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SOME REMARKS ON THE MASLOV INDEX

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Abstract: It is a classical fact that the Kashiwara–Wall index of a triplet of Lagrangians in a symplectic space over a field $\Bbbk$ defines a 2-cocycle μ_{KW} on the associated symplectic group with values in the Witt group of $\Bbbk$. Moreover, modulo the square of the fundamental ideal this is a trivial 2-cocycle. In this work we revisit this fact from the viewpoint of the theory of Sturm sequences and Sylvester matrices developed by J. Barge and J. Lannes in [**1**]. We define a refinement by a factor of 2 of μ_{KW} and use the technology of Sylvester matrices to give an explicit formula for the coboundary associated to the mod I^2 reduction of the cocycle which is valid for any field of characteristic different from 2. Finally, we explicitly compute the values of the coboundary on standard elements of the symplectic group.

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1. Introduction

The Maslov index is an object that has permeated a wide range of mathematical disciplines. It appeared in mathematical physics in the work of Maslov [**11**] and found its way into dynamical systems (see for instance [**16**]), as well as algebraic topology through the seminal work of Wall [**19**] to mention a few. For surveys on this vast topic we refer the interested reader for example to the work of Cappell, Lee, and Miller [**3**] or Ghys and Ranicki [**4**].

The strategy to approach the Maslov index in the present work is rather classical: it will appear as an additivity defect associated to an invariant in a Witt group associated to a pair of Lagrangians. This is for instance the geometric interpretation in the classical integer-valued case provided by Wall's work [**19**], where the Maslov index appears as a defect of additivity for the signature of a $4n$-manifold with boundary with respect to gluing along parts of the boundary. A similar approach appears in connection with gluing properties of the so-called η invariant for Dirac operators; see for instance [**2**]. In the present case we will approach the Maslov index by studying the paths in the so-called opposition graph of the Lagrangians, which has as vertices the Lagrangians and edges joining any two transverse Lagrangians. An analogous approach of the Maslov index can also be found for instance in [**7**]. The novelty here is the observation, which we borrow from [**1**], that to any such path α, i.e. any finite sequence of Lagrangians where two consecutives are transverse, one can attach a canonical symmetric bilinear form $S(\alpha)$, called by Barge and Lannes the Sylvester matrix of α. This matrix then defines a class in the Witt group $W(\Bbbk)$ of the underlying field $\Bbbk$, the *Maslov index of the path* α. The Maslov index of a path is crucially not additive with respect to the operation of concatenation of paths, and the *Maslov index of a triple of Lagrangians* is a measure of this deviation. More precisely, in

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Proposition 4.5 we show that given three Lagrangians Λ_0, Λ_1, Λ_2 and two paths α_{01} and α_{12} joining Λ_0 to Λ_1 and Λ_1 to Λ_2 respectively, the Witt class of the orthogonal sum

$$S(\alpha_{01} * \alpha_{12})\perp - S(\alpha_{01})\perp - S(\alpha_{12})$$

is independent of the chosen paths, and usually non-trivial; this is our proposed definition of the Maslov index $\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2)$. To show that it is indeed a Maslov index we check the axiomatic properties of the invariant as stated in [**3**], and the resolution of the multiplicative ambiguity in the axiomatic definition by a straightforward computation shows that $2\mu_{BL} = \mu_{KW}$, where the latter stands for Wall's index as generalized in [**6**].

In the final part of this work we focus on one of the crucial properties of this index, namely that it is a cocycle: given four Lagrangians Λ_0, Λ_1, Λ_2, Λ_3, we have:

$$\mu_{BL}(\Lambda_1, \Lambda_2, \Lambda_3) - \mu_{BL}(\Lambda_0, \Lambda_2, \Lambda_3) + \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_3) - \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = 0.$$

As a consequence, if we fix a Lagrangian Λ, then μ_{BL} defines a 2-cocycle on the symplectic group

$$\begin{aligned} \mu\colon Sp_{2g}(\Bbbk) \times Sp_{2g}(\Bbbk) &\longrightarrow W(\Bbbk) \\ (A, B) &\longmapsto \mu_{BL}(\Lambda, A\Lambda, AB\Lambda) \end{aligned}$$

and hence a central extension of $Sp_{2g}(\Bbbk)$. When $\Bbbk = \mathbb{R}$ this extension has attracted much attention; see for instance [**10**]. It is known, see for instance [**13**], that this Maslov cocycle is trivial when reduced modulo I^2, the square of the fundamental ideal. In particular there is a unique function $\Phi\colon Sp_{2g}(\Bbbk) \to W(\Bbbk)/I^2$ such that, for any $A, B \in W(\Bbbk)$,

$$\Phi(AB) - \Phi(A) - \Phi(B) = \mu(A, B) \bmod I^2.$$

The case $\Bbbk = \mathbb{R}$ is classic, and the function Φ was explicitly computed in this case by Turaev [**18**], although his method does not seem to generalize to other fields. We will show that our definition of μ via the Maslov index of paths leads to a natural definition of Φ valid over any field.

