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ON THE STRONG SUBDIFFERENTIABILITY OF HOMOGENEOUS POLYNOMIALS AND (SYMMETRIC) TENSOR PRODUCTS

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Abstract: We study the (uniform) strong subdifferentiability of norms of Banach spaces $\mathcal{P}(^N X, Y^*)$ of all continuous N-homogeneous polynomials and tensor products of Banach spaces, namely $X\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X$ and $\widehat{\otimes}_{\pi_s,N} X$. Among other results, we characterize when the norms of spaces $\mathcal{P}(^N\ell_p, \ell_q)$, $\mathcal{P}(^N l_{M_1}, l_{M_2})$, and $\mathcal{P}(^N d(w,p), l_{M_2})$ are strongly subdifferentiable. Analogous results for multilinear mappings are also obtained. Since strong subdifferentiability of a dual space implies reflexivity, we improve some known results in [**38, 48, 49**] (in the spirit of Pitt's compactness theorem) on the reflexivity of spaces of N-homogeneous polynomials and N-linear mappings. Concerning the projective (symmetric) tensor norms, we provide positive results by considering the subsets U and U_s of elementary tensors on the unit spheres of $X\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X$ and $\widehat{\otimes}_{\pi_s,N} X$, respectively. Specifically, we prove that the norms of $\widehat{\otimes}_{\pi_s,N} \ell_2$ and $\ell_2 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_2$ are uniformly strongly subdifferentiable on U_s and U, and that the norms of $c_0 \widehat{\otimes}_{\pi_s} c_0$ and $c_0 \widehat{\otimes}_\pi c_0$ are strongly subdifferentiable on U_s and U in the complex case.

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1. Introduction

The main aim of this paper is to study the strong subdifferentiability of the norm of Banach spaces of N-homogeneous polynomials $\mathcal{P}(^N X)$ and (their predual) symmetric tensor products $\widehat{\otimes}_{\pi_s,N} X$. To do so, the following characterization of strong subdifferentiability given in [**43**, Theorem 1.2] (see also [**47**]) is a very useful tool: the norm $\|\cdot\|$ of a Banach space X is strongly subdifferentiable at a point $x \in S_X$ if and only if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ such that

$$\text{dist}(x^*, D(x)) < \varepsilon \quad \text{whenever } x^* \in B_{X^*} \text{ satisfies } x^*(x) > 1 - \delta, \tag{1.1}$$

where $D(x) := \{x^* \in S_{X^*} : x^*(x) = 1\}$. For more information and examples concerning Banach spaces with strongly subdifferentiable norm, see Subsection 2.1.

As we will detail in the subsequent sections, the characterization (1.1) appears in a natural way as *a kind of* Bishop–Phelps–Bollobás property for functionals in X^*. The relation between the denseness of norm-attaining mappings and the geometry of the underlying spaces arises in the area from the very beginning. In his pioneer work [**55**], Lindenstrauss exhibits some geometrical properties of Banach spaces X and Y, which guarantee that the set of norm-attaining linear operators in $\mathcal{L}(X,Y)$ is dense in the whole space. He also showed that the lack of extreme points of the unit ball of the domain space (which is, of course, a geometrical property of the space)

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plays a fundamental role when trying to obtain examples of spaces for which the set of norm-attaining operators is not dense in the whole space. Since then, this relation between the theory of norm-attaining mappings and the geometry of Banach spaces has appeared naturally [**15, 53**]. In particular, it appears in the context of Bishop–Phelps–Bollobás type theorems. Roughly speaking, the Bishop–Phelps–Bollobás theorem (the *quantitative* version of the well known result of Bishop and Phelps [**13**] due to Bollobás [**14**]) states that, whenever $(x_0^*, x_0) \in S_{X^*} \times S_X$ satisfies $x_0^*(x_0) \approx 1$, there exists $(x_1^*, x_1) \in S_{X^*} \times S_X$ such that $x_1^*(x_1) = 1$, $x_1^* \approx x_0^*$, and $x_1 \approx x_0$.

Acosta, Aron, García, and Maestre ([**3**]) were the first to study a possible extension of the Bishop–Phelps–Bollobás theorem to the context of linear operators. As expected, some geometrical properties of the spaces (like the *approximate hyperplane series property* defined and studied there) appear as sufficient conditions for the validity of a Bishop–Phelps–Bollobás theorem for linear operators. For more information on Bishop–Phelps–Bollobás type results (in the linear, multilinear, and polynomial context), we refer the reader to [**1, 3, 4, 5, 20, 21, 24, 64**] and the references therein. We also send the reader to two recent surveys on the topic [**2, 25**].

Recently, in the framework of Bishop–Phelps–Bollobás type theorems for linear operators and multilinear mappings, strong subdifferentiability, uniform smoothness, and uniform convexity of the norm of a Banach space have been considered (see, for instance, [**31, 32, 33, 54**]). Having this in mind, we intend to relate the strong subdifferentiability of the norms of $\mathcal{P}(^N X)$ and $\widehat{\otimes}_{\pi_s,N} X$ with some Bishop–Phelps–Bollobás type properties for polynomials (all the definitions will be given in Section 2). In Theorem A we show that, under certain hypotheses on the underlying space X, the strong subdifferentiability of the space $\mathcal{P}(^N X)$ of N-homogeneous polynomials is equivalent to a *polynomial* version of (1.1). Moreover, a vector-valued version of this equivalence is addressed. We use this result to characterize the strong subdifferentiability of the norms of spaces of polynomials between some classical sequence spaces, such as ℓ_p, Orlicz spaces, and Lorentz spaces. Following the same ideas, we replicate the process in the multilinear context (see Theorem B and Corollary B). These results can be seen as a continuation in the line of study of the classical *Pitt's compactness theorem* in the polynomial and multilinear setting. Let us briefly explain this last assertion: Pitt's theorem (for reflexive ℓ_q spaces) affirms that every bounded linear operator from ℓ_p into ℓ_q is compact if and only if $1 < q < p < \infty$, and this is also equivalent to the reflexivity of the space $\mathcal{L}(\ell_p, \ell_q)$ (this last result can be found, for instance, in [**62**]). In [**10**], some versions of Pitt's theorem for operators between reflexive Lorentz and Orlicz sequence spaces are established; as in the classical scenario, the fact that every bounded linear operator between Lorentz and Orlicz sequence spaces is compact is equivalent to the reflexivity of the space of linear operators. In the polynomial and multilinear context, *Pitt's type theorems* have been addressed by many authors (see [**6, 38, 48, 49, 52, 60**], among others). In this setting, it is natural to replace the *compactness* with *weak sequential continuity* of polynomials and multilinear mappings (recall that these notions are equivalent for linear operators defined on reflexive spaces). For instance, in [**60**] (see also [**6**]) it is proved that every N-homogeneous polynomial from ℓ_p into ℓ_q is weakly sequentially continuous if and only if $Nq < p$, and this is equivalent to the reflexivity of the space $\mathcal{P}(^N \ell_p, \ell_q)$ of N-homogeneous polynomials from ℓ_p into ℓ_q. Some results in this line were also addressed in [**38, 48, 49**] in the more general setting of Lorentz and Orlicz sequence spaces. Now, as strong subdifferentiability of a dual space is a stronger property than reflexivity (see Remark 2.6 below), it seems natural to ask if the weak sequential continuity of every N-homogeneous polynomial from ℓ_p into ℓ_q is equivalent to the strong subdifferentiability of the space $\mathcal{P}(^N\ell_p,\ell_q)$. The same question makes sense for N-homogeneous polynomials between Lorentz and Orlicz sequence spaces. Our results show that this is indeed the case, not only in the polynomial but also in the multilinear context.

In Section 4, we study the *dual* version of the *polynomial* property (1.1) and obtain some results on the strong subdifferentiability of elementary tensors on projective symmetric tensor products. As the involved tools and reasonings also apply to the multilinear context, we obtain positive results on the strong subdifferentiability of elementary tensors on the projective tensor product of classical spaces such as c_0 and ℓ_2.

Our results should be compared with those in [**16, 40**], where the authors studied Gâteaux and Fréchet differentiability of the norm of spaces of symmetric tensor products and homogeneous polynomials in relation to w^*-(strongly) exposing and strongly norm-attaining points in these spaces.

2. Preliminary material and main results

In this section, we give the basic concepts we will use throughout the paper and state our main results. First, we set the notation and recall some known properties that will appear in what follows. All Banach spaces considered here are over the real field $\mathbb{R}$ or over the complex field $\mathbb{C}$. We denote B_X, S_X, and X^*, the closed unit ball, the unit sphere, and the topological dual of a Banach space X. We denote by $\mathcal{P}(^NX,Y)$ the Banach space of all continuous N-homogeneous polynomials from X into Y endowed with the supremum norm, while $P_{\text{wsc}}(^NX,Y)$ stands for the Banach space of all *weakly sequentially continuous* N-homogeneous polynomials from X into Y. The Banach space of all continuous N-linear mappings from $X_1\times\cdots\times X_N$ into Y, endowed with the supremum norm, will be denoted by $\mathcal{L}(X_1\times\cdots\times X_N,Y)$, while the space of all continuous N-linear symmetric mappings from $X\times\cdots\times X$ to Y will be denoted by $\mathcal{L}_s(^NX,Y)$. When $Y=\mathbb{K}$ is the scalar field, we will omit it and write $\mathcal{P}(^NX)$, $\mathcal{P}_{\text{wsc}}(^NX)$, $\mathcal{L}(X_1\times\cdots\times X_N)$, and $\mathcal{L}_s(^NX)$. Given $P\in\mathcal{P}(^NX,Y)$ we consider $P^t\colon Y^*\to\mathcal{P}(^NX,\mathbb{K})$, the *transpose of* P, given by $(P^ty^*)(x):=y^*P(x)$ for every $x\in X$ and every $y^*\in Y^*$. For $P\in\mathcal{P}(^NX,Y)$, we set $\mathrm{NA}(P):=\{x\in S_X:\|P(x)\|=\|P\|\}$.

We recall the following well known properties of Banach spaces. A Banach space X is *strictly convex* (SC, for short) if

$$\left\|\frac{x+y}{2}\right\|<1\quad\text{whenever } x,y\in S_X,\ x\neq y.$$

The *modulus of convexity* of a Banach space X is defined for each $\varepsilon\in(0,2]$ by

$$\delta_X(\varepsilon):=\inf\left\{1-\left\|\frac{x+y}{2}\right\|:x,y\in B_X,\ \|x-y\|\geqslant\varepsilon\right\},$$

and X is said to be *uniformly convex* (UC, for short) if $\delta_X(\varepsilon)>0$ for $\varepsilon\in(0,2]$. The *modulus of smoothness* of a Banach space X is defined for each $\tau>0$ by

$$\rho_X(\tau):=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1:x,y\in X,\ \|x\|=1,\ \|y\|=\tau\right\},$$

and X is said to be *uniformly smooth* (US, for short) if $\lim_{\tau\to0}\rho_X(\tau)/\tau=0$. It is a well known result of Šmulian that X is uniformly convex if and only if X^* is uniformly smooth and both properties imply reflexivity. We say that a Banach space X has the *Kadec–Klee property* if the weak and norm topologies coincide on S_X. Analogously, X^* has the w^*-*Kadec–Klee property* if the weak* and norm topologies coincide on S_{X^*}.

We will say that X has the *sequential Kadec–Klee property* if $\|x_n - x_0\| \to 0$ whenever $\|x_n\| \to \|x_0\|$ and $x_n \xrightarrow{w} x_0$ (analogously, we define the *sequential w^*-Kadec–Klee property*). It is worth mentioning that the reader could find the above definition a little bit *confusing* since, in the literature, many authors refer to the sequential Kadec–Klee property simply as the Kadec–Klee property.

A Banach space X is said to have the *approximation property* (for short, AP) (respectively, *compact approximation property* (for short, CAP)) if for every compact subset C of X and every $\varepsilon > 0$ there is a finite-rank (respectively, compact) operator $T\colon X \to X$ such that $\|Tx - x\| \leqslant \varepsilon$ for every $x \in C$. It is immediate that the AP implies the CAP. However, it is known that there exists a Banach space that has the CAP but does not have the AP [**68**].

In the next subsections, we focus on some relevant properties and tools.

2.1. Strong subdifferentiability. The following notions (specifically, Definitions 2.1 and 2.3 below) are the central ones in the present paper.

Definition 2.1. We say that the norm $\|\cdot\|$ of a Banach space X is *strongly subdifferentiable* (SSD, for short) at $x \in S_X$ when the one-sided limit

$$\lim_{t\to 0^+} \frac{\|x+th\|-1}{t} \tag{2.1}$$

exists uniformly in $h \in B_X$. When it holds for every x in a subset $U \subseteq S_X$ we say that *X is SSD on U*, and when it holds for every $x \in S_X$ we simply say that *X is SSD*. A careful reader should realize immediately that a norm is Fréchet differentiable if and only if it is both Gâteaux and SSD. In other words, SSD is what is missing in Gâteaux to be Fréchet.

It is well known that, for an arbitrary $x \in S_X$, the above limit exists in every direction h and that

$$\tau(x,h) := \lim_{t\to 0^+} \frac{\|x+th\|-1}{t} = \max\{\operatorname{Re} x^*(h) : x^* \in D(x)\} \quad (h \in X), \tag{2.2}$$

where $D(x)$ is the set of all normalized support functionals for B_X at x. That is,

$$D(x) := \{x^* \in S_{X^*} : x^*(x) = 1\}.$$

This means that the norm of X is SSD at $x \in S_X$ if and only if

$$\lim_{t\to 0^+} \sup\left\{ \frac{\|x+th\|-1}{t} - \tau(x,h) : h \in B_X \right\} = 0. \tag{2.3}$$

As we mentioned in the introduction, Franchetti and Payá ([**43**]) proved the following result, which is an analogue of the characterization for Fréchet differentiability proved by Šmulian ([**66**]).

Theorem 2.2 ([**43**, Theorem 1.2])**.** *Let X be a Banach space. The norm of X is SSD at $x \in S_X$ if and only if, given $\varepsilon > 0$, there exists $\delta(\varepsilon, x) > 0$ such that whenever $x^*(x) > 1 - \delta(\varepsilon, x)$ for some $x^* \in B_{X^*}$, the distance* $\operatorname{dist}(x^*, D(x)) < \varepsilon$.

In other words, the norm of X is SSD at $x \in S_X$ if and only if x *strongly exposes* the set $D(x)$. That is, the distance $\operatorname{dist}(x_n^*, D(x))$ tends to zero for any sequence $(x_n^*)_{n=1}^\infty \subseteq B_{X^*}$ with $\operatorname{Re} x_n^*(x) \to 1$ as $n \to \infty$.

For background on the study of strong subdifferentiability of the norm, we refer the reader to [**8, 22, 23, 42, 45, 46, 47, 50**]. For a systematic study on the topic, we suggest [**36, 43**]. Here, we shall only mention some examples of classical Banach spaces with SSD norm: it is known that every finite-dimensional Banach space is SSD. Since a Banach space X is uniformly smooth if and only if its norm is uniformly Fréchet differentiable on S_X, it is clear that every uniformly smooth Banach space is SSD; for instance, the sequence spaces ℓ_p with $1 < p < \infty$ are SSD. It is worth mentioning that if a dual space X^* is SSD, then X is reflexive (see [**43**, Theorem 3.3]); hence ℓ_1 and ℓ_∞ are not SSD. The sequence spaces c_0 and $d_*(w, 1)$ (predual of Lorentz sequence space) are examples of non-reflexive SSD Banach spaces. There are other examples of non-reflexive spaces with an SSD norm, for instance, the predual of the Hardy space H^1 and the predual of the Lorentz space $L_{p,1}(\mu)$. Moreover, if X is a predual of a Banach space with the w^*-Kadec–Klee property, then X is SSD (see [**32**, Proposition 2.6]). On the other hand, it is known that if X is SSD, then X is an Asplund space (see [**43, 47**]). It is worth mentioning that there are spaces whose norms are nowhere SSD [**36**, Theorem III.1.9 and Proposition III.4.5] (the second one states that $H^1(D)$ is nowhere SSD except at 0). Also, the ℓ_1-norm is SSD exactly at finitely supported vectors ([**45**, Example 1.1] and [**42**, Theorem 6]). By [**28**, Theorem A], for every space with a fundamental biorthogonal system, there exists a dense subspace with an SSD norm (see [**27**] for more instances in this line).

We now deal with a uniform and, at the same time, localized version of strong subdifferentiability. It was proved in [**43**, Proposition 4.1] that a Banach space X is uniformly smooth if and only if the limit in (2.1) is also uniform in $x \in S_X$ (we already mentioned in the above paragraph that if X is uniformly smooth, then X is SSD). Since uniform smoothness is a quite restrictive property, we consider the case where the limit in (2.1) is uniform in $x \in U$ for some subset $U \subset S_X$.

Definition 2.3. Given a set $U \subseteq S_X$, we say that the norm of a Banach space X is *uniformly strongly subdifferentiable on* U (USSD on U, for short) if the limit (2.1) is uniform for $h \in B_X$ and $x \in U$. In other words, the norm of X is USSD on U if and only if

$$\lim_{t \to 0^+} \sup\left\{ \frac{\|x + th\| - 1}{t} - \tau(x, h) : h \in B_X,\, x \in U \right\} = 0. \tag{2.4}$$

A relation between numerical range and uniform strong subdifferentiability was established in [**61**]. Also, the uniform strong subdifferentiability of the norm of JB*-triples was studied in [**12**] (see also [**11**]), where it is proved that if X is a JB*-triple, then X is USSD on the set of non-zero tripotents of X. However, we could not find a systematic study of this property in the literature.

