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UNCOUNTABLE INTERVAL OF VARIETIES OF INVOLUTION SEMIGROUPS SHARING A COMMON SEMIGROUP VARIETY REDUCT

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Abstract: A pair of involution semigroups sharing a common semigroup reduct of order eight is constructed with the property that the varieties they generate bound an interval that contains an uncountable chain. Consequently, there exist uncountably many non-finitely generated varieties of involution semigroups sharing a common semigroup variety reduct that is finitely generated.

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1. Introduction

Recall that a *unary semigroup* $\langle \mathcal{S}, {}^* \rangle$ is a semigroup $\mathcal{S}$ equipped with a unary operation *. A unary semigroup $\langle \mathcal{S}, {}^* \rangle$ that satisfies the equations

$$(x^*)^* \approx x \quad \text{and} \quad (xy)^* \approx y^* x^* \tag{1}$$

is an *involution semigroup*. Common examples of involution semigroups include groups $\langle \mathcal{G}, {}^{-1} \rangle$ under inversion $^{-1}$ and matrix semigroups $\langle \mathcal{M}_n, {}^T \rangle$ under the usual transposition ^{T}. An involution semigroup $\langle \mathcal{S}, {}^* \rangle$ and its semigroup *reduct* $\mathcal{S}$ are similar in many ways, but the varieties they generate can satisfy very contrasting properties. For instance, the variety $\mathsf{Var}\langle \mathcal{S}, {}^* \rangle$ and its semigroup variety *reduct* $\mathsf{Var}\,\mathcal{S}$ can satisfy very different equational properties; see, for example, [**3, 5, 7, 8, 11, 12, 13, 16**].

The lattice $\mathscr{L}_{\mathsf{inv}}$ of varieties of involution semigroups and the lattice $\mathscr{L}_{\mathsf{sem}}$ of varieties of semigroups are also well known to be highly incompatible [**2, 15**]. Most notably, inclusions in $\mathscr{L}_{\mathsf{inv}}$ need not resemble those from $\mathscr{L}_{\mathsf{sem}}$; for example, there exist an abundance of pairs of finite involution semigroups $\langle \mathcal{S}, {}^* \rangle$ and $\langle \mathcal{T}, {}^* \rangle$ such that $\mathsf{Var}\langle \mathcal{S}, {}^* \rangle \nsubseteq \mathsf{Var}\langle \mathcal{T}, {}^* \rangle$ and $\mathsf{Var}\,\mathcal{S} \subseteq \mathsf{Var}\,\mathcal{T}$ [**9**]. Even in cases when the inclusion $\mathsf{Var}\langle \mathcal{S}, {}^* \rangle \subseteq \mathsf{Var}\langle \mathcal{T}, {}^* \rangle$ holds, the intervals $[\mathsf{Var}\langle \mathcal{S}, {}^* \rangle, \mathsf{Var}\langle \mathcal{T}, {}^* \rangle]$ and $[\mathsf{Var}\,\mathcal{S}, \mathsf{Var}\,\mathcal{T}]$ need not be similar. This is well illustrated by the following example.

Example 1 (Lee [**14**])**.** There exist involution semigroups $\langle \mathcal{S}, {}^* \rangle$ and $\langle \mathcal{T}, {}^* \rangle$ of order four such that the interval $[\mathsf{Var}\langle \mathcal{S}, {}^* \rangle, \mathsf{Var}\langle \mathcal{T}, {}^* \rangle]$ contains an infinite chain even though its semigroup variety reduct $[\mathsf{Var}\,\mathcal{S}, \mathsf{Var}\,\mathcal{T}]$ is just the chain $\mathsf{Var}\,\mathcal{S} \subset \mathsf{Var}\,\mathcal{T}$ of order two.

It is of fundamental interest to ask if there exists an example possessing more extreme properties: either $[\mathsf{Var}\langle \mathcal{S}, {}^* \rangle, \mathsf{Var}\langle \mathcal{T}, {}^* \rangle]$ is uncountable or $[\mathsf{Var}\,\mathcal{S}, \mathsf{Var}\,\mathcal{T}]$ is trivial, that is, $\mathsf{Var}\,\mathcal{S} = \mathsf{Var}\,\mathcal{T}$. Surprisingly, the answer to this seemingly elusive question is affirmative, and the construction of such an example is the main goal of the present article.