The plan of this work is the following. In Section 2 we will first recall basic results on the Witt group for a field, their construction, and the elementary descriptions of the first quotients of the fundamental ideal by its powers. Then we recall some elementary facts about the set $\mathcal{L}_g$ of all Lagrangians on a symplectic vector space over $\Bbbk$ of dimension $2g$. In particular we will recall the definition of two sets of functions introduced by Leray [**9**], a linear map $\beta_{\Lambda,M}$ defined for any pair of Lagrangians, and a difference map, defined for any two Lagrangians that are transverse to a given third one and will prove a number of algebraic relations between these maps. Section 3 introduces the main tool of this work; building on the different maps defined in the previous section we show that to any Lagrangian path α, i.e. a finite sequence of Lagrangians where any two consecutives are transverse, corresponds a canonical symmetric bilinear map on the direct sum of the Lagrangians in the path, its Sylvester matrix $S(\alpha)$. This is a slight generalization of the main object introduced by Barge and Lannes in [**1**, Chapter 2]. The Witt class of $S(\alpha)$ is the Maslov index of the Lagrangian path, and the form $S(\alpha)$ encodes a number of properties of α. In particular we will give an alternative proof of [**1**, Proposition-Définition A.2.1].

Corollary 1.1. *Let $\alpha : (\Lambda_0, \Lambda_1, \dots, \Lambda_n, \Lambda_{n+1})$ be a Lagrangian path and let $S(\alpha)$ denote its Sylvester matrix. The following are equivalent:*

1. *The bilinear form $S(\alpha)$ is non-degenerate.*
2. *The two Lagrangians Λ_0 and Λ_{n+1} are transverse.*

The bulk of Section 3 is the proof of our main tool to manipulate or compute Sylvester matrices.

Lemma 1.2 (Shortcut lemma)**.** *Let $\Lambda_0, \ldots, \Lambda_{n+1}$ be a Lagrangian path, and let S denote its Sylvester matrix. Assume that there exist two indices $0 \leq i < j \leq n+1$ such that $\Lambda_i \pitchfork \Lambda_j$. Then we have two new Lagrangian paths:*

1. *The sub-sequence $\Lambda_i, \ldots, \Lambda_j$, with associated Sylvester matrix $S(\Lambda_i, \ldots, \Lambda_j)$.*
2. *The shortened sequence $\Lambda_0, \ldots, \Lambda_i, \Lambda_j, \ldots, \Lambda_{n+1}$, whose Sylvester matrix is $S(\Lambda_0, \ldots, \Lambda_i, \Lambda_j, \ldots, \Lambda_{n+1})$.*

Then,

$$S \text{ is isometric to } S(\Lambda_i, \ldots, \Lambda_j) \perp S(\Lambda_0, \ldots, \Lambda_i, \Lambda_j, \ldots, \Lambda_{n+1}),$$

where $\perp$ stands for the orthogonal sum of bilinear forms.

In Section 4 we use the shortcut lemma to prove that given three Lagrangians Λ_0, Λ_1, Λ_2, if we choose two Lagrangian paths α_{01} and α_{12} respectively starting at Λ_0 to Λ_1 and Λ_1 to Λ_2, then the class in $W(\Bbbk)$ of the bilinear form

$$\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = S(\alpha_{01} * \alpha_{12}) \perp - S(\alpha_{01}) \perp - S(\alpha_{12})$$

is independent of the choice of paths. We then show that the index μ_{BL} satisfies the characteristic properties of the Maslov index (see [**3**, Section 1]) and compare it to the Kashiwara–Wall index.

Theorem 1.3. *The index μ_{BL} satisfies the following properties:*

1. *If any two of the three Lagrangians Λ_0, Λ_1, Λ_2 are equal, then $\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = 0$.*
2. *If $\phi \in Sp_{2g}(\Bbbk)$, then $\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = \mu_{BL}(\phi \cdot \Lambda_0, \phi \cdot \Lambda_1, \phi \cdot \Lambda_2)$.*
3. *The index μ is a 2-cocycle; if Λ_0, Λ_1, Λ_2, Λ_3 are four Lagrangians, then*

$$\mu_{BL}(\Lambda_1, \Lambda_2, \Lambda_3) - \mu_{BL}(\Lambda_0, \Lambda_2, \Lambda_3) + \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_3) - \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = 0.$$

4. *If $\sigma \in \mathfrak{S}_3$ is a permutation of the indices 0, 1, and 2, then:*

$$\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = \varepsilon(\sigma) \mu_{BL}(\Lambda_\sigma(0), \Lambda_\sigma(1), \Lambda_\sigma(2)).$$

5. *If μ_{KW} denotes the Kashiwara–Wall index of three Lagrangians, then in $W(\Bbbk)$*

$$2\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = \mu_{WK}(\Lambda_0, \Lambda_1, \Lambda_2).$$

