Publ. Mat. **69** (2025), 195–216
DOI: 10.5565/PUBLMAT6912509

REVISITING THE MARCINKIEWICZ THEOREM FOR NON-COMMUTATIVE MAXIMAL FUNCTIONS

Léonard Cadilhac and Éric Ricard

Abstract: We give an alternative proof of a Marcinkiewicz interpolation theorem for non-commutative maximal functions and positive maps and refine earlier versions of the statement. The main novelty is that it provides a substitute for the maximal function of a martingale in L_p, $1 < p \leqslant \infty$, losing very little on numerical constants. For non-positive maps, the above mentioned theorem fails but we can still obtain some interpolation results by weakening the maximal norm that we consider.

2020 Mathematics Subject Classification: 46L51, 46B70, 46L52.

Key words: non-commutative integration, real interpolation, maximal inequalities, weak type inequalities.

1. Introduction

In classical analysis, a maximal function M is simply the supremum of a given family of functions $(f_i)_{i\in I}$,

$$M := \sup_{i\in I} |f_i|.$$

Obtaining bounds for certain meaningful maximal functions is a fundamental theme in harmonic analysis, with a variety of applications ranging from Lebesgue's differentiation theorem to ergodic theory and the convergence of Fourier series. When trying to adapt this notion to non-commutative analysis, where functions are replaced by operators, an immediate difficulty appears: in general, a family of operators does not admit a supremum. Nonetheless, alternative ways to formulate maximal inequalities exist and can be traced back at least to the 70's and the early days of non-commutative martingale and ergodic theory ([**6**], [**15**], [**21**]). A systematic approach to non-commutative maximal functions was initiated two decades ago in works of Pisier [**19**] and Junge [**11**] and has been widely adopted since.

In this paper, we revisit a result of Junge and Xu [**13**] that extends Marcinkiewicz interpolation in the following way: let $p_0 < p < p_1$; if a non-commutative positive maximal operator is weakly bounded on L_{p_0} and L_{p_1}, then it is strongly bounded on L_p. Interpolation generally transfers well from classical to non-commutative function spaces so this may appear to be a routine result but – details will be given shortly – weak and strong types have here been defined in different ways and independently (weak type in [**6**] and [**21**] and strong type in [**19**]), making this theorem, if not surprising, quite notable. Our first goal is to present a new, very simple proof of this theorem and then to explore its potential extensions to operators that are not necessarily positive. A motivation discussed along the way is to compare different definitions of maximal norms and how real interpolation might apply to them. Note that simplifications and refinements of Junge and Xu's original result can already be found in [**1**], [**8**]. In particular, Junge and Xu always assume strong type (p_1, p_1) and the result at the beginning of the paragraph was only proved later by Dirksen [**8**].

Let us now recall the definition of maximal norms as introduced by Pisier and Junge. Let $(\mathcal{N}, \tau)$ be a von Neumann algebra equipped with a normal semifinite faithful trace. For $0 < p \leqslant \infty$, the space $L_p(\mathcal{N}; \ell_\infty)$ consists of all sequences of operators $x = (x_n)_{n \geqslant 0}$ for which there exist operators $a, b \in L_{2p}(\mathcal{N})$ and contractions $(u_n)_{n \geqslant 0}$ in $\mathcal{N}$ such that

$$\forall n \geqslant 1, \quad x_n = a u_n b. \tag{1.1}$$

We set

$$\|x\|_{L_p(\mathcal{N}; \ell_\infty)} = \inf \|a\|_{2p} \|b\|_{2p},$$

where the infimum runs over all decompositions as above. Then $L_p(\mathcal{N}; \ell_\infty)$ is a Banach space for $p \geqslant 1$ and a quasi-Banach for $p < 1$. We refer to [**11**] for further details. When x consists of positive operators, we have a simpler description

$$\|x\|_{L_p(\mathcal{N}; \ell_\infty)} = \inf\{\|a\|_p \mid \forall n \geqslant 0,\, 0 \leqslant x_n \leqslant a\}.$$

This explains why $\|x\|_{L_p(\mathcal{N}; \ell_\infty)}$ plays the role of the norm of the maximal function. It is proved in [**12**] that this infimum is achieved but it may depend on p if x belongs to $L_p(\mathcal{N}; \ell_\infty)$ for multiple values of p. Advantages of these definitions include: a good behaviour with respect to complex interpolation [**11**], [**20**], a nice duality theory, and the ability to carry out in the non-commutative setting several classical applications of maximal functions. A drawback is that these spaces do not form a real interpolation scale [**14**].

We will also need a weak version that is commonly used in non-commutative analysis. The space $\Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)$ consists of all sequences of operators $x = (x_n)_{n \geqslant 0}$ for which the following quantity is finite:

$$\|x\|_{\Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)} = \sup_{\lambda > 0} \lambda \inf_e \{(\tau(1-e))^{1/p} \mid \forall n \geqslant 0,\, \|e x_n e\| \leqslant \lambda\},$$

where the infimum runs over all self-adjoint projections in $\mathcal{N}$. Thus, if $x \in \Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)$ with $\|x\|_{\Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)} < C$, then for all $\lambda > 0$, there is a projection $e \in \mathcal{N}$ with

$$\tau(1-e) \leqslant \frac{C^p}{\lambda^p}, \quad \forall n \geqslant 0,\, \|e x_n e\| \leqslant \lambda. \tag{1.2}$$

The space $\Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)$ is a quasi-Banach space.

At this stage, we point out that it does not match a more natural definition as for commutative measure spaces commonly denoted by $L_{p,\infty}(\mathcal{N}; \ell_\infty)$. If x consists of positive operators, $x \in \Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)$ does not imply that there exists $a \in L_{p,\infty}(\mathcal{N})$ such that $0 \leqslant x_n \leqslant a$. This requirement would provide better constants for interpolation (see Remark 3.9) but is too strong in practice to formulate Doob's weak type inequality for non-commutative martingales or a weak type inequality for ergodic theory. In the following, $\mathcal{M}$ denotes another von Neumann algebra equipped with an n.s.f. trace.

Definition 1.1. Let $S = (S_n)_{n \geqslant 0}$, where $S_n \colon L_p(\mathcal{M}) \to L_0(\mathcal{N})$ is a sequence of maps

- S is said to be of strong type (p,p) with constant C if S is bounded from $L_p(\mathcal{M})$ to $L_p(\mathcal{N}; \ell_\infty)$ with constant C.
- S is said to be of weak type (p,p) if S is bounded from $L_p(\mathcal{M})$ to $\Lambda_{p,\infty}(\mathcal{N}; \ell_\infty)$.

  More precisely, we say that S is of weak type (p,p) with constant C, if for any $x \in L_p(\mathcal{M})$ and $\lambda > 0$ there exists a projection $e \in \mathcal{N}$ such that

$$\tau(1-e) \leqslant \frac{C^p \|x\|_p^p}{\lambda^p}, \quad \forall n \geqslant 0,\, \|e S_n(x) e\| \leqslant \lambda. \tag{1.3}$$

Note that when $p=\infty$, the weak and strong types are equivalent and simply mean that the family (S_n) is uniformly bounded from $\mathcal{M}$ to $\mathcal{N}$.

The family (S_n) is of weak type (p,p) with constant C iff $(S_n)\colon L_p(\mathcal{M})\to\Lambda_{p,\infty}(\mathcal{N};\ell_\infty)$ with norm C.

We are now able to state the Marcinkiewicz interpolation theorem for non-commutative maximal functions.

Theorem 1.2 (Junge, Xu, Dirksen)**.** *Let $1\leqslant p_0<p_1\leqslant\infty$. Let $S=(S_n)_{n\geqslant 0}$ be a sequence of positive linear maps from $L_{p_0}(\mathcal{M})+L_{p_1}(\mathcal{M})$ to $L_0(\mathcal{N})$. Assume that S is of weak type (p_0,p_0) with constant C_0 and of weak type (p_1,p_1) with constant C_1. Let $\theta\in(0,1)$ and p be determined by $1/p=(1-\theta)/p_0+\theta/p_1$. Then S is of strong type (p,p) with constant less than*

$$CC_0^{1-\theta}C_1^{\theta}\alpha^2_{\theta,p_0,p_1},$$

where C is a universal constant and

$$\alpha_{\theta,p_0,p_1}=\left(\frac{1}{p}-\frac{1}{p_0}\right)^{-1}+\left(\frac{1}{p_1}-\frac{1}{p}\right)^{-1}.$$

Compared to earlier approaches ([**13**], [**1**], [**7**], [**8**]) ours has two main advantages: first, it identifies clearly the non-commutative part of the proof, which is reduced to a single lemma (Lemma 3.1), the rest of the argument consisting of manipulations of singular values; second, it allows us to construct an element in $L_p(\mathcal{N})$ playing the role of a maximal function. Keeping the notations of Theorem 1.2, we have

Theorem 1.3. *Let $x\in L_{p_0}(\mathcal{M})\cap L_{p_1}(\mathcal{M})$. Then, there exists $a\in(L_1+L_\infty)(\mathcal{N})^+$ and contractions $(u_n)_{n\geqslant 0}$ such that*

$$\forall n\geqslant 0,\quad S_n(x)=au_na.$$

Moreover, for any $p\in(p_0,p_1)$

$$\|a\|^2_{2p}\leqslant CC_0^{1-\theta}C_1^{\theta}\alpha^2_{\theta,p_0,p_1}(\ln\alpha_{\theta,p_0,p_1})^2,$$

where p and θ still verify $1/p=(1-\theta)/p_0+\theta/p_1$.

As previously mentioned, in general, the minimizing factorization $(x_n)=(ay_nb)$ in (1.1) may depend on p. This is an important difference from the commutative case and can be considered to be at the root of the failure of real interpolation for non-commutative maximal norms. We show that in this particular case, if we are willing to lose an exponent in the constants, this phenomenon cannot occur and a concrete maximal function can be exhibited. In particular, this applies to $(S_n)_{n\geqslant 0}$ a family of conditional expectation to define a maximal function for non-commutative martingales. We state the results for sequences of maps for convenience; they hold as well for general families $S=(S_i)_{i\in I}$.

In the rest of the paper, we explore variants and limitations of this theorem. First, in the remainder of Section 3, we show that the method of proof extends to some asymmetric versions of the maximal norm, already considered in [**11**], [**9**], and [**20**]. For those, the range of values of p for which the theorem applies has to be restricted (see Theorem 3.13 and Proposition 3.14).

We begin Section 4 by noting that without the positivity hypothesis on the maps (S_n), the theorem fails even with $\mathcal{M}=\mathbb{C}$. We introduce weaker maximal quasi-norms taking their inspiration in the definition of weak type rather than strong type. We show that they form a real interpolation scale and do give rise to Banach spaces for some parameters. They seem to be the most well-adapted kind of maximal inequalities beyond the positive case. They can be useful if one's goal is only to study questions of pointwise convergence in tracial von Neumann algebras. We also show that one of them exactly corresponds to the real interpolated spaces for the couple $(L_p(\mathcal{M};\ell_\infty^c), L_\infty(\mathcal{M};\ell_\infty^c))$ of one-sided maximal function spaces. We conclude with some basic bounds related to asymmetric factorizations of the form:

$$x_n = ay_n + z_nb \quad \text{with} \quad (y_n),(z_n) \in L_\infty(\mathcal{M};\ell_\infty) \text{ and } a,b \in L_p(\mathcal{M}).$$

This is close in spirit to what Junge and Xu did in [**13**] for symmetric factorizations. Unfortunately, one cannot improve them using the machinery of interpolation as the real method fails for $(L_p(\mathcal{M};\ell_\infty^c))_{p\geqslant 1}$.

2. Preliminaries

We assume that the reader is familiar with non-commutative integration. This section briefly recalls some standard notations and definitions (see also [**19**]), and presents a simple decomposition lemma.

2.1. Non-commutative integration. We will denote by $(\mathcal{M},\tau_\mathcal{M})$ or $(\mathcal{N},\tau_\mathcal{N})$ non-commutative measure spaces, meaning that $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras and $\tau_\mathcal{M}$ (resp. $\tau_\mathcal{N}$) is a normal semifinite faithful trace on $\mathcal{M}$ (resp. on $\mathcal{N}$). In practice there will be no ambiguity on the trace used and we will write τ instead of $\tau_\mathcal{N}$ or $\tau_\mathcal{M}$. The space of τ-measurable operators affiliated with $\mathcal{M}$ is denoted by $L_0(\mathcal{M})$ and the non-commutative L_p-spaces associated with $(\mathcal{M},\tau)$ by $L_p(\mathcal{M})$, $p\in(0,\infty]$. For $x\in L_0(\mathcal{M})$, $\mu(x)$ designates the singular value function of x given by, for $t>0$,

$$\mu_t(x) = \inf_{\tau(1-e)\leqslant t} \|xe\|,$$

where the infimum runs over projections $e\in\mathcal{M}$. The first step of the main proofs of this paper is to obtain estimates for finite projections. The following simple lemma is essential to extend those estimates to more general operators.