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Theorem 2. *There exist involution semigroups $\langle \mathcal{S}_8, {}^* \rangle$ and $\langle \mathcal{S}_8, {}^{\circledast} \rangle$, sharing a common semigroup reduct $\mathcal{S}_8$ of order eight, such that the interval $[\mathsf{Var}\langle \mathcal{S}_8, {}^* \rangle, \mathsf{Var}\langle \mathcal{S}_8, {}^{\circledast} \rangle]$ contains an uncountable chain.*

It follows that the variety $\mathsf{Var}\langle \mathcal{S}_8, {}^{\circledast} \rangle$ has uncountably many subvarieties. In contrast, the variety $\mathsf{Var}\langle \mathcal{S}_8, {}^* \rangle$ has only four subvarieties.

Since there exist only countably many finitely generated varieties, uncountably many varieties in the interval $[\mathsf{Var}\langle \mathcal{S}_8, {}^* \rangle, \mathsf{Var}\langle \mathcal{S}_8, {}^{\circledast} \rangle]$ are non-finitely generated.

Corollary 3. *There exist uncountably many non-finitely generated varieties of involution semigroups whose semigroup variety reduct coincides with the finitely generated variety* $\mathsf{Var}\,\mathcal{S}_8$.

Background information is first given in Section 2. A sufficient condition is then established in Section 3 under which an interval in $\mathscr{L}_{\mathsf{inv}}$ contains an uncountable chain. Using this condition, the involution semigroups $\langle \mathcal{S}_8, {}^* \rangle$ and $\langle \mathcal{S}_8, {}^{\circledast} \rangle$ in Theorem 2 are constructed in Section 4.

More details on the differences between the lattices $\mathscr{L}_{\mathsf{inv}}$ and $\mathscr{L}_{\mathsf{sem}}$ can be found in [**15**, Section 1.5].

2. Preliminaries

Acquaintance with rudiments of universal algebra is assumed. Refer to the monograph of Burris and Sankappanavar [**1**] for more information.

2.1. Words and terms. Let $\mathscr{X}$ be a countably infinite alphabet and $\mathscr{X}^* = \{x^* \mid x \in \mathscr{X}\}$ be its disjoint copy. Elements of $\mathscr{X} \cup \mathscr{X}^*$ are called *variables*. The *free involution monoid* over $\mathscr{X}$ is the free semigroup $(\mathscr{X} \cup \mathscr{X}^*)^+$, together with the empty word $\varnothing$, with unary operation * given by $(x^*)^* = x$ for all $x \in \mathscr{X}$,

$$(x_1 x_2 \cdots x_m)^* = x_m^* x_{m-1}^* \cdots x_1^*$$

for all $x_1, x_2, \ldots, x_m \in \mathscr{X} \cup \mathscr{X}^* \cup \{\varnothing\}$, and $\varnothing^* = \varnothing$. Elements of the involution monoid $(\mathscr{X} \cup \mathscr{X}^*)^+ \cup \{\varnothing\}$ are called *words*, while words in the monoid $\mathscr{X}^+ \cup \{\varnothing\}$ are said to be *plain*.

The set of *terms* over $\mathscr{X}$, denoted by $\mathsf{T}(\mathscr{X})$, is the smallest set such that $\mathscr{X} \cup \{\varnothing\} \subseteq \mathsf{T}(\mathscr{X})$; if $\mathbf{t}_1, \mathbf{t}_2 \in \mathsf{T}(\mathscr{X})$, then $\mathbf{t}_1\mathbf{t}_2 \in \mathsf{T}(\mathscr{X})$; and if $\mathbf{t} \in \mathsf{T}(\mathscr{X})$, then $\mathbf{t}^* \in \mathsf{T}(\mathscr{X})$. The *subterms* of a term $\mathbf{t}$ are then recursively defined as follows: $\mathbf{t}$ is a subterm of $\mathbf{t}$; if $\mathbf{s}_1\mathbf{s}_2$ is a subterm of $\mathbf{t}$ where $\mathbf{s}_1, \mathbf{s}_2 \in \mathsf{T}(\mathscr{X})$, then so are $\mathbf{s}_1$ and $\mathbf{s}_2$; if $\mathbf{s}^*$ is a subterm of $\mathbf{t}$ where $\mathbf{s} \in \mathsf{T}(\mathscr{X})$, then so is $\mathbf{s}$. The proper inclusion $(\mathscr{X} \cup \mathscr{X}^*)^+ \subset \mathsf{T}(\mathscr{X})$ holds and the involution axioms (1) can be used to convert any term $\mathbf{t} \in \mathsf{T}(\mathscr{X})\backslash\{\varnothing\}$ into a unique word $\lfloor \mathbf{t} \rfloor \in (\mathscr{X} \cup \mathscr{X}^*)^+$. For instance, $\lfloor x(x^3(yx^*)^*)^* zy^* \rfloor = xy(x^*)^4 zy^*$.