Finally, in Section 5 we proceed to study the associated 2-cocycle on the symplectic group, given by fixing a Lagrangian L and defining for any $A, B \in Sp_{2g}(\Bbbk)$, $\mu(A, B) = \mu_{BL}(L, AL, ABL)$. As we have fixed L, the symplectic space is isometric to $L \oplus L^*$ with the standard symplectic form. Let $\mathcal{S}_L \subseteq Sp_{2g}(\Bbbk)$ (resp. $\mathcal{S}_{L^*}$) be the stabilizer of L (resp. L^*) and consider the canonical evaluation and $E\colon \mathcal{S}_L * \mathcal{S}_{L^*} \to Sp_{2g}(\Bbbk)$ from the free product of the two stabilizers to the symplectic group. Almost by definition, an element in $\mathcal{S}_L * \mathcal{S}_{L^*}$ is a sequence of quadratic forms $q_n, q_{n+1}, \ldots, q_m$ alternatively defined on L and on L^*. Such a sequence is called a Sturm sequence in [**1**] and Barge and Lannes show there how to associate to a Sturm sequence a Lagrangian path and hence a Sylvester matrix.

The technology of Sturm sequences and Sylvester matrices allows us to define four canonical functions f_{00}, f_{01}, f_{11}, f_{10} on the free product $\mathcal{S}_L * \mathcal{S}_{L^*}$. The behavior of these functions with respect to the free product shows that there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "f_{01}"] & \mathcal{S}_L * \mathcal{S}_{L^*} \arrow[r] \arrow[dl, "f_{00}"] \arrow[d] & Sp_{2g}(\Bbbk) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & W(\Bbbk) \arrow[r] & \Gamma \arrow[r] & Sp_{2g}(\Bbbk) \arrow[r] & 1
\end{tikzcd}$$

where f_{01}, restricted to the kernel K, is a homomorphism but the retraction f_{00} *is not.* By standard group cohomology arguments, f_{00} defines a 2-cocycle for the bottom extension, and by construction of the functions this is in fact μ_{BL}. Finally, by analyzing the image of f_{00}, we show that its reduction mod I^2, where I is the fundamental ideal in the Witt group, is a homomorphism, therefore f_{00} mod I^2 is our desired trivialization. We finally compute explicitly this function for typical elements in the symplectic group.

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2. General background

In all this work we fix a field $\Bbbk$ of characteristic different from 2.

2.1. Witt monoid and Witt group. We recall here some elementary facts about the Witt group of $\Bbbk$; for more ample information and in particular for the proofs of the results presented we refer the reader to [**12**] or [**8**].

Let V denote a finite dimensional $\Bbbk$-vector space and denote by $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ its dual. Let $\mathrm{Sym}(\Bbbk)$ denote the set of all symmetric bilinear forms defined on finite dimensional $\Bbbk$-vector spaces up to isometry. An element in $\mathrm{Sym}(\Bbbk)$ is represented by a pair (P, q), where P is a $\Bbbk$-vector space of finite dimension and $q\colon P \to P^*$ is a $\Bbbk$-linear map that coincides with its dual map $q^*\colon P^{**} \to P^*$ up to the canonical identification $P \simeq P^{**}$. By definition, the space P is the *support* of q; if clear from the context, we will omit the support from the notation and write simply q for (P, q).

Recall that a symmetric bilinear form (P, q) is *non-degenerate* if and only if $q\colon P \to P^*$ is an isomorphism, and that a symmetric bilinear form (P, q) is *neutral* if and only if it is non-degenerate and there exists a sub-vector space $I \subset P$ that coincides with its own orthogonal $I = I^{\perp}$.

The orthogonal sum of symmetric bilinear forms, which we denote by $\perp$, endows the set $\mathrm{Sym}(\Bbbk)$ with the structure of a commutative monoid. The isometry classes of neutral forms determine a sub-monoid $\mathrm{Neut}(\Bbbk) \subseteq \mathrm{Sym}(\Bbbk)$; the quotient $MW(\Bbbk) = \mathrm{Sym}(\Bbbk)/\mathrm{Neut}(\Bbbk)$ is by definition the *Witt monoid* of $\Bbbk$. More precisely two symmetric bilinear forms (P_1, q_1) and (P_2, q_2) in $\mathrm{Sym}(\Bbbk)$ are equivalent if and only if there exist two neutral forms (N_1, n_1) and (N_2, n_2) such that $q_1 \perp n_1$ and $q_2 \perp n_2$ are isometric.

The Witt *group* $W(\Bbbk)$ is the image in $MW(\Bbbk)$ of the sub-monoid of $\mathrm{Sym}(\Bbbk)$ generated by the symmetric non-degenerate bilinear forms. Given that for a non-degenerate form q the orthogonal sum $q\perp -q$ is neutral, we get indeed a group structure: the inverse of q is $-q$. In addition, the tensor product of bilinear forms endows $W(\Bbbk)$ with a commutative multiplication compatible with the orthogonal sum and endows $W(\Bbbk)$ with the structure of a unital commutative ring.