2.2. (Symmetric) tensor products. The *projective tensor product* between the Banach spaces $X_1, \ldots, X_N$, denoted by $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$, is defined as the completion of the algebraic tensor product $X_1 \otimes \cdots \otimes X_N$ endowed with the norm

$$\|z\|_\pi := \inf\left\{ \sum_{i=1}^n \|x_i^1\| \cdots \|x_i^N\| : z = \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^N \right\},$$

where the infimum is taken over all representations of z of the form $\sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^N$. It is well known that the tensor product between $X_1, \ldots, X_N$ linearizes continuous N-linear mappings on $X_1 \times \cdots \times X_N$. Indeed, we have the isometric isomorphism

$$(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N)^* = \mathcal{L}(X_1 \times \cdots \times X_N),$$

where the duality is given by

$$L_A(z) = \langle z, A \rangle = \sum_{i=1}^\infty A(x_i^1, \ldots, x_i^N),$$

for $A \in \mathcal{L}(X_1 \times \cdots \times X_N)$ and $z = \sum_{i=1}^{\infty} x_i^1 \otimes \cdots \otimes x_i^N \in X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$. Moreover, for multilinear mappings with values in a dual space Y^* we have the isometric isomorphism

$$((X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N) \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X_1 \times \cdots \times X_N, Y^*),$$

with the duality given by

$$L_A(z) = \langle z, A\rangle = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} A(x_{j,i}^1, \ldots, x_{j,i}^N)(y_j),$$

for $A \in \mathcal{L}(X_1 \times \cdots \times X_N, Y^*)$ and $z = \sum_{j=1}^{\infty} v_j \otimes y_j$, where $(y_j)_j \subset Y$ and $(v_j)_j \subset X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$ with $v_j = \sum_{i=1}^{\infty} x_{j,i}^1 \otimes \cdots \otimes x_{j,i}^N$. It is well known that the closed unit ball of $X_1 \widehat{\otimes}_\pi X_2$ is the closed convex hull of $B_{X_1} \otimes B_{X_2}$. That is, $B_{X_1 \widehat{\otimes}_\pi X_2} = \overline{\mathrm{co}}(B_{X_1} \otimes B_{X_2})$.

On the other hand, the *symmetric projective tensor product* of X, denoted by $\widehat{\otimes}_{\pi_s,N} X$, is the completion of the linear space $\otimes_{s,N} X$ generated by $\{\otimes^N x : x \in X\}$ (here, $\otimes^N x$ stands for the elementary tensor $x \otimes \overset{N}{\cdots} \otimes x$) endowed with the norm

$$\|z\|_{\pi_s,N} := \inf\left\{\sum_{i=1}^{n} |\lambda_i| \|x_i\|^N : z = \sum_{i=1}^{n} \lambda_i \otimes^N x_i\right\},$$

where the infimum is taken over all the possible representations of z of that form. The symmetric projective tensor product linearizes continuous homogeneous polynomials. In general, the identity

$$((\widehat{\otimes}_{\pi_s,N} X) \widehat{\otimes}_\pi Y)^* = \mathcal{P}(^N X, Y^*)$$

holds isometrically, with the duality given by

$$L_P(z) = \langle z, P\rangle = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_{j,i} P(x_{j,i})(y_j)$$

for $P \in \mathcal{P}(^N X, Y^*)$ and $z = \sum_{j=1}^{\infty} v_j \otimes y_j$ for $(y_j)_j \subset Y$ and $(v_j)_j \subset \widehat{\otimes}_{\pi_s,N} X$ with $v_j = \sum_{i=1}^{\infty} \lambda_{j,i} \otimes^N x_{j,i}$. We also have that $B_{\widehat{\otimes}_{\pi_s,N} X} = \overline{\mathrm{aco}}(\{\otimes^N x : x \in S_X\})$, where $\mathrm{aco}(C)$ stands for the absolute convex hull of the set C.

We refer the reader to the first chapters of the books [**35, 63**] for an introduction to tensor products (see also [**37**]), and to Floret's survey article [**41**] for symmetric tensor products.

2.3. Orlicz and Lorentz sequence spaces. We briefly recall the definitions and some properties of Orlicz and Lorentz sequence spaces. These spaces, as well as ℓ_p spaces, will provide examples of applications of our main results. An *Orlicz function* M is a continuous non-decreasing and convex function defined for $t \geqslant 0$ such that $M(0) = 0$, $M(t) > 0$ for every $t > 0$, and $\lim_{t\to\infty} M(t) = \infty$. The *Orlicz sequence space* l_M associated to an Orlicz function M is the space of all sequences of scalars $x = (a_i)_i$ with $\sum_i M(|a_i|/\rho) < \infty$ for some $\rho > 0$. The space l_M equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{i=1}^{\infty} M(|a_i|/\rho) \leqslant 1\right\}$$

is a Banach space. An Orlicz function M is said to satisfy the Δ_2-*condition at zero* if

$$\limsup_{t\to 0} \frac{M(2t)}{M(t)} < \infty.$$

The canonical vectors $\{e_n\}_n$ form a symmetric basic sequence in l_M and a basis for the subspace $h_M \subset l_M$ consisting of those sequences $x = (a_i)_i \in l_M$ such that $\sum_i M(|a_i|/\rho) < \infty$ for every $\rho > 0$. The equality $l_M = h_M$ holds if and only if M satisfies the Δ_2-condition at zero. Let

$$\alpha_M = \sup\left\{p > 0 : \sup_{0<t,\lambda\leqslant 1} \frac{M(\lambda t)}{M(\lambda)t^p} < \infty\right\} \tag{2.5}$$

and

$$\beta_M = \inf\left\{q > 0 : \inf_{0<t,\lambda\leqslant 1} \frac{M(\lambda t)}{M(\lambda)t^q} > 0\right\}. \tag{2.6}$$

It is known that $1 \leqslant \alpha_M \leqslant \beta_M \leqslant \infty$, and $\beta_M < \infty$ if and only if M satisfies the Δ_2-condition at zero. Moreover, the space l_M is reflexive if and only if $\beta_M < \infty$ and $\alpha_M > 1$ or, equivalently, M and its dual function $M^*(u) = \max\{tu - M(t) : 0 < t < \infty\}$ satisfy the Δ_2-condition at zero. It is also known that l_M has the (uniform) Kadec–Klee property if and only if M satisfies the Δ_2-condition at zero. A detailed study of these and other properties of Orlicz sequence spaces can be found, for instance, in [**58**].

Other properties in which we are particularly interested are uniform convexity and uniform smoothness. In [**19**, Theorem 2.38] it is shown that the space l_M endowed with the Orlicz norm

$$\|x\|^0 = \sup\left\{\sum_{i=1}^{\infty} a_i b_i : \sum_{i=1}^{\infty} M(|b_i|) \leqslant 1\right\}$$

is uniformly convex if and only if M satisfies the Δ_2-condition at zero and M is uniformly convex on $[0, \pi_M(1)]$, where $\pi_M(\alpha) = \inf\{t > 0 : M^*(p(t)) \geqslant \alpha\}$ (here, p is the right derivative of M), i.e., given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$M\left(\frac{t+s}{2}\right) \leqslant (1-\delta)\frac{M(t) + M(s)}{2}$$

for all $s, t \in [0, \pi_M(1)]$ satisfying $|s - t| \geqslant \varepsilon \max\{s, t\}$. Since $(l_M, \|\cdot\|) = (l_{M^*}, \|\cdot\|^0)^*$ isometrically, we obtain necessary and sufficient conditions for the uniform smoothness of l_M endowed with the Luxemburg norm.

We now focus our attention on Lorentz sequence spaces. Let $1 \leqslant p < \infty$ and $v = (v_i)_i$ be a non-increasing sequence of positive numbers such that $v_1 = 1$, $\lim_i v_i = 0$, and $\sum_i v_i = \infty$. The *Lorentz sequence space* $d(v, p)$ is the Banach space of all sequences $x = (a_i)_i$ such that

$$\|x\| = \sup_{\pi}\left(\sum_{i=1}^{\infty} v_i\,|a_{\pi(i)}|^p\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all permutations π of the set of positive integers. It is well known that $d(v, p)$ is reflexive if and only if $1 < p < \infty$. Note that the canonical vectors $\{e_n\}_n$ form a symmetric basic sequence in $d(v, p)$. In [**18**, Theorem 2] it is proved that $d(v, p)$ has the sequential Kadec–Klee property if $1 < p < \infty$. Moreover, in [**7**] it is shown that $d(v, p)$ $(1 < p < \infty)$ is uniformly convex if and only if

$$\inf_n \frac{\sum_{i=1}^{2n} v_i}{\sum_{i=1}^{n} v_i} = k > 1.$$

For basic properties of Lorentz sequence spaces, we refer the reader to [**58**].

As we will see below, Theorem A relates the strong subdifferentiability of the space of homogeneous polynomials with the study of weakly sequentially continuous polynomials. In that sense, the lower and upper indices of a Banach space X defined by Gonzalo and Jaramillo in [**49**] will appear naturally in our context, since they are closely related to the study of weakly sequentially continuous polynomials. We are particularly interested in the values of these indices for Orlicz and Lorentz sequence spaces, computed by Gonzalo in [**48**]. First, we recall the definition of lower and upper indices of a Banach space X. A sequence $(x_n)_n$ in X is said to have an *upper p-estimate* ($1 \leqslant p < \infty$) if there exists a constant C such that

$$\left\|\sum_{n=1}^{n} a_n x_n\right\| \leqslant C\left(\sum_{n=1}^{n} |a_n|^p\right)^{\frac{1}{p}}$$

for every n-tuple of scalars $a_1, \dots, a_n$. A Banach space X has *property S_p* if every weakly null semi-normalized basic sequence in X has a subsequence with an upper p-estimate. The *lower index of X* is defined as

$$l(X) = \sup\{p \geqslant 1 : X \text{ has property } S_p\}.$$

Analogously, using lower q-estimates (instead of upper p-estimates) the *property T_q* can be defined and the *upper index of X* is defined as

$$u(X) = \inf\{q \geqslant 1 : X \text{ has property } T_q\}.$$

It is not difficult to see that $l(\ell_p) = u(\ell_p) = p$ for $1 < p < \infty$. As we already mentioned, in [**48**] the author computes the values of lower and upper indices for Orlicz and Lorentz sequence spaces. In the case of Orlicz spaces, it is shown that $l(h_M) = \alpha_M$ and $u(h_M) = \beta_M$, where α_M and β_M are the lower and upper Boyd indices defined in (2.5) and (2.6). Since we are interested in reflexive Orlicz spaces and, in that case, the equality $l_M = h_M$ holds, we will use that $l(l_M) = \alpha_M$ and $u(l_M) = \beta_M$. For Lorentz sequence spaces, we believe that the exact values of both lower and upper indices are not known. On the one hand, it is known that $l(d(v,p)) = p$ for $1 < p < \infty$. On the other hand $u(d(v,p)) \geqslant r^*(v)p$, where

$$r(v) = \inf\{s \in [1,\infty] : v \in \ell_s\} \quad \text{and} \quad \frac{1}{r(v)} + \frac{1}{r^*(v)} = 1.$$

2.4. Motivation and tools. Recall that the Bishop–Phelps–Bollobás theorem states that, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $(x_0^*, x_0) \in S_{X^*} \times S_X$ satisfies $|x_0^*(x_0)| > 1 - \eta(\varepsilon)$ there exists $(x_1^*, x_1) \in S_{X^*} \times S_X$ such that

$$|x_1^*(x_1)| = 1, \quad \|x_1^* - x_0^*\| < \varepsilon, \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

Note that the characterization of SSD stated in Theorem 2.2 is *a kind of* Bishop–Phelps–Bollobás theorem in which the point x_0 is fixed and the η in the definition depends not only on $\varepsilon > 0$ but also on the fixed point x_0. In that sense, this property is referred to as the *local Bishop–Phelps–Bollobás point property*, since we *fix* a point and the η is *localized*. This property and its *dual* counterpart (where, instead of a point, a linear functional is fixed and the function η depends on ε and the fixed functional) were defined and studied in the context of linear and multilinear mappings.

For the sake of clarity, let us define the local Bishop–Phelps–Bollobás properties in the context of N-linear mappings. As we are going to deal with the SSD of spaces of polynomials and symmetric projective tensor products, we also define the polynomial versions of such properties. We follow the notation in [**32, 33, 34**].

Definition 2.4. Let $N \in \mathbb{N}$ and $X, X_1, \ldots, X_N, Y$ be Banach spaces.

(i) The *local Bishop–Phelps–Bollobás point property* ($\mathbf{L}_{p,p}$, for short).

The pair $(X_1 \times \cdots \times X_N, Y)$ has the $\mathbf{L}_{p,p}$ if, given $\varepsilon > 0$ and $(x_1, \ldots, x_N) \in S_{X_1} \times \cdots \times S_{X_N}$, there exists $\eta(\varepsilon, x_1, \ldots, x_N) > 0$ such that whenever $A \in \mathcal{L}(X_1 \times \cdots \times X_N, Y)$ with $\|A\| = 1$ satisfies

$$\|A(x_1, \ldots, x_N)\| > 1 - \eta(\varepsilon, x_1, \ldots, x_N),$$

there exists $B \in \mathcal{L}(X_1 \times \cdots \times X_N, Y)$ with $\|B\| = 1$ such that

$$\|B(x_1, \ldots, x_N)\| = 1 \quad \text{and} \quad \|B - A\| < \varepsilon.$$

The pair (X, Y) has the *N-homogeneous polynomial* $\mathbf{L}_{p,p}$ if, given $\varepsilon > 0$ and $x \in S_X$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $P \in S_{\mathcal{P}(^N X, Y)}$ satisfies $\|P(x)\| > 1 - \eta(\varepsilon, x)$, there exists $Q \in \mathcal{P}(^N X, Y)$ such that $\|Q(x)\| = 1$ and $\|P - Q\| < \varepsilon$.

(ii) The *local Bishop–Phelps–Bollobás operator property* ($\mathbf{L}_{o,o}$, for short).

The pair $(X_1 \times \cdots \times X_N, Y)$ has the $\mathbf{L}_{o,o}$ if, given $\varepsilon > 0$ and $A \in \mathcal{L}(X_1 \times \cdots \times X_N, Y)$ with $\|A\| = 1$, then there exists $\eta(\varepsilon, A) > 0$ such that whenever $(x_1, \ldots, x_N) \in S_{X_1} \times \cdots \times S_{X_N}$ satisfies

$$\|A(x_1, \ldots, x_N)\| > 1 - \eta(\varepsilon, A),$$

there exists $(x_1^0, \ldots, x_N^0) \in S_{X_1} \times \cdots \times S_{X_N}$ such that

$$\|A(x_1^0, \ldots, x_N^0)\| = 1 \quad \text{and} \quad \|x_i^0 - x_i\| < \varepsilon$$

for every $i = 1, \ldots, N$.

The pair (X, Y) has the *N-homogeneous polynomial* $\mathbf{L}_{o,o}$ if, given $\varepsilon > 0$ and $P \in \mathcal{P}(^N X, Y)$ with $\|P\| = 1$, there exists $\eta(\varepsilon, P) > 0$ such that whenever $x \in S_X$ satisfies $\|P(x)\| > 1 - \eta(\varepsilon, P)$, there exists $x_0 \in S_X$ such that $\|P(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

Let us briefly explain the connection between these local Bishop–Phelps–Bollobás type properties and the geometry of the underlying Banach spaces. In the first place, as an easy consequence of Theorem 2.2, we have the following:

- X is SSD if and only if the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$;
- X^* is SSD if and only if the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$.

When dealing with vector-valued linear and multilinear operators, we have only one of the implications in the above equivalences.

- If the pair $(X_1 \times \cdots \times X_N, Y)$ has the $\mathbf{L}_{p,p}$, then X_i is SSD for every $i = 1, \ldots, N$ (see [**33**, Proposition 2.3]). The converse does not hold (see [**32**, Remark 3.3]).
- If the pair $(X_1 \times \cdots \times X_N, Y)$ has the $\mathbf{L}_{o,o}$, then X_i^* is SSD for every $i = 1, \ldots, N$ (see [**33**, Proposition 2.3]). The converse does not hold (see [**24**, Theorem 2.1]).

At this point, we are ready to point out our major motivation in the study of SSD of the spaces of N-homogeneous polynomials and symmetric tensor products. Since the projective tensor product of two Banach spaces X_1 and X_2 *linearizes* the space of bilinear forms on $X_1 \times X_2$, the following questions come up naturally:

(Q1) Does the pair $(X_1 \times X_2, \mathbb{K})$ have the $\mathbf{L}_{p,p}$ if and only if $X_1 \widehat{\otimes}_\pi X_2$ is SSD?

(Q2) Does the pair $(X_1 \times X_2, \mathbb{K})$ have the $\mathbf{L}_{o,o}$ if and only if $(X_1 \widehat{\otimes}_\pi X_2)^*$ is SSD?

Concerning these questions, we observe the following facts.

Fact 2.5. *Let X and $X_1, \ldots, X_N$ be Banach spaces.*

(i) *By the linearization property of $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$ (or $\widehat{\otimes}_{\pi_s,N} X$), it is straightforward that if $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$ (respectively, $\widehat{\otimes}_{\pi_s,N} X$) is SSD, then the pair $(X_1 \times \cdots \times X_N, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ (respectively, $(X, \mathbb{K})$ has the N-homogeneous polynomial $\mathbf{L}_{p,p}$). The converse does not hold: consider $N = 2$ and $X_1 = X_2 = \ell_2$* [**33**, Corollary 2.8].

(ii) *Suppose that X_1 has the AP. If X_1 is strictly convex or has the sequential Kadec–Klee property, then $(X_1 \times X_2, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if the pair $(X_1 \widehat{\otimes}_\pi X_2, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (equivalently, $\mathcal{L}(X_1, X_2^*)$ is SSD)* [**34**].

Our goal is to obtain *differentiability properties* of symmetric tensor products and their dual spaces, which are the spaces of homogeneous polynomials. In that sense, a *polynomial version* of Fact 2.5(ii) would be helpful to obtain a relation between the SSD of the norm of $(\widehat{\otimes}_{\pi_s,N} X)^* = \mathcal{P}(^N X)$ and the N-homogeneous polynomial $\mathbf{L}_{o,o}$. A similar argument could be reproduced to obtain the SSD of the norm of the symmetric tensor product $\widehat{\otimes}_{\pi_s,N} X$ from a pair $(X, \mathbb{K})$ having the N-homogeneous polynomial $\mathbf{L}_{p,p}$. Unfortunately (or not), we cannot expect that since, as we state in Fact 2.5(i), even in the case when $N = 2$ the SSD of the (symmetric) projective tensor product cannot be deduced from the $\mathbf{L}_{p,p}$ properties. This point is where the SSD (or USSD) on certain subsets are considered as the link between a differentiability property of the symmetric tensor product and the (uniform) N-homogeneous polynomial $\mathbf{L}_{p,p}$.