Remark 4. For any subterm $\mathbf{s}$ of a term $\mathbf{t}$, either $\lfloor \mathbf{s} \rfloor$ or $\lfloor \mathbf{s}^* \rfloor$ is a factor of $\lfloor \mathbf{t} \rfloor$.

2.2. Equations, deducibility, and satisfiability. An *equation* is an expression $\mathbf{s} \approx \mathbf{t}$ formed by terms $\mathbf{s}, \mathbf{t} \in \mathsf{T}(\mathscr{X})\backslash\{\varnothing\}$. Specifically, a *word equation* is an equation $\mathbf{u} \approx \mathbf{v}$ formed by words $\mathbf{u}, \mathbf{v} \in (\mathscr{X} \cup \mathscr{X}^*)^+$ and a *plain equation* is an equation $\mathbf{u} \approx \mathbf{v}$ formed by plain words $\mathbf{u}, \mathbf{v} \in \mathscr{X}^+$.

An equation $\mathbf{s} \approx \mathbf{t}$ is *directly deducible* from an equation $\mathbf{u}_1 \approx \mathbf{u}_2$ if there exist a substitution $\varphi\colon \mathscr{X} \to \mathsf{T}(\mathscr{X})\backslash\{\varnothing\}$ and distinct $i, j \in \{1, 2\}$ such that $\varphi\mathbf{u}_i$ is a subterm of $\mathbf{s}$, and replacing this particular subterm $\varphi\mathbf{u}_i$ of $\mathbf{s}$ with $\varphi\mathbf{u}_j$ results in the term $\mathbf{t}$. An equation $\mathbf{s} \approx \mathbf{t}$ is *deducible* from a set Σ of equations if there exists a finite sequence

$$\mathbf{s} = \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r = \mathbf{t}$$

of distinct terms such that each equation $\mathbf{t}_i \approx \mathbf{t}_{i+1}$ is directly deducible from some equation in Σ.

An involution semigroup $\langle \mathcal{S}, {}^* \rangle$ *satisfies* an equation $\mathbf{s} \approx \mathbf{t}$, or $\mathbf{s} \approx \mathbf{t}$ is *satisfied* by $\langle \mathcal{S}, {}^* \rangle$ if, for any substitution $\varphi\colon \mathscr{X} \to \mathcal{S}$, the elements $\varphi\mathbf{s}$ and $\varphi\mathbf{t}$ of $\mathcal{S}$ coincide. Satisfaction of equations by semigroups is similarly defined. Note that equations of semigroups are necessarily plain. A class of (involution) semigroups *satisfies* an equation if every (involution) semigroup in it satisfies the equation.

2.3. Equational theories and bases. For any class $\mathfrak{K}$ of involution semigroups, the set of equations satisfied by every involution semigroup in $\mathfrak{K}$, denoted by $\mathsf{Eq}\,\mathfrak{K}$, is the *equational theory* of $\mathfrak{K}$. The set of word equations in $\mathsf{Eq}\,\mathfrak{K}$ is denoted by $\mathsf{Eq_W}\,\mathfrak{K}$. An *equational basis* for $\mathfrak{K}$ is any subset Σ of $\mathsf{Eq}\,\mathfrak{K}$ such that every equation in $\mathsf{Eq}\,\mathfrak{K}$ is deducible from Σ.

3. Sufficient condition for continuum of subvarieties

A term $\mathbf{u}$ *divides* a term $\mathbf{s}$ if some substitution $\varphi\colon \mathscr{X} \to \mathsf{T}(\mathscr{X})\backslash\{\varnothing\}$ exists such that $\varphi\mathbf{u}$ is a subterm of $\mathbf{s}$. It follows that a non-trivial equation $\mathbf{s} \approx \mathbf{t}$ cannot be directly deducible from an equation $\mathbf{u}_1 \approx \mathbf{u}_2$ if neither $\mathbf{u}_1$ nor $\mathbf{u}_2$ divides $\mathbf{s}$.