By definition, there is an injection $W(\Bbbk) \hookrightarrow MW(\Bbbk)$ and, as $\Bbbk$ is a field, we have a canonical retraction $MW(\Bbbk) \to W(\Bbbk)$ called the *regularization* map; it sends the symmetric bilinear form (P,q) onto the induced form on the quotient $P/P^\perp$, where $P^\perp = \ker q$ is the *radical* of α. For a general symmetric bilinear form (P,q), its class $\overline{(P,q)} \in W(\Bbbk)$ will always refer to the class of its regularized form. Finally, given a unit $a \in \Bbbk^\times$ we will denote by $\langle a\rangle$ the bilinear form on $\Bbbk$ with associated matrix $[a]$.

The ring $W(\Bbbk)$ has a unique maximal ideal I such that $W(\Bbbk)/I = \mathbb{Z}/2$, its *fundamental ideal*. This is the kernel of the map "rank mod 2": $I = \ker(W(\Bbbk) \to \mathbb{Z}/2)$, that sends a bilinear form onto the mod 2 reduction of the dimension of its support; as neutral forms have even rank this is indeed a well-defined map. It is known that the fundamental ideal I is generated by the Pfister forms $\langle 1, -\lambda\rangle = \langle 1\rangle \perp \langle -\lambda\rangle$, for λ a unit in $\Bbbk$, [**8**, p. 316].

In this work we will mainly be interested in the quotient $W(\Bbbk)/I^2$, which is by definition part of an extension of abelian groups

$$(*) \qquad 0 \longrightarrow I/I^2 \longrightarrow W(\Bbbk)/I^2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The kernel I/I^2 is isomorphic to the multiplicative group of units in $\Bbbk$ up to the squares, $I/I^2 = \Bbbk^\times/(\Bbbk^\times)^2$ via the discriminant map, $\mathrm{dis}\colon W(\Bbbk) \to \Bbbk^\times/(\Bbbk^\times)^2$ that sends a non-degenerate bilinear form q of rank r onto $(-1)^{\frac{r(r-1)}{2}} \det(q)$. An elementary but important fact for the present work is that this exact sequence does not usually split. For instance, the quadratic form reduction by Gauss's method implies that $W(\mathbb{R}) \simeq \mathbb{Z}$; then $I = 2\mathbb{Z}$ and in this case $(*)$ is:

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Even so, for any field $\Bbbk$ the pull-back of the extension $(*)$ along the canonical map $\mathbb{Z}/4 \to \mathbb{Z}/2$ always splits:

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, equal] & \widetilde{W} \arrow[r] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d] \arrow[dl, dotted] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r] & W(\Bbbk)/I^2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

A section of the pull-back is induced by the dotted morphism $\mathbb{Z}/4 \to W(\Bbbk)/I^2$ that sends $n \in \mathbb{Z}/4$ to the diagonal form $n\langle 1\rangle$. To check that this map is well defined, observe that for all integers m the rank of $4m\langle 1\rangle$ is even, and its discriminant is $(-1)^{\frac{4m\times(4m-1)}{2}} = 1$.

Combined with the isomorphism $I/I^2 \simeq \Bbbk^\times/(\Bbbk^\times)^2$, the section induces a surjective morphism of groups:

$$\begin{aligned} F\colon \Bbbk^\times/(\Bbbk^\times)^2 \oplus \mathbb{Z}/4 &\longrightarrow W(\Bbbk)/I^2 \\ (\lambda, n) &\longmapsto \langle 1, -\lambda\rangle \oplus n\langle 1\rangle. \end{aligned}$$

We will use this map in our explicit computations of the trivialization of the Maslov cocycle.

2.2. Lagrangian combinatorics. Fix an integer $g \geq 1$ and let $L = \Bbbk^g$. Denote by $H(L)$ the $\Bbbk$-vector space $L \oplus L^*$ together with the alternating bilinear form ω, known as the *symplectic form*,

$$\omega((x,\xi),(y,\eta)) = \xi(y) - \eta(x).$$

The group of isometries of ω is by definition the symplectic group $Sp_{2g}(\Bbbk)$. Recall that a *Lagrangian* in the symplectic space $H(L)$ is a sub-vector space $\Lambda \subseteq H(L)$ that coincides with its own orthogonal $\Lambda = \Lambda^{\perp}$. In the set $\mathcal{L}_g$ of all Lagrangians in $H(L)$, we have two canonical elements: L and L^*. If X and X' are two Lagrangians in $H(L)$, we will say that X and X' are *transverse* if and only if $X + X' = H(L)$; we will then write $X \pitchfork X'$. For dimensional reasons this is equivalent to the fact that $X \cap X' = \{0\}$.