2.5. Main results. We now state our main results, which will be proved in Sections 3 and 4. Although we are mainly interested in differentiability properties of spaces of homogeneous polynomials and symmetric tensor products, we also state some results in the context of multilinear mappings and (full, non-symmetric) projective tensor products. We focus first on the relation between the SSD of the norm of the space of homogeneous polynomials and the N-homogeneous polynomial $\mathbf{L}_{o,o}$ (see Definition 2.4).

Theorem A. *Let $N \in \mathbb{N}$, let X be a Banach space with the CAP and the sequential Kadec–Klee property, and let Y be a uniformly convex Banach space. Then, the following are equivalent.*

(a) *$\mathcal{P}(^N X, Y^*)$ is SSD.*
(b) *The pair $((\widehat{\otimes}_{\pi_s,N} X) \widehat{\otimes}_\pi Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (for linear functionals).*
(c) *$\mathcal{P}(^N X, Y^*)$ is reflexive.*
(d) *$\mathcal{P}(^N X, Y^*) = \mathcal{P}_{\mathrm{wsc}}(^N X, Y^*)$.*
(e) *The pair (X, Y^*) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$.*

As a consequence of the previous equivalence, we deduce the SSD of spaces of homogeneous polynomials between ℓ_p, Lorentz and Orlicz sequence spaces. In view of Theorem A, necessary and sufficient conditions for Orlicz and Lorentz sequence spaces to be reflexive, sequential Kadec–Klee property, uniformly convex or uniformly smooth were stated in Subsection 2.3.

Corollary A. *Let $1 < p, q < \infty$ and let M_1, M_2 be Orlicz functions such that $1 < \alpha_{M_i}, \beta_{M_i} < \infty$ for $i = 1, 2$. Suppose that l_{M_2} is uniformly smooth.*

(i) *$\mathcal{P}(^N \ell_p)$ is SSD if and only if $N < p$.*
(i) *$\mathcal{P}(^N \ell_p, \ell_q)$ is SSD if and only if $Nq < p$.*
(iii) *$\mathcal{P}(^N l_{M_1})$ is SSD if and only if $N < \alpha_{M_1}$.*
(iv) *$\mathcal{P}(^N l_{M_1}, l_{M_2})$ is SSD if and only if $N\beta_{M_2} < \alpha_{M_1}$.*

(v) $\mathcal{P}(^N d(w,p))$ *is SSD if and only if* $N < p$.
(vi) $\mathcal{P}(^N d(w,p), l_{M_2})$ *is SSD if and only if* $N\beta_{M_2} < p$.

Note that the equivalence between (a) and (b) in Theorem A follows immediately from the fact that the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if X^* is SSD. The implication (b) $\Rightarrow$ (c) is trivial since, as we already mentioned in Definition 2.1, strongly subdifferentiable dual spaces are reflexive. The equivalence (c) $\Leftrightarrow$ (d) is essentially contained in [**59**] (see also [**52**]), where it is proved that, if X, Y are reflexive and X has the CAP, then $\mathcal{P}(^N X, Y)$ is reflexive if and only if $\mathcal{P}(^N X, Y) = \mathcal{P}_{\mathrm{wsc}}(^N X, Y)$. The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (a) are left for the next section; in the first one we need the sequential Kadec–Klee property of the space X, while the second holds for every reflexive Banach space X and every uniformly convex Banach space Y.

Remark 2.6. Let us make some relevant observations related to the previous theorem.

(i) In view of Theorem A and Corollary A, it is natural to ask if there exists some reflexive Banach space which is not SSD. In [**65**, Example 2], there is an example of a reflexive Banach space Z (a renorming of ℓ_2) which is strictly convex but is not *midpoint locally uniformly rotund* (for its definition, see the article mentioned above). Then Z^* is reflexive and is not SSD. Indeed, by [**32**, Theorem 2.5], if a dual space X^* is SSD, then X is strictly convex if and only if X is midpoint locally uniformly rotund. This example shows that strong subdifferentiability is a stronger property than reflexivity for dual spaces.
(ii) Note that any of the statements in the previous theorem imply that the space X is reflexive. It is worth mentioning that the reflexivity of the space Y is necessary for (e) $\Rightarrow$ (a). Indeed, if X is a finite-dimensional space, then the pair (X, Y) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$ for every Banach space Y (the proof is analogous to [**24**, Theorem 2.4], where the statement is proved for linear operators). Then, if X is finite-dimensional and Y is non-reflexive, the pair (X, Y) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$ and $\mathcal{P}(^N X, Y)$ is not SSD. Moreover, in order to obtain examples of spaces X and Y such that $\mathcal{P}(^N X, Y^*)$ is SSD, the norms of X^* and Y^* necessarily need to be SSD (see, for instance, the proof of [**34**, Theorem B]).
(iii) In view of the previous remark, it is natural to ask if, in Theorem A, the uniform convexity of Y can be relaxed to "Y^* is SSD". We do not know if we can change the uniform convexity hypothesis, which we use in the proof of (e) $\Rightarrow$ (a).
(iv) If Y is reflexive and every $P \in \mathcal{P}(^N X, Y^*)$ attains its norm, then $\mathcal{P}(^N X, Y^*)$ is reflexive by James' theorem.
(v) A bounded linear operator defined on a reflexive Banach space is compact if and only if it maps weakly convergent sequences into norm convergent sequences. Having this in mind, it might be worth mentioning that item (d) in Theorem A is not equivalent to saying that every $P \in \mathcal{P}(^N X, Y^*)$ is a *compact polynomial*, provided that $N \geqslant 2$ (recall that $P \in \mathcal{P}(^N X, Y^*)$ is compact if it maps the unit ball of X into a relatively compact set of Y^*). For instance, N-homogeneous polynomials from ℓ_2 to $\mathbb{K}$ are compact but $\mathcal{P}(^N \ell_2)$ is not reflexive, provided that $N \geqslant 2$.
(vi) As we already mentioned in the introduction, our results should be compared with those in [**16, 40**]. For instance, in [**40**, Theorem 2.4] Ferrera proved that, given a real Banach space X, the norm of $\mathcal{P}(^N X)$ is Fréchet differentiable at $P \in S_{\mathcal{P}(^N X)}$ if and only if P *strongly attains its norm*, which means that there exists $x_0 \in S_X$ such that if $(x_n)_n \subseteq S_X$ satisfies $|P(x_n)| \to 1$ as $n \to \infty$, then $\operatorname{dist}(x_n, x_0)$ or $\operatorname{dist}(x_n, -x_0)$ tends to zero. Similar results were obtained in [**16**],

where Boyd and Ryan characterized Fréchet and Gâteaux differentiability of the norm of spaces of homogeneous polynomials (and symmetric tensor products) in terms of w^*-(strong) exposition of points in the unit ball. Note from the equivalence (a) $\Leftrightarrow$ (e) in Theorem A that, assuming that X has the CAP and the sequential Kadec–Klee property, the norm of $\mathcal{P}(^N X)$ is SSD at *every* norm-one $P \in \mathcal{P}(^N X)$ if and only if $|P(x_n)| \to 1$ as $n \to \infty$ implies $\operatorname{dist}(x_n, NA(P)) \to 0$ for *every* norm-one $P \in \mathcal{P}(^N X)$.

We now state the multilinear counterpart of Theorem A and Corollary A.

Theorem B. *Let $N \in \mathbb{N}$ and $X_1, \ldots, X_N$ be reflexive Banach spaces with Schauder bases such that $X_1, \ldots, X_{N-1}$ have the sequential Kadec–Klee property and X_N is uniformly convex. Then, the following are equivalent.*

(a) *The norm of $\mathcal{L}(X_1 \times \cdots \times X_N) = \mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.*
(b) *The pair $(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (for linear functionals).*
(c) *$\mathcal{L}(X_1 \times \cdots \times X_N)$ is reflexive.*
(d) *$\mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*) = \mathcal{L}_{\mathrm{wsc}}(X_1 \times \cdots \times X_{N-1}, X_N^*)$.*
(e) *The pair $(X_1 \times \cdots \times X_N, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (for multilinear forms).*

Theorem B provides the following characterizations applied for some specific Banach spaces. We refer the reader again to Subsection 2.3 for background on these spaces.

Corollary B. *Let $1 < p_1, \ldots, p_N, q < \infty$ and let $M_1, \ldots, M_{N+1}$ be Orlicz functions satisfying the Δ_2-condition and $1 < \alpha_{M_1}, \beta_{M_1}, \ldots, \alpha_{M_{N+1}}, \beta_{M_{N+1}} < \infty$. Suppose also that $l_{M_{N+1}}$ is uniformly smooth.*

(i) *$\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N})$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.*
(ii) *$\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$.*
(iii) *$\mathcal{L}(l_{M_1} \times \cdots \times l_{M_N})$ is SSD if and only if $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < 1$.*
(iv) *$\mathcal{L}(l_{M_1} \times \cdots \times l_{M_N}, l_{M_{N+1}})$ is SSD if and only if $\frac{1}{\alpha_{M_1}} + \cdots + \frac{1}{\alpha_{M_N}} < \frac{1}{\beta_{M_{N+1}}}$.*
(iv) *$\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N))$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < 1$.*
(vi) *$\mathcal{L}(d(w_1, p_1) \times \cdots \times d(w_N, p_N), l_{M_{N+1}})$ is SSD if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{\beta_{M_{N+1}}}$.*

As we did following the statement of Theorem A, we now make some observations regarding the proof of the equivalence in Theorem B. The equivalence (a) $\Leftrightarrow$ (b) and the implication (b) $\Rightarrow$ (c) are, as in the polynomial case, immediate. Note that, in contrast with Theorem A, we require that $X_1, \ldots, X_N$ have Schauder bases, which is stronger than the compact approximation property hypothesis. The reason is that the equivalence (c) $\Leftrightarrow$ (d) is proved in [**38**, Theorem 1 and Corollary 2] under this stronger assumption. Hence, we only need to prove implications (d) $\Rightarrow$ (e) $\Rightarrow$ (a), which we leave for the next section.

We now focus on the (uniform) strong subdifferentiability of symmetric (respectively, full) projective tensor products from the *polynomial* (respectively, *multilinear*) $\mathbf{L}_{p,p}$. We consider the following subsets of $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$ and $\widehat{\otimes}_{\pi_s, N} X$, respectively,

$$U := \{x_1 \otimes \cdots \otimes x_N : \|x_1\| = \cdots = \|x_N\| = 1\} \subseteq S_{X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N}$$

and

$$U_s := \{\otimes^N x : \|x\| = 1\} \subseteq S_{\widehat{\otimes}_{\pi_s, N} X},$$

and we invoke Definitions 2.1 and 2.3 on the (uniform) SSD at a subset of the unit sphere.

Theorem C. *In the symmetric projective tensor setting, the following statements hold.*

(i) $\widehat{\otimes}_{\pi_s,N}\ell_2$ *is USSD on* U_s *for any* $N \in \mathbb{N}$.

(ii) $c_0\widehat{\otimes}_{\pi_s}c_0$ *is SSD on* U_s *(in the complex case).*

In the (full, non-symmetric) projective tensor setting, we have the following.

(iii) $\ell_2\widehat{\otimes}_{\pi}\overset{N}{\cdots}\widehat{\otimes}_{\pi}\ell_2$ *is USSD on* U *for any* $N \in \mathbb{N}$.

(iv) $c_0\widehat{\otimes}_{\pi}c_0$ *is SSD on* U *(in the complex case).*

3. On the strong subdifferentiability of $\mathcal{P}(^N X, Y^*)$ and $\mathcal{L}(X_1 \times \cdots \times X_N, Y^*)$

In this section, we prove Theorems A and B and their respective corollaries. In Subsection 3.2, we show that, under the assumption of X being uniformly convex (we require neither the CAP nor the sequential Kadec–Klee property), the N-homogeneous polynomial $\mathbf{L}_{o,o}$ is equivalent to each N-homogeneous polynomial strongly exposing a certain set (a property which is formally stronger than the SSD of the norm of $\mathcal{P}(^N X, Y^*)$). Finally, in Subsection 3.3, we make a diagram showing the implications between all the properties appearing in the previous sections, and the hypotheses needed in each implication.

3.1. Proofs of Theorem A and Theorem B. We focus first on the proofs of Theorems A and B. As we already mentioned, we only need to prove implications (d) $\Rightarrow$ (e) $\Rightarrow$ (a) in both theorems.

Proof of Theorem A: We begin with (d) $\Rightarrow$ (e). We are going to prove that if X is reflexive and has the sequential Kadec–Klee property, then the pair (X, Y^*) has the *N-homogeneous polynomial* $\mathbf{L}_{o,o}$ *for weakly sequentially continuous polynomials.* That is, given $\varepsilon > 0$ and $P \in \mathcal{P}_{\mathrm{wsc}}(^N X, Y^*)$ with $\|P\| = 1$, there exists $\eta(\varepsilon, P) > 0$ such that whenever $x \in S_X$ satisfies $\|P(x)\| > 1 - \eta(\varepsilon, P)$ there exists $x_0 \in S_X$ such that

$$\|P(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

We argue by contradiction. Suppose that there are $\varepsilon_0 > 0$, $P_0 \in \mathcal{P}_{\mathrm{wsc}}(^N X, Y^*)$ with $\|P_0\| = 1$, and $(x_n)_{n\in\mathbb{N}} \subseteq B_X$ such that

$$1 \geqslant \|P_0(x_n)\| \geqslant 1 - \frac{1}{n} \quad \text{and} \quad \operatorname{dist}(x_n, \mathrm{NA}(P_0)) \geqslant \varepsilon_0 > 0. \tag{3.1}$$

Since X is reflexive, we may assume that there exists $x_0 \in B_X$ such that $x_n \xrightarrow{w} x_0$. Given that P_0 is weakly sequentially continuous, we have $P_0(x_n) \xrightarrow{\|\cdot\|} P_0(x_0)$. By using (3.1), we get that $\|P_0(x_0)\| = 1$ and, therefore, $x_0 \in S_X$. As X has the sequential Kadec–Klee property, $(x_n)_{n=1}^{\infty}$ converges to x_0 in norm, which is a contradiction.

Now we focus on the implication (e) $\Rightarrow$ (a). Given a norm-one polynomial $P \in \mathcal{P}(^N X, Y^*)$ and $\varepsilon > 0$, we want to find $\delta > 0$ such that

$$\frac{\|P + tQ\| - 1}{t} - \tau(P, Q) < \varepsilon$$

for every $0 < t < \delta$ and every $Q \in \mathcal{P}(^N X, Y^*)$ with $\|Q\| = 1$. Since Y is uniformly convex, in view of the characterization of uniform convexity given in [**54**, Theorem 2.1], we can take $0 < \tilde{\eta}(\varepsilon) < \varepsilon$ such that, if $(y^*, y_0) \in S_{Y^*} \times S_Y$ satisfies $|y^*(y_0)| > 1 - \tilde{\eta}(\varepsilon)$, then there exists $y_1 \in S_Y$ such that $|y^*(y_1)| = 1$ and $\|y_1 - y_0\| < \varepsilon$. We will see that

$$\delta = \frac{\eta\big(2^{-1}\tilde{\eta}\big(\frac{\varepsilon}{2}\big), P\big)}{2}$$

works for our purposes, where $\eta(\varepsilon, P) > 0$ is the one in hypothesis (e). Observe that, without loss of generality, we may assume that $0 < \eta(\varepsilon, P) < \varepsilon$. In particular, $\delta < 2^{-2}\tilde{\eta}(\frac{\varepsilon}{2})$. For any Q with $\|Q\| = 1$ and $0 < t < \delta$ fixed, take $x_t \in S_X$ such that $\|(P + tQ)(x_t)\| = \|P + tQ\|$. Such an x_t exists because hypothesis (e) implies that every polynomial attains its norm. Then, we have

$$\begin{aligned}\|P(x_t)\| &= \|(P + tQ - tQ)(x_t)\| = \|(P + tQ)(x_t) - tQ(x_t)\| \\ &\geqslant \|P + tQ\| - t \geqslant 1 - t - t > 1 - 2\delta = 1 - \eta\left(2^{-1}\tilde{\eta}\left(\frac{\varepsilon}{2}\right), P\right),\end{aligned}$$

and, by hypothesis, there is $z \in S_X$ such that

$$\|P(z)\| = 1 \quad \text{and} \quad \|x_t - z\| < 2^{-1}\tilde{\eta}\left(\frac{\varepsilon}{2}\right).$$

Consider $y_t \in S_Y$ such that

$$(P + tQ)(x_t)(y_t) = \|(P + tQ)(x_t)\| = \|P + tQ\|.$$

Then,

$$\begin{aligned}\frac{\|P + tQ\| - 1}{t} - \tau(P, Q) &\leqslant \frac{\mathrm{Re}[(P + tQ)(x_t)(y_t)] - \mathrm{Re}[P(x_t)(y_t)]}{t} - \tau(P, Q) \\ &= \mathrm{Re}[Q(x_t)(y_t)] - \tau(P, Q).\end{aligned} \tag{3.2}$$

Now, from the inequalities

$$\begin{aligned}\mathrm{Re}[P(x_t)(y_t)] &= \mathrm{Re}[(P + tQ - tQ)(x_t)(y_t)] \\ &= \mathrm{Re}[(P + tQ)(x_t)(y_t)] - \mathrm{Re}[tQ(x_t)(y_t)] \\ &\geqslant \|P + tQ\| - t \geqslant 1 - 2t > 1 - 2\delta > 1 - 2^{-1}\tilde{\eta}\left(\frac{\varepsilon}{2}\right)\end{aligned}$$

and

$$|\mathrm{Re}\,[P(z)(y_t)] - \mathrm{Re}[P(x_t)(y_t)]| \leqslant \|P(z) - P(x_t)\| \leqslant \|z - x_t\| \leqslant 2^{-1}\tilde{\eta}\left(\frac{\varepsilon}{2}\right),$$

we deduce that

$$|P(z)(y_t)| \geqslant \mathrm{Re}[P(z)(y_t)] > \mathrm{Re}[P(x_t)(y_t)] - 2^{-1}\tilde{\eta}\left(\frac{\varepsilon}{2}\right) > 1 - \tilde{\eta}\left(\frac{\varepsilon}{2}\right).$$

Then, by uniform convexity of Y, there exists $y \in S_Y$ such that $P(z)(y) = \|P(z)\| = 1$ and $\|y - y_t\| < \varepsilon/2$. Finally, since

$$\tau(P, Q) \geqslant \mathrm{Re}[Q(z)(y)],$$

going back to (3.2) we see that

$$\begin{aligned}\frac{\|P + tQ\| - 1}{t} - \tau(P, Q) &\leqslant \mathrm{Re}[Q(x_t)(y_t)] - \mathrm{Re}[Q(z)(y)] \\ &\leqslant \mathrm{Re}[Q(x_t)(y_t)] - \mathrm{Re}[Q(z)(y_t)] + \mathrm{Re}[Q(z)(y_t)] - \mathrm{Re}[Q(z)(y)] \\ &\leqslant \|x_t - z\| + \|y_t - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

whenever $0 < t < \delta$, which is the desired statement. □

Proof of Theorem B: Let us begin with the implication (d) ⇒ (e). Arguing by contradiction, exactly as in the proof of Theorem A, we observe that the pair $(X_1 \times \cdots \times X_{N-1}, X_N^*)$ has the $\mathbf{L}_{o,o}$ *for weakly sequentially continuous mappings*. That is, the statement in Definition 2.4(ii) holds for every $A \in \mathcal{L}_{\mathrm{wsc}}(X_1 \times \cdots \times X_{N-1}, X_N^*)$.