Lemma 2.1. *Let $p\in(0,\infty)$ and $x\in L_p^+(\mathcal{M})$. There are finite projections $(r_n)_{n\in\mathbb{Z}}\in\mathcal{M}$ so that $x=\sum_{n\in\mathbb{Z}}2^{-n}r_n\in L_p(\mathcal{M})$ and for all $\alpha>0$*

$$\sum_{n\in\mathbb{Z}} 2^{-n\alpha}1_{[0,\tau(r_n)]} \leqslant \frac{1}{1-2^{-\alpha}}\mu(x^\alpha) \quad \textit{and} \quad \sum_{n\in\mathbb{Z}} (|n|+1)2^{-n}1_{[0,\tau(r_n)]} \leqslant C\mu(x(|\ln(x)|+1)).$$

Proof: Actually, this is a commutative result. Let φ_n be the indicator function on $\mathbb{R}$ of the set of reals whose $-n$th digit in base 2 is 1. Clearly for all $t\geqslant 0$, $t=\sum_{n\in\mathbb{Z}}2^{-n}\varphi_n(t)$. Set $r_n=\varphi_n(x)$, which is a finite projection as $x\geqslant 2^{-n}r_n$, thus $\tau(r_n)\leqslant 2^{np}\|x\|_p^p$. By Lebesgue's dominated convergence theorem $x=\sum_{n\in\mathbb{Z}}2^{-n}r_n$ holds in $L_p(\mathcal{M})$.

Let $s>0$. If for all $k\in\mathbb{Z}$, $\tau(r_k)<s$, then $\sum_{k\in\mathbb{Z}}2^{-k\alpha}1_{[0,\tau(r_k)]}(s)=0$ and there is nothing to prove. Otherwise set $k_s=\min\{k \mid \tau(r_k)\geqslant s\}$, which is well defined as $\tau(r_k)\leqslant 2^{kp}\|x\|_p^p$. Then $\sum_{k\in\mathbb{Z}}2^{-k\alpha}1_{[0,\tau(r_k)]}(s)=\sum_{k\geqslant k_s}2^{-k\alpha}=\frac{2^{-\alpha k_s}}{1-2^{-\alpha}}$. But $x^\alpha\geqslant 2^{-k_s\alpha}r_{k_s}$ (recall that they commute), thus $\mu(x^\alpha)(s)\geqslant 2^{-k_s\alpha}\mu(r_{k_s})(s)=2^{-k_s\alpha}$. For the second inequality, follow the same proof and note that $\sum_{k\geqslant k_s}(|k|+1)2^{-k}\leqslant(|k_s|+1)2^{-k_s}\sum_{k\geqslant 0}(k+1)2^{-k}$. □

Remark 2.2. Note that Lemma 2.1 still holds for any $x\in L_0(\mathcal{M})^+$ but in general the sum $\sum_{n\in\mathbb{Z}}2^{-n}r_n$ only converges in $L_0(\mathcal{M})$.

2.2. Hardy–Littlewood majorization. Let f and g be two non-increasing functions from $(0,\infty)$ to $\mathbb{R}^+$. We say that g *majorizes* f and write $f \preceq g$ if

$$\forall t > 0, \quad \int_0^t f \leqslant \int_0^t g.$$

We will use the following properties. First, let $p \in [1,\infty]$ and $x, y \in L_p(\mathcal{M})$, then

$$\mu(x) \preceq \mu(y) \Longrightarrow \|x\|_p \leqslant \|y\|_p.$$

Second, assume that $x = \sum_{n\in\mathbb{Z}} x_n$, where the sum converges in $L_p(\mathcal{M})$. Then

$$\mu(x) \preceq \sum_{n\in\mathbb{Z}} \mu(x_n). \tag{2.1}$$

One way to justify this inequality is to note that, for any $t > 0$, $x \mapsto \int_0^t \mu(x) = \|x\|_{L_1+tL_\infty}$ is a norm and use the triangle inequality.

3. Interpolation for positive maps

Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra equipped with an n.s.f. trace.

3.1. Majorization and factorization. We start with two easy lemmas that capture the non-commutative aspects of the proof of our main theorem.

Lemma 3.1. *Let $N \in \mathbb{N}$ and $(q_k)_{|k|\leqslant N}$ be a sequence of disjoint projections in $\mathcal{M}$ with $e = \sum_{|k|\leqslant N} q_k$. Let $(d_k)_{|k|\leqslant N}$ be a sequence of strictly positive reals. Then for any $x \in L_0(\mathcal{M},\tau)^+$:*

$$0 \leqslant exe \leqslant \Big(\sum_{|k|\leqslant N} 1/d_k\Big) \sum_{|k|\leqslant N} d_k q_k x q_k.$$

Proof: The matrix $(1/\sqrt{d_i d_j})_{-N\leqslant i,j\leqslant N}$ corresponds to a rank-one positive operator with norm $C = \sum_{|k|\leqslant N} 1/d_k$. Thus the matrix $(C\delta_{i,j} - 1/\sqrt{d_i d_j})_{-N\leqslant i,j\leqslant N}$ is positive, hence conjugating by $(\sqrt{d_i}\delta_{i,j})$, $(Cd_i\delta_{i,j} - 1)_{-N\leqslant i,j\leqslant N}$ also is. We can find $T \in \mathbb{N}$ and families of complex numbers $(a_{i,t})_{|i|\leqslant N, t\leqslant T}$ such that $Cd_i\delta_{i,j} - 1 = \sum_{t=1}^T \overline{a_{i,t}} a_{j,t}$. Hence

$$C\sum_{|k|\leqslant N} d_k q_k x q_k - exe = \sum_{t=1}^T \Big(\sum_{|i|\leqslant N} a_{i,t} q_i\Big)^* x \Big(\sum_{|j|\leqslant N} a_{j,t} q_i\Big) \geqslant 0. \qquad \square$$

The following is standard by using polar decompositions in $\mathcal{M}\overline{\otimes}B(\ell_2(\mathbb{Z}))$:

Lemma 3.2. *Let $(a_i)_{i\in\mathbb{Z}}$ and $(b_i)_{i\in\mathbb{Z}}$ be sequences of elements in L_r and L_s such that $\sum_{i\in\mathbb{Z}} a_i a_i^* \in L_{r/2}$ and $\sum_{i\in\mathbb{Z}} b_i^* b_i \in L_{s/2}$ with $r, s > 0$. For any sequence of contractions $(u_i)_{i\in\mathbb{Z}}$ in $\mathcal{M}$, there is a contraction $u \in \mathcal{M}$ such that, in $L_{rs/(r+s)}$,*

$$\sum_{i\in\mathbb{Z}} a_i u_i b_i = \Big(\sum_{i\in\mathbb{Z}} a_i a_i^*\Big)^{1/2} u \Big(\sum_{i\in\mathbb{Z}} b_i^* b_i\Big)^{1/2}.$$

3.2. The Marcinkiewicz theorem. The first version of the Marcinkiewicz interpolation for maximal functions is given by [**13**, Theorem 3.1]: assuming weak type (p_0, p_0) and strong type (p_1, p_1), the authors obtain strong type (p, p) for $p_1 < p < p_0$. This has been extended in several directions in [**7, 1**]. A very satisfactory statement was obtained by Dirksen in [**8**]. Our approach is similar to Dirksen's, the novelties being in the second statement and in the simplicity of the proof, which is "almost commutative" apart from Lemmas 3.1 and 3.2. Actually, with a careful look at [**8**], fixing $p_0 < r < s < p_1$, one can find a depending only on r, s such that (3.1) hold for $r < p < s$.

Theorem 3.3. *Let* $1 \leqslant p_0 < p_1 \leqslant \infty$ *and* $S = (S_n)$ *be a sequence of linear positive maps from* $L_{p_0}(\mathcal{M}) + L_{p_1}(\mathcal{M})$ *to* $L_0(\mathcal{N})$. *Assume that* S *is of weak type* (p_0, p_0) *and* (p_1, p_1) *with constants* C_0 *and* C_1. *Let* $p \in (p_0, p_1)$ *and* $\theta \in (0, 1)$ *such that* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Set*

$$\alpha_{\theta,p_0,p_1} = \left(\frac{1}{p} - \frac{1}{p_0}\right)^{-1} + \left(\frac{1}{p_1} - \frac{1}{p}\right)^{-1}.$$

Then S *is of strong type* (p, p) *with constant*

$$C_{\theta,p_0,p_1} \leqslant C C_0^{1-\theta} C_1^{\theta} \alpha_{\theta,p_0,p_1}^2.$$

Moreover, there is a constant C'_{θ,p_0,p_1} *such that for any* $x \in L_{p_0}(\mathcal{M}) \cap L_{p_1}(\mathcal{M})$ *there exist* $a \in (L_1 + L_\infty)^+$ *and contractions* $u_n \in \mathcal{N}$ *independent of* p *or* θ *with*

$$S_n(x) = a u_n a, \quad \|a\|_{2p}^2 \leqslant C'_{\theta,p_0,p_1} \|x\|_p, \tag{3.1}$$

and

$$C'_{\theta,p_0,p_1} \leqslant C C_0^{1-\theta} C_1^{\theta} (\alpha_{\theta,p_0,p_1} \ln(\alpha_{\theta,p_0,p_1}))^2.$$

To prove Theorem 3.3, we proceed in two steps. First, in Lemma 3.4, assuming that x is a projection, we give a construction of a majoring element based on Lemma 3.1. Second, for a general x, we decompose x into a dyadic combination of projections $x = \sum 2^n x_n$, apply the first step to each projection and put them back together.

Scaling the trace, we can and will assume $C_0 = C_1$ and by homogeneity $C_0 = 1$.

Assume $p_0 < p < p_1$ fixed. Let us introduce notations that will play an important role in the proof of Theorem 3.3. Set $I = \mathbb{Z}$ if $p_1 \neq \infty$ and $I = \mathbb{Z}_{\geqslant 0}$ if $p_1 = \infty$. Let $(d_k)_{k \in I}$ be a sequence of positive real numbers such that:

$$C_d = \sum_{k \in I} \frac{1}{d_k} < \infty.$$

Set

$$\tilde{d}_k = 4 C_d d_k 2^{-k/p_0} \text{ if } k \geqslant 1 \text{ and } \tilde{d}_k = 4 C_d d_k 2^{-k/p_1} \text{ if } k \leqslant 0.$$

We also assume that

$$\sum_{k \in I} \tilde{d}_k 2^{k/p} < \infty. \tag{3.2}$$

For $s > 0$, define the dilation operator D_s by

$$\begin{aligned} D_s \colon L_0(\mathbb{R}^+) &\longrightarrow L_0(\mathbb{R}^+) \\ f &\longmapsto [t \mapsto f(t/s)]. \end{aligned}$$

The connection between dilation operators and Marcinkiewicz interpolation was made explicit by Boyd in [**3**]. A closely related way to approach interpolation of weak type inequalities was developed earlier by Calderón [**4**]. We found Boyd's formulation to be more convenient in this paper but the two are essentially equivalent (as shown for example in [**16**]).

Lemma 3.4. *Let $r \in \mathcal{M}$ be a finite projection. There exists an element $z \in L_p(\mathcal{N})$ (depending only on r and the choice of $(d_k)_{k\in\mathbb{Z}}$) such that*

$$\forall n \geqslant 0, \quad 0 \leqslant S_n(r) \leqslant z, \quad \mu(z) \preceq \sum_{k\in I} \tilde{d}_k D_{2^k}(\mu(r)).$$

Proof: Let $r \in \mathcal{M}$ with $\tau(r) = t$.