For any set $\mathscr{P} \subseteq (\mathscr{X} \cup \mathscr{X}^*)^+$ and any words $\mathbf{p}, \mathbf{q} \in (\mathscr{X} \cup \mathscr{X}^*)^+$, the triple $(\mathscr{P}; \mathbf{p}, \mathbf{q})$ is an *inside-outside triple* if $\mathbf{p} \in \mathscr{P}$ and $\mathbf{q} \notin \mathscr{P}$. A system $\mathfrak{S} = \{(\mathscr{P}_i; \mathbf{p}_i, \mathbf{q}_i) \mid i \in I\}$ of inside-outside triples is *independent* if for any distinct $j, k \in I$ neither $\mathbf{p}_j$ nor $\mathbf{q}_j$ divides any term $\mathbf{t}$ with $\lfloor \mathbf{t} \rfloor \in \mathscr{P}_k$.

Theorem 5. *Let $\mathfrak{S} = \{(\mathscr{P}_i; \mathbf{p}_i, \mathbf{q}_i) \mid i \in I\}$ be any countably infinite independent system of inside-outside triples. Suppose that $\mathbf{V}$ is any variety of involution semigroups such that for each $i \in I$ the following implication holds for any word $\mathbf{w} \in (\mathscr{X} \cup \mathscr{X}^*)^+$:*

$$\mathbf{p}_i \approx \mathbf{w} \in \mathsf{Eq_W}\,\mathbf{V} \implies \mathbf{w} \in \mathscr{P}_i. \tag{2}$$

Then $\mathbf{V}$ contains an uncountable chain of subvarieties. Further, if some subvariety $\mathbf{U}$ of $\mathbf{V}$ satisfies the equations $\{\mathbf{p}_i \approx \mathbf{q}_i \mid i \in I\}$, then the uncountable chain is contained in the interval $[\mathbf{U}, \mathbf{V}]$.

Proof: For any set $N \subseteq I$, let $\mathbf{V}_N$ denote the subvariety of $\mathbf{V}$ defined by $\{\mathbf{p}_i \approx \mathbf{q}_i \mid i \in N\}$. Since $\mathbf{q}_i \notin \mathscr{P}_i$ by the definition of an inside-outside triple, it follows from (2) that $\mathbf{p}_i \approx \mathbf{q}_i \notin \mathsf{Eq_W}\,\mathbf{V}$. In other words,

(a) $\mathbf{p}_i \approx \mathbf{q}_i \notin \mathsf{Eq}\,\mathbf{V}$ for all $i \in I$.

Therefore, $\mathbf{V} \neq \mathbf{V}_N$ for all $N \subseteq I$. Seeking a contradiction, suppose that $\mathbf{p}_m \approx \mathbf{q}_m \in \mathsf{Eq}\,\mathbf{V}_N$ for some $m \notin N$. Then there exists a finite sequence $\mathbf{p}_m = \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r = \mathbf{q}_m$ of distinct terms such that each equation $\mathbf{t}_j \approx \mathbf{t}_{j+1}$ is directly deducible from some equation in the equational basis $\mathsf{Eq}\,\mathbf{V} \cup \{\mathbf{p}_i \approx \mathbf{q}_i \mid i \in N\}$ for $\mathbf{V}_N$. If every equation $\mathbf{t}_j \approx \mathbf{t}_{j+1}$ is directly deducible from some equation in $\mathsf{Eq}\,\mathbf{V}$, then $\mathbf{p}_m \approx \mathbf{q}_m \in \mathsf{Eq}\,\mathbf{V}$ and (a) is contradicted. Therefore, some equation $\mathbf{t}_j \approx \mathbf{t}_{j+1}$ is directly deducible from some equation in $\{\mathbf{p}_i \approx \mathbf{q}_i \mid i \in N\}$. Let ℓ be the least index such that $\mathbf{t}_\ell \approx \mathbf{t}_{\ell+1}$ is directly deducible from $\mathbf{p}_n \approx \mathbf{q}_n$ for some $n \in N$. Then

(b) either $\mathbf{p}_n$ or $\mathbf{q}_n$ divides $\mathbf{t}_\ell$.

The minimality of ℓ implies that each equation in $\{\mathbf{t}_j \approx \mathbf{t}_{j+1} \mid 1 \leq j < \ell\}$ is directly deducible from some equation in $\mathsf{Eq}\,\mathbf{V}$, whence $\mathbf{t}_1 \approx \mathbf{t}_\ell \in \mathsf{Eq}\,\mathbf{V}$. But since $\lfloor \mathbf{t}_1 \rfloor \approx \lfloor \mathbf{t}_\ell \rfloor \in \mathsf{Eq}_\mathsf{W}\,\mathbf{V}$ with $\lfloor \mathbf{t}_1 \rfloor = \mathbf{p}_m$, it follows from (2) that

(c) $\lfloor \mathbf{t}_\ell \rfloor \in \mathscr{P}_m$.