The symplectic group has a natural left action on the set $\mathcal{L}_g$ that is transitive, cf. for instance [**17**, §2]; the proof for $\Bbbk = \mathbb{R}$ is valid for any field. Let $\mathcal{S}_L$ denote the point-wise stabilizer of the Lagrangian L; by direct inspection, elements in this group, when written by blocks according to the decomposition $H(L) = L \oplus L^*$, have the shape

$$E(q) = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix},$$

where $q\colon L^* \to L$ satisfies $q = q^*$. In particular $\mathcal{S}_L$ is canonically isomorphic to the $\Bbbk$-vector space of symmetric bilinear forms with support L^*. Similarly, $\mathcal{S}_{L^*}$, the point-wise stabilizer of L^*, consists of those matrices of the form

$$E(q') = \begin{pmatrix} 1 & 0 \\ q' & 1 \end{pmatrix},$$

where $q'\colon L \to L^*$ is symmetric and is isomorphic to the vector space of symmetric bilinear forms with support L.

Let Λ be a fixed Lagrangian, and denote by $\mathcal{L}_{g,\pitchfork\Lambda}$ the set of those Lagrangians that are transverse to Λ. Because the canonical action of the symplectic group preserves transversality, we have induced canonical actions of $\mathcal{S}_L$ on $\mathcal{L}_{g,\pitchfork L}$ and of $\mathcal{S}_{L^*}$ on $\mathcal{L}_{g,\pitchfork L^*}$. These actions are both simple transitive, and hence the bijections

$$\begin{array}{rcl} \mathcal{S}_L & \longrightarrow & \mathcal{L}_{g,\pitchfork L} \\ \left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right) & \longmapsto & \left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right) L^* \end{array}, \qquad \begin{array}{rcl} \mathcal{S}_{L^*} & \longrightarrow & \mathcal{L}_{g,\pitchfork L^*} \\ \left(\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right) & \longmapsto & \left(\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right) L \end{array}$$

endow the sets $\mathcal{L}_{g,\pitchfork L}$ and $\mathcal{L}_{g,\pitchfork L^*}$ of the structure of an affine set over the corresponding stabilizer group. If we fix an arbitrary Lagrangian Λ, the above discussion leads to two different affine structures on $\mathcal{L}_{g,\pitchfork\Lambda}$ over the vector space of symmetric bilinear forms with support Λ^*: one where elements in $\mathcal{S}_\Lambda$ are written according to the decomposition $H = \Lambda \oplus \Lambda^*$, and one where they are written according to the decomposition $H = \Lambda^* \oplus \Lambda$. To fix this ambiguity we chose the action corresponding to the decomposition $\Lambda^* \oplus \Lambda$. As a consequence, when considering bilinear forms with support L, we have to conjugate the described action of $\mathcal{S}_{L^*}$ on $\mathcal{L}_{g,\pitchfork L^*}$ by the symplectic matrix $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$ and this introduces a sign; with this convention the Lagrangian $\left(\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right) L$ is the translation of L along $-q$ and $\left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right) L^*$ is the translation of $L^* \in \mathcal{L}_{g,\pitchfork L}$ by p.

The affine space structure tells us that given two elements $M, N \in \mathcal{L}_{g,\pitchfork\Lambda}$, their difference $M - N = d_\Lambda(N, M)$ is a well-defined symmetric bilinear form with support Λ^*. For any two Lagrangians there is yet another natural map, first considered by Leray [**9**].

Lemma 2.1. *Let Λ and M be two Lagrangians. The evaluation map*

$$\begin{aligned} e_{\Lambda,M}\colon \Lambda\times M &\longrightarrow \Bbbk \\ (\ell,m) &\longmapsto \omega(\ell,m) \end{aligned}$$

is a bilinear form; it induces by adjunction a linear map $\beta_{\Lambda,M}\colon \Lambda\to M^*$ that satisfies the following properties:

(1) $\beta_{\Lambda,M}=-\beta^*_{M,\Lambda}$.
(2) $\operatorname{Im}\beta_{\Lambda,M}=H(L)/(\Lambda+M)$.
(3) $\ker\beta_{\Lambda,M}=\Lambda\cap M$.
(4) $\beta_{\Lambda,M}$ *is invertible if and only if* $\Lambda\pitchfork M$.
(5) *If* $\phi\in Sp_{2g}(\Bbbk)$, *then* $\beta_{\Lambda,M}=(\phi|_M)^*\circ\beta_{\phi\Lambda,\phi M}\circ\phi|_\Lambda$.

Proof: These are all immediate computations. □

The following result explains the links between the maps β and d.