The map S is of weak type (p_0, p_0) with constant 1. Hence, using (1.3) for $\lambda = 2^{-(k-2)/p_0}$ and $k \geqslant 1$, we obtain projections $(e_k)_{k\geqslant 1}$ such that

$$\forall k \geqslant 1\, \forall n \geqslant 0, \quad \|e_k S_n(r) e_k\| \leqslant 2^{-(k-2)/p_0}, \quad \tau(1-e_k) \leqslant 2^{k-2}t.$$

Similarly the weak type (p_1, p_1) of S gives projections $(e_k)_{k\leqslant 0}$ such that

$$\forall k \leqslant 0\, \forall n \geqslant 0, \quad \|e_k S_n(r) e_k\| \leqslant 2^{-(k-2)/p_1}, \quad \tau(1-e_k) \leqslant 2^{k-2}t.$$

Considering $1-\bigvee_{i\leqslant k}(1-e_i)$ instead of e_i, we get a decreasing family of projections such that

$$\forall n \geqslant 0, \quad \|e_k S_n(r) e_k\| \leqslant 2^{2/p_0} 2^{-k/p_0} 1_{k>0} + 2^{2/p_1} 2^{-k/p_1} 1_{k\leqslant 0}, \quad \tau(1-e_k) \leqslant 2^{k-1}t.$$

Set $q_k = e_k - e_{k+1}$ and $q_\infty = \bigwedge_k e_k$, $E_N = \sum_{|k|\leqslant N} q_k$; we have $\sum_{k\in\mathbb{Z}} q_k = 1 - q_\infty$ and $\tau(q_k) \leqslant 2^k t$. Note that if $p_1 = \infty$, we can take $e_k = 1$ for $k \leqslant 0$ so that $q_k = 0$ for $k < 0$.

By Lemma 3.1, for all $n, N \geqslant 0$

$$\begin{aligned} 0 \leqslant E_N S_n(r) E_N &\leqslant C_d \sum_{|k|\leqslant N} d_k q_k S_n(r) q_k \\ &\leqslant 4C_d \left(\sum_{k=1}^{N} d_k 2^{-k/p_0} q_k + \sum_{k=0}^{N} d_{-k} 2^{k/p_1} q_{-k} \right) = z_N. \end{aligned}$$

By (3.2), the increasing sequence $(z_N)_{N\geqslant 1}$ converges in $L_p(\mathcal{N})$ (using that $\tau(q_k) \leqslant 2^k t$). Denote by $z \in L_p(\mathcal{N})$ its limit. Then, as $q_\infty S_n(r) q_\infty = 0$, we get

$$\forall n \geqslant 0, \quad 0 \leqslant S_n(r) \leqslant z = \sum_{k\in I} \tilde{d}_k q_k.$$

The inequality $\tau(q_k) \leqslant 2^k t$ yields $\mu(q_k) \leqslant D_{2^k}(1_{[0,t]})$. Hence by (2.1)

$$\mu(z) \preceq \sum_{k\in I} \tilde{d}_k D_{2^k}(\mu(r)),$$

with appropriate changes if $p_1 = \infty$. □

Proof of Theorem 3.3: Writing any element $x \in L_p(\mathcal{M})$ as a linear combination of four positive elements with coefficients i^k, each having norm less than $\|x\|_p$ and using Lemma 3.2, it suffices to consider only $x \geqslant 0$. This will only change the constant C.

Let $x \in L_p^+$; we use Lemma 2.1 to write $x = \sum_{m\in\mathbb{Z}} 2^{-m} r_m$. We apply Lemma 3.4 to each r_m to obtain some element $z_m \in L_p(\mathcal{M})$ with

$$\forall n \geqslant 0, \quad S_n(r_m) \leqslant z_m \quad \text{and} \quad \mu(z_m) \preceq \sum_{k\in I} \tilde{d}_k D_{2^k}(\mu(r_m)), \tag{3.3}$$

where $I = \mathbb{Z}$ if $p_1 < \infty$ and $I = \mathbb{Z}_{\geqslant 0}$ if $p_1 = \infty$. For any finite subset $J \subset \mathbb{Z}$,

$$\forall\, n \geqslant 0, \quad 0 \leqslant S_n\left(\sum_{m\in J} 2^{-m} r_m \right) \leqslant z_J := \sum_{m\in J} 2^{-m} z_m. \tag{3.4}$$

By (2.1),

$$\mu(z_J) \preceq \sum_{m\in J} 2^{-m} \mu(z_m) \preceq \sum_{m\in J} \sum_{k\in I} 2^{-m} \tilde{d}_k D_{2^k}(\mu(r_m)) \preceq \sum_{k\in I} \tilde{d}_k D_{2^k}\left(\sum_{m\in J} 2^{-m} \mu(r_m) \right). \tag{3.5}$$

By the construction of Lemma 2.1, we have $\sum_{m\in\mathbb{Z}} 2^{-m}\mu(r_m) \leqslant 2\mu(x)$. Thus the sum $\sum 2^{-m}\mu(r_m)$ converges in $L_p(0,\infty)$. Combined with (3.5) and (3.2), this implies that the sum $\sum 2^{-m}z_m$ converges in $L_p(\mathcal{N})$. Set $z = \sum_{m\in\mathbb{Z}} 2^{-m}z_m$; we have

$$\mu(z) \preceq \sum_{k\in I} \tilde{d}_k D_{2^k}\Big(\sum_{m\in\mathbb{Z}} 2^{-m}\mu(r_m)\Big) \leqslant 2\sum_{k\in I} \tilde{d}_k D_{2^k}(\mu(x)).$$

In particular,

$$\|z\|_p \leqslant 2\sum_{k\in I} \tilde{d}_k \|D_{2^k}(\mu(x))\|_p = 2\|x\|_p \sum_{k\in I} \tilde{d}_k 2^{k/p}. \tag{3.6}$$

Moreover, note that by real interpolation each S_n is continuous on L_p since it is of weak type (p_0,p_0) and (p_1,p_1), so

$$\forall n \geqslant 0,\quad S_n(x) = S_n\Big(\sum_{m\in\mathbb{Z}} 2^{-m}r_m\Big) \leqslant \sum_{m\in\mathbb{Z}} z_m = z.$$

To get the first statement, we choose $d_k = 2^{k(1/p_0-1/p)/2}$ for $k \geqslant 1$ and $d_k = 2^{k(1/p_1-1/p)/2}$ for $k\leqslant 0$. We get that $\tilde{d}_k 2^{k/p} = 4C_d/d_k$ for $k\in I$, where $C_d = \sum_{k\in I} 1/d_k$. Hence, computations in (3.6) give

$$\|z\|_p \leqslant 8\Big(\frac{1}{1-2^{(1/p-1/p_0)/2}} + \frac{2}{1-2^{(1/p_1-1/p)/2}}\Big)^2 \|x\|_p;$$

when $p_1 = \infty$, we get only 1 for the second term in the sum.

To prove the second statement we choose $d_k = |k|(\ln|k|)^2 + 1$. Thus, given $x \in L_{p_0}(\mathcal{M}) \cap L_{p_1}(\mathcal{M})$ thanks to (3.3), we can still define $z = \sum_{m\in\mathbb{Z}} 2^{-m}z_m$, which makes sense in $\bigcap_{p_0<p<p_1} L_p(\mathcal{N}) \subset L_{p_0}(\mathcal{N}) + L_{p_1}(\mathcal{N})$. It is independent of p and (3.6) becomes

$$\|z\|_p \leqslant C\|x\|_p \sum_{k\geqslant 0} (k(\ln|k|)^2+1)(2^{-k(1/p_0-1/p)} + 2^{-k(1/p-1/p_1)}). \tag{3.7}$$

Up to an absolute factor, $\sum_{k\geqslant 0}(k(\ln|k|)^2+1)2^{-ks}$, $s>0$ is controlled by

$$\int_2^\infty 2^{-ts} t|\ln t|^2\,dt \lesssim \frac{|\ln s|^2+1}{s^2}.$$

This yields the estimate. □

Remark 3.5. When $p_1 = \infty$, the constant $C_{\theta,p_0,\infty}$ actually remains bounded when $\theta \to 1$. One can also see from the proof that z and a are bounded when x is bounded. Indeed, by construction the family (z_m) is uniformly bounded by a constant C_{p_0}. But then in $x = \sum_{m\mathbb{Z}} 2^{-m}r_m$, r_m is 0 as soon as $2^{-m} > \|x\|$, and $z = \sum_{2^{-m}\leqslant\|x\|} 2^{-m}z_m$ is bounded by $2C_{p_0}$.

Remark 3.6. The proof only uses estimates for the norm of D_{2^k} on L_p. Thus, we may replace the space L_p in the arguments above with any symmetric function space $E \subset L_{p_0} + L_{p_1}$ with lower Boyd index p_0 and upper Boyd index $q_E < p_1$. This way, we can conclude that S is also bounded from $E(\mathcal{M})$ to $E(\mathcal{N};\ell_\infty)$. We refer to any of [**3, 7, 8, 16**] for details.

Remark 3.7. Both in the commutative case and in [**8**], it is also possible to relax to $0 < p_0 < p_1 \leqslant \infty$ with worse constants. Let us keep the notations introduced in the proof of Theorem 3.3 and write $z_m = \sum_{k\in I} \tilde{d}_k q_{k,m}$ the sequence obtained in Lemma 3.4. We have $z = \sum_{m\in\mathbb{Z}} 2^{-m} \sum_{k\in I} \tilde{d}_k q_{k,m}$ and to estimate its norm, one has to use the p-triangular inequality if $p < 1$ rather than the Hardy–Littlewood majorization. Indeed, we have

$$\|z\|_p^p \leqslant \sum_{k\in I, m\in\mathbb{Z}} 2^{-mp}\tilde{d}_k^p\|q_{k,m}\|_p^p \leqslant \sum_{k\in I}\tilde{d}_k^p 2^k \sum_{m\in\mathbb{Z}} 2^{-mp}\tau(r_m) \leqslant \frac{\|x\|_p^p}{1-2^{-p}}\sum_{k\in I}\tilde{d}_k^p 2^k.$$

To obtain the last inequality, we used Lemma 2.1 with $\alpha = p$. Then, one has to choose d_k as before so that the series converges. Unfortunately, we have no relevant non-commutative applications for the moment.

Remark 3.8. Both in the commutative case and in [**8**], we can merely assume that S_n is sub-additive or even that $\mu(S_n(f+g)) \preceq \mu(S_n(f))+\mu(S_n(g))$. Indeed, we just use linearity in (3.4), where sub-additivity is enough if one replaces $S_n(r_m)$ with $S_n(2^m r_m)$ in the arguments of Lemma 3.4, and there is no need to use homogeneity either. Moreover, we only used the hypothesis for projections, so we can change weak type to restricted weak type as in [**8**].

Remark 3.9. Assume the following stronger form of weak type inequality: for any projection r there is $a \in L_{p_0,\infty}$ with $\|a\|_{p_0,\infty}^{p_0} \leqslant \tau(r)$ and $0 \leqslant S_n(p) \leqslant a$, and similarly for p_1 with an operator b. Then the conclusion of Theorem 3.3 holds with a better constant, namely it is possible to remove the square in the expression of C_{θ,p_0,p_1}. Indeed, using the same proof, one can take e_k to be a spectral projection of b for $k \leqslant 0$ and a spectral projection of $e_0 a e_0$ for $k > 0$. The assumption yields $0 \leqslant S_n(r) \leqslant 2((1-e_0)b(1-e_0) + e_0 a e_0)$ without Lemma 3.4 being needed so that we simply end up with $0 \leqslant S_n(p) \leqslant 8\sum_k \min(2^{-k/p_0}, 2^{-k/p_1})q_k$.

Remark 3.10. The constant in Theorem 1.3 (or the second part of Theorem 3.3) for $p_0 = 1$ and $p_1 = \infty$ cannot be improved to $CC_0^{1-\theta}C_1^{\theta}\alpha_{\theta,p_0,p_1}^2$ as in Theorem 1.2. More precisely, if Theorem 1.3 holds with constants $C'_{\theta,1,\infty}$ when $\theta \in (0,1)$, then defining $f\colon (1,\infty) \to \mathbb{R}^+$ for $p \in (1,\infty)$ by

$$C'_{1-\frac{1}{p},1,\infty} = C_0^{\frac{1}{p}} C_1^{1-\frac{1}{p}} f(p)$$

one must have

$$\lim_{p\to 1^+} (p-1)^2 f(p) = \infty.$$

This will be proved at the end of the section. We do not know if the constant can be improved for other values of the indices.

Our main application is about martingale theory to recover the Doob maximal inequality of [**11**]. Assume $(\mathcal{M},\tau)$ to be endowed with an increasing filtration $(\mathcal{M}_n)_{n\geqslant 0}$ and associated conditional expectations $(\mathcal{E}_n)_{n\geqslant 0}$.

Cuculescu's construction gives that $S = (\mathcal{E}_n)_{n\geqslant 0}$ is of weak type $(1,1)$, but it is obviously of strong type (∞,∞). Then Theorem 3.3 provides a substitute for maximal functions of martingales.

Corollary 3.11. *Let $x \in (L_p(\mathcal{M}) \cap L_1(\mathcal{M}))^+$ for some $1 < p \leqslant \infty$. Then there is $z \in L_p(\mathcal{M})^+$ such that*

$$0 \leqslant \mathcal{E}_n(x) \leqslant z \quad \textit{and} \quad \forall 1 < q \leqslant p,\ \|z\|_q \leqslant C_q\|x\|_q,$$

with $C_q = O(\ln(q-1)/(q-1))^2$ when $q \to 1$ and $C_q = O(1)$ when $q \to \infty$.