Assuming (d), this implies that the pair $(X_1 \times \cdots \times X_{N-1}, X_N^*)$ has the $\mathbf{L}_{o,o}$. Take $\varepsilon > 0$ and a norm-one N-linear form $A \in \mathcal{L}(X_1 \times \cdots \times X_N)$, and let $\eta(\cdot, \tilde{A}) > 0$ be the one in the definition of property $\mathbf{L}_{o,o}$ for the pair $(X_1 \times \cdots \times X_{N-1}, X_N^*)$, where $\tilde{A}(x_1, \ldots, x_{N-1})(x_N) = A(x_1, \ldots, x_N)$. As X_N is uniformly convex, there exists $0 < \tilde{\eta}(\varepsilon) < \varepsilon$ such that if $(x_N^*, x_N) \in S_{X_N^*} \times S_{X_N}$ satisfies $|x_N^*(x_N)| > 1 - \tilde{\eta}(\varepsilon)$, then there exists $x_N^1 \in S_{X_N}$ such that $|x_N^*(x_N^1)| = 1$ and $\|x_N^1 - x_N\| < \varepsilon$. Suppose that

$$|A(x_1, \ldots, x_N)| > 1 - \frac{1}{2}\eta\bigg(\frac{\tilde{\eta}(\varepsilon)}{2N}, \tilde{A}\bigg).$$

We have

$$\|\tilde{A}(x_1, \ldots, x_{N-1})\| \geqslant |\tilde{A}(x_1, \ldots, x_{N-1})(x_N)| > 1 - \eta\bigg(\frac{\tilde{\eta}(\varepsilon)}{2N}, \tilde{A}\bigg)$$

and, by the $\mathbf{L}_{o,o}$ property for the pair $(X_1 \times \cdots \times X_{N-1}, X_N^*)$, there exists a norm-one element $(x_1^1, \ldots, x_{N-1}^1) \in S_{X_1} \times \cdots \times S_{X_{N-1}}$ such that

$$\|\tilde{A}(x_1^1, \ldots, x_{N-1}^1)\| = 1 \quad \text{and} \quad \|x_i - x_i^1\| < \frac{\tilde{\eta}(\varepsilon)}{2N}, \quad i = 1, \ldots, N-1.$$

As a consequence,

$$\begin{aligned}
\big||\tilde{A}(x_1^1, \ldots, x_{N-1}^1)(x_N)| - |\tilde{A}(x_1, \ldots, x_{N-1})(x_N)|\big| \\
&\leqslant |A(x_1^1, \ldots, x_{N-1}^1, x_N) - A(x_1, \ldots, x_{N-1}, x_N)| \\
&\leqslant \|x_{N-1}^1 - x_{N-1}\| + \cdots + \|x_1^1 - x_1\| \\
&\leqslant \frac{\tilde{\eta}(\varepsilon)}{2}
\end{aligned}$$

and, hence,

$$|\tilde{A}(x_1^1, \ldots, x_{N-1}^1)(x_N)| > |\tilde{A}(x_1, \ldots, x_{N-1})(x_N)| - \frac{\tilde{\eta}(\varepsilon)}{2} > 1 - \tilde{\eta}(\varepsilon).$$

Then, there exists $x_N^1 \in S_{X_N}$ such that

$$|\tilde{A}(x_1^1, \ldots, x_{N-1}^1)(x_N^1)| = 1 \quad \text{and} \quad \|x_N - x_N^1\| < \varepsilon.$$

In summation,

$$|A(x_1^1, \ldots, x_{N-1}^1, x_N^1)| = 1 \quad \text{and} \quad \|x_i - x_i^1\| < \varepsilon, \quad i = 1, \ldots, N,$$

as desired.

Now we prove (e) $\Rightarrow$ (a). Given $\varepsilon > 0$ and $A \in \mathcal{L}(X_1 \times \cdots \times X_N)$, we will find $\delta > 0$ such that

$$\frac{\|A + tL\| - 1}{t} - \tau(A, L) < N\varepsilon$$

for every $0 < t < \delta$ and every $L \in \mathcal{L}(X_1 \times \cdots \times X_N)$ with $\|L\| = 1$ (recall (2.2) and (2.3)). We will see that $\delta := \frac{\eta(\varepsilon, A)}{2} > 0$ does the job, where $\eta(\varepsilon, A) > 0$ is the one in the definition of property $\mathbf{L}_{o,o}$ (see Definition 2.4(ii)). For any L and $0 < t < \delta$ fixed, take $\mathbf{x}_t = (x_1^t, \ldots, x_N^t) \in S_{X_1} \times \cdots \times S_{X_N}$ such that $(A + tL)(\mathbf{x}_t) = \|A + tL\|$. Then, we have

$$\begin{aligned}
\operatorname{Re} A(\mathbf{x}_t) &= \operatorname{Re}[(A + tL)(\mathbf{x}_t)] - \operatorname{Re}[tL(\mathbf{x}_t)] \\
&\geqslant \|A + tL\| - t \geqslant 1 - t - t > 1 - \eta(\varepsilon, A),
\end{aligned}$$

and, by hypothesis, there exists $\mathbf{z}_t = (z_1^t, \dots, z_N^t) \in S_{X_1} \times \cdots \times S_{X_N}$ with $\|x_i^t - z_i^t\| < \varepsilon$ and $A(\mathbf{z}_t) = 1$. That is, the linear functional defined as the evaluation in $\mathbf{z}_t$ belongs to the set $D(A)$ of support functionals at A. Then,

$$\begin{aligned}\frac{\|A + tL\| - 1}{t} - \tau(A, L) &= \frac{\mathrm{Re}[(A + tL)(\mathbf{x}_t)] - 1}{t} - \tau(A, L) \\ &\leqslant \frac{\mathrm{Re}[(A + tL)(\mathbf{x}_t)] - \mathrm{Re}[A(\mathbf{x}_t)]}{t} - \tau(A, L) \\ &= \mathrm{Re}[L(\mathbf{x}_t)] - \tau(A, L) \leqslant \mathrm{Re}[L(\mathbf{x}_t)] - \mathrm{Re}[L(\mathbf{z}_t)] < N\varepsilon,\end{aligned}$$

which proves the desired statement. □

Now, we move towards the proof of Corollary A. In view of the equivalence (a) ⇔ (d) in Theorem A, and taking into account that all the Banach spaces considered in Corollary A satisfy the hypotheses of the theorem (see Subsection 2.3), we only need to check that the space of N-homogeneous polynomials coincides with the space of weakly sequentially continuous N-homogeneous polynomials. The following remark will be useful in the proof of the corollary.

Remark 3.1. In [**51**, Remark 3] the authors show that if X has a quotient isomorphic to ℓ_p and $N \geqslant p$, then $\mathcal{P}(^N X) \neq \mathcal{P}_{\mathrm{wsc}}(^N X)$. Following the same ideas we can see that, if ℓ_p is isomorphic to a quotient of X, ℓ_q is isomorphic to a subspace of Y and $Nq \geqslant p$, then $\mathcal{P}(^N X, Y) \neq \mathcal{P}_{\mathrm{wsc}}(^N X, Y)$. Indeed, let $\pi\colon X \to \ell_p$ be a quotient map and take a bounded sequence $(x_n)_n$ in X such that $\pi(x_n) = e_n$, where $\{e_n\}$ is the canonical basis of ℓ_p. Since X does not contain a copy of ℓ_1, by Rosenthal's theorem we know that $(x_n)_n$ admits a weakly Cauchy subsequence $(x_{n_j})_j$. Consider the weakly null sequence in X given by $y_j = x_{n_{2j}} - x_{n_{2j+1}}$ and the polynomial $Q \in \mathcal{P}(^N \ell_p, \ell_q)$ defined by

$$Q(a_1, a_2, \dots, a_j, \dots) = (a_{n_2}^N, a_{n_4}^N, \dots, a_{n_{2j}}^N, \dots)$$

(here we use the fact that $Nq \geqslant p$). Finally, let $i\colon \ell_q \to Y$ be an isomorphism onto its image and consider $P = i \circ Q \circ \pi \in \mathcal{P}(^N X, Y)$. Noting that $\|P(y_j)\| = \|i(Q(e_{n_{2j}} - e_{n_{2j+1}}))\| \not\to 0$, we conclude that P is not weakly sequentially continuous.

Proof of Corollary A: For ℓ_p-spaces it is known that $\mathcal{P}(^N \ell_p) = \mathcal{P}_{\mathrm{wsc}}(^N \ell_p)$ if and only if $N < p$ and that $\mathcal{P}(^N \ell_p, \ell_q) = \mathcal{P}_{\mathrm{wsc}}(^N \ell_p, \ell_q)$ if and only if $Nq < p$ (see for example [**39**, Chapter 2.4]). This gives items (i) and (ii).

Let us prove items (iii) and (iv). On the one hand, by [**49**, Theorem 2.5 and Corollary 2.6] we have that $\mathcal{P}(^N l_{M_1}) = \mathcal{P}_{\mathrm{wsc}}(^N l_{M_1})$ if $N < l(l_{M_1}) = \alpha_{M_1}$ and that $\mathcal{P}(^N l_{M_1}, l_{M_2}) = \mathcal{P}_{\mathrm{wsc}}(^N l_{M_1}, l_{M_2})$ if $N\beta_{M_2} = Nu(l_{M_2}) < l(l_{M_1}) = \alpha_{M_1}$. This gives the "if" implication in items (iii) and (iv). On the other hand, suppose that $N \geqslant \alpha_{M_1}$. Putting $p = \alpha_{M_1}$, we have that ℓ_p is isomorphic to a quotient space of l_{M_1} (see [**57**, Theorem 1 and Corollary 1]). Hence, from Remark 3.1, we deduce that $\mathcal{P}(^N l_{M_1}) \neq \mathcal{P}_{\mathrm{wsc}}(^N l_{M_1})$ and, consequently, $\mathcal{P}(^N l_{M_1})$ is not SSD. In the vector-valued case, suppose that $N\beta_{M_2} \geqslant \alpha_{M_1}$ and put $p = \alpha_{M_1}$ and $q = \beta_{M_2}$. Again by [**57**, Theorem 1 and Corollary 1] we have that ℓ_p is isomorphic to a quotient space of l_{M_1} and ℓ_q is isomorphic to a subspace of l_{M_2}. Then, by virtue of Remark 3.1 we have $\mathcal{P}(^N l_{M_1}, l_{M_2}) \neq \mathcal{P}_{\mathrm{wsc}}(^N l_{M_1}, l_{M_2})$, which is the desired statement.

Finally, we sketch the proof of items (v) and (vi). If $N < l(d(w,p)) = p$, then $\mathcal{P}(^N d(w,p)) = \mathcal{P}_{\mathrm{wsc}}(^N d(w,p))$ by virtue of the cited results in [**49**]. When $N \geqslant p$, given that $d(w,p)$ has a quotient isomorphic to ℓ_p (see [**56**, Proposition 4]), by Remark 3.1 we have that $\mathcal{P}(^N d(w,p)) \neq \mathcal{P}_{\mathrm{wsc}}(^N d(w,p))$. The proof of (vi) is analogous to that of (iv).

Alternatively, for the "only if" parts, one could use that under the hypotheses of Remark 3.1, $\mathcal{P}(^N\ell_p, \ell_q)$ is a subspace of $\mathcal{P}(^N X, Y)$. Thus, if $Nq \geqslant p$, this space cannot be reflexive. □

Proof of Corollary B: As in the proof of Corollary A, we only need to check that, in each case, the space of N-linear mappings coincides with the space of weakly sequentially continuous N-linear mappings (or, equivalently, that the space of N-linear mappings is reflexive). Note that, as in Corollary A, the spaces considered satisfy the hypotheses of Theorem B. Items (i) and (ii) follow from the fact that $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q) = \mathcal{L}_{\mathrm{wsc}}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ if and only if $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$ (see [**39**, Chapter 2.4]). The "if" part of items (iii) and (iv) follows from [**38**, Lemma], where it is proved that if

$$\frac{1}{l(X_1)} + \cdots + \frac{1}{l(X_N)} < \frac{1}{u(X_{N+1})},$$

then every N-linear mapping in $\mathcal{L}(X_1 \times \cdots \times X_N, X_{N+1})$ is weakly sequentially continuous.

The "only if" implication follows applying a *multilinear* version of Remark 3.1. Specifically, it can be proved that if ℓ_{p_i}, $i = 1, \ldots, N$, is isomorphic to a quotient of X_i, ℓ_q is isomorphic to a subspace of X_{N+1} and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_N} \geqslant \frac{1}{q},$$

then $\mathcal{L}(X_1 \times \cdots \times X_N, X_{N+1}) \neq \mathcal{L}_{\mathrm{wsc}}(X_1 \times \cdots \times X_N, X_{N+1})$. Or, as before, one could see that $\mathcal{L}(\ell_{p_1} \times \cdots \times \ell_{p_N}, \ell_q)$ is a subspace of $\mathcal{L}(X_1 \times \cdots \times X_N, X_{N+1})$. The details are left to the reader. Finally, items (v) and (vi) follow by applying the same arguments. □

3.2. Strongly exposing polynomials. Before carrying out a deeper analysis of the existing relation between the N-homogeneous polynomial $\mathbf{L}_{o,o}$ and strong subdifferentiability, let us recall some definitions needed for this subsection. For an N-homogeneous polynomial $P \in \mathcal{P}(^N X, Y^*)$ with $\|P\| = 1$, we define the set

$$C(P) := \overline{\mathrm{co}}\{(\otimes^N x) \otimes y : x \in S_X,\ y \in S_Y,\ \text{and } P(x)(y) = 1\},$$

which clearly satisfies $C(P) \subseteq D(P) = \{\varphi \in S_{\mathcal{P}(^N X, Y^*)^*} : \varphi(P) = 1\}$. Note that when $N = 1$, X is reflexive and $Y = \mathbb{K}$ we have that $C(P) = D(P)$, but the equality does not hold in general. As we already mentioned in Subsection 2.1, the norm of $\mathcal{P}(^N X, Y^*)$ is SSD at P if and only if P strongly exposes $D(P)$. Our aim in this section is to establish some relations between the N-homogeneous polynomial $\mathbf{L}_{o,o}$, strong exposition of the set $C(P)$ and strong subdifferentiability of the norm of $\mathcal{P}(^N X, Y^*)$. As a byproduct, we obtain a result on the denseness of norm-attaining symmetric tensor products, in the same line as [**26**].

Remark 3.2. Let X, Y be Banach spaces and $P \in S_{\mathcal{P}(^N X, Y^*)}$. If P strongly exposes $C(P)$, then the norm of $\mathcal{P}(^N X, Y^*)$ is SSD at P since $C(P) \subseteq D(P)$. In particular, if P strongly exposes $C(P)$ for every $P \in S_{\mathcal{P}(^N X, Y^*)}$, then the norm of $\mathcal{P}(^N X, Y^*)$ is SSD.

Implication (e) ⇒ (a) of Theorem A shows that, without any assumption on the space X, the N-homogeneous polynomial $\mathbf{L}_{o,o}$ of the pair $(X, \mathbb{K})$ imply that $\mathcal{P}(^N X)$ is SSD. In other words, the N-homogeneous polynomial $\mathbf{L}_{o,o}$ is stronger than strong subdifferentiability. In view of the previous remark, it is natural to ask if there is a relation between the N-homogeneous polynomial $\mathbf{L}_{o,o}$ and the situation when every

$P \in S_{\mathcal{P}(^N X, Y^*)}$ strongly exposes the set $C(P)$. In Theorem 3.4 below, we prove that if the underlying spaces are uniformly convex, these properties are in fact equivalent. In order to do this, we prove an auxiliary lemma which relates the N-homogeneous polynomial $\mathbf{L}_{o,o}$ with the denseness of norm-attaining symmetric tensors. This result, interesting on its own, should be compared with [**26**, Theorem 3.8]. Recall that $z \in \widehat{\otimes}_{\pi_s,N} X$ *attains its projective symmetric norm* if there are bounded sequences $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{K}$ and $(x_n)_{n=1}^\infty \subseteq B_X$ such that $\|z\|_{\pi_s,N} = \sum_{n=1}^\infty |\lambda_n|$ and $z = \sum_{n=1}^\infty \lambda_n \otimes^N x_n$. In such a case, we say that the tensor z is norm-attaining. We denote by $\mathrm{NA}_\pi(\widehat{\otimes}_{\pi_s,N} X)$ the set of all $z \in \widehat{\otimes}_{\pi_s,N} X$ such that z attains its projective symmetric norm. Analogously, $z \in (\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y$ *attains its projective norm* if there are bounded sequences $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{K}$, $(x_n)_{n=1}^\infty \subseteq B_X$ and $(y_n)_{n=1}^\infty \subseteq B_Y$ such that $z = \sum_{n=1}^\infty \lambda_n (\otimes^N x_n) \otimes y_n$ with $\|z\|_\pi = \sum_{n=1}^\infty |\lambda_n|$. As expected, we denote by $\mathrm{NA}_\pi((\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y)$ the set of norm-attaining tensors.

