left(\frac{\bar{f}_*(x)}{|f_*(x)|+\epsilon}\nabla f_*(x)\right)\right),$$

for almost every x in Ω. Therefore, given the assumptions, it follows from Proposition B.7 that $v_\epsilon \in H^1(\Omega)$ and the net $(v_\epsilon)_{\epsilon\in(0,1)}$ is uniformly bounded in $H^1(\Omega)$. Also,

$$\lim_{\epsilon \searrow 0} v_\epsilon(x) = |f_*(x)|^{p-2}f_*(x)$$

for all $x \in \Omega$.

Therefore, by Lemma B.10, the function $|f_*|^{p-2}f_*$ belongs to $H^1(\Omega)$, as required to finish the proof of the lemma.

We remark that since $\nabla f_* = 0$ almost everywhere on $\{f_*=0\}$, we conclude that

$$\lim_{\epsilon \searrow 0} \nabla v_\epsilon(x) = |f_*(x)|^{p-2}(\mathrm{Re}(\mathrm{sign}(\bar{f}_*(x))\nabla f_*(x))\nabla f_*(x))\chi_{\{f_*\neq 0\}}(x),$$

for almost every $x \in \Omega$. Hence, by Lemma B.10, we deduce the equality (2.2), which was established earlier in the proof, under the assumption that $|f|^{p-2}f \in H^1(\Omega)$. □

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