Proof: If $x \in \mathcal{M}^+$, the statement follows directly from Theorem 3.3 when $q < \infty$. For $q = \infty$, this is Remark 3.5.

When $x \in L_p^+$, $1 < p < \infty$, this is justified by (3.7). □

Remark 3.12. Since the optimal behaviour of the constant in the Doob maximal inequality is known to be of order $(p-1)^{-2}$ when p goes to 1, it follows that it is not possible to strengthen the weak $(1,1)$-inequality as in Remark 3.9.

3.3. The asymmetric Marcinkiewicz theorem. In [**11**], asymmetric maximal inequalities were considered, and they can be deduced in the same way.

For convenience, we recall the definition of asymmetric maximal function spaces. For $0 < p \leqslant \infty$ and $0 < \gamma < 1$, $L_p(\mathcal{N};\ell_\infty^\gamma)$ consists of all sequences of operators $x = (x_n)_{n\geqslant 0}$ for which there exist operators $a, b \in L_p(\mathcal{N})^+$ and contractions $(u_n)_{n\geqslant 0}$ in $\mathcal{N}$ such that

$$\forall n \geqslant 1, \quad x_n = a^\gamma u_n b^{1-\gamma}.$$

The associated norm is $\|x\|_{L_p(\mathcal{N};\ell_\infty)} = \inf \|a\|_p\|b\|_p$, where the infimum runs over all decompositions as above. Then $L_p(\mathcal{N};\ell_\infty^\gamma)$ is a Banach space for $p \geqslant \max\{2\gamma, 2(1-\gamma)\}$ and a quasi-Banach otherwise. Of course, we recover $L_p(\mathcal{N};\ell_\infty)$ when $\gamma = \frac{1}{2}$. The limit cases $\gamma = 0, 1$ correspond to the column and row maximal function spaces $L_p(\mathcal{N};\ell_\infty^c)$ and $L_p(\mathcal{N};\ell_\infty^c)$ that are recalled at the beginning of Section 4.

For a sequence of maps $S = (S_n)$ as before, we say that it is of strong γ-asymmetric type (p,p) if S is bounded from $L_p(\mathcal{M})$ to $L_p(\mathcal{N};\ell_\infty^\gamma)$. We easily get

Theorem 3.13. *Let $1 \leqslant p_0 < p_1 \leqslant \infty$ and $S = (S_n)$ be a sequence of linear positive maps from $L_{p_0}(\mathcal{M}) + L_{p_1}(\mathcal{M})$ to $L_0(\mathcal{N})$. Assume that S is of weak type (p_0,p_0) and (p_1,p_1) with constants C_0 and C_1.*

Then for any $0 < \theta, \gamma < 1$, S is of strong γ-asymmetric (p,p) type, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ if $p > \max\{2\gamma, 2(1-\gamma)\}$.

Moreover, under these conditions, there is a constant $C_{\theta,\gamma,p_0,p_1}$ such that for any $x \in L_{p_0}(\mathcal{M}) \cap L_{p_1}(\mathcal{M})$ there exist $a_\gamma, b_\gamma \in L_p^+$ and contractions $u_n \in \mathcal{M}$ (all independent of θ) with

$$S_n(x) = a_\gamma^\gamma u_n b_\gamma^{1-\gamma}, \quad \|a_\gamma\|_p, \|b_\gamma\|_p \leqslant C_{\theta,\gamma,p_0,p_1}\|x\|_p.$$

Proof: This is just a variation on the previous argument. We use the notation from the proof of Theorem 3.3.

We assume $x \in L_{p_0}(\mathcal{M}) \cap L_{p_1}(\mathcal{M})$ is positive. Write $x = \sum_{m\in\mathbb{Z}} 2^{-m} r_m$. We fix $d_k = |k|(\ln|k|)^2 + 1$ to construct the elements z_m in Lemma 3.4.

We have that there exist contractions $c_{m,n} \in \mathcal{M}$ such that $S_n(r_m) = z_m^{1/2} c_{m,n} z_m^{1/2}$. We use Lemma 3.2 to get contractions $v_{m,n}$ such that

$$S_n(x) = \sum_{m\in\mathbb{Z}} 2^{-\gamma m} z_m^{1/2} c_{m,n} 2^{-(1-\gamma)m} z_m^{1/2} = \Big(\sum_{m\in\mathbb{Z}} 2^{-2\gamma m} z_m\Big)^{1/2} v_{m,n} \Big(\sum_{m\in\mathbb{Z}} 2^{-2(1-\gamma)m} z_m\Big)^{1/2}.$$

We set $a_\gamma = \big(\sum_{m\in\mathbb{Z}} 2^{-2\gamma m} z_m\big)^{1/(2\gamma)}$ and $b_\gamma = \big(\sum_{m\in\mathbb{Z}} 2^{-2(1-\gamma)m} z_m\big)^{1/(2(1-\gamma))}$. We now justify that they are in L_p, which also legitimates the use of Lemma 3.2. Thanks to Lemma 2.1

$$\mu(a_\gamma)^{2\gamma} \preceq \sum_{m\in\mathbb{Z}} 2^{-2m\gamma}\mu(z_m) \preceq \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-2m\gamma}\tilde{d}_k D_{2^k}(\mu(r_m)) \leqslant C_\gamma \sum_{k\in\mathbb{Z}} \tilde{d}_k D_{2^k}(\mu(x^{2\gamma})).$$

And since $p/(2\gamma) \geqslant 1$, we can use the triangular inequality to get

$$\|a_\gamma\|_p = \|a_\gamma^{2\gamma}\|_{p/(2\gamma)}^{1/(2\gamma)} \leqslant \Big(C_\gamma \sum_{k\in\mathbb{Z}} \tilde{d}_k 2^{k/p} \|x^{2\gamma}\|_{p/(2\gamma)}\Big)^{1/(2\gamma)} \leqslant C_{\theta,\gamma,p_0,p_1}\|x\|_p.$$

One deals with $b_\gamma = a_{1-\gamma}$ in the same way. □

The condition $p > \max\{2\gamma, 2(1-\gamma)\}$ cannot be removed in general, and we provide an easy example for $p_0 = 1$ and $p_1 = \infty$.

We choose $\mathcal{M} = \mathbb{C}$ and $\mathcal{N} = \mathbb{B}(\ell_2)$ with their natural traces. Let $S_n(\lambda) = \lambda T_n$ with $T_n = e_{1,1} + 1/\sqrt{n}(e_{1,n} + e_{n,1}) + 1/n e_{n,n} \geqslant 0$.

It is clear that $(T_n)_{n\geqslant 1}$ is bounded in $\mathbb{B}(\ell_2)$, thus S is of strong (∞,∞) type.

For any $t>0$, the projection $r = \sum_{k>1/t} e_{k,k}$ satisfies $\tau(1-r) \leqslant 1/t$ and $\|rT_n r\|_\infty \leqslant 2t$. Thus S is also of weak $(1,1)$ type.

Proposition 3.14. *For $\theta > 1/2$, S is not of strong θ-asymmetric $(2\theta, 2\theta)$ type.*

Proof: Otherwise, there would exist $a, b \in S_{2\theta}^+$ and contractions u_n such that $T_n = a^\theta u_n b^{1-\theta}$. Denoting by (δ_i) the canonical basis of ℓ_2, we would have

$$\langle a^\theta \delta_n, u_n b^{1-\theta}\delta_1\rangle = 1/\sqrt{n} \leqslant \|a^\theta \delta_n\| \cdot \|b^{1-\theta}\delta_1\|.$$

Thus $\|a^\theta \delta_n\| \geqslant C/\sqrt{n}$, yielding $\|a^\theta\|_2^2 \geqslant \sum_{n\geqslant 1} C/n = \infty$ and $a \notin S_{2\theta}$, a contradiction. □

We conclude by noting that Theorem 3.13 also holds for $\gamma = 0, 1$ in the case $p_0 \geqslant 1$ and $p_1 = \infty$ if $p > 2p_0$. Indeed, if $x \in L_p^+$, then $S_n(x)^2 \leqslant C_1 S_n(x^2)$ (because $p_1 = \infty$) and by Theorem 3.3, $S_n(x)^2 \leqslant a^2$ for some $a \in L_p$. Thus we may conclude that $S_n(x) = u_n a = a u_n^*$ for some contractions u_n using the polar decomposition. This is similar to [**11**, Section 5]. It is not possible to go down to $p = 2p_0$ with the previous counterexample when $p_0 = 1$ (or a variation if $p_0 > 1$).

3.4. Constructing counterexamples. We conclude this section by presenting a proof of Remark 3.10 and a lemma that will be used to construct other counterexamples in the next section. Recall than Lemma 3.1 introduced a way to construct a supremum for families of operators for which a diagonal is controlled. The following lemma shows that this construction is, in a certain sense, optimal.

Lemma 3.15. *Let $\mathcal{H}$ be a Hilbert space. Let $(p_i)_{i\leqslant N}$ be a finite family of orthogonal projections in $\mathcal{B}(\mathcal{H})$ and $(\alpha_i)_{i\leqslant N}$ positive real numbers. Let $a \in B(\mathcal{H})^+$ be such that $a^2 \geqslant b^*b$ for any b of the form*

$$b = \sum_{i\leqslant N} \alpha_i c_i p_i, \quad \|c_i\| \leqslant 1.$$

Set $p = \sum_{i\leqslant N} p_i$. Then there exists an invertible contraction C and a sequence of positive real numbers $(\lambda_i)_{i\leqslant N}$ such that

$$ap = \frac{1}{2} C^{-1}\left(\sum_{i\leqslant N} \lambda_i^{-1/2}\alpha_i p_i\right) \quad \text{and} \quad \sum_{i\leqslant N}\lambda_i = 1.$$

Proof: For $i \leqslant N$, set $\mathcal{H}_i = p_i\mathcal{H}$. Without loss of generality, we can assume that $\mathcal{H} = \bigoplus_{i\leqslant N}\mathcal{H}_i$. Set $\tilde{a} = a\left(\sum_{i\leqslant N}\alpha_i^{-1}p_i\right)$ and note that $|\tilde{a}|^2 \geqslant b^*b$ for any b of the form $b = \sum_{i\leqslant N} c_i p_i$, where c_i are contractions.

Let $\xi = (\xi_i)_{i\leqslant N} \in \mathcal{H}$. By choosing contractions c_i such that

$$\Big\|\sum_{i\leqslant N} c_i\xi_i\Big\|_{\mathcal{H}} = \sum_{i\leqslant N}\|\xi_i\|_{\mathcal{H}_i},$$

we obtain

$$\|a\xi\|_{\mathcal{H}} \geqslant \sum_{i\leqslant N}\|\xi_i\|_{\mathcal{H}_i}.$$

Consequently $\tilde{a}^2 \geqslant 1$ so $\tilde{a}$ is invertible and its inverse $\tilde{a}^{-1}$ can be regarded as a contraction from $\mathcal{H}$ to $\ell_1((\mathcal{H}_i)_{i\leqslant N})$. Denote by c the anti-adjoint of $\tilde{a}^{-1}$; it is contractive from $\ell_\infty((\mathcal{H}_i)_{i\leqslant N})$ to $\mathcal{H}$. Since $\mathcal{H}_i$ can sit as a 1-complemented subspace in $\mathcal{B}(\mathcal{H}_i)$, c can be extended to a contraction from $A = \ell_\infty((\mathcal{B}(H_i))_{i\leqslant N})$, which is a C^*-algebra, to $\mathcal{H}$. By the little Grothendieck theorem [**18**, Theorem 9.4], there exists a state φ on A such that for any $x = (x_i)_{i\leqslant N} \in A$,

$$\|c(x)\|_{\mathcal{H}}^2 \leqslant 2\varphi(xx^* + x^*x).$$

The state φ can be decomposed as $\varphi(x) = \sum_{i\leqslant N} \lambda_i \varphi_i(x_i)$, where φ_i's are states on $\mathcal{B}(\mathcal{H}_i)$ and λ_i's are positive real numbers such that $\sum_{i\leqslant N} \lambda_i = 1$. Then

$$\|c(x)\|_{\mathcal{H}}^2 \leqslant 2\sum_{i\leqslant N} \lambda_i \varphi_i(x_i x_i^* + x_i^* x_i) \leqslant 4\sum_{i\leqslant N} \lambda_i \|x_i\|_{B(\mathcal{H}_i)}^2.$$

Now consider c as a bounded operator from $\mathcal{H} \subset A$ to $\mathcal{H}$. The previous inequality shows that $c^*c \leqslant 4d^2$, where d is the diagonal operator $d = \sum_{i\leqslant N} \sqrt{\lambda_i} p_i$. So c can be factorized as $c = 2Cd$, where C is a contraction. Hence, $\tilde{a}^{-1}$ admits a factorization of the form $\tilde{a}^{-1} = 2dC'$, which concludes the proof going back to a. □

Proof of Remark 3.10: Set $\mathcal{M} = \mathbb{C}$ and $\mathcal{N} = \mathcal{B}(\ell_2)$. For any $n \geqslant 0$, let

$$q_n = \sum_{i=2^n-1}^{2^{n+1}-2} e_{i,i} \in \mathcal{B}(\ell_2) \quad \text{and} \quad Q_N = \sum_{n=0}^{N} q_n.$$

Let $X \subset \mathcal{B}(\ell_2)^+$ be the set of operators x of the form

$$x = b^*b \quad \text{with} \quad b = \sum_{i=0}^{\infty} 2^{-i/2} c_i q_i, \quad \|c_i\| \leqslant 1.$$

Consider the family of operators $S = (S_x)_{x\in X} \colon \mathbb{C} \to \mathcal{B}(\ell_2)$ defined by

$$S_x \colon t \in \mathbb{C} \longmapsto tx.$$

By basic inequalities $\|Q_n^\perp x Q_n^\perp\| \leqslant 2^{-n}$ and $\operatorname{tr} Q_n = 2^{n+1} - 1$. Hence, one S is of weak type $(1,1)$ and of strong type (∞,∞). Let $a \in \mathcal{B}(\ell_2)$ be such that $a \geqslant S_x(1)$ for any $x \in X$, meaning $a \geqslant x$ for any $x \in X$. Let us prove a lower estimate for $\|a\|_p$ as p goes to 1.