Lemma 3.3. *Let X and Y be reflexive Banach spaces.*

(i) *If the pair $(X, \mathbb{K})$ has the N-homogeneous polynomial $\mathbf{L}_{o,o}$, then*

$$\overline{\mathrm{NA}_\pi(\widehat{\otimes}_{\pi_s,N} X)}^{\|\cdot\|_{\pi,s,N}} = \widehat{\otimes}_{\pi_s,N} X.$$

In fact, given $\varepsilon > 0$ and $z \in S_{\widehat{\otimes}_{\pi_s,N} X}$, if $P_0 \in S_{\mathcal{P}(^N X)}$ satisfies $1 = \langle P_0, z\rangle$, there exists $w \in \mathrm{NA}_\pi(\widehat{\otimes}_{\pi_s,N} X)$ such that $\|w - z\|_{\pi_s,N} < \varepsilon$ and $\|w\|_{\pi_s,N} = \langle P_0, w\rangle$.

(ii) *If Y is uniformly convex and the pair (X, Y^*) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$, then*

$$\overline{\mathrm{NA}_\pi((\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = (\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y.$$

In fact, given $\varepsilon > 0$ and $z \in S_{(\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y}$, if $P_0 \in S_{\mathcal{P}(^N X, Y^*)}$ satisfies $1 = \langle P_0, z\rangle$, there exists $w \in \mathrm{NA}_\pi((\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y)$ such that $\|w - z\|_\pi < \varepsilon$ and $\|w\|_\pi = \langle P_0, w\rangle$.

Proof: For the proof of (i), we follow ideas from [**29**, Proposition 4.3]. Let $z \in \widehat{\otimes}_{\pi_s,N} X$ with $\|z\| = 1$ and $\varepsilon > 0$ be given and fix $\delta > 0$ (which will be chosen appropriately later). We can find $P_0 \in \mathcal{P}(^N X)$ such that $\|P_0\| = \langle P_0, z\rangle = 1$. Let $z' = \sum_{j=1}^m \lambda_j \otimes^N x_j$ with $\lambda_j \geqslant 0$, $(x_j)_j \subset B_X$ and $m \in \mathbb{N}$ be such that

$$\sum_{j=1}^m \lambda_j \leqslant 1 + \eta(\delta, P_0)^2 \quad \text{and} \quad \|z - z'\| < \eta(\delta, P_0)^2,$$

where $\eta(\delta, P_0) > 0$ is the one given in the definition of the N-homogeneous polynomial $\mathbf{L}_{o,o}$ of the pair $(X, \mathbb{K})$. Note that

$$1 + \eta(\delta, P_0)^2 \geqslant \sum_{j=1}^m \lambda_j \geqslant \mathrm{Re} \sum_{j=1}^m \lambda_j \langle P_0, \otimes^N x_j\rangle > 1 - \eta(\delta, P_0)^2,$$

which implies that $\sum_{j=1}^m \lambda_j (1 - \mathrm{Re}\langle P_0, \otimes^N x_j\rangle) < 2\eta(\delta, P_0)^2$. Now, defining

$$A = \{i \in \{1, \ldots, m\} : 1 - \mathrm{Re}\langle P_0, \otimes^N x_i\rangle < \eta(\delta, P_0)\}$$

and noting that

$$\eta(\delta, P_0) \sum_{j \in \{1,\ldots,m\}\setminus A} \lambda_j \leqslant \sum_{j \in \{1,\ldots,m\}\setminus A} \lambda_j (1 - \mathrm{Re}\langle P_0, \otimes^N x_j\rangle) < 2\eta(\delta, P_0)^2,$$

we deduce $\sum_{j\in\{1,\dots,m\}\setminus A}\lambda_j < 2\eta(\delta,P_0)$. By the N-homogeneous polynomial $\mathbf{L}_{o,o}$ of the pair $(X;\mathbb{K})$, for each $j\in A$ we can take $u_j\in S_X$ so that

$$|P_0(u_j)|=1 \quad \text{and} \quad \|u_j-x_j\|<\delta.$$

Write $P_0(u_j)=\theta_j\in\mathbb{T}$ for each $j\in A$. Note that $\|\otimes^N u_j-\otimes^N x_j\|<\frac{N^{N+1}}{N!}\delta$ for each $j\in A$ (see, for instance, [**26**, Lemma 2.2]). Moreover, for each $j\in A$,

$$\operatorname{Re}\theta_j=\operatorname{Re}\langle P_0,\otimes^N u_j\rangle>(1-\eta(\delta,P_0))-\frac{N^{N+1}}{N!}\delta,$$

which implies that $|1-\theta_j|<\sqrt{2\big(\eta(\delta,P_0)+\frac{N^{N+1}}{N!}\delta\big)}$ for each $j\in A$ (if we consider the real scalar field, then θ_j would be 1). If we let $w=\sum_{j\in A}\lambda_j\theta_j^{-1}\otimes^N u_j$, then

$$\begin{aligned}
\|w-z\| &\leqslant \|w-z'\|+\|z'-z\| \\
&\leqslant \Big\|\sum_{j\in A}\lambda_j\theta_j^{-1}\otimes^N u_j-\sum_{j=1}^m\lambda_j\otimes^N x_j\Big\|+\eta(\delta,P_0)^2 \\
&\leqslant \Big\|\sum_{j\in A}\lambda_j\theta_j^{-1}\otimes^N u_j-\sum_{j\in A}\lambda_j\otimes^N u_j\Big\| \\
&\quad+\Big\|\sum_{j\in A}\lambda_j(\otimes^N u_j-\otimes^N x_j)\Big\|+2\eta(\delta,P_0)+\eta(\delta,P_0)^2 \\
&< (1+\eta(\delta,P_0)^2)\left(\sqrt{2\left(\eta(\delta,P_0)+\frac{N^{N+1}}{N!}\delta\right)}+\frac{N^{N+1}}{N!}\delta\right) \\
&\qquad\qquad+2\eta(\delta,P_0)+\eta(\delta,P_0)^2.
\end{aligned}$$

On the one hand, choosing $\delta>0$ small enough we obtain $\|w-z\|<\varepsilon$. On the other hand, noticing that $\|w\|=\langle P_0,w\rangle=\sum_{j\in A}\lambda_j$ we deduce that w attains its norm.

We briefly sketch the proof of (ii), which is analogous to the previous one. In what follows, we denote $\|\cdot\|$ both projective and symmetric projective norms, since it is clear by context. Let $\varepsilon>0$ and $z\in S_{(\widehat{\otimes}_{\pi_s,N}X)\widehat{\otimes}_\pi Y}$ be given, and consider $P_0\in S_{\mathcal{P}(^N X,Y^*)}$ such that $1=\langle P_0,z\rangle$. Take the element $z'=\sum_{j=1}^m\big(\sum_{i=1}^n\lambda_{j,i}\otimes^N x_{j,i}\big)\otimes y_j$ with $\lambda_{j,i}>0$, $x_{j,i}\subset B_X$, and $y_j\in B_Y$ such that

$$\sum_{j=1}^m\sum_{i=1}^n\lambda_{j,i}\leqslant 1+\min\left\{\eta\left(\frac{\delta_Y(\delta/2)}{2},P_0\right),\frac{\delta_Y(\delta/2)}{2}\right\}^2=:1+\tilde{\eta}(\delta,P_0)^2$$

and $\|z-z'\|<\tilde{\eta}(\delta,P_0)^2$, where $\delta>0$ is fixed (and chosen appropriately below) and $\delta_Y(\cdot)$ is the modulus of uniform convexity of Y. It can be seen that

$$\sum_{j=1}^m\sum_{i=1}^n\lambda_{j,i}(1-\operatorname{Re}P_0(x_{j,i})(y_j))<2\tilde{\eta}(\delta,P_0)^2.$$

Then, defining

$$A=\{(j,i)\in\{1,\dots,m\}\times\{1,\dots,n\}:1-\operatorname{Re}P_0(x_{j,i})(y_j)<\tilde{\eta}(\delta,P_0)\}$$

we have

$$\sum_{(j,i)\notin A}\lambda_{j,i}<2\tilde{\eta}(\delta,P_0).$$

Now, on the one hand, by the N-homogeneous polynomial $\mathbf{L}_{o,o}$ of the pair (X,Y^*), for each $(j,i)\in A$ we can take $u_{j,i}\in S_X$ so that

$$\|P_0(u_{j,i})\|=1 \quad \text{and} \quad \|u_{j,i}-x_{j,i}\|<\frac{\delta_Y(\delta/2)}{2}.$$

On the other hand, for each $(j,i)\in A$ we have

$$\begin{aligned}|P_0(u_{j,i})(y_j)| &> |P_0(x_{j,i})(y_j)|-\|u_{j,i}-x_{j,i}\|\\ &\geqslant \operatorname{Re} P_0(x_{j,i})(y_j)-\|u_{j,i}-x_{j,i}\|\\ &> 1-\tilde{\eta}(\delta,P_0)-\frac{\delta_Y(\delta/2)}{2}\geqslant 1-\frac{\delta_Y(\delta/2)}{2}-\frac{\delta_Y(\delta/2)}{2}\end{aligned}$$

and, since Y is uniformly convex, by [**54**, Theorem 2.1] there exists $v_j\in S_Y$ such that

$$|P_0(u_{j,i})(v_j)|=1 \quad \text{and} \quad \|v_j-y_j\|<\delta.$$

Then, putting $\theta_{j,i}=P_0(u_{j,i})(v_j)$, the norm-attaining tensor which approximates z is

$$w=\sum_{(j,i)\in A}\left(\sum_{i=1}^{n}\lambda_{j,i}\theta_{j,i}^{-1}\otimes^N u_{j,i}\right)\otimes v_j. \qquad \square$$

Now, we are ready to prove the mentioned equivalence between the N-homogeneous polynomial $\mathbf{L}_{o,o}$ and strong exposition of the set $C(P)$.

Theorem 3.4. *Let X and Y be uniformly convex Banach spaces. The pair (X,Y^*) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$ if and only if P strongly exposes $C(P)$ for every $P\in S_{\mathcal{P}(^N X,Y^*)}$.*

Proof: Suppose first that (X,Y^*) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$. In view of implication (e) $\Rightarrow$ (a) of Theorem A, $\mathcal{P}(^N X,Y^*)$ is SSD (here we use the uniform convexity of Y). Hence, it is enough to show that $C(P)$ and $D(P)$ coincide for each $P\in\mathcal{P}(^N X,Y^*)$ with $\|P\|=1$. To this end, let $\phi\in D(P)$. Since $\mathcal{P}(^N X,Y^*)$ is reflexive we have $\phi\in(\widehat{\otimes}_{\pi_s,N}X)\widehat{\otimes}_\pi Y$ and, by Lemma 3.3, we know that given $\varepsilon>0$ there exists $\phi'=\sum_{j=1}^m\big(\sum_{i=1}^n\lambda_{j,i}\otimes^N u_{j,i}\big)\otimes v_j$ in $(\widehat{\otimes}_{\pi_s,N}X)\widehat{\otimes}_\pi Y$ satisfying

$$\|\phi'\|=\langle P,\phi'\rangle=\sum_{j=1}^m\sum_{i=1}^n\lambda_{j,i} \quad \text{and} \quad \|\phi-\phi'\|<\varepsilon,$$

where $\lambda_{j,i}>0$, $u_{j,i}\in S_X$, and $v_j\in S_Y$ for each $(j,i)\in\{1,\dots,m\}\times\{1,\dots,n\}$. This implies that $\phi'':=\frac{\phi'}{\|\phi'\|}=\sum_{j=1}^m\sum_{i=1}^n\big(\frac{\lambda_{j,i}}{\|\phi'\|}\otimes^N u_{j,i}\big)\otimes v_j\in C(P)$ and

$$\begin{aligned}\|\phi''-\phi\| &= \left\|\frac{\phi'}{\|\phi'\|}-\frac{\phi}{\|\phi'\|}-\phi\left(1-\frac{1}{\|\phi'\|}\right)\right\|\\ &\leqslant \frac{\|\phi'-\phi\|}{\|\phi'\|}+\|\phi\|\left(\frac{|\|\phi'\|-1|}{\|\phi'\|}\right)<\frac{\varepsilon}{1-\varepsilon}+\frac{\varepsilon}{1-\varepsilon}=\frac{2\varepsilon}{1-\varepsilon}.\end{aligned}$$

Thus, $D(P)\subseteq C(P)$ and we are done.

Now, let us see the reverse implication. Suppose, by contradiction, that (X,Y^*) does not satisfy the N-homogeneous polynomial $\mathbf{L}_{o,o}$. Then, there exists $P\in\mathcal{P}(^N X,Y^*)$ with $\|P\|=1$ and $\varepsilon_0>0$ such that, for every $n\in\mathbb{N}$, there exists $x_n\in S_X$ with

$$1-\frac{1}{n}\leqslant\|P(x_n)\|\leqslant 1 \quad \text{and} \quad \operatorname{dist}(x_n,\operatorname{NA}(P))\geqslant\varepsilon_0>0. \tag{3.3}$$

Take $y_n\in S_Y$ and $\theta_n\in\mathbb{T}$ so that

$$|P(x_n)(y_n)|=P(x_n)(\theta_n y_n)>1-\frac{1}{n}.$$

Since P strongly exposes $C(P)$, we have that $\operatorname{dist}((\otimes^N x_n)\otimes\theta_n y_n, C(P))\to 0$. Without loss of generality, we assume that, for every $n\in\mathbb{N}$,

$$\operatorname{dist}((\otimes^N x_n)\otimes\theta_n y_n, C(P))\leqslant \frac{1}{n}.$$

For each $n\in\mathbb{N}$, take $\lambda_{1,n},\dots,\lambda_{s_n,n}>0, u_{1,n},\dots,u_{s_n,n}\in S_X$ and $v_{1,n},\dots,v_{s_n,n}\in S_Y$ to be such that

(I) $\displaystyle\sum_{j=1}^{s_n}\lambda_{j,n}=1,$

(II) $P(u_{j,n})(v_{j,n})=1$ for every $j=1,\dots,s_n$, and

(III) $\displaystyle\left\|(\otimes^N x_n)\otimes\theta_n y_n-\sum_{j=1}^{s_n}\lambda_{j,n}(\otimes^N u_{j,n})\otimes v_{j,n}\right\|<\frac{1}{n}.$

Now, let us take $x_n^*\in S_{X^*}$ and $y_n^*\in S_{Y^*}$ to be such that $x_n^*(x_n)=1$ and $y_n^*(y_n)=\theta_n^{-1}$ for every $n\in\mathbb{N}$. Then

$$\begin{aligned}\operatorname{Re}\sum_{j=1}^{s_n}\lambda_{j,n}x_n^*(u_{j,n})^N y_n^*(v_{j,n}) &= \operatorname{Re}\Big\langle (x_n^*)^N\otimes y_n^*, \sum_{j=1}^{s_n}\lambda_{j,n}(\otimes^N u_{j,n})\otimes v_{j,n}\Big\rangle\\ &\overset{\text{(III)}}{\geqslant} \operatorname{Re}\langle (x_n^*)^N\otimes y_n^*, (\otimes^N x_n)\otimes\theta_n y_n\rangle-\frac{1}{n}=1-\frac{1}{n}.\end{aligned}$$

By a standard convex combination argument, there exists $t_n\in\{1,\dots,s_n\}$ such that $z_n:=u_{t_n,n}\in S_X$ and $w_n:=v_{t_n,n}\in S_Y$ satisfy

$$\operatorname{Re}x_n^*(z_n)^N y_n^*(w_n)\geqslant 1-\frac{1}{n}.$$

Taking $n_0\in\mathbb{N}$ large enough so that

$$\left(1-\frac{1}{n_0}\right)^{\frac{1}{N}}>1-\delta_X(\varepsilon_0/2),$$

we have

$$|x_{n_0}^*(z_{n_0})|>1-\delta_X(\varepsilon_0/2),$$

which implies, by the uniform convexity of X, that $\|\theta z_{n_0}-x_{n_0}\|<\varepsilon_0$ for some $\theta\in\mathbb{T}$. But $\|P(\theta z_{n_0})\|=1$ from (II) above, which yields a contradiction with (3.3). □

Remark 3.5. It is worth noting that if X has the CAP and the sequential Kadec–Klee property, and Y is uniformly convex, then the following are equivalent:

(a) $\mathcal{P}(^N X, Y^*)$ is SSD.

(b) (X,Y^*) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$.

(c) P strongly exposes $C(P)$ for every $P\in S_{\mathcal{P}(^N X,Y^*)}$.

Indeed, implication (a) ⇒ (b) follows from Theorem A (here we use the CAP and Kadec–Klee properties of the space X), while (b) ⇒ (c) follows from the first implication in the proof of Theorem 3.4. Finally, implication (c) ⇒ (a) is Remark 3.2. In view of Theorem 3.4, we have that the equivalence (b) ⇔ (c) holds whenever Y is uniformly convex and X is uniformly convex or has the CAP and the sequential Kadec–Klee property. Since there exist uniformly convex spaces failing the CAP (see, for instance, [**67**]), Theorem 3.4 applies to Banach spaces which are not in the hypotheses of Theorem A.