Lemma 2.2. *Let X be a Lagrangian in $H(L)$, and let $\Lambda,M\in\mathcal{L}_{g,\pitchfork X}$. The difference between these two Lagrangians is $d_X(\Lambda,M)\in\mathcal{S}_{X^*}$ and we have that*

$$d_X(\Lambda,M)=-(\beta_{X,M})^{-1}\beta_{\Lambda,M}\beta^{-1}_{\Lambda,X}.$$

Proof: Consider the commutative diagram

$$\begin{tikzcd} & M \arrow[d, "i_M"'] \arrow[dr, "\beta_{M,X}"] & \\ X \arrow[r, "i_X"] & H(L) \arrow[r, "p_X"] & X^* \\ & \Lambda \arrow[u, "i_\Lambda"] \arrow[ur, "\beta_{\Lambda,X}"'] & \end{tikzcd}$$

Since X is transverse to both M and Λ, the morphisms $\beta_{M,X}$ and $\beta_{\Lambda,X}$ are invertible, and this gives us two sections of the middle exact sequence: $\sigma_M=i_M\circ\beta^{-1}_{M,X}$ and $\sigma_\Lambda : i_\Lambda\circ\beta^{-1}_{\Lambda,X}$. Because Λ and M are both Lagrangians, the symplectic form pulled back along any of the above two sections to X^* is the zero form.

Then, by definition, $d_X(\Lambda,M)=i_M\circ\beta^{-1}_{M,X}-i_\Lambda\circ\beta^{-1}_{\Lambda,X}\colon X^*\to X$. More explicitly, if $\alpha\in X^*$, there exists a unique couple $(m,\ell)\in M\times\Lambda$ such that $\alpha=\omega(m,-)=\omega(\ell,-)$ and $d_X(\Lambda,M)(\alpha)=m-\ell$. By direct inspection, the map

$$\beta_{X,M}d_X(\Lambda,M)\beta_{\Lambda,X}\colon \Lambda\longrightarrow M^*$$

sends an element $\ell\in\Lambda$ to the linear map that on $\mu\in M$ evaluates to $\omega(m-\ell,\mu)=-\omega(\ell,\mu)$.

Otherwise said, we have a commutative diagram

$$\begin{tikzcd} X^*\times X^* \arrow[r, "d"] & \Bbbk \arrow[d, equal] \\ \Lambda\times M \arrow[u, "\beta_{\Lambda,X}\times\beta_{M,X}"] \arrow[r, "e"', "\Lambda{,}M"] & \Bbbk \end{tikzcd}$$

where $d\colon X^*\times X^*\to R$ is the adjoint to $d_X(\Lambda,M)\colon X^*\to X$. □

Lemma 2.3. *Let Λ_1 and Λ_2 be two transverse Lagrangians and X a third arbitrary Lagrangian. Denote by p_1 (resp. p_2) the projection map from X onto Λ_1 (resp. Λ_2) parallel to Λ_2 (resp. Λ_1). Then*

$$p_1 = (\beta_{\Lambda_1,\Lambda_2})^{-1}\beta_{X,\Lambda_2} \quad \textit{and} \quad p_2 = (\beta_{\Lambda_2,\Lambda_1})^{-1}\beta_{X,\Lambda_1}.$$

Proof: Direct computation. □

Lemma 2.4. *Let Λ_1, Λ_2, Λ_3 denote three Lagrangians such that $\Lambda_1 \pitchfork \Lambda_2$ and $\Lambda_2 \pitchfork \Lambda_3$. Let X be an arbitrary Lagrangian. Then*

$$\beta_{X,\Lambda_3} = \beta_{\Lambda_1,\Lambda_3}(\beta_{\Lambda_1,\Lambda_2})^{-1}\beta_{X,\Lambda_2} + \beta_{\Lambda_2,\Lambda_3}(\beta_{\Lambda_2,\Lambda_1})^{-1}\beta_{X,\Lambda_1}.$$

Proof: By direct inspection, with the same notations as in Lemma 2.3,

$$\beta_{X,\Lambda_3} = \beta_{\Lambda_1,\Lambda_3}p_1 + \beta_{\Lambda_2,\Lambda_3}p_2$$

and replacing p_1 and p_2 with their expressions we get the result. □

We now arrive at the fundamental relation.

Proposition 2.5. *With the same hypothesis as in Lemma 2.4,*

$$-(\beta_{\Lambda_2,\Lambda_1})^{-1}\beta_{X,\Lambda_1} + d_{\Lambda_2}(\Lambda_1,\Lambda_3)\beta_{X,\Lambda_2} + (\beta_{\Lambda_2,\Lambda_3})^{-1}\beta_{X,\Lambda_3} = 0.$$

Proof: By composing the relation from Lemma 2.4 by $(\beta_{\Lambda_2,\Lambda_3})^{-1}$ to the left we get:

$$(\beta_{\Lambda_2,\Lambda_3})^{-1}\beta_{X,\Lambda_3} - (\beta_{\Lambda_2,\Lambda_1})^{-1}\beta_{X,\Lambda_1} - (\beta_{\Lambda_2,\Lambda_3})^{-1}\beta_{\Lambda_1,\Lambda_3}(\beta_{\Lambda_1,\Lambda_2})^{-1}\beta_{X,\Lambda_2} = 0$$

and from Lemma 2.2

$$-(\beta_{\Lambda_2,\Lambda_3})^{-1}\beta_{\Lambda_1,\Lambda_3}(\beta_{\Lambda_1,\Lambda_2})^{-1} = d_{\Lambda_2}(\Lambda_1,\Lambda_2),$$

whence the result. □

3. Lagrangian paths and Sturm sequences

In this section we reformulate some results from Appendix A in [**1**] in terms of Lagrangian paths based at either L or L^*.