By Lemma 3.15, for any $N > 0$, there exists a sequence of positive reals $(\lambda_{i,N})_{0\leqslant i\leqslant N}$ such that

$$4Q_N a Q_N \geqslant \sum_{i=0}^{N} 2^{-i} \lambda_{i,N}^{-1} q_i \quad \text{and} \quad \sum_{i=0}^{N} \lambda_{i,N} = 1.$$

Fix an ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and let $\lambda_i = \lim_{N,\mathfrak{U}} \lambda_{i,N}$. We have $\sum_{i=0}^{\infty} \lambda_i \leqslant 1$. Since $\lambda_{i,N}^{-1} \leqslant 2^{i+2}\|a\|_p$, we can also conclude that $\lambda_i > 0$ and that for all N, $4Q_N a Q_N \geqslant \sum_{i=0}^{N} 2^{-i}\lambda_i^{-1} q_i$.

Consequently, we have that $4^p\|a\|_p^p \geqslant \sum_{i\geqslant 0} \lambda_i^{-p} 2^{i(1-p)}$. Thus using basic inequalities and the Cauchy–Schwarz inequality

$$\|a\|_p^p \gtrsim \sum_{i=1/2(p-1)}^{1/(p-1)} \lambda_i^{-p} \gtrsim \frac{1}{(p-1)^2} \Big(\sum_{i\geqslant 1/2(p-1)} \lambda_i^p \Big)^{-1} \geqslant \frac{R_p^{-1}}{(p-1)^2},$$

where $R_p = \sum_{i\geqslant 1/2(p-1)} \lambda_i$.

Thus, with the notation of Remark 3.10, we must have $(p-1)^2 f(p)^p \gtrsim R_p^{-1}$, so that $\lim_{p\to 1}(p-1)^2 f(p) = \infty$. □

A more careful examination of the estimates above yields a slightly stronger quantitative version of Remark 3.10. Formulating this version in full generality would result in a convoluted statement of little use so we prefer to briefly illustrate how to optimize the computations with a ln factor. Keeping the notations of the proof above, for all $m \geqslant 1$, since $0 < \lambda_i < 1$:

$$\|a\|_p^p \gtrsim 2^{m(1-p)} \sum_{i=0}^{m} \lambda_i{}^{-p} \geqslant 2^{m(1-p)} \sum_{i=0}^{m} \lambda_i{}^{-1} = 2^{m(1-p)} S_m.$$

We claim that for infinitely many m we have $S_m \geqslant m^2 \ln m$. Otherwise $S_m < m^2 \ln m$ when $m \geqslant m_0$ for some m_0. Let $(\tilde{\lambda}_i)_{i\geqslant 0}$ be the non-increasing rearrangement of $(\lambda_i)_{i\geqslant 0} \in \ell_1$. We have that for $m \geqslant m_0$:

$$4m^2 \ln(2m) \geqslant S_{2m} \geqslant \tilde{S}_{2m} = \sum_{i=0}^{2m} \tilde{\lambda}_i^{-1} \geqslant m\tilde{\lambda}_m^{-1}.$$

Thus $\tilde{\lambda}_m \gtrsim \frac{1}{m \ln m}$, which contradicts $\sum_{i=0}^{\infty} \tilde{\lambda}_m \leqslant 1$.

Let m_k be an increasing sequence with $S_{m_k} \geqslant m_k^2 \ln m_k$ and choose $p_k = 1 + m_k^{-1}$. Thus, we must have $(p_k - 1)^2 f(p_k)^{p_k} \gtrsim \ln(p_k - 1)$, so that $\limsup_{p\to 1} \frac{(p-1)^2}{\ln(p-1)} f(p) > 0$.

4. Beyond positivity: Λ-spaces

The positivity assumption for S cannot be removed in Theorem 3.3. Indeed, consider $\mathcal{M} = \mathbb{C}$ and $\mathcal{N} = \mathbb{B}(\ell_2)$ with their natural traces as above and set $S_n(\lambda) = \lambda T_n$ with $T_n = e_{n,1} + e_{1,n}$. The map $S = (S_n)$ is clearly of strong type (∞, ∞) and weak type $(1, 1)$. If S were of strong type (p, p), this would imply that there is $A \in S_p^+$ such that $-A \leqslant T_n \leqslant A$. This would force that $A_{1,1}A_{n,n} \geqslant 1$ for all $n \geqslant 1$, but this is impossible as we must also have that $A_{n,n}$ goes to 0.

It is, however, possible to obtain some positive results with weaker factorizations using the row and column ℓ_∞-valued Lorentz spaces and other variations of maximal norms. Our reference is [**20**].

Let $1 \leqslant p, q \leqslant \infty$; a sequence $x = (x_n)$ of elements in $L_{p,q}(\mathcal{N})$ belongs to $L_{p,q}(\mathcal{N}; \ell_\infty^c)$ (or simply $L_{p,q}(\ell_\infty^c)$) if there exists $a \in L_{p,q}(\mathcal{N})^+$ and contractions $u_n \in \mathcal{N}$ such that $x_n = u_n a$. This is equivalent to saying that $x_n^* x_n \leqslant a^2$ for all $n \geqslant 0$. The infimum of $\|a\|_{p,q}$ over all possible a defines the quantity $\|x\|_{L_{p,q}(\ell_\infty^c)}$. We obtain a Banach space if $p > 2$ or if $p = 2$ and $q \geqslant 2$.

The row version $L_{p,q}(\mathcal{N}; \ell_\infty^r)$ is obtained by taking adjoints.

The couples $(L_q(\mathcal{N}; \ell_\infty^c), L_r(\mathcal{N}; \ell_\infty^c))$ are compatible in the sense of interpolation theory (we assume $r > q$). One of the results in [**17**], generalized in [**20**], is that they behave well with respect to the complex interpolation method (in the Banach spaces range).

We will also need a weaker version of non-commutative maximal inequalities. The construction naturally extends the definition of weak type (p, p) for maximal operators and still coincides with standard maximal inequalities in the commutative case. It has the advantage of retaining its relation with almost uniform and bilateral almost uniform convergence so it may be used as a substitute for the strong version of maximal inequalities when studying pointwise convergence questions occurring in ergodic theory and Fourier analysis in tracial von Neumann algebras.

Our starting point is the following notions of non-increasing rearrangement for a sequence $(x_n)_{n\geqslant 0}$ that should be thought of as a non-commutative analogue of $\mu(\sup_{n\geqslant 0} |x_n|)$.

Definition 4.1. Given a sequence $x = (x_n)_{n\geqslant 0}$ in $L_0(\mathcal{N})$, we define three non-increasing functions $\mu(x), \mu_c(x), \mu_r(x)\colon \mathbb{R}^{+*} \to \mathbb{R}^+$ as, for $t > 0$,

$$\mu(x,t) = \inf_{\tau(1-e)\leqslant t} \sup_{n\geqslant 0} \|ex_ne\|, \quad \mu_c(x,t) = \inf_{\tau(1-e)\leqslant t} \sup_{n\geqslant 0} \|x_ne\|, \quad \mu_r((x_n)) = \mu_c((x_n^*)),$$

where the infimum runs over all projections $e \in \mathcal{N}$.

When x is a constant self-adjoint sequence, $x_n = a = a^*$ for all $n \geqslant 0$, we recover $\mu(x) = \mu(a)$. We could have used pairs of projections e, f with $\tau(1-e), \tau(1-f) \leqslant t$ and $\|fx_ne\|$ to fully recover $\mu(a)$ for general a but this would have made no significant difference. Note also that the following fundamental inequality is still verified for any $s, t > 0$ and $x = (x_n), y = (y_n) \in L_0(\mathcal{N})^{\mathbb{N}}$:

$$\mu_\sharp(x + y, t + s) \leqslant \mu_\sharp(x, t) + \mu_\sharp(y, s), \tag{4.1}$$

with $\sharp = c, r, \emptyset$. This also holds with $s = 0$ replacing $\mu_\sharp(y)(s)$ with $\|y\|$.

The definition is motivated by weak type maximal inequalities and made so that $\|(x_n)\|_{\Lambda_{p,\infty}(\mathcal{N};\ell_\infty)} = \sup_{t>0} t^{1/p}\mu((x_n))(t)$. Indeed, one just needs to set $\lambda = \frac{C}{t^{1/p}}$ in (1.2). Thus

$$\Lambda_{p,\infty}(\mathcal{N};\ell_\infty) = \{(x_n) \in L_p(\mathcal{N})^{\mathbb{N}} \mid \|(x_n)\|_{\Lambda_{p,\infty}} = \sup_{t>0} t^{1/p}\mu((x_n))(t) < \infty\}.$$

It is natural to extend the definition to Lebesgue and Lorentz spaces.

Definition 4.2. For $p, q > 0$, we define for $\sharp = c, r, \emptyset$

$$\Lambda_{p,q}(\mathcal{N};\ell_\infty^\sharp) = \{(x_n) \in L_p(\mathcal{N})^{\mathbb{N}} \mid \|(x_n)\|_{\Lambda_{p,q}^\sharp} = \|\mu_\sharp((x_n))\|_{p,q} < \infty\}.$$

As usual, we will write Λ_p instead of $\Lambda_{p,p}$. In order to lighten notations, we may also write $\Lambda_{p,q}^\sharp$ instead of $\Lambda_{p,q}(\mathcal{N};\ell_\infty^\sharp)$. Of course $\Lambda_\infty^\sharp = L_\infty(\mathcal{N};\ell_\infty^\sharp) = \ell_\infty(\mathcal{N})$. Let us collect a few simple properties of these spaces.

Proposition 4.3. *Let $p, q \in (0, \infty]$.*

(1) *$\Lambda_{p,q}(\mathcal{N};\ell_\infty^\sharp)$ is a quasi-Banach space,*

(2) *for any sequence $x \in L_p(\mathcal{M};\ell_\infty^\sharp)$,*

$$\|x\|_{\Lambda_p(\mathcal{N};\ell_\infty^\sharp)} \leqslant 2^{1/p}\|x\|_{L_p(\mathcal{M};\ell_\infty^\sharp)}.$$

Proof: Point (1) is clear using (4.1). Let us prove (2). Let $X = (x_n)_{n\geqslant 0}$ be a sequence in $L_p(\mathcal{N};\ell_\infty)$. Let $a, b \in L_{p/2}(\mathcal{N})$ and $(y_n)_{n\geqslant 0}$ in $\mathcal{N}$ a sequence of contractions such that for any $n \geqslant 0$, $x_n = ay_nb$. Let $t > 0$. We can find a projection $e \in \mathcal{N}$ such that $\tau(e_1) \leqslant t$ and $\|ea\|_\infty \leqslant \mu(a, t)$. Similarly, we choose a projection e_2 such that $\tau(e_2) \leqslant t$ and $\|be_2\|_\infty \leqslant \mu(b, t)$. Set $e = e_1 \vee e_2$. It is clear that $\tau(e_2) \leqslant 2t$ and for any $n \geqslant 0$, $\|ex_ne\|_\infty \leqslant \mu(a, t)\mu(b, t)$. Therefore

$$\mu(X, 2t) \leqslant \mu(a, t)\mu(b, t),$$

which implies the desired inequality by integrating over t and Hölder's inequality.