3.3. Diagram with implications. In order to sum up all the properties we have discussed in the previous subsections, we provide diagrams that show the connections between them and the required hypotheses.

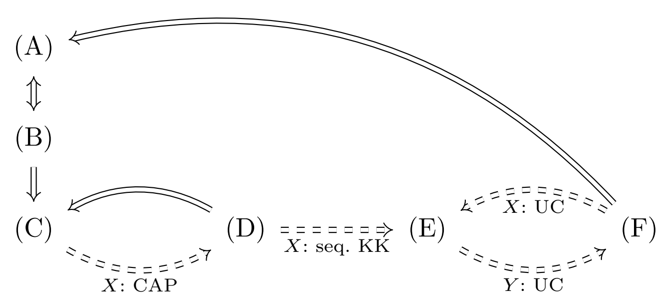
Let $X, Y, X_1, \ldots, X_N$ be reflexive Banach spaces, and consider the following statements:

(A) $\mathcal{P}(^N X, Y^*)$ is SSD.
(B) The pair $((\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (for linear functionals).
(C) $\mathcal{P}(^N X, Y^*)$ is reflexive.
(D) $\mathcal{P}(^N X, Y^*) = \mathcal{P}_{\mathrm{wsc}}(^N X, Y^*)$.
(E) The pair (X, Y^*) has the N-homogeneous polynomial $\mathbf{L}_{o,o}$.
(F) P strongly exposes $C(P)$ for every $P \in S_{\mathcal{P}(^N X, Y^*)}$,

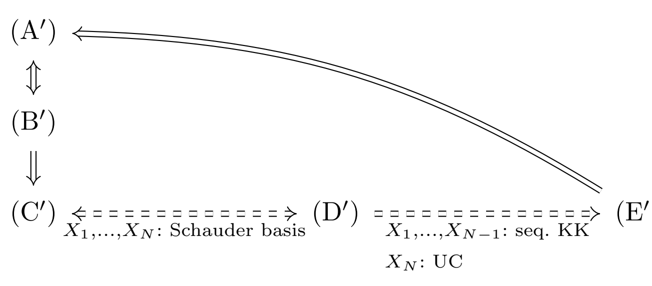
and

(A′) The norm of $\mathcal{L}(X_1 \times \cdots \times X_N) = \mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*)$ is SSD.
(B′) The pair $(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (for linear functionals).
(C′) $\mathcal{L}(X_1 \times \cdots \times X_N)$ is reflexive.
(D′) $\mathcal{L}(X_1 \times \cdots \times X_{N-1}, X_N^*) = \mathcal{L}_{\mathrm{wsc}}(X_1 \times \cdots \times X_{N-1}, X_N^*)$.
(E′) The pair $(X_1 \times \cdots \times X_N, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (for multilinear forms).

Then the following implications hold:



and



4. On the (uniform) strong subdifferentiability of $X\widehat{\otimes}_\pi Y$ and $\widehat{\otimes}_{\pi_s,N} X$

In this section, we establish a connection between the strong subdifferentiability of (symmetric) projective tensor products and the Bishop–Phelps–Bollobás point type properties, which are the dual counterpart of $\mathbf{L}_{o,o}$ properties. Let us briefly clarify the *point properties* we will deal with throughout the section. In [**31**] (see also [**30**]), the authors defined and studied the *Bishop–Phelps–Bollobás point property* (BPBpp, for short) for linear and bilinear operators. We state this property in the next definition and extend it to the polynomial setting.

Definition 4.1. Let $N \in \mathbb{N}$ and $X, X_1, \dots, X_N, Y$ be Banach spaces. We say that the pair $(X_1 \times \cdots \times X_N, Y)$ has the *BPBpp* if, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $A \in \mathcal{L}(X_1 \times \cdots \times X_N, Y)$ with $\|A\| = 1$ and $(x_1, \dots, x_N) \in S_{X_1} \times \cdots \times S_{X_N}$ satisfy

$$\|A(x_1, \dots, x_N)\| > 1 - \eta(\varepsilon),$$

there is a new N-linear mapping $B \in \mathcal{L}(X_1 \times \cdots \times X_N, Y)$ with $\|B\| = 1$ such that

$$\|B(x_1, \dots, x_N)\| = 1 \quad \text{and} \quad \|B - A\| < \varepsilon.$$

Analogously, we say that the pair (X, Y) has the *N-homogeneous polynomial BPBpp* if, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $P \in \mathcal{P}(^N X, Y)$ with $\|P\| = 1$ and $x \in S_X$ satisfy $\|P(x)\| > 1 - \eta(\varepsilon)$, there exists $Q \in \mathcal{P}(^N X, Y)$ with $\|Q\| = 1$ such that $\|Q(x)\| = 1$ and $\|P - Q\| < \varepsilon$.

Note that these properties are the *uniform* versions of properties $\mathbf{L}_{p,p}$ from Definition 2.4, in the sense that the η does not depend on the points $(x_1, \dots, x_N) \in S_{X_1} \times \cdots \times S_{X_N}$ and $x \in S_X$ but only on $\varepsilon > 0$. Using these *Bishop–Phelps–Bollobás point type properties* as tools, we will derive some strong subdifferentiability results for the Banach spaces $\ell_2 \widehat{\otimes}_\pi \overset{N}{\cdots} \widehat{\otimes}_\pi \ell_2$, $\widehat{\otimes}_{\pi_s, N} \ell_2$, $c_0 \widehat{\otimes}_\pi c_0$, and $c_0 \widehat{\otimes}_{\pi_s} c_0$ from BPBpp and $\mathbf{L}_{p,p}$ type results (see Subsection 4.2 below).

It is not difficult to see that, both in the above definition and in Definition 2.4(i), we can take $\|A\|$ and $\|P\|$ less than or equal to one (not necessarily $\|A\| = \|P\| = 1$) by making a standard change of parameters. We will make use of this fact without any explicit mention.

4.1. The tools. We start this section by proving the tool we need to prove the results in Theorem C. It is known that the pair $(X, \mathbb{K})$ has the BPBpp for linear functionals if and only if X is uniformly smooth (see [**31**, Proposition 2.1]). Note that the uniform smoothness of X is equivalent to saying that the norm of X is USSD on $U = S_X$ (recall Definition 2.3). Our next result is a *localization* of the above mentioned characterization.

Proposition 4.2. *Let X be a Banach space and $U \subseteq S_X$. Then the following are equivalent.*

(a) *The norm of X is USSD on U.*
(b) *The pair $(X, \mathbb{K})$ has the* BPBpp for the set U. *That is, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $x_1^* \in S_{X^*}$ and $u \in U$ satisfy $|x_1^*(u)| > 1 - \eta(\varepsilon)$, there exists $x_2^* \in S_{X^*}$ such that $|x_2^*(u)| = 1$ and $\|x_1^* - x_2^*\| < \varepsilon$.*

Proof: Suppose that the norm of X is USSD on the set U. Given $\varepsilon > 0$, let $\delta > 0$ be such that, if $0 < t < \delta$, then

$$\frac{\|u + tz\| - 1}{t} - \tau(u, z) < \frac{\varepsilon}{2}$$

for every $(u, z) \in U \times B_X$ (recall (2.4)). We will show that the pair $(X, \mathbb{K})$ has the BPBpp for the set U with $\eta(\varepsilon) := \frac{\delta\varepsilon}{4} > 0$. Suppose that this is not the case. Then, there exist $u \in U$ and $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(u) > 1 - \eta(\varepsilon)$ and $\|x^* - \tilde{x}^*\| > \varepsilon$ for every $\tilde{x}^* \in S_{X^*}$ satisfying $\tilde{x}^*(u) = 1$. Then, $D(u)$ and $x^* + \varepsilon B_{X^*}$ are w^*-compact, convex, and disjoint sets. Now, by the Hahn–Banach separation theorem, there exists $z \in S_X$ such that

$$\begin{aligned}\tau(u, z) &= \max\{\operatorname{Re} \tilde{x}^*(z) : \tilde{x}^* \in D(u)\} \\ &\leqslant \min\{\operatorname{Re}(x^* + \varepsilon z^*)(z) : z^* \in B_{X^*}\} = \operatorname{Re} x^*(z) - \varepsilon.\end{aligned}$$

Then, for $t = \frac{\delta}{2}$ we have

$$\begin{aligned}\frac{\varepsilon}{2} > \frac{\|u+tz\|-1}{t} - \tau(u,z) &\geqslant \frac{\operatorname{Re} x^*(u+tz)-1}{t} - \operatorname{Re} x^*(z) + \varepsilon \\ &= \frac{\operatorname{Re} x^*(u)-1}{t} + \varepsilon \\ &\geqslant \frac{1-\eta-1}{t} + \varepsilon = \frac{-\eta}{t} + \varepsilon = \frac{-\delta\varepsilon}{4}\frac{2}{\delta} + \varepsilon = \frac{\varepsilon}{2},\end{aligned}$$

which is a contradiction. The other implication is analogous to [**31**, Proposition 2.1]. □

Although Proposition 4.2 may seem artificial at first glance, it is useful to relate the BPBpp (for multilinear operators and homogeneous polynomials) with the geometry of the (symmetric) tensor products. From now on, given $X, X_1, \dots, X_N$ Banach spaces, we fix notation as follows:

$$\begin{aligned}U &:= \{x_1 \otimes \cdots \otimes x_N : \|x_j\| = 1\} \subseteq S_{X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N};\\ U_s &:= \{\otimes^N x : \|x\| = 1\} \subseteq S_{\widehat{\otimes}_{\pi_s,N} X}.\end{aligned}$$

Proposition 4.3. *Let $X, X_1, \dots, X_N$ be Banach spaces.*

(i) *$X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N$ is USSD on U if and only if $(X_1 \times \cdots \times X_N, \mathbb{K})$ has the BPBpp.*
(ii) *$\widehat{\otimes}_{\pi_s,N} X$ is USSD on U_s if and only if $(X, \mathbb{K})$ has the N-homogeneous polynomial BPBpp.*

Proof: A simple linearizing argument shows that $(X_1 \times \cdots \times X_N, \mathbb{K})$ has the BPBpp if and only if $(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N, \mathbb{K})$ has the BPBpp for the set U. This means that the statement follows from Proposition 4.2. A similar argument can be applied in the polynomial context. □

When dealing with (non-necessarily uniform) strong subdifferentiability of tensor products, we have the analogous *local* version of Proposition 4.3. Recall the definition of the N-homogeneous polynomial $\mathbf{L}_{p,p}$ in Definition 2.4.

Proposition 4.4. *Let $X, X_1, \dots, X_N$ be Banach spaces.*

(i) *$X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N$ is SSD on U if and only if $(X_1 \times \cdots \times X_N, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.*
(ii) *$\widehat{\otimes}_{\pi_s,N} X$ is SSD on U_s if and only if $(X, \mathbb{K})$ has the N-homogeneous polynomial $\mathbf{L}_{p,p}$.*

Proof: The proofs of items (i) and (ii) are analogous. Hence we only prove (i). The pair $(X_1 \times \cdots \times X_N, \mathbb{K})$ fails the $\mathbf{L}_{p,p}$ if and only if there is $(x_1, \dots, x_N) \in S_{X_1} \times \cdots \times S_{X_N}$ and a sequence of norm-one N-linear forms $L_n \colon X_1 \times \cdots \times X_N \to \mathbb{K}$, such that

$$L_n(x_1, \dots, x_N) \longrightarrow 1 \quad \text{and} \quad \operatorname{dist}(L_n, D(x_1, \dots, x_N)) \not\longrightarrow 0,$$

where $D(x_1, \dots, x_N) = \{L \in \mathcal{L}(X_1 \times \cdots \times X_N, \mathbb{K}) : L(x_1, \dots, x_N) = \|L\| = 1\}$. In terms of projective tensor products, there is an element $u = x_1 \otimes \cdots \otimes x_N \in U$ and a sequence of norm-one linear functionals $\varphi_n \in (X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N)^*$ (each φ_n is the functional associated to L_n) such that

$$\varphi_n(u) \longrightarrow 1 \quad \text{and} \quad \operatorname{dist}(\varphi_n, D(u)) \not\longrightarrow 0,$$

which is equivalent to the norm of $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N$ not being SSD at u. □

4.2. Proof of Theorem C. Now we are ready to proceed to the proof of Theorem C. On the one hand, in Theorem 4.5 we prove that if $X, X_1, \ldots, X_N$ are Banach spaces with micro-transitive norms (see the definition in the paragraph below), then $(X_1 \times \cdots \times X_N, \mathbb{K})$ and $(X, \mathbb{K})$ have the multilinear and N-homogeneous polynomial BPBpp, respectively. Since Hilbert spaces have micro-transitive norms, this result together with Proposition 4.3 gives items (i) and (iii) of Theorem C. On the other hand, we prove that the pair $(c_0, \mathbb{C})$ has the 2-homogeneous $\mathbf{L}_{p,p}$ by observing that, roughly speaking, a 2-homogeneous polynomial P on c_0 is *almost* finite whenever it *almost* attains its norm. In fact, we will provide a slightly stronger result with codomain a finite-dimensional Hilbert space. This, together with Proposition 4.4, proves item (ii). Item (iv) follows from the result in [**20**] where it is proved that $(c_0 \times c_0, \mathbb{C})$ has the bilinear $\mathbf{L}_{p,p}$. It might be worth mentioning that $\ell_1^N \widehat{\otimes}_\pi Y$ is SSD if and only if Y is SSD since $\ell_1^N \widehat{\otimes}_\pi Y = \ell_1^N(Y)$ (see, for instance, [**43**, Proposition 2.2]).

4.2.1. The BPBpp on spaces with micro-transitive norms. Given a Hausdorff topological group G with identity e and a Hausdorff space T, we say an action $G \times T \to T$ is micro-transitive if for every $x \in T$ and every neighborhood U of e in G the orbit Ux is a neighborhood of x in T. In terms of Banach spaces, we say that the norm of a Banach space is *micro-transitive* if its group of surjective isometries acts micro-transitively on its unit sphere. Equivalently, we have that the norm of a Banach space is micro-transitive if and only if there is a function $\beta\colon (0,2) \to \mathbb{R}^+$ such that if $x, y \in S_X$ satisfy $\|x - y\| < \beta(\varepsilon)$, then there is a surjective isometry $T \in \mathcal{L}(X, X)$ satisfying $T(x) = y$ and $\|T - \mathrm{Id}\| < \varepsilon$ (see [**17**, Proposition 2.1]).

Theorem 4.5. *Let $X, X_1, \ldots, X_N$ be Banach spaces with micro-transitive norms and Z an arbitrary Banach space. Then the following results hold.*

(i) *The pair $(X_1 \times \cdots \times X_N, Z)$ has the BPBpp.*
(ii) *The pair (X, Z) has the N-homogeneous polynomial BPBpp.*

Proof: Let us first prove item (i). For simplicity, we prove the case $N = 2$. By [**17**, Collorary 2.13], X_1 and X_2 are uniformly convex and uniformly smooth. Then, it follows from [**5**, Theorem 2.2] that $(X_1 \times X_2, Z)$ has the Bishop–Phelps–Bollobás property with $\varepsilon \mapsto \eta(\varepsilon)$. Let $A \in \mathcal{L}(X_1 \times X_2, Z)$ with $\|A\| = 1$ and $\|A(x_1, x_2)\| > 1 - \eta'(\varepsilon)$ for some $x_1 \in S_{X_1}$ and $x_2 \in S_{X_2}$, where

$$\eta'(\varepsilon) = \eta\left(\min\left\{\frac{\varepsilon}{3}, \beta_{X_1}\left(\frac{\varepsilon}{3}\right), \beta_{X_2}\left(\frac{\varepsilon}{3}\right)\right\}\right).$$

Here, $\beta_{X_1}, \beta_{X_2}\colon (0,2) \to \mathbb{R}^+$ are functions induced from the micro-transitivity of the norms X_1 and X_2, respectively. Then there are $\widetilde{B} \in \mathcal{L}(X_1 \times X_2, Z)$ and $(\tilde{x_1}, \tilde{x_2}) \in S_{X_1} \times S_{X_2}$ such that

- $\|\widetilde{B}\| = \|\widetilde{B}(\tilde{x_1}, \tilde{x_2})\| = 1$,
- $\max\{\|x_1 - \tilde{x_1}\|, \|x_2 - \tilde{x_2}\|\} < \min\{\beta_{X_1}(\frac{\varepsilon}{3}), \beta_{X_2}(\frac{\varepsilon}{3})\}$,
- $\|\widetilde{B} - A\| < \frac{\varepsilon}{3}$.

Let $T_j \in \mathcal{L}(X_1, X_1)$, $j = 1, 2$, be surjective isometries such that

$$T_j(x_j) = \tilde{x_j} \quad \text{and} \quad \|T_j - \mathrm{Id}_{X_j}\| < \frac{\varepsilon}{3} \quad (j = 1, 2).$$

Define $B(x, y) = \widetilde{B}(T_1(x), T_2(y))$ for every $(x, y) \in X_1 \times X_2$. Then it is routine to check that $\|B\| = \|B(x_1, x_2)\| = 1$ and $\|A - B\| \leqslant \varepsilon$.