Definition 3.1. Let Λ and M be two Lagrangians. A *Lagrangian path* of length n joining Λ to M is a sequence of $n+2$ Lagrangians

$$\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_{n+1} = M$$

such that for each $0 \leq i \leq n$ we have $\Lambda_i \pitchfork \Lambda_{i+1}$.

If $\Lambda = M$, we call this a *Lagrangian loop* based at Λ.

If $\alpha : (\Lambda_0, \ldots, \Lambda_n, M)$ and $\beta : (M, \Lambda_1', \ldots, \Lambda_{m+1})$ are two Lagrangian paths, the first ending at M and the second starting at M, then their concatenation is the Lagrangian path $\alpha * \beta : (\Lambda_0, \ldots, \Lambda_n, M, \Lambda_1', \ldots, \Lambda_{m+1})$.

An arbitrary Lagrangian path has an associated Sylvester matrix:

Definition 3.2. Let $\alpha : (\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_{n+1})$ be a Lagrangian path. The Sylvester matrix of α is the matrix $S(\alpha)$ of the symmetric bilinear map with support $\Lambda_1^* \oplus \cdots \oplus \Lambda_n^*$ whose block coefficients are given as follows:

$$\begin{cases} a_{i,i} = d_{\Lambda_i}(\Lambda_{i-1}, \Lambda_{i+1}) & \text{for } i = 1, \ldots, n, \\ a_{i+1,i} = (\beta_{\Lambda_{i+1},\Lambda_i})^{-1} \colon \Lambda_i^* \longrightarrow \Lambda_{i+1} & \text{for } i = 1, \ldots, n-1, \\ a_{i,i+1} = a_{i+1,i}^* = -(\beta_{\Lambda_i,\Lambda_{i+1}})^{-1}, & \end{cases}$$

and all other coefficients are zero.

Writing $b_{\Lambda_{i+1},\Lambda_i} = (\beta_{\Lambda_{i+1},\Lambda_i})^{-1}$, the matrix $S(\alpha)$ is a trigonal matrix of the following form:

$$\begin{pmatrix} d_{\Lambda_1}(\Lambda_0,\Lambda_2) & b_{\Lambda_1,\Lambda_2} & 0 & \cdots & 0 \\ -b_{\Lambda_2,\Lambda_1} & d_{\Lambda_2}(\Lambda_1,\Lambda_3) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{\Lambda_{n-1},\Lambda_n} \\ 0 & \cdots & 0 & -b_{\Lambda_n,\Lambda_{n-1}} & d_{\Lambda_n}(\Lambda_{n-1},\Lambda_{n+1}) \end{pmatrix}.$$

The map that associates its Sylvester matrix to a Lagrangian path has an obvious equivariance property, which we formally state for future reference. The proof is trivial.

Lemma 3.3. *Let $\alpha : (\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_{n+1})$ be a Lagrangian path and $\phi \in Sp_{2g}(\Bbbk)$, then $\phi_*(\alpha) : (\phi(\Lambda_0), \phi(\Lambda_1), \ldots, \phi(\Lambda_n), \phi(\Lambda_{n+1}))$ is again a Lagrangian path, and the map ϕ induces a canonical isometry between the associated Sylvester matrices $S(\alpha)$ and $S(\phi_*(\alpha))$.*

The following is a first result showing how much of the properties of a Lagrangian path is encoded in its Sylvester matrix.

Proposition 3.4. *Let $\alpha : (\Lambda_0, \ldots, \Lambda_{n+1})$ be a Lagrangian path and denote by $S(\alpha)$ its Sylvester matrix. To reduce the amount of notation, we write $\beta_{i,j}$ for $\beta_{\Lambda_i,\Lambda_j}$ and we consider the morphism*

$$E_0 \colon \Lambda_0 \longrightarrow \Lambda_1^* \oplus \cdots \oplus \Lambda_n^*,$$

whose matrix is the transpose of the row

$$\begin{bmatrix} \beta_{0,1} & \beta_{0,2} & \cdots & \beta_{0,n} \end{bmatrix}$$

and

$$T_n \colon \Lambda_{n+1}^* \longrightarrow \Lambda_1 \oplus \cdots \oplus \Lambda_n,$$

whose matrix is the transpose of the row

$$\begin{bmatrix} 0 & \cdots & 0 & -(\beta_{n,n+1})^{-1} \end{bmatrix}.$$

Then

1. *Both morphisms E_0 and T_n are injective.*
2. *The following diagram, viewed as a chain map between horizontal chain complexes, is a homotopy equivalence.*

$$\begin{tikzcd} \Lambda_0 \arrow[r, "\beta_{0,n+1}"] \arrow[d, "E_0"] & \Lambda_{n+1}^* \arrow[d, "T_n"] \\ \Lambda_1^* \oplus \cdots \oplus \Lambda_n^* \arrow[r, "S(\alpha)"'] & \Lambda_1 \oplus \cdots \oplus \Lambda_n \end{tikzcd}$$