Now $m \neq n$ because $m \notin N$ and $n \in N$. Therefore, the independence of $\mathfrak{S}$ is contradicted by (b) and (c).

Consequently, $\mathbf{p}_i \approx \mathbf{q}_i \in \mathsf{Eq}\,\mathbf{V}_N$ if and only if $i \in N$. It follows that $\mathbf{V}_N \supseteq \mathbf{V}_{N'}$ if and only if $N \subseteq N'$, whence the set $\{\mathbf{V}_N \mid N \subseteq I\}$ consists of uncountably many subvarieties of $\mathbf{V}$. A standard argument shows that this set—which is isomorphic to the dual of the power set of I—contains an uncountable chain. Indeed, select any bijection φ from I onto the rational numbers $\mathbb{Q}$ and, for any real number $r \in \mathbb{R}$, define $N(r) = \{i \in I \mid \varphi i < r\}$. Then $N(r) \subset N(r')$ if and only if $r < r'$; in other words, $\mathbf{V}_{N(r)} \supset \mathbf{V}_{N(r')}$ if and only if $r < r'$.

Now if some subvariety $\mathbf{U}$ of $\mathbf{V}$ satisfies the equations $\{\mathbf{p}_i \approx \mathbf{q}_i \mid i \in I\}$, then $\mathbf{U} \subseteq \mathbf{V}_N$ for all $N \subseteq I$. Therefore, the uncountable chain is contained in$[\mathbf{U}, \mathbf{V}]$. □

4. Construction of $\langle \mathcal{S}_8, {}^* \rangle$ and $\langle \mathcal{S}_8, {}^{\circledast} \rangle$

Let $\mathbb{N}$ denote the positive integers. For each $i \in \mathbb{N}$, define the word

$$\mathbf{p}_i = x_0 t_1 t_2 x_0 \cdot x_1 y_1 x_1 \cdot x_2 y_2 x_2 \cdots x_i y_i x_i \cdot x_{i+1} t_3 t_4 x_{i+1}.$$

Let $\mathbf{q}_i = x_0 x_0^* \mathbf{p}_i$ and $\mathscr{P}_i = \{\mathbf{p}_i\}$, so that $(\mathscr{P}_i; \mathbf{p}_i, \mathbf{q}_i)$ is an inside-outside triple.

Lemma 6. *The system* $\{(\mathscr{P}_i; \mathbf{p}_i, \mathbf{q}_i) \mid i \in \mathbb{N}\}$ *of inside-outside triples is independent.*

Proof: Consider any $j, k \in \mathbb{N}$ such that $j \neq k$. Since $\mathbf{q}_j$ contains the variable x_0 thrice but every variable occurs at most twice in $\mathbf{p}_k$, it is impossible for $\mathbf{q}_j$ to divide any term $\mathbf{t}$ with $\lfloor \mathbf{t} \rfloor \in \mathscr{P}_k = \{\mathbf{p}_k\}$.

Seeking a contradiction, suppose that $\mathbf{p}_j$ divides some term $\mathbf{t}$ with $\lfloor \mathbf{t} \rfloor \in \mathscr{P}_k = \{\mathbf{p}_k\}$. Then by Remark 4 there exists some substitution $\varphi\colon \mathscr{X} \to \mathsf{T}(\mathscr{X})\backslash\{\varnothing\}$ such that either $\lfloor \varphi \mathbf{p}_j \rfloor$ or $\lfloor (\varphi \mathbf{p}_j)^* \rfloor$ is a factor of $\lfloor \mathbf{t} \rfloor = \mathbf{p}_k$.

Case 1: $\lfloor \varphi \mathbf{p}_j \rfloor$ is a factor of $\mathbf{p}_k$. Note that

$$\lfloor \varphi \mathbf{p}_j \rfloor = \lfloor \varphi x_0 \rfloor \lfloor \varphi t_1 \rfloor \lfloor \varphi t_2 \rfloor \lfloor \varphi x_0 \rfloor \cdot \lfloor \varphi x_1 \rfloor \lfloor \varphi y_1 \rfloor \lfloor \varphi x_1 \rfloor \cdot \lfloor \varphi x_2 \rfloor \lfloor \varphi y_2 \rfloor \lfloor \varphi x_2 \rfloor \cdots$$
$$\cdots \lfloor \varphi x_j \rfloor \lfloor \varphi y_j \rfloor \lfloor \varphi x_j \rfloor \cdot \lfloor \varphi x_{j+1} \rfloor \lfloor \varphi t_3 \rfloor \lfloor \varphi t_4 \rfloor \lfloor \varphi x_{j+1} \rfloor$$

and

$$\mathbf{p}_k = x_0 t_1 t_2 x_0 \cdot x_1 y_1 x_1 \cdot x_2 y_2 x_2 \cdots x_k y_k x_k \cdot x_{k+1} t_3 t_4 x_{k+1}.$$