When $\sharp = c, r$, the proof is simpler and there is actually no factor $2^{1/p}$. □

Controlling the Λ_p norm of a sequence $(x_n)_{n\geqslant 0}$ is much weaker than controlling its standard maximal norm. In particular, it does not allow us to exhibit an element in $L_p(\mathcal{N})$ that would play the role of $\sup_{n\geqslant 0} x_n$, even for positive sequences.

Recall that we use the classical notation from [**2**] concerning interpolation theory. Since every $\Lambda_{p,q}^\sharp$ can be continuously embedded in the topological vector space $L_0(\mathcal{N})^{\mathbb{N}}$, these spaces are all compatible in the sense of interpolation. The following proposition asserts that the $\Lambda_{p,q}^\sharp$ form a (real) interpolation scale.

Proposition 4.4. *For* $0 < p < \infty$, $t > 0$, *and any* $x = (x_n) \in \Lambda_p^\sharp + \Lambda_\infty^\sharp$*:*

$$K(t, x, \Lambda_p^\sharp, \Lambda_\infty^\sharp) \simeq_p K(t, \mu_\sharp(x), L_p, L_\infty).$$

Consequently, for $0 < \theta < 1$, $0 < p_0 < p_1 \leqslant \infty$, *and* $0 < q \leqslant \infty$ *such that* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$(\Lambda_{p_0,q_0}^\sharp, \Lambda_{p_1,q_1}^\sharp)_{\theta,q} \simeq \Lambda_{p,q}^\sharp.$$

Proof: We detail the proof only in the case $\sharp = \emptyset$; the others are similar.

First, fix $t > 0$, $\varepsilon > 0$, and choose a decomposition $x = a + b$ such that $\|a\|_{\Lambda_p} + t\|b\|_{\Lambda_\infty} \leqslant (1+\varepsilon)K(t, x, \Lambda_p, \Lambda_\infty)$. Since $\|b\|_{\Lambda_\infty} = \|b\|_\infty$, we have that $\mu(x)(t) \leqslant \mu(a)(t) + \|b\|_\infty$. Hence, we can decompose $\mu(x) = \alpha + \beta$ with $\alpha \leqslant \mu(a)$ and $\beta \leqslant \|b\|_\infty$. It follows that $K(t, \mu(x), L_p, L_\infty) \leqslant \|a\|_{\Lambda_p} + t\|b\|_{\Lambda_\infty}$. Thus $K(t, \mu(x), L_p, L_\infty) \leqslant K_t(t, x, \Lambda_p, \Lambda_\infty)$.

For the reverse inequality, we use the Holmstedt formula $K(t, \mu(x), L_p, L_\infty) \simeq_p \big(\int_0^{t^p} \mu(x)^p(s)\, ds\big)^{1/p}$ for $t > 0$. Fix $t > 0$; we can find a projection $e \in \mathcal{N}$ such that $\tau(1-e) \leqslant t^p$ and for any $n \in \mathbb{N}$, $\|ex_ne\| \leqslant 2\mu(x)(t^p)$. We decompose $x = exe + ((1-e)x + ex(1-e)) = b + a$. If $s > t^p$, then clearly $\mu(a)(s) = 0$, and otherwise, $\mu(a)(s) \leqslant \mu(x)(s) + \mu(b)(0) \leqslant \mu(x)(s) + 2\mu(x)(t^p)$. We get

$$\|\mu(a)\|_p^p \lesssim_p \int_0^{t^p} \mu(x)^p(s)\, ds + t^p \mu(x)^p(t^p) \leqslant 2 \int_0^{t^p} \mu(x)^p(s)\, ds.$$

Thus $K_t(t, x, \Lambda_p, \Lambda_\infty) \leqslant \|\mu(a)\|_p + t2\mu(x)(t^p) \lesssim_p \big(\int_0^{t^p} \mu(x)^p(s)\, ds\big)^{1/p}$. This concludes the proof of the K-functional equivalence.

The last statement follows from the estimate for the K-functional, the corresponding result for commutative L_p, and the reiteration principle. □

The following version of the Marcinkiewicz theorem is now clear:

Proposition 4.5. *Assume that* S *is of weak type* (p_0, p_0) *and* (p_1, p_1) *with* $0 < p_0, p_1 \leqslant \infty$. *Then* $S\colon L_p(\mathcal{M}) \to \Lambda_p(\mathcal{N}; \ell_\infty)$ *is bounded.*

Quite surprisingly, the spaces $\Lambda_{p,q}^c$ are connected to $L_{p,q}(\ell_\infty^c)$ via real interpolation. To make this precise, we start with an effective characterization of $\Lambda_p^\sharp$.

To improve clarity, we use capital letters for sequences. Thus recall that if $X = (x_n)$, for an element $n \in \mathcal{N}$, we set $Xq = (x_nq)$ and similarly on the left. We will consider the weighted Lorentz spaces $\ell_{p,q}^\omega$ associated to the measure on $\mathbb{Z}$ given by $\omega(\{n\}) = 2^n$.

Lemma 4.6. *Let* $0 < p < \infty$ *and* $0 < q \leqslant \infty$. *We have:*

(1) *A sequence* $X \in \Lambda_{p,q}^c$ *if and only if there exist a sequence of disjoint projections* $q_k \in \mathcal{N}$ *with* $\tau(q_k) \leqslant 2^k$, $(a_k) \in \ell_{p,q}^{\omega,+}$, *and contractions* $U_k = (u_{n,k}) \in \ell_\infty(\mathcal{N})$ *for* $k \in \mathbb{Z}$ *such that* $X = \sum_{k\in\mathbb{Z}} a_k U_k q_k$. *Moreover,*

$$\|X\|_{\Lambda_{p,q}^c} \simeq_{p,q} \inf_{(a_k)_{k\in\mathbb{Z}}} \|(a_k)\|_{p,q,\omega},$$

where the infimum over all the sequences $(a_k)_{k\in\mathbb{Z}}$ *such that a decomposition of the form* $X = \sum_{k\in\mathbb{Z}} a_k U_k q_k$ *exists.*

(2) $\Lambda_{p,q} = \Lambda_{p,q}^c + \Lambda_{p,q}^r$ *with equivalent semi-norms.*

Proof: We start with the left to right implication in (1). We may assume that $\|X\|_{\Lambda_{p,q}^c} = 1$. From the definition of μ_c, we may find projections f_k such that $\tau(1-f_k) \leqslant 2^k$ and $\|Xf_k\| \leqslant \mu_c(X)(2^k) + 2^{-k^2} = a_k$. As we did in Lemma 3.4, we may replace (f_k) with a sequence of smaller decreasing projections (e_k) such that $\tau(1-e_k) \leqslant 2^{k+1}$. Since

$\sum_{k\in\mathbb{Z}}(\mu_c(2^k)+2^{-k^2})1_{(2^k,2^{k+1})}$ on $\mathbb{R}$ and (a_k) on $(\mathbb{Z},\omega)$ have the same distribution up to a dilation by 2, we deduce that $\|(a_k)\|_{p,q,\omega}\leqslant C_p$, where the constant C_p depends only on p and goes to infinity only when p goes to 0. Set $q_k=e_k-e_{k-1}$, and note that these projections are disjoint. We clearly have the decomposition with $U_k=a_k^{-1}Xq_k$. The convergence of the series holds in $L_0(\mathcal{N})^{\mathbb{N}}$ for instance.

For the other implication, we prove it first when $q=p$ and $p>1$. Assume that $X=\sum_{k\in\mathbb{Z}}a_kU_kq_k$. Clearly

$$\mu_c(X,2^l)\leqslant\Big\|\sum_{k\geqslant l}a_kU_kq_k\Big\|\leqslant\sum_{k\geqslant l}a_k\leqslant\Big(\sum_{k\geqslant l}2^ka_k^p\Big)^{1/p}\Big(\sum_{k\geqslant l}2^{-k/(p-1)}\Big)^{(p-1)/p},$$

thanks to the Hölder inequality. We get

$$\|\mu_c(X)\|_p^p\leqslant\sum_{l\in\mathbb{Z}}2^l\mu_c(X,2^l)^p\leqslant\frac{1}{(1-2^{-1/(p-1)})^{p-1}}\sum_{l\in\mathbb{Z}}\sum_{k\geqslant l}2^ka_k^p\leqslant K_p\sum_{k\in\mathbb{Z}}2^ka_k^p.$$

Note that $K_p^{1/p}$ remains bounded when $p\to1$ but goes to ∞ with p.

If $p\leqslant1$, one just needs to use $\big(\sum_{k\geqslant l}a_k\big)^p\leqslant\sum_{k\geqslant l}a_k^p$ to get the same estimate with $C_p=2$.

The Lorentz case follows easily by interpolation using the linear map $(a_k)\mapsto\sum_{k\in\mathbb{Z}}a_kU_kq_k$.

To deal with (2), first the bounded inclusion $\Lambda^c_{p,q}+\Lambda^r_{p,q}\subset\Lambda_{p,q}$ is clear as $\mu\leqslant\mu_c,\mu_r$. We use (1) for the reverse. Let $X\in\Lambda_{p,q}$ with norm 1. Just as we did, we can find a sequence of decreasing projections f_k such that $\tau(1-f_k)\leqslant2^k$ and $\|f_kXf_k\|\leqslant\mu(X)(2^k)+2^{-k^2}=a_k$. Set $q_k=f_k-f_{k-1}$ and as $X=\sum_{k\in\mathbb{Z}}f_kXq_k+f_{k+1}Xq_k$, decompose

$$X=\sum_{k\in\mathbb{Z}}a_k(a_k^{-1}f_kXq_k)q_k+a_kq_k(a_k^{-1}q_kXf_{k+1})=\sum_{k\in\mathbb{Z}}a_k(U_kq_k+q_kV_k).$$

Clearly U_k, V_k are contractions. By the first point $\sum_{k\in\mathbb{Z}}a_kU_kq_k\in\Lambda^c_{p,q}$ with norm less than $C_{p,q}$ and similarly for $\sum_{k\in\mathbb{Z}}a_kq_kV_k\in\Lambda^r_{p,q}$. □

The version of the Marcinkiewicz theorem for Λ-spaces included in Proposition 4.4 can now be written as

Corollary 4.7. *Assume that S is of weak type (p_0,p_0) and (p_1,p_1) with $0<p_0<p_1\leqslant\infty$. Let $p_0<p<p_1$; then for any $X\in L_p(\mathcal{M})$, there exist $Z\in\Lambda_p(\ell^c_\infty)$, $Y\in\Lambda_p(\ell^r_\infty)$ so that $S(x)=Z+Y$ and*

$$\|Z\|_{\Lambda^c_p}+\|Y\|_{\Lambda^r_p}\leqslant C_p\|X\|_p.$$

Actually, Z and Y only depend on X, not on p.

Remark 4.8. We choose to use (2^k) for simplicity but the above lemma works more generally for any geometric sequence $(2^{\alpha k})$ with $\alpha>0$. Namely, $X\in\Lambda^c_{p,q}$ iff it can be decomposed as $X=\sum_{k\in\mathbb{Z}}a_kU_kq_k$, where (q_k) is a sequence of disjoint projections with $\tau(q_k)\leqslant2^{\alpha k}$, $(a_k)\in\ell^{\omega^\alpha,+}_{p,q}$, and $U_k=(u_{n,k})\in\ell_\infty(\mathcal{N})$ are contractions for $k\in\mathbb{Z}$. Moreover, $\|X\|_{\Lambda^c_{p,q}}\simeq_{p,q,\alpha}\|(a_k)\|_{p,q,\omega^\alpha}$.