The proof of (ii) follows the same line as the previous one. Using again the fact that a Banach space X with micro-transitive norm is uniformly convex and taking [**4**, Theorem 3.1] into account, we deduce that (X, Z) has the Bishop–Phelps–Bollobás property with $\varepsilon \mapsto \eta(\varepsilon)$. Hence, if $P \in \mathcal{P}(^N X, Z)$ with $\|P\| = 1$ is such that

$$\|P(x_0)\| > 1 - \eta\left(\min\left\{\frac{\varepsilon}{N+1}, \beta_X\left(\frac{\varepsilon}{N+1}\right)\right\}\right)$$

for some $x_0 \in S_X$, then there exist $\widetilde{Q} \in \mathcal{P}(^N X, Z)$, $\|\widetilde{Q}\| = 1$, and $\tilde{x_0} \in S_X$ such that

$$\|\widetilde{Q}(\tilde{x_0})\| = 1, \quad \|x_0 - \tilde{x_0}\| < \frac{\varepsilon}{N+1}, \quad \text{and} \quad \|P - \widetilde{Q}\| < \frac{\varepsilon}{N+1}.$$

Finally, letting $T \in \mathcal{L}(X, X)$ be a surjective isometry such that $T(x_0) = \tilde{x_0}$ and $\|T - \mathrm{Id}_X\| < \frac{\varepsilon}{N+1}$, we consider $Q(x) = \widetilde{Q}(T(x))$, which attains its norm at $x_0 \in S_X$ and approximates P. The details are left to the reader. □

The previous theorem together with Proposition 4.3 shows that if $X, X_1, \ldots, X_N$ are Banach spaces with micro-transitive norms, then $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_N$ is USSD on U and $\widehat{\otimes}_{\pi_s,N} X$ is USSD on U_s. Although we stated the result for Banach spaces with micro-transitive norms, the existence of Banach spaces with micro-transitive norms other than Hilbert spaces remains an open problem.

4.2.2. The $\mathbf{L}_{p,p}$ for 2-homogeneous polynomials on c_0. In order to prove the 2-homogeneous polynomial $\mathbf{L}_{p,p}$ for the pair (c_0, H), with H a finite-dimensional complex Hilbert space, we need the following key result which is motivated by [**9**, Proposition 2]. For a subset A of $\mathbb{N}$, let us denote by π_A the natural projection from c_0 onto ℓ_∞^A.

Proposition 4.6. *Consider the complex space c_0 and a complex Hilbert space H. Given $x_0 \in S_{c_0}$ and $\varepsilon > 0$, there exists $\eta(\varepsilon, x_0) > 0$ such that $\|P - P \circ \pi_A\| < \varepsilon$ for any $P \in \mathcal{P}(^2 c_0, H)$ with $\|P\| = 1$ and $\|P(x_0)\| > 1 - \eta(\varepsilon, x_0)$, where $A := \{i \in \mathbb{N} : |x_0(i)| = 1\}$.*

Proof: Suppose that $\|P(x_0)\| > 1 - \varepsilon_0$. We will see that $\varepsilon_0 > 0$ can be chosen depending on x_0 and $\varepsilon > 0$ in such a way that $\|P - P \circ \pi_A\| < \varepsilon$. For simplicity, and without loss of generality, we will suppose that $A = \{1, \ldots, n\}$. Now, consider

$$y = (0, \ldots, 0, y_{n+1}, y_{n+2}, \ldots) \in B_{c_0}.$$

Then, for every $\lambda \in \mathbb{C}$, $|\lambda| = 1 - \max\{|x_0(i)| : i > n\}$, we have

$$\|P(x_0) \pm 2\check{P}(x_0, \lambda y) + \lambda^2 P(y)\| = \|P(x_0 \pm \lambda y)\| \leqslant 1. \tag{4.1}$$

Then,

$$\|P(x_0) + \lambda^2 P(y)\| \leqslant 1.$$

Note that

$$\|P(x_0)\|^2 + 2\,\mathrm{Re}\langle P(x_0), \lambda^2 P(y)\rangle + |\lambda|^4 \|P(y)\|^2 = \|P(x_0) + \lambda^2 P(y)\|^2 \leqslant 1,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on H. By choosing λ so that $\langle P(x_0), \lambda^2 P(y)\rangle$ is purely imaginary we deduce

$$(\|P(x_0)\|^2 + |\lambda|^4 \|P(y)\|^2)^{\frac{1}{2}} \leqslant 1.$$

Since $\|P(x_0)\| > 1 - \varepsilon_0$, we have $((1 - \varepsilon_0)^2 + |\lambda|^4 \|P(y)\|^2)^{\frac{1}{2}} \leqslant 1$ and, consequently,

$$\|P(y)\| \leqslant \frac{\sqrt{2\varepsilon_0 - \varepsilon_0^2}}{|\lambda|^2} =: \beta_1(\varepsilon_0). \tag{4.2}$$

Now, returning to (4.1),

$$\|P(x_0)+2\lambda\check{P}(x_0,y)\|-\|\lambda^2P(y)\|\leqslant\|P(x_0)\pm2\check{P}(x_0,\lambda y)+\lambda^2P(y)\|\leqslant 1.$$

Then, from the estimate (4.2), we deduce that

$$\|P(x_0)+2\lambda\check{P}(x_0,y)\|\leqslant 1+|\lambda|^2\beta_1(\varepsilon_0).$$

Note again that

$$\|P(x_0)\|^2+2\operatorname{Re}\langle P(x_0),2\lambda\check{P}(x_0,y)\rangle+4|\lambda|^2\|\check{P}(x_0,y)\|^2\leqslant(1+|\lambda|^2\beta_1(\varepsilon_0))^2.$$

Choosing λ so that $\langle P(x_0),2\lambda\check{P}(x_0,y)\rangle$ is purely imaginary we have

$$\|P(x_0)\|^2+4|\lambda|^2\|\check{P}(x_0,y)\|^2\leqslant(1+|\lambda|^2\beta_1(\varepsilon_0))^2,$$

from which we deduce (using again the fact that $\|P(x_0)\|>1-\varepsilon_0$) that

$$\|\check{P}(x_0,y)\|<\frac{1}{2|\lambda|}((1+|\lambda|^2\beta_1(\varepsilon_0))^2-(1-\varepsilon_0)^2)^{\frac12}=:\beta_2(\varepsilon_0). \tag{4.3}$$

Given any $x\in c_0$, let us call $x^A=\pi_A(x)=(x_1,\dots,x_n,0,\dots)$ and $x^{A^c}=x-x^A$. Note that

$$P(x_0^A)=P(x_0-x_0^{A^c})=P(x_0)-2\check{P}(x_0,x_0^{A^c})+P(x_0^{A^c})$$

and, hence,

$$1-\varepsilon_0<\|P(x_0)\|=\|P(x_0^A)+2\check{P}(x_0,x_0^{A^c})-P(x_0^{A^c})\|<\|P(x_0^A)\|+2\beta_2(\varepsilon_0)+\beta_1(\varepsilon_0).$$

This gives $\|P(x_0^A)\|>1-\varepsilon_0-2\beta_2(\varepsilon_0)-\beta_1(\varepsilon_0)=:1-\beta_3(\varepsilon_0)$. In particular, this shows that $\|P\circ\pi_A\|>1-\beta_3(\varepsilon_0)$. Using the finite dimensionality of ℓ_∞^A, we take $u=(u_1,\dots,u_n,0,0,\dots)$ an element of S_{c_0} such that $\|(P\circ\pi_A)(u)\|=\|P\circ\pi_A\|$. Applying the maximum modulus principle, we may assume that $|u_1|=\cdots=|u_n|=1$. Moreover, by a simple change of variables we may assume $u=(1,\dots,1,0,0,\dots)$.

Using the fact that $\|P(u)\|>1-\beta_3(\varepsilon_0)$ and arguing as in (4.3), we can prove that $\|\check{P}(u^A,y)\|<\beta_4(\varepsilon_0)$ for every $y=(0,\dots,0,y_{n+1},\dots)\in B_{c_0}$ for some $\beta_4(\varepsilon_0)>0$ satisfying that $\beta_4(\varepsilon_0)\to0$ as $\varepsilon_0\to0$.

Next, let us consider the basis of $\mathbb{C}^n$,

$$\begin{aligned} z_1&=(1,\dots,1),\\ z_2&=(1,-n+1,1,\dots,1),\\ &\ \vdots\\ z_n&=(1,\dots,1,-n+1),\end{aligned}$$

and $\overline{z_j}=(z_j,0,\dots)\in c_0$. Given $(x_1,\dots,x_n)\in\mathbb{C}^n$ it can be checked that

$$(x_1,\dots,x_n)=\frac1n(x_1+\cdots+x_n)z_1+\frac1n\sum_{j=2}^n(x_1-x_j)z_j. \tag{4.4}$$

Now, for any $(x_1,\dots,x_n)\in B_{\ell_\infty^n}$ and any $y=(0,\dots,0,y_{n+1},\dots)\in B_{c_0}$,

$$P(x_1,\dots,x_n,y_{n+1},\dots)=P(x_1,\dots,x_n,0\dots)+2\check{P}((x_1,\dots,x_n,0\dots),y)+P(y).$$

By virtue of (4.4),

$$\check{P}((x_1,\dots,x_n,0\dots),y)=\frac1n(x_1+\cdots+x_n)\check{P}(\overline{z_1},y)+\frac1n\sum_{j=2}^n(x_1-x_j)\check{P}(\overline{z_j},y).$$

If we call $\psi_j(\cdot) = \frac{2}{n}\check{P}(\overline{z_j},\cdot)$, then we have

$$\begin{aligned} P(x_1,\dots,x_n,y_{n+1},\dots) &= P(x_1,\dots,x_n,0,\dots) + (x_1+\cdots+x_n)\psi_1(y) \\ &\quad + \sum_{j=2}^{n}(x_1-x_j)\psi_j(y) + P(y). \end{aligned} \tag{4.5}$$

There is a little abuse of notation: we write $\psi_j(y)$ instead of $\psi_j(y_{n+1},y_{n+2},\dots)$ (and we will keep this notation).

Let us prove that $\|\psi_j\|$ is small for $j=2,\dots,n$. We only consider the case $j=2$, the other cases being analogous. From equation (4.5), choosing $x_2 = e^{i\theta}$ (for any real θ) and $x_j = 1$ if $j \neq 2$, we deduce

$$\begin{aligned} &P(1,e^{i\theta},1,\dots,1,0,\dots) + (1-e^{i\theta})\psi_2(y) \\ &\qquad = P(1,e^{i\theta},1,\dots,1,y_{n+1},\dots) - (n-1+e^{i\theta})\psi_1(y) - P(y). \end{aligned}$$

Then,

$$\begin{aligned} &\|P(1,e^{i\theta},1,\dots,1,0,\dots) + (1-e^{i\theta})\psi_2(y)\| \\ &\qquad \leqslant 1 + n\|\psi_1(y)\| + \|P(y)\| < 1 + 2\beta_4(\varepsilon_0) + \beta_1(\varepsilon_0). \end{aligned}$$

That is,

$$\begin{aligned} &\|P(1,e^{i\theta},1,\dots,1,0,\dots)\|^2 + 2\operatorname{Re}\langle P(1,e^{i\theta},1,\dots,1,0,\dots),(1-e^{i\theta})\psi_2(y)\rangle \\ &\qquad + \|(1-e^{i\theta})\psi_2(y)\|^2 \leqslant (1+2\beta_4(\varepsilon_0)+\beta_1(\varepsilon_0))^2. \end{aligned}$$

As we can vary the argument of y (independent of θ), we deduce that

$$\|P(1,e^{i\theta},1,\dots,1,0,\dots)\|^2 + \|(1-e^{i\theta})\psi_2(y)\|^2 \leqslant (1+2\beta_4(\varepsilon_0)+\beta_1(\varepsilon_0))^2$$

and, since it holds for every $y \in B_{c_0}$, then

$$\|P(1,e^{i\theta},1,\dots,1,0,\dots)\|^2 + |1-e^{i\theta}|^2\|\psi_2\|^2 \leqslant (1+2\beta_4(\varepsilon_0)+\beta_1(\varepsilon_0))^2.$$

Therefore,

$$\begin{aligned} |1-e^{i\theta}|^2\,\|\psi_2\|^2 &\leqslant (1+2\beta_4(\varepsilon_0)+\beta_1(\varepsilon_0))^2 - \|P(1,e^{i\theta},1,\dots,1,0,\dots)\|^2 \\ &= \underbrace{1 - \|P(1,\dots,1,0,\dots)\|^2}_{\text{(I)}} \\ &\quad + \|P(1,\dots,1,0,\dots)\|^2 - \|P(1,e^{i\theta},1,\dots,1,0,\dots)\|^2 + \beta_5(\varepsilon_0), \end{aligned}$$

where $1+\beta_5(\varepsilon_0) := (1+2\beta_4(\varepsilon_0)+\beta_1(\varepsilon_0))^2$. Given that $\|P(1,\dots,1,0,\dots)\| = \|P\circ\pi_A\| > 1-\beta_3(\varepsilon_0)$, we have (I) $< 2\beta_3(\varepsilon_0) - \beta_3(\varepsilon_0)^2$.

Define $f(\theta) = \|P(1,\dots,1,0,\dots)\|^2 - \|P(1,e^{i\theta},1,\dots,1,0,\dots)\|^2$ and $g(\theta) = |1-e^{i\theta}|$, and note that $g(\theta) = 2\sin(\theta/2)$ for $\theta \geqslant 0$. It is worth noting that $\theta \mapsto \|P(1,e^{i\theta},1,\dots,1,0,\dots)\|$ is differentiable at $\theta=0$ (because P is holomorphic and $P(1,e^{i\theta},1,\dots,1,0,\dots) \not\to 0$ when $\theta \to 0$). Note that from L'Hôpital's rule we have

$$\lim_{\theta\to 0^+}\frac{f(\theta)}{g(\theta)^2} = \lim_{\theta\to 0^+}\frac{-2\|P(1,e^{i\theta},1,\dots,1,0,\dots)\|\big(\frac{d}{d\theta}\|P(1,e^{i\theta},1,\dots,1,0,\dots)\|\big)}{4\sin(\theta/2)\cos(\theta/2)}. \tag{4.6}$$

On the one hand, it is clear that

$$\lim_{\theta\to 0^+} \frac{\frac{d}{d\theta}\|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)\|}{\cos(\theta/2)} = 0$$

since $\|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)\|$ has a local maximum at $\theta = 0$ and, consequently,

$$\frac{d}{d\theta}\|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)\|\Big|_{\theta=0} = 0.$$

On the other hand, applying again L'Hôpital's rule we have

$$\lim_{\theta\to 0^+} \frac{\|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)\|}{\sin(\theta/2)} = \lim_{\theta\to 0^+} \frac{2\frac{d}{d\theta}\|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)\|}{\cos(\theta/2)} = 0.$$

Then, going back to (4.6) we obtain that

$$\lim_{\theta\to 0^+} \frac{f(\theta)}{g(\theta)^2} = 0. \tag{4.7}$$

Take $\theta_0 > 0$ such that $|1 - e^{i\theta_0}|^2 = 4\sin^2(\theta_0/2) = \gamma(\varepsilon_0) := (2\beta_3(\varepsilon_0) - \beta_3(\varepsilon_0)^2 + \beta_5(\varepsilon_0))^{\frac{1}{2}}$. Then we obtain

$$\gamma(\varepsilon_0)\|\psi_2\|^2 \leqslant \gamma(\varepsilon_0)^2 + f(2\arcsin(\gamma(\varepsilon_0)^{1/2}/2)),$$

from which we deduce that

$$\|\psi_2\|^2 \leqslant \gamma(\varepsilon_0) + \frac{f(2\arcsin(\gamma(\varepsilon_0)^{1/2}/2))}{\gamma(\varepsilon_0)} \xrightarrow[\varepsilon_0\to 0]{} 0.$$

The limit

$$\frac{f(2\arcsin(\gamma(\varepsilon_0)^{1/2}/2))}{\gamma(\varepsilon_0)} = \frac{f(\theta_0)}{g(\theta_0)^2} \xrightarrow[\varepsilon_0\to 0]{} 0$$

follows from (4.7) and the fact that θ_0 goes to 0 as $\varepsilon_0 \to 0$. As we already mentioned, we can obtain the same bounds for $\psi_3, \dots, \psi_n$. Then, looking at (4.5), we conclude that there is some $\varepsilon_0 = \eta(\varepsilon, x_0) > 0$ such that

$$\|P(x_1, \dots, x_n, y_{n+1}, \dots) - P(x_1, \dots, x_n, 0, \dots)\| < \varepsilon$$

for every $(x_1, \dots, x_n) \in B_{\ell_\infty^n}$ and $y = (0, \dots, 0, y_{n+1}, \dots) \in B_{c_0}$. This proves the statement. $\square$

Now, we are ready to state and prove the main result of this subsection. Before that, let us note that if X and Y are finite-dimensional Banach spaces, the pair (X, Y) has the 2-homogeneous polynomial $\mathbf{L}_{p,p}$. Indeed, it follows by contradiction using the compactness of the unit ball $B_{\mathcal{P}(^2X,Y)}$.

Theorem 4.7. *For a finite-dimensional Hilbert space H, the pair (c_0, H) has the 2-homogeneous polynomial $\mathbf{L}_{p,p}$ in the complex case.*

Proof: Let $\varepsilon > 0$ and $x_0 \in S_{c_0}$ be fixed. Consider the finite set $A := \{i \in \mathbb{N} : |x_0(i)| = 1\}$. Since ℓ_∞^A and H are finite-dimensional, we can consider $\tilde{\eta}(\varepsilon, x_0^A) > 0$ from the 2-homogeneous polynomial $\mathbf{L}_{p,p}$ for the pair (ℓ_∞^A, H). We may assume that $(1-\varepsilon)\tilde{\eta}(\varepsilon, x_0^A) < \varepsilon$. Suppose that

$$\|P(x_0)\| > 1 - \min\left\{\eta\left(\min\left\{\frac{(1-\varepsilon)\tilde{\eta}(\varepsilon, x_0^A)}{2}, \varepsilon\right\}, x_0\right), \frac{(1-\varepsilon)\tilde{\eta}(\varepsilon, x_0^A)}{2}\right\},$$

where $\eta(\cdot, x_0) > 0$ is chosen from Proposition 4.6. Then we have that

$$\|P - P \circ \pi_A\| < \min\left\{\frac{(1-\varepsilon)\tilde{\eta}(\varepsilon, x_0^A)}{2}, \varepsilon\right\}.$$

Thus,

$$\begin{aligned}\|P\circ\pi_A(x_0)\| &= \|P(x_0)+(P\circ\pi_A(x_0)-P(x_0))\|\\ &> \left(1-\frac{(1-\varepsilon)\tilde{\eta}(\varepsilon,x_0^A)}{2}\right)-\|P-P\circ\pi_A\|\\ &> \left(1-\frac{(1-\varepsilon)\tilde{\eta}(\varepsilon,x_0^A)}{2}\right)-\frac{(1-\varepsilon)\tilde{\eta}(\varepsilon,x_0^A)}{2}=1-(1-\varepsilon)\tilde{\eta}(\varepsilon,x_0^A).\end{aligned}$$

This implies that there exists $\overline{Q}\in\mathcal{P}(^2\ell_\infty^A,H)$ with $\|\overline{Q}\|=1$ such that

$$\|\overline{Q}(x_0^A)\|=1 \quad\text{and}\quad \|\overline{Q}-P\circ\pi_A\|<\varepsilon,$$

where $P\circ\pi_A$ is viewed as an element of $\mathcal{P}(^2\ell_\infty^A,H)$. Define $Q\in\mathcal{P}(^2c_0,H)$ as the natural extension of $\overline{Q}$ to c_0. That is, $Q(x)=\overline{Q}(x^A)$. Note that $\|Q\|=\|Q(x_0)\|=1$ and

$$\|Q-P\|\leqslant\|\overline{Q}-P\circ\pi_A\|+\|P\circ\pi_A-P\|<2\varepsilon,$$

which completes the proof. □

Notice that the hypothesis of H being finite-dimensional was only used to affirm that the pair (ℓ_∞^A,H) has the 2-homogeneous polynomial $\mathbf{L}_{p,p}$. In other words, if (ℓ_∞^A,H) has the 2-homogeneous polynomial for a finite set A and an infinite-dimensional Hilbert space H, then Theorem 4.7 would also hold for infinite-dimensional Hilbert spaces.