If one of the factors $\lfloor \varphi x_0 \rfloor, \lfloor \varphi x_1 \rfloor, \ldots, \lfloor \varphi x_{j+1} \rfloor$ of $\lfloor \varphi \mathbf{p}_j \rfloor$ is not a single variable, then the word $\lfloor \varphi \mathbf{p}_j \rfloor$ is of the form $\cdots z_1 z_2 \cdots z_1 z_2 \cdots$ for some $z_1, z_2 \in \mathscr{X} \cup \mathscr{X}^*$ and so cannot be a factor of $\mathbf{p}_k$. Therefore, every one of $\lfloor \varphi x_0 \rfloor, \lfloor \varphi x_1 \rfloor, \ldots, \lfloor \varphi x_{j+1} \rfloor$ is a single variable. Now since the prefix $\lfloor \varphi x_0 \rfloor \lfloor \varphi t_1 \rfloor \lfloor \varphi t_2 \rfloor \lfloor \varphi x_0 \rfloor$ and the suffix $\lfloor \varphi x_{j+1} \rfloor \lfloor \varphi t_3 \rfloor \lfloor \varphi t_4 \rfloor \lfloor \varphi x_{j+1} \rfloor$ of $\lfloor \varphi \mathbf{p}_j \rfloor$ are words of length at least four that begin and end with the same variable, they must coincide with the prefix $x_0 t_1 t_2 x_0$ and the suffix $x_{k+1} t_3 t_4 x_{k+1}$ of $\mathbf{p}_k$, respectively. It follows that

$$\lfloor \varphi x_1 \rfloor \lfloor \varphi y_1 \rfloor \lfloor \varphi x_1 \rfloor \cdot \lfloor \varphi x_2 \rfloor \lfloor \varphi y_2 \rfloor \lfloor \varphi x_2 \rfloor \cdots \lfloor \varphi x_j \rfloor \lfloor \varphi y_j \rfloor \lfloor \varphi x_j \rfloor = x_1 y_1 x_1 \cdot x_2 y_2 x_2 \cdots x_k y_k x_k,$$

which is impossible due to $j \neq k$.

Case 2: $\lfloor(\varphi\mathbf{p}_j)^*\rfloor$ is a factor of $\mathbf{p}_k$. Since $\lfloor(\varphi\mathbf{p}_j)^*\rfloor$ has the same form as $\lfloor\varphi\mathbf{p}_j\rfloor$, the argument in Case 1 can be repeated to show that the present case is also impossible. □

Construction of the involution semigroups $\langle \mathcal{S}_8, {}^*\rangle$ and $\langle \mathcal{S}_8, {}^{\circledast}\rangle$ requires the following matrix semigroups under the usual matrix multiplication: the Brandt monoid

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and its subsemilattice

$$\mathcal{S}\ell = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

These matrix semigroups are closed under both the usual transposition ^{T} and the *skew transposition* ^{S} across the secondary diagonal:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^S = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Note that $x^T = x$ for all $x \in \mathcal{S}\ell$.

Lemma 7. *Suppose that* $\mathbf{u} \approx \mathbf{v} \in \mathsf{Eq_W}\langle \mathcal{S}\ell, {}^S\rangle$. *Then the set of variables occurring in* $\mathbf{u}$ *coincides with the set of variables occurring in* $\mathbf{v}$.

Proof: Suppose that x is a variable that occurs in $\mathbf{u}$ but not in $\mathbf{v}$. Generality is not lost by assuming that $x \in \mathscr{X}$. Let $\varphi\colon \mathscr{X} \to \mathcal{S}\ell$ be the substitution that maps x to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and any other variable to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then

$$\varphi\mathbf{u} = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } x^* \text{ is in } \mathbf{u}, \\[2ex] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \text{otherwise;} \end{cases} \qquad \varphi\mathbf{v} = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } x^* \text{ is in } \mathbf{v}, \\[2ex] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{otherwise.} \end{cases}$$

Therefore, the contradiction $\varphi\mathbf{u} \neq \varphi\mathbf{v}$ is obtained. □

Lemma 8. *Let* $i \in \mathbb{N}$ *be odd. Suppose that* $\mathbf{p}_i \approx \mathbf{w} \in \mathsf{Eq}\,\mathcal{B}$ *for some* $\mathbf{w} \in \mathscr{X}^+$. *Then* $\mathbf{w} = \mathbf{p}_i$.