Corollary 4.9. *Let $0<p<\infty$, $0<q\leqslant\infty$, $0<\theta<1$, and set $\frac{1}{p_\theta}=\frac{1-\theta}{p}$. Then with equivalent semi-norms:*

$$(L_p(\ell^c_\infty),L_\infty(\ell^c_\infty))_{\theta,q}=\Lambda_{p_\theta,q}(\ell^c_\infty).$$

Proof: As we saw in Proposition 4.3, the Markov inequality gives contractive inclusions $L_p(\ell_\infty^c) \subset \Lambda_p(\ell_\infty^c)$ for $p > 0$. Thus interpolation immediately gives a bounded inclusion

$$(L_p(\ell_\infty^c), L_\infty(\ell_\infty^c))_{\theta,q} \subset (\Lambda_p(\ell_\infty^c), \Lambda_\infty(\ell_\infty^c))_{\theta,q} \simeq \Lambda_{p_\theta,q}(\ell_\infty^c).$$

For the reverse, first take $X \in \Lambda_{p_\theta}(\ell_\infty^c)$ and consider the decomposition from Lemma 4.6(1) and Remark 4.8 with $\alpha = p$. We have written $X = \sum a_k U_k q_k$ in $L_0(\mathcal{N})^{\mathbb{N}}$ with

$$J(2^k, X_k, L_p(\ell_\infty^c), L_\infty(\ell_\infty^c)) = \max\{\|a_k U_k q_k\|_{L_p(\ell_\infty^c)}, 2^k \|a_k U_k q_k\|_{L_\infty(\ell_\infty^c)}\} \leqslant a_k 2^k.$$

Moreover, $\big(\sum_{k\in\mathbb{Z}} 2^{kp} a_k^{p_\theta}\big)^{1/p_\theta} \lesssim \|X\|_{\Lambda_{p_\theta}^c}$. Since

$$\sum_{k\in\mathbb{Z}} (2^{-k\theta} J(2^k, X_k, L_p(\ell_\infty^c), L_\infty(\ell_\infty^c))^{p_\theta} \leqslant \sum_{k\in\mathbb{Z}} 2^{kp} a_k^{p_\theta},$$

we conclude that $X \in (L_p(\ell_\infty^c), L_\infty(\ell_\infty^c))_{\theta,p_\theta}$ by the equivalence between the J- and K-methods (for quasi-normed spaces). Thus $(L_p(\ell_\infty^c), L_\infty(\ell_\infty^c))_{\theta,p_\theta} \simeq \Lambda_{p_\theta}(\ell_\infty^c)$. The general statement follows by the reiteration theorem. □

More generally, choosing the correct weight ω^α, the same proof gives that with $0 < p_0 < p_1$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{1}{p_1}$ and $0 < q \leqslant \infty$:

$$(L_{p_0}(\ell_\infty^c), L_{p_1}(\ell_\infty^c))_{\theta,q} = \Lambda_{p_\theta,q}(\ell_\infty^c).$$

Corollary 4.10. *For $2 < p < \infty$ and $1 \leqslant q \leqslant \infty$, $\Lambda_{p,q}(\ell_\infty^\sharp)$ has an equivalent norm.*

Proof: For $\sharp = c$, this is clear from Corollary 4.9 as $L_{p,q}(\ell_\infty^c)$ is a Banach space. The case $\sharp = r$ is obtained by taking adjoints. The remaining case then follows using Lemma 4.6(2). □

At this point, it is worth justifying that $\Lambda_p(\ell_\infty^c) \neq L_p(\ell_\infty^c)$:

Proposition 4.11. *Set $\mathcal{N} = \mathbb{B}(\ell_2)$. Let $0 < p < \infty$, and $q > 2$. The formal identity map on $L_p(\ell_\infty^c)$ is not bounded from $\Lambda_{p,q}(\ell_\infty^c)$ to $L_{p,\infty}(\ell_\infty^c)$, nor from $\Lambda_p(\ell_\infty^c)$ to $L_p(\ell_\infty^c)$.*

Proof: Set $N > 0$ and let $p_i = \sum_{k=2^{i-1}}^{2^i-1} e_{k,k}$ for $i = 1, \dots, N$. Choose some families of contractions $(u_{n,i})_{n\geqslant 0}$, $i \leqslant N$, such that $\{(u_{n,1}, \dots, u_{n,N});\, n \geqslant 0\}$ is dense in B^N, where B is the unit ball of compact operators. Set $X_i = 2^{-i/p}(u_{n,i}p_i)_{n\geqslant 0} \in L_p(\ell_\infty^c) \cap L_\infty(\ell_\infty^c)$.

By Lemma 4.6, the norm of $X = \sum_{i=1}^N X_i$ in $\Lambda_{p,q}^c$ is controlled by $C_{p,q}N^{1/q}$. Similarly, we also have $\|X\|_{\Lambda_p^c} \lesssim_p N^{1/p}$.

Next we estimate the norm of X in $L_{p,\infty}(\ell_\infty^c)$ and in $L_p(\ell_\infty)$.

By definition, we can find $a \in S_{p,\infty}^+$ (or $a \in S_p$) such that $\|a\|_{p,\infty} \leqslant 2\|X\|_{L_{p,\infty}(\ell_\infty^c)}$ (or $\|a\|_p \leqslant 2\|X\|_{L_p(\ell_\infty^c)}$) and

$$\left|\sum_{i=1}^N 2^{-i/p} u_{n,i} p_i\right|^2 = \left(\sum_{i=1}^N 2^{-i/p} p_i\right) \left|\sum_{i=1}^n u_{n,i} p_i\right|^2 \left(\sum_{i=1}^N 2^{-i/p} p_i\right) \leqslant a^2.$$

We may apply Lemma 3.15 to get $\lambda_i \geqslant 0$ with $\sum_{i=1}^N \lambda_i = 1$ and C an invertible contraction such that $a = (2C)^{-1} \sum_{i=1}^N 2^{-i/p} \lambda_i^{-1/2} p_i$ for some contraction C.

Note that at least one of the λ_i is smaller than $\frac{1}{N}$. It follows that $\|a\|_{p,\infty} \gtrsim N^{1/2} 2^{-i/p} \|p_i\|_{p,\infty} = N^{1/2}$, whereas $\|X\|_{\Lambda_{p,q}^c} \lesssim N^{1/q}$.

We also have the estimate $2\|a\|_p \geqslant \big(\sum_{i=1}^N \lambda_i^{-p/2}\big)^{1/p}$. Thanks to the Hölder inequality with $q = 1 + \frac{p}{2}$,

$$N = \sum_{i=1}^N \frac{\lambda_i^{p/(2q)}}{\lambda_i^{p/(2q)}} \leqslant \Big(\sum_{i=1}^N \lambda_i^{-p/2}\Big)^{1/q} \Big(\sum_{i=1}^N \lambda_i\Big)^{1/q'}.$$

Thus, similarly $2\|a\|_p \geqslant N^{1/p+1/2}$, whereas $\|X\|_{\Lambda_p^c} \lesssim_p N^{1/p}$. □

Remark 4.12. The same proposition applies to row spaces by taking adjoints. Actually, it is easy to see that the formal identity map on $L_p(\ell_\infty^c) \cap L_\infty(\ell_\infty^c)$ is not bounded from $\Lambda_{p,q}$ to $L_{p,\infty}(\ell_\infty^c) + L_{p,\infty}(\ell_\infty^r)$ using the same counterexample. Indeed, let $s \in \mathbb{B}(\ell_2)$ be the shift operator: if $(e_i)_{i\geqslant 0}$ is the canonical basis of ℓ_2, then $se_i = e_{i+1}$ for every $i \geqslant 0$. We can assume that the set $\{(u_{n,i}); n \geqslant 0\}$ is also stable by left multiplication by s. Now, assume that $X = (x_n)_n$ in the proof above decomposes as $Y + Z$ with $Y = (w_n)a$, $Z = b(v_n)$ with (v_n), (w_n) bounded and $a, b \in L_{p,\infty}$. For any $k \geqslant 0$, by assumption on $(u_{n,i})$, $s^k X$ is a subsequence of X so there are indices n_k such that $s^k x_n = w_{n_k} a + b v_{n_k}$. Thus $x_n = (s^*)^k w_{n_k} a + (s^*)^k b v_{n_k}$ for all n. Since b is in the Schatten (p,∞)-class, $s^{*k} b$ goes to 0 in the (p,∞)-norm and we can assume by weak-$*$ compacity that $x_n = \tilde{w}_n a$ for some bounded sequence $(\tilde{w}_n)$. This implies that $\|X\|_{L_p(\ell_\infty^c)+L_p(\ell_\infty^r)} = \|X\|_{L_p(\ell_\infty^c)}$.

Remark 4.13. When the second index $q \leqslant 2$ and $2 < p < \infty$, we have an inclusion

$$\Lambda_{p,q}(\ell_\infty^c) \subset L_{p,\infty}(\ell_\infty^c).$$

One can see it using duality. The space $L_{p,q}(\ell_\infty^c)$ is the anti-dual of $L_{p',q'}(\ell_1^c)$ when $q \geqslant 2$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. It consists of sequences (x_n) such that there are column contractions (u_n) and (v_n) in $\mathcal{N}$ and $\alpha \in L_2(\mathcal{N})$ and $\beta \in L_{p^\#,q^\#}(\mathcal{N})$, where $1/2 + 1/q^\# = 1/q'$ and $1/2 + 1/p^\# = 1/p'$ with $x_n = \alpha u_n^* v_n \beta$. The norm is obtained by taking the infimum of $\|\alpha\|_2 \cdot \|\beta\|_{p^\#,q^\#}$ over all possible decompositions.

Using $L_{p^\#,q^\#} = (L_{2^\#}, L_2)_{1-\frac{2}{p},q^\#}$, it follows that the inclusion

$$L_{p',q'}(\ell_1^c) \subset (L_2(\ell_1^c), L_1(\ell_1^c))_{1-\frac{2}{p},q^\#}$$

is bounded. This yields a bounded inclusion

$$(L_2(\ell_\infty^c), L_\infty(\ell_\infty^c))_{1-\frac{2}{p},\frac{2q}{q+2}} \subset L_{p,q}(\ell_\infty^c).$$

In particular, $\Lambda_{\theta,q}(\ell_\infty^c) \subset \Lambda_{\theta,2}(\ell_\infty^c) \subset L_{p,\infty}(\ell_\infty^c)$.

Remark 4.14. The previous arguments can be used to actually prove that one cannot replace $\Lambda_p^c + \Lambda_p^r$ with $L_p(\ell_\infty^c) + L_p(\ell_\infty^r)$ in this corollary when $0 < p_1 \leqslant \infty$ in general. Indeed, using the notation from Proposition 4.11, set $U_i = (u_{n,i})_{i\geqslant 0}$ and define a map $S\colon \ell_0^{\omega^2} \to B(\ell_2)^{\mathbb{N}}$ by $S((x_i)) = \sum_{i\geqslant 0}^N x_i U_i p_i$. By Lemma 4.6, it is bounded from $\ell_q^{\omega^2}$ to Λ_q^c for all $q < \infty$ and thus of weak type (q,q) with constant C_q independent of N. For $q = \infty$, its norm is controlled by $\sqrt{N}$. Taking $x = (2^{-2i/p})$ as in Proposition 4.11, we have $\|x\|_p = N^{1/p}$ but Remark 4.12 yields

$$\|S(x)\|_{L_p(\ell_\infty^c)+L_p(\ell_\infty^r)} = \|S(x)\|_{L_p(\ell_\infty^c)} \gtrsim N^{1/2+1/p}.$$

This covers the case $p_1 < \infty$ choosing $p \in (\max\{p_0, 2\}, p_1)$. For $p_1 = \infty$, one just needs to note that homogeneity would imply that for such p and $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0}$:

$$\sqrt{N} \lesssim \|S\|_{\ell_p^{\omega^2} \to L_p(\ell_\infty^c)+L_p(\ell_\infty^r)} \lesssim C_{p_0}^{1-\theta} \sqrt{N}^\theta.$$

The previous remark shows that we cannot expect the spaces $L_p(\ell_\infty^c) + L_p(\ell_\infty^r)$ to form a real interpolation scale or satisfy a Marcinkiewicz-type theorem for non-positive maps. We can nonetheless formulate some positive results involving those (quasi)-norms by employing variations of the argument used in the previous section.

Lemma 4.15. *Assume that S is of weak type (p_0,p_0) and (p_1,p_1) with constants 1 with $p_0,p_1 \geqslant 1$. Then for any finite projection $r \in \mathcal{M}$, there exist disjoint projections q_k, $k \in \mathbb{Z}$, with $\tau(q_k) \leqslant 2^k\tau(r)$ sequences $u_n, v_n \in \mathcal{N}$ and a constant $C = C_{p_0,p_1} > 0$*

$$\forall n \geqslant 0, \quad S_n(r) = zu_n + v_n z, \quad \textit{with } z = \sum_k c_k q_k, \quad \textit{and } \|u_n\|, \|v_n\| \leqslant C,$$

where $c_k = 2^{k/p_1}(|k|+1)$ for $k \leqslant 0$ and $c_k = 2^{-k/p_0}k$ for $k \geqslant 1$.