4.3. Vector-valued polynomial $\mathbf{L}_{p,p}$. In the previous subsection, we deduce some differentiability properties of projective (symmetric) tensor products from $\mathbf{L}_{p,p}$ properties for scalar-valued polynomials and multilinear operators. In this subsection, we focus on the N-homogeneous polynomial $\mathbf{L}_{p,p}$ in the vector-valued case, although we cannot always get differentiability properties of tensor products from the vector-valued $\mathbf{L}_{p,p}$ (see the comment below Corollary 4.9). Recall that a Banach space Y has the *property* β with constant $0\leqslant\rho<1$ if there exist $\{y_i : i\in I\}\subset S_Y$ and $\{y_i^* : i\in I\}\subset S_{Y^*}$ such that

- $y_i^*(y_i)=1$ for all $i\in I$,
- $|y_i^*(y_j)|\leqslant\rho<1$ for all $i,j\in I$ with $i\neq j$,
- $\|y\|=\sup_{i\in I}|y_i^*(y)|$ for all $y\in Y$.

Classic examples of Banach spaces satisfying the property β are c_0 and ℓ_∞. This property was introduced by Lindenstrauss in [**55**], in order to obtain examples of spaces Y such that the set of norm-attaining operators on $\mathcal{L}(X,Y)$ is dense in the whole space, for every Banach space X. In [**32**, Proposition 2.8] it is proved that if X is SSD and Y has property β, then the pair (X,Y) has the $\mathbf{L}_{p,p}$. Next, we prove the polynomial version of this result.

Proposition 4.8. *Let X, Y be Banach spaces. If Y has property β and $(X,\mathbb{K})$ has the N-homogeneous polynomial $\mathbf{L}_{p,p}$, then (X,Y) has the N-homogeneous polynomial $\mathbf{L}_{p,p}$.*

Proof: Let $\varepsilon\in(0,1)$ and $x_0\in S_X$ be fixed. Suppose that $P\in\mathcal{P}(^NX,Y)$ with $\|P\|=1$ satisfies

$$\|P(x_0)\|>1-\min\{\eta(\tilde{\varepsilon},x_0),\tilde{\varepsilon}\},$$

where η is the one in the definition of the N-homogeneous polynomial $\mathbf{L}_{p,p}$ for the pair $(X, \mathbb{K})$ and

$$\tilde{\varepsilon} := \left(\frac{1-\rho}{4}\right)\varepsilon > 0,$$

where $0 \leqslant \rho < 1$ is the constant in the definition of property β. Let us take $\alpha_0 \in \Lambda$ such that

$$|(P^t y^*_{\alpha_0})(x_0)| = |y^*_{\alpha_0}(P(x_0))| > 1 - \eta(\tilde{\varepsilon}, x_0).$$

Then, there exists $Q \in \mathcal{P}(^N X, \mathbb{K})$ with $\|Q\| = 1$ such that

$$|Q(x_0)| = 1 \quad \text{and} \quad \|Q - P^t y^*_{\alpha_0}\| < \tilde{\varepsilon}.$$

Let us define $\tilde{P} \colon X \to Y$ by

$$\tilde{P}(x) := P(x) + ((1+\varepsilon)Q - P^t y^*_{\alpha_0})(x) y_{\alpha_0}$$

and note that $\|\tilde{P} - P\| < \varepsilon + \tilde{\varepsilon}$. We will now prove that $\tilde{P}$ attains its norm at x_0. For every $x \in X$, we have that

$$[\tilde{P}^t y^*_{\alpha_0}](x) = y^*_{\alpha_0}(\tilde{P}(x)) = (1+\varepsilon)Q(x),$$

which shows that $\tilde{P}^t y^*_{\alpha_0} = (1+\varepsilon)Q$. On the other hand, if $\alpha \neq \alpha_0$ and $x \in B_X$, then

$$\begin{aligned}\|[\tilde{P}^t y^*_{\alpha}](x)\| &\leqslant \|P\| + |y^*_{\alpha}(y_{\alpha_0})|(\varepsilon\|Q\| + \|Q - P^t y^*_{\alpha_0}\|)\\ &< 1 + \rho(\varepsilon + 2\tilde{\varepsilon}) = 1 + \varepsilon\left(\rho + \frac{(1-\rho)\rho}{2}\right) < 1 + \varepsilon.\end{aligned}$$

This shows that $\|\tilde{P}\| = \|\tilde{P}^t y^*_{\alpha_0}\| = |y^*_{\alpha_0}(\widetilde{P}(x_0))|$; hence $\tilde{P}$ attains its norm at x_0 as desired. Therefore, the pair (X, Y) has the N-homogeneous polynomial $\mathbf{L}_{p,p}$. □

As an immediate consequence of the previous proposition, we have the following.

Corollary 4.9. *Let $N \in \mathbb{N}$ be given.*

(i) *If X is a finite-dimensional Banach space, then the pairs (X, c_0) and (X, ℓ_∞) have the N-homogeneous polynomial $\mathbf{L}_{p,p}$.*
(ii) *The pairs (ℓ_2, c_0) and (ℓ_2, ℓ_∞) have the N-homogeneous polynomial $\mathbf{L}_{p,p}$.*
(iii) *The pairs (c_0, c_0) and (c_0, ℓ_∞) have the 2-homogeneous polynomial $\mathbf{L}_{p,p}$ in the complex case.*

In view of the isometry $((\widehat{\otimes}_{\pi_s,N}\ell_2)\widehat{\otimes}_\pi \ell_1)^* = \mathcal{P}(^N \ell_2, \ell_\infty)$, the results in Propositions 4.3 and 4.4, and the fact that (ℓ_2, ℓ_∞) has the N-homogeneous polynomial $\mathbf{L}_{p,p}$, it is natural to ask if $(\widehat{\otimes}_{\pi_s,N}\ell_2)\widehat{\otimes}_\pi \ell_1$ is SSD on the set of elementary tensors. However, it is easy to see that this is not possible since the norm of ℓ_1 is not SSD. Although in general these notions cannot be related in the vector-valued case, next we show that when the codomain is a Banach space with micro-transitive norm they do have a relation.

Proposition 4.10. *Let X be a Banach space and Y a Banach space with micro-transitive norm. The pair (X, Y^*) has the N-homogeneous polynomial $\mathbf{L}_{p,p}$ if and only if $\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y$ is SSD on the set $V := \{\otimes^N x \otimes y : \|x\| = \|y\| = 1\}$.*

Proof: First, we will show that the N-homogeneous polynomial $\mathbf{L}_{p,p}$ property implies that the space $\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y$ is SSD on V. Analogously to what we did in Proposition 4.2 with the uniform strong subdifferentiability, it can be proved that $\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y$ is SSD on V if and only if $(\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ for the set V.

Let η be the one in the definition of the N-homogeneous polynomial $\mathbf{L}_{p,p}$ for the pair (X,Y^*), $\tilde{\eta}$ the one in the definition of the BPBpp for the pair $(Y,\mathbb{K})$, and $\beta(\varepsilon)$ the one in the definition of the micro-transitivity property of Y^* (see, for instance, [**17**, Proposition 3.4]). Given ε, let $\varphi \in (\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y)^*$, $\|\varphi\| = 1$, and $\otimes^N x_0 \otimes y_0 \in V$ be such that

$$\varphi(\otimes^N x_0 \otimes y_0) > 1 - \min\Big\{\eta\Big(\frac{\beta(\varepsilon)}{2}, x_0\Big), \tilde{\eta}\Big(\frac{\beta(\varepsilon)}{2}\Big)\Big\}. \tag{4.8}$$

We want to show that there is $\psi \in (\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y)^*$, $\|\psi\| = 1$, with

$$\psi(\otimes^N x_0 \otimes y_0) = 1 \quad \text{and} \quad \|\psi - \varphi\| < 2\varepsilon.$$

Because of the duality $((\widehat{\otimes}_{\pi_s,N} X)\widehat{\otimes}_\pi Y)^* = \mathcal{P}(^N X, Y^*)$, there is a norm-one polynomial $P \in \mathcal{P}(^N X, Y^*)$ such that $P(x)(y) = \varphi(\otimes^N x \otimes y)$. By (4.8) we have that $P(x_0)(y_0) > 1 - \tilde{\eta}\big(\frac{\beta(\varepsilon)}{2}\big)$ and, since the pair $(Y,\mathbb{K})$ has the BPBpp, there exists $y_0^* \in S_{Y^*}$ such that

$$y_0^*(y_0) = 1 \quad \text{and} \quad \|y_0^* - P(x_0)\| < \frac{\beta(\varepsilon)}{2}.$$

On the other hand, given that $\|P(x_0)\| > 1 - \eta\big(\frac{\beta(\varepsilon)}{2}, x_0\big)$, there exists a norm-one polynomial $Q \in \mathcal{P}(^N X, Y^*)$ such that

$$\|Q(x_0)\| = 1 \quad \text{and} \quad \|Q - P\| < \frac{\beta(\varepsilon)}{2}.$$

Therefore, we have $\|y_0^* - Q(x_0)\| < \beta(\varepsilon)$ and this implies that there exists a surjective isometry $T\colon Y^* \to Y^*$ such that $T(Q(x_0)) = y_0^*$ and $\|T - \mathrm{Id}_{Y^*}\| < \varepsilon$. Finally, define $\tilde{Q}\colon X \to Y^*$ by $\tilde{Q}(x) = T(Q(x))$ and note that

$$\tilde{Q}(x_0)(y_0) = 1 \quad \text{and} \quad \|\tilde{Q} - P\| \leqslant \|\tilde{Q} - Q\| + \|Q - P\| < 2\varepsilon.$$

Thus, if $\psi \in (\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y)^*$, $\|\psi\| = 1$, is the linear functional associated to $\tilde{Q}$, we have

$$\psi(\otimes^N x_0 \otimes y_0) = 1 \quad \text{and} \quad \|\psi - \varphi\| < 2\varepsilon,$$

which is the desired statement.

Now, suppose that the pair (X,Y^*) does not have the N-homogeneous polynomial $\mathbf{L}_{p,p}$. We want to show that $(\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y, \mathbb{K})$ does not have the $\mathbf{L}_{p,p}$ for the set V. By hypothesis, there are $x_0 \in S_X$, $\varepsilon > 0$, and $(P_j)_j \subseteq \mathcal{P}(^N X, Y^*)$ norm-one polynomials such that

$$\|P_j(x_0)\| \to 1 \quad \text{and} \quad \mathrm{dist}(P_j, \{P \in S_{\mathcal{P}(^N X, Y^*)} : \|P(x_0)\| = 1\}) > \varepsilon. \tag{4.9}$$

Composing each P_n with a suitable isometry $T_n\colon Y^* \to Y^*$, we may assume that $P_n(x_0)$ is a multiple of a fixed $y_0^* \in S_{Y^*}$. Choose $y_0 \in S_Y$ so that $y_0^*(y_0) = 1$. For each $j \in \mathbb{N}$ define $\varphi_j \in (\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y)^*$ as the linearization of P_j. Then $\|\varphi_j\| = \|P_j\| = 1$, $|\varphi_j(\otimes^N x_0 \otimes y_0)| = \|P_j(x_0)\| \to 1$, and equation (4.9) implies that

$$\mathrm{dist}(\varphi_j, \{\psi \in (\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y)^* : |\psi(\otimes^N x_0 \otimes y_0)| = 1\}) > \varepsilon.$$

Therefore, $(\widehat{\otimes}_{\pi_s,N} X \widehat{\otimes}_\pi Y, \mathbb{K})$ does not have the $\mathbf{L}_{p,p}$ for the set V, as we wished. □

As a consequence, we obtain the following corollary.

Corollary 4.11. *If H is a Hilbert space, the following results hold.*

(i) *The space $(\widehat{\otimes}_{\pi_s,N} H)\widehat{\otimes}_\pi H$ is SSD on the set $V = \{\otimes^N x \otimes y : \|x\| = \|y\| = 1\}$.*

(ii) *If, in addition, H is complex and finite-dimensional, the space $(c_0 \widehat{\otimes}_{\pi_s} c_0)\widehat{\otimes}_\pi H$ is SSD on the set $V = \{\otimes^2 x \otimes y : \|x\| = \|y\| = 1\}$.*

4.4. A negative result on bilinear symmetric forms. It is a well known fact that the polarization formula gives an isomorphism between the space of N-homogeneous polynomials $\mathcal{P}(^N X, Z)$ and the space of N-linear symmetric mappings $\mathcal{L}_s(^N X, Z)$. Moreover, in some spaces this isomorphism is in fact an isometry. This is the case when X is a Hilbert space. Then, it is natural to ask if it is possible to obtain similar results to the ones obtained before when we deal with symmetric forms instead of polynomials. In this short subsection, we will show, with a simple counterexample, that in Proposition 4.3(ii) we cannot replace the N-homogeneous polynomial BPBpp with a similar property using the N-linear symmetric BPBpp.

Let us begin with some proper definitions and remarks. We say that the pair (X, Z) has the *N-linear symmetric Bishop–Phelps–Bollobás point property* (N-linear symmetric BPBpp, for short) if, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathcal{L}_s(^N X, Z)$, $\|A\| = 1$, and $(x_1, \dots, x_N) \in S_X \times \cdots \times S_X$ satisfy

$$\|A(x_1, \dots, x_N)\| > 1 - \eta(\varepsilon),$$

there exists $B \in \mathcal{L}_s(^N X, Z)$ with $\|B\| = 1$ such that

$$\|B(x_1, \dots, x_N)\| = 1 \quad \text{and} \quad \|B - A\| < \varepsilon.$$

When dealing with symmetric multilinear forms, we have the linear isometry

$$\mathcal{L}_s(^N X) = (\widehat{\otimes}_{\pi_s, N} X)^*,$$

where we endow the N-fold symmetric tensor product with the (full, non-symmetric) projective norm π. Thus, it is reasonable to wonder if an analogous result to Proposition 4.3 holds, and we can relate the N-linear symmetric BPBpp with USSD on the set $U_s = \{\otimes^N x : \|x\| = 1\}$ (considering the projective norm π). By Theorem C, $\widehat{\otimes}_{\pi_s, N} \ell_2$ is USSD on the set U_s (recall that for Hilbert spaces the projective norm and the symmetric projective norm coincide). However, as shown below, ℓ_2 does not enjoy the BPBpp for N-linear symmetric mappings. Therefore, a proposition similar to Proposition 4.3 replacing polynomials with symmetric multilinear mapping cannot be obtained.

Example 4.12. The pair $(\ell_2, \mathbb{K})$ fails the bilinear symmetric BPBpp. Moreover, the pair $(\ell_2^2, \mathbb{K})$ fails this property.

Proof: Suppose, on the contrary, that the pair $(\ell_2^2, \mathbb{K})$ has the bilinear symmetric BPBpp and consider $A\colon \ell_2^2 \times \ell_2^2 \to \mathbb{K}$ the symmetric bilinear form given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Given $0 < \varepsilon < 1$, let $0 < \eta(\varepsilon) < 1$ be the one in the definition of the bilinear symmetric BPBpp. Let $a, b > 0$ be such that $a^2 + b^2 = 1$ and

$$A((a, b), (a, -b)) = a^2 - b^2 > 1 - \eta(\varepsilon).$$

Then, there is a symmetric norm-one bilinear form B with

$$|B((a, b), (a, -b))| = 1 \quad \text{and} \quad \|A - B\| < \varepsilon.$$

Now, let

$$\begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix}$$

be the matrix associated with B. Since $\|B\| = 1$, we have that $|d_1|$ and $|d_2|$ cannot exceed 1. Therefore,

$$|d_1\, a^2 - d_2\, b^2| = |B((a, b), (a, -b))| = 1 = a^2 + b^2$$

implies that $d_1 = -d_2$ and $|d_1| = |d_2| = 1$. Then, $\|A - B\| \geqslant 1$, which is the desired contradiction. □

In contrast with this negative result, it is worth mentioning that complex Hilbert spaces have the BPBpp for several classes of operators: self-adjoints, anti-symmetric, unitary, normal, compact normal, compact, and Schatten–von Neumann operators (see [**21**, Theorem 3.1]). They also have the Bishop–Phelps–Bollobás property for symmetric bilinear mappings and Hermitian bilinear mappings and, in the real case, they have the Bishop–Phelps–Bollobás property for symmetric bilinear mapping (see [**44**]). Also, although Hilbert spaces fail to have the bilinear symmetric BPBpp, the pair $(\ell_2^d, \mathbb{K})$ enjoys a local Bishop–Phelps–Bollobás type property for symmetric bilinear forms by compactness.

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