Proof: A plain word $\mathbf{u} \in \mathscr{X}^+$ is an *isoterm* for a semigroup $\mathcal{S}$ if the following implication holds for every word $\mathbf{w} \in \mathscr{X}^+$:

$$\mathbf{u} \approx \mathbf{w} \in \mathsf{Eq}\,\mathcal{S} \implies \mathbf{u} = \mathbf{w}.$$

Jackson ([**4**, proof of Theorem 4.1]) has shown that $\mathbf{p}_i$ is an isoterm for the semigroup $\mathcal{B}' \times \mathcal{B}''$, where $\mathcal{B}' = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right\}$ and $\mathcal{B}'' = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right\}$ are subsemigroups of $\mathcal{B}$. It follows that $\mathbf{p}_i$ is also an isoterm for $\mathcal{B}$. □

Lemma 9. *Let* $i \in \mathbb{N}$ *be odd. Suppose that* $\mathbf{p}_i \approx \mathbf{w} \in \mathsf{Eq_W}\{\langle \mathcal{B}, {}^T\rangle, \langle \mathcal{S}\ell, {}^S\rangle\}$ *for some* $\mathbf{w} \in (\mathscr{X} \cup \mathscr{X}^*)^+$. *Then* $\mathbf{w} = \mathbf{p}_i$.

Proof: Since $\mathbf{p}_i \approx \mathbf{w} \in \mathsf{Eq_W}\langle \mathcal{S}\ell, {}^S\rangle$ and the word $\mathbf{p}_i$ is plain, the word $\mathbf{w}$ is also plain by Lemma 7. Then $\mathbf{p}_i \approx \mathbf{w} \in \mathsf{Eq}\,\mathcal{B}$, so that $\mathbf{w} = \mathbf{p}_i$ by Lemma 8. □

Lemma 10. *The interval* $\left[\mathsf{Var}\langle \mathcal{B}, {}^T\rangle, \mathsf{Var}\{\langle \mathcal{B}, {}^T\rangle, \langle \mathcal{S}\ell, {}^S\rangle\}\right]$ *contains an uncountable chain.*

Proof: By Lemma 6, the system $\mathfrak{S} = \{(\mathscr{P}_i; \mathbf{p}_i, \mathbf{q}_i) \mid i = 1, 3, 5, \dots\}$ of inside-outside triples is independent. By Lemma 9, the implication (2) holds with $\mathbf{V} = \mathsf{Var}\{\langle \mathcal{B}, {}^T\rangle, \langle \mathcal{S}\ell, {}^S\rangle\}$. Therefore, by Theorem 5, the variety $\mathbf{V}$ contains an uncountable chain of subvarieties. Further, since the subvariety $\mathbf{U} = \mathsf{Var}\langle \mathcal{B}, {}^T\rangle$ of $\mathbf{V}$ satisfies the equation $xx^*x \approx x$ and so also the equation $\mathbf{p}_i \approx \mathbf{q}_i$ for any i, the uncountable chain is contained in $[\mathbf{U}, \mathbf{V}]$. □

Remark 11. The variety $\mathsf{Var}\langle \mathcal{B}, {}^T\rangle$ has only four subvarieties [**6**] while the variety $\mathsf{Var}\{\langle \mathcal{B}, {}^T\rangle, \langle \mathcal{S}\ell, {}^S\rangle\}$ has uncountably many subvarieties [**10**].

Define the direct products

$$\langle \mathcal{P}, {}^*\rangle = \langle \mathcal{B}, {}^T\rangle \times \langle \mathcal{S}\ell, {}^T\rangle \quad \text{and} \quad \langle \mathcal{P}, {}^{\circledast}\rangle = \langle \mathcal{B}, {}^T\rangle \times \langle \mathcal{S}\ell, {}^S\rangle;$$

in other words, $\mathcal{P} = \mathcal{B} \times \mathcal{S}\ell$ is an involution semigroup under any of the unary operations $(x, y)^* = (x^T, y^T)$ and $(x, y)^{\circledast} = (x^T, y^S)$.