In particular, for $p \in (p_0,p_1)$

$$\|S(r)\|_{L_p(\ell_\infty^c)+L_p(\ell_\infty^r)} \leqslant C_{p,p_0,p_1}\|r\|_p.$$

Proof: With the notation from the proof of Lemma 3.4,

$$S_n(r) = \sum_{k\in\mathbb{Z}} (q_k S_n(r) e_{k+1} + e_k S_n(r) q_k).$$

For $k \in \mathbb{Z}$, set $b_k = \min(2^{k/p_0}, 2^{k/p_1})$ and put

$$u_n = \sum_{k\in\mathbb{Z}} \frac{b_k}{|k|+1} q_k S_n(r) e_{k+1}, \quad v_n = \sum_{k\in\mathbb{Z}} \frac{b_k}{|k|+1} e_k S_n(r) q_k, \quad z = \sum_{k\in\mathbb{Z}} b_k(|k|+1)q_k.$$

Then one easily checks $S_n(r) = zu_n + v_n z$, $\|u_n\|, \|v_n\| \leqslant C_{p_0,p_1}$ and the estimate for z as well as $S_n(r) = zu_n + v_n z$. The last estimate is also clear. □

Remark 4.16. For $a \in L_{2p}^+$, $p \geqslant 1$, and any contraction $u \in \mathcal{M}$, one can always find a contraction $v \in \mathcal{M}$ such that $aua = \frac{1}{2}(a^2 v + va^2)$; this follows for instance from the Cauchy formula for the holomorphic function $F(z) = a^{2(1-z)}ua^{2z}$ with values in L_p. Thus, when S is positive, Lemma 3.4 implies this one.

For general elements, we can get

Proposition 4.17. *Assume that S is of weak type (p_0,p_0) and (p_1,p_1) with constants 1. Then for any $x \in L_{p_0}(\mathcal{M}) \cap L_{p_1}(\mathcal{M})$, there exist $z \in \bigcap_{p_0<p<p_1} L_p(\mathcal{N})^+$ and sequences $u_n, v_n \in \mathcal{N}$ such that*

$$\forall n \geqslant 0, \quad S_n(x) = zu_n + v_n z, \ \|u_n\|, \|v_n\| \leqslant 1,$$

and

$$\forall p_0 < p < p_1, \quad \|z\|_p \leqslant C_p \|x\|_{p_0}^{1-\theta} \|x\|_{p_1}^{\theta},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof: Decomposing x as a combination of four positive elements and using the fact that $L_p(\mathcal{N};\ell_\infty^c)$ and $L_p(\mathcal{N};\ell_\infty^r)$ are quasi-Banach spaces, we can assume that $x \in L_p^+$.

Using homogeneity, we may assume that $\|x\|_p = 1$. We use Lemma 2.1 to write $x = \sum_{m\in\mathbb{Z}} 2^{-m} r_m$ for some finite projections r_m.

We can apply Lemma 4.15 to each r_m to get

$$S_n(x) = \sum_{m\in\mathbb{Z}} 2^{-m}(z_m u_{m,n} + v_{m,n} z_m),$$

with $\|u_{m,n}\|, \|v_{m,n}\| \leqslant 1$ and $\mu(z_m) \preceq \sum_{k\in\mathbb{Z}} 2^{-k} c_k D_{2^k}(\mu(r_m))$.

Using Lemma 3.2, we can find elements u_n and v_n in $\mathcal{N}$ such that

$$\sum_{m\in\mathbb{Z}} 2^{-m} z_m u_{m,n} = \Big(\sum_{m\in\mathbb{Z}} 2^{-2m} m^2 z_m^2\Big)^{1/2} u_n,$$

$$\sum_{m\in\mathbb{Z}} 2^{-m} v_{m,n} z_m = v_n \Big(\sum_{m\in\mathbb{Z}} 2^{-2m} m^2 z_m^2\Big)^{1/2},$$

with $\|u_n\|$, $\|v_n\|$ bounded by an absolute constant C.

Set $z = \big(\sum_{m\in\mathbb{Z}} 2^{-2m} m^2 z_m^2\big)^{1/2}$. Then $z \otimes e_{0,0}$ is the modulus of $\sum_{m\in\mathbb{Z}} 2^{-m} m z_m \otimes e_{m,0}$ in $L_p(\mathcal{N}\overline{\otimes}\mathcal{B}(\ell_2))$. Thus

$$\mu(z) \preceq \sum_{m\in\mathbb{Z}} 2^{-m} m \sum_{k\in\mathbb{Z}} c_k D_{2^k}(\mu(r_m)) = \sum_{k\in\mathbb{Z}} c_k D_{2^k}\Big(\sum_{m\in\mathbb{Z}} 2^{-m} m \mu(r_m)\Big).$$

By Lemma 2.1, $\sum_{m\in\mathbb{Z}} 2^{-m} m\mu(r_m) \leqslant C\mu(x(|\ln(x)|+1))$ for some constant $C > 0$. We get

$$\|z\|_p \lesssim \sum_{k\in\mathbb{Z}} c_k 2^{k/p} \|x(|\ln(x)|+1)\|_p,$$

where we used that D_{2^k} has norm less than $2^{k/p}$ on L_p.

We have the inequalities $t^p(|\ln(t)|+1)^p \lesssim_{p,p_0} t^{p_0}$ for $0 < t < 1$ and $t^p(|\ln(t)|+1)^p \lesssim_{p,p_1} t^{p_1}$ for $t \geqslant 1$. They yield that $\|x(|\ln(x)|+1)\|_p^p \lesssim \|x\|_{p_0}^{p_0} + \|x\|_{q_1}^{q_1}$.

Hence we have that $\|z\|_p \lesssim \|x\|_{p_0}^{p_0} + \|x\|_{q_1}^{q_1}$. The conclusion follows using homogeneity as usual. ☐

Unfortunately, trying to improve the above estimate using reiteration as in [**13**] does not provide anything more than Corollary 4.7.

Let us conclude by highlighting the link between Λ-spaces and bilateral almost uniform convergence. Recall that a sequence of operators (x_n) converges *bilaterally almost uniformly* (b.a.u.) to x if and only if for any $\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that

$$\tau(1-e) \leqslant \varepsilon \quad \text{and} \quad \|e(x_n - x)e\| \longrightarrow 0.$$

This notion was introduced to serve as one of the non-commutative analogues of pointwise convergence (see [**10**] for more details and the early developments related to this form of convergence). Let $(S_n)_{n\geqslant 0}$ be a sequence of operators defined on $L_1(\mathcal{M})+\mathcal{M}$. A standard approach to prove the convergence of $(S_n)_{n\geqslant 0}$ to some limit operator S_∞ on $L_p(\mathcal{M})$ is the following: first, prove convergence on a dense subset of $L_p(\mathcal{M})$ (usually taken to be $L_1(\mathcal{M}) \cap \mathcal{M}$); second, prove a maximal inequality and use a form of Banach principle to extend the convergence on all of $L_p(\mathcal{M})$.

Proposition 4.18. *Let $p \in [1,\infty)$ and $S = (S_n)$ be a sequence of maps from $L_p(\mathcal{M})$ to $L_p(\mathcal{N})$ such that S is bounded from $L_p(\mathcal{M})$ to $\Lambda_p(\mathcal{N};\ell_\infty)$. Assume that there exists a bounded linear map S_∞ from $L_p(\mathcal{M})$ to $L_p(\mathcal{N})$ and a dense subset $E \subset L_p(\mathcal{M})$ such that for any $x \in E$, $(S_n(x))$ converges b.a.u. to $S_\infty(x)$. Then $(S_n(x))$ converges b.a.u. to $S_\infty(x)$ for any $x \in L_p(\mathcal{M})$.*

As the topic of b.a.u. convergence is not central to this paper, the proof is left to the reader. The intended application for such a result would be to the convergence of sequences of operators that are not positive, that verify weak type maximal inequalities, and for which the strong type cannot be proved. But no natural example of such a behaviour has been discovered so far, as every proof of maximal inequality relies in one way or another on a reduction to positive operators. A good candidate would be the matrix-valued Carleson operator. In the meantime, we believe that the proposition (or more refined versions that would involve symmetric spaces) complements results of b.a.u. convergence in tracial von Neumann algebras in the spirit of [**5**].

Acknowledgement. The authors are supported by ANR-19-CE40-0002. The authors thank the Lorentz Center, in which discussions about this subject were initiated.

References

[1] T. N. Bekjan, Z. Chen, and A. Osękowski, Noncommutative maximal inequalities associated with convex functions, *Trans. Amer. Math. Soc.* **369(1)** (2017), 409–427. DOI: 10.1090/tran/6663.

[2] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer-Verlag, Berlin-New York, 1976. DOI: 10.1007/978-3-642-66451-9.

[3] D. W. Boyd, Indices of function spaces and their relationship to interpolation, *Canadian J. Math.* **21** (1969), 1245–1254. DOI: 10.4153/CJM-1969-137-x.

[4] A.-P. Calderón, Spaces between L_1 and L_∞ and the theorem of Marcinkiewicz, *Studia Math.* **26** (1966), 273–299. DOI: 10.4064/sm-26-3-301-304.

[5] V. Chilin and S. Litvinov, On individual ergodic theorems for semifinite von Neumann algebras, *J. Math. Anal. Appl.* **495(1)** (2021), Paper no. 124710, 16 pp. DOI: 10.1016/j.jmaa.2020.124710.

[6] I. Cuculescu, Martingales on von Neumann algebras, *J. Multivariate Anal.* **1(1)** (1971), 17–27. DOI: 10.1016/0047-259X(71)90027-3.

[7] S. Dirksen, Noncommutative Boyd interpolation theorems, *Trans. Amer. Math. Soc.* **367(6)** (2015), 4079–4110. DOI: 10.1090/S0002-9947-2014-06185-0.

[8] S. Dirksen, Weak-type interpolation for noncommutative maximal operators, *J. Operator Theory* **73(2)** (2015), 515–532. DOI: 10.7900/jot.2014mar12.2022.

[9] G. Hong, M. Junge, and J. Parcet, Algebraic Davis decomposition and asymmetric Doob inequalities, *Comm. Math. Phys.* **346(3)** (2016), 995–1019. DOI: 10.1007/s00220-016-2581-3.

[10] R. Jajte, *Strong Limit Theorems in Non-Commutative Probability*, Lecture Notes in Math. **1110**, Springer-Verlag, Berlin, 1985. DOI: 10.1007/BFb0101453.

[11] M. Junge, Doob's inequality for non-commutative martingales, *J. Reine Angew. Math.* **549** (2002), 149–190. DOI: 10.1515/crll.2002.061.

[12] M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities, *Ann. Probab.* **31(2)** (2003), 948–995. DOI: 10.1214/aop/1048516542.

[13] M. Junge and Q. Xu, Noncommutative maximal ergodic theorems, *J. Amer. Math. Soc.* **20(2)** (2007), 385–439. DOI: 10.1090/S0894-0347-06-00533-9.

[14] M. Junge and Q. Xu, Notes on real interpolation of operator L_p-spaces, *Acta Math. Sci. Ser. B (Engl. Ed.)* **41(6)** (2021), 2173–2182. DOI: 10.1007/s10473-021-0622-2.

[15] E. C. Lance, Ergodic theorems for convex sets and operator algebras, *Invent. Math.* **37(3)** (1976), 201–214. DOI: 10.1007/BF01390319.

[16] S. J. Montgomery-Smith, The Hardy operator and Boyd indices, in: *Interaction between Functional Analysis, Harmonic Analysis, and Probability* (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math. **175**, Marcel Dekker, Inc., New York, 1996, pp. 359–364.

[17] M. Musat, Interpolation between non-commutative BMO and non-commutative L_p-spaces, *J. Funct. Anal.* **202(1)** (2003), 195–225. DOI: 10.1016/S0022-1236(03)00099-5.

[18] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Ser. in Math. **60**, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. DOI: 10.1090/cbms/060.

[19] G. Pisier, Non-commutative vector valued L_p-spaces and completely p-summing maps, *Astérisque* **247** (1998), 131 pp.

[20] L. WU, R. XIA, AND D. ZHOU, Asymmetric Burkholder inequalities in noncommutative symmetric spaces, Preprint (2022). arXiv:2201.05474.

[21] F. J. YEADON, Ergodic theorems for semifinite von Neumann algebras—I, *J. London Math. Soc. (2)* **16(2)** (1977), 326–332. DOI: 10.1112/jlms/s2-16.2.326.

Léonard Cadilhac
Institut de Mathématiques de Jussieu, Sorbonne Université - Paris, France
E-mail address: cadilhac@imj-prg.fr

Éric Ricard
UNICAEN, CNRS, LMNO, 14000 Caen, France
E-mail address: eric.ricard@unicaen.fr

Received on July 3, 2023.
Accepted on June 4, 2024.