Lemma 12. *The inclusion* $\mathsf{Var}\langle \mathcal{P}, {}^*\rangle \subseteq \mathsf{Var}\langle \mathcal{P}, {}^{\circledast}\rangle$ *holds and consequently the interval* $[\mathsf{Var}\langle \mathcal{P}, {}^*\rangle, \mathsf{Var}\langle \mathcal{P}, {}^{\circledast}\rangle]$ *contains an uncountable chain.*

Proof: Since $\langle \mathcal{S}\ell, {}^T\rangle$ is an involution subsemigroup of $\langle \mathcal{B}, {}^T\rangle$,

$$\mathsf{Var}\langle \mathcal{P}, {}^*\rangle = \mathsf{Var}\langle \mathcal{B}, {}^T\rangle \subseteq \mathsf{Var}\{\langle \mathcal{B}, {}^T\rangle, \langle \mathcal{S}\ell, {}^S\rangle\} = \mathsf{Var}\langle \mathcal{P}, {}^{\circledast}\rangle.$$

The result then follows from Lemma 10. □

Now let $\mathcal{S}_8$ be the subset of $\mathcal{P} = \mathcal{B} \times \mathcal{S}\ell$ consisting of the elements

$$\textsc{o} = (\mathbf{0}, \mathbf{0}), \qquad \textsc{a} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{0}\right), \quad \textsc{b} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{0}\right), \quad \textsc{c} = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{0}\right),$$

$$\textsc{d} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{0}\right), \quad \textsc{e} = \left(\mathbf{0}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), \quad \textsc{f} = \left(\mathbf{0}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right), \quad \textsc{i} = (\mathbf{1}, \mathbf{1}),$$

where $\mathbf{0} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\mathbf{1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then $\mathcal{S}_8$ is a subsemigroup of $\mathcal{P}$ that is closed under both of the unary operations * and $^{\circledast}$. It is clear that $\langle \mathcal{S}_8, {}^*\rangle$ is an amalgamation of

$$\langle \mathcal{B}, {}^T\rangle \cong \langle \{\textsc{o}, \textsc{a}, \textsc{b}, \textsc{c}, \textsc{d}, \textsc{i}\}, {}^*\rangle \quad \text{and} \quad \langle \mathcal{S}\ell, {}^T\rangle \cong \langle \{\textsc{o}, \textsc{e}, \textsc{f}, \textsc{i}\}, {}^*\rangle,$$

and that $\langle \mathcal{S}_8, {}^{\circledast}\rangle$ is an amalgamation of

$$\langle \mathcal{B}, {}^T\rangle \cong \langle \{\textsc{o}, \textsc{a}, \textsc{b}, \textsc{c}, \textsc{d}, \textsc{i}\}, {}^{\circledast}\rangle \quad \text{and} \quad \langle \mathcal{S}\ell, {}^S\rangle \cong \langle \{\textsc{o}, \textsc{e}, \textsc{f}, \textsc{i}\}, {}^{\circledast}\rangle.$$

It follows that $\mathsf{Var}\langle \mathcal{S}_8, {}^*\rangle = \mathsf{Var}\langle \mathcal{P}, {}^*\rangle$ and $\mathsf{Var}\langle \mathcal{S}_8, {}^{\circledast}\rangle = \mathsf{Var}\langle \mathcal{P}, {}^{\circledast}\rangle$. Theorem 2 is thus a consequence of Lemma 12.

Remark 13. Since the variety $\mathsf{Var}\langle \mathcal{B}, {}^S\rangle$ has uncountably many subvarieties [**10**], it is natural to question whether or not Lemma 12 is also true if the varieties $\mathsf{Var}\langle \mathcal{P}, {}^*\rangle$ and $\mathsf{Var}\langle \mathcal{P}, {}^{\circledast}\rangle$ are replaced by $\mathsf{Var}\langle \mathcal{B}, {}^T\rangle$ and $\mathsf{Var}\langle \mathcal{B}, {}^S\rangle$. But since $\langle \mathcal{B}, {}^T\rangle$ does not satisfy the equation $xx^* \approx x^*x$ of $\langle \mathcal{B}, {}^S\rangle$ while $\langle \mathcal{B}, {}^S\rangle$ does not satisfy the equation $xx^*x \approx x$ of $\langle \mathcal{B}, {}^T\rangle$, the varieties $\mathsf{Var}\langle \mathcal{B}, {}^T\rangle$ and $\mathsf{Var}\langle \mathcal{B}, {}^S\rangle$ exclude one another, and certainly do not bound a non-empty interval.

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