Proof: (1) Since $\Lambda_0 \pitchfork \Lambda_1$, the morphism $\beta_{0,1}$ is an isomorphism, and as it is the first component of the matrix of E_0 this is an injective morphism. The morphism T_0 is also injective: its only non-zero component is an isomorphism because the underlying Lagrangians that determine it are transverse. We now check the commutativity of the diagram, and for this we first compute $S(\alpha)E_0$. Again to reduce the amount of notation, we write $d_j(j-1, j+1)$ instead of $d_{\Lambda_j}(\Lambda_{j-1}, \Lambda_{j+1})$.

The first coefficient of the column matrix $S(\alpha)E_0$ is

$$d_1(0,2)\beta_{0,1} + \beta_{1,2}^{-1}\beta_{0,2}.$$

As $\Lambda_0 \pitchfork \Lambda_1$, we know that $\beta_{0,1}$ is invertible, and Lemma 2.2 shows that the coefficient is in fact zero.

For $2 \leq i \leq n-1$, the i-th coefficient is

$$-\beta_{i,i-1}^{-1}\beta_{0,i-1} + d_i(i-1,i+1)\beta_{0,i} + \beta_{i,i+1}^{-1}\beta_{0,i+1},$$

which is zero according to Proposition 2.5.

Finally, the last coefficient is equal to

$$-\beta_{n,n-1}\beta_{0,n-1} + d_n(n-1,n+1)\beta_{0,n},$$

which, by Proposition 2.5, is equal to

$$-\beta_{n,n+1}^{-1}\beta_{0,n+1}.$$

And this is exactly the unique non-zero coefficient of the column matrix $T_n\beta_{0,n+1}$.

(2) Now consider the projection over the first $n-1$ coordinates:

$$p\colon \Lambda_1 \oplus \cdots \oplus \Lambda_n \longrightarrow \Lambda_1 \oplus \cdots \oplus \Lambda_{n-1}$$

and the map

$$q\colon \Lambda_1^* \oplus \cdots \oplus \Lambda_n^* \longrightarrow \Lambda_2^* \oplus \cdots \oplus \Lambda_n^*,$$

which is given by the matrix

$$\begin{bmatrix} -\beta_{0,2}\beta_{0,1}^{-1} & 1 & 0 & \cdots & 0 \\ -\beta_{0,3}\beta_{0,1}^{-1} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\beta_{0,n}\beta_{0,1}^{-1} & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The identity block on the right-hand side of the matrix shows that q is surjective and a direct computation shows that $qE_0 = 0$, hence for dimensional reasons $q = \operatorname{coker} E_0$. Furthermore, trivially, $p = \operatorname{coker} T_n$.

Finally, let $s\colon \Lambda_2^* \oplus \cdots \oplus \Lambda_n^* \to \Lambda_1 \oplus \cdots \oplus \Lambda_{n-1}$ be the matrix obtained by erasing the first row and the first column in $S(\alpha)$. We then have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \Lambda_0 \arrow[r, "E_0"] \arrow[d, "\beta_{0,n+1}"] & \Lambda_1^* \oplus \cdots \oplus \Lambda_n^* \arrow[r, "q"] \arrow[d, "S(\alpha)"] & \Lambda_2^* \oplus \cdots \oplus \Lambda_n^* \arrow[r] \arrow[d, "s"] & 0 \\ 0 \arrow[r] & \Lambda_{n+1}^* \arrow[r, "T_n"] & \Lambda_1 \oplus \cdots \oplus \Lambda_n \arrow[r, "p"] & \Lambda_1 \oplus \cdots \oplus \Lambda_{n-1} \arrow[r] & 0 \end{tikzcd}$$

The snake lemma then shows that the morphism of chain complexes (E_0, T_n) is a quasi-isomorphism if and only if s is an isomorphism. Since the complexes we consider are complexes of $\Bbbk$-vector spaces, any quasi-isomorphism is in fact a homotopy equivalence.

We are left with showing that s is invertible. The identity block on the right-hand side of the matrix q imposes, by direct computation, that s has to be the matrix that we get from $S(\alpha)$ by erasing its first column and last row:

$$s = \begin{pmatrix} \beta_{1,2}^{-1} & 0 & \cdots & \cdots & 0 \\ d_2(1,3) & \beta_{2,3}^{-1} & 0 & \cdots & \vdots \\ -\beta_{3,2}^{-1} & d_3(2,4) & \beta_{3,4}^{-1} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \cdots & 0 & -\beta_{n-1,n-2}^{-1} & d_{n-1}(n-2,n) & \beta_{n-1,n}^{-1} \end{pmatrix}.$$

Let us now show that $pS(\alpha) = sq$. The matrix for $pS(\alpha)$ is the matrix for $S(\alpha)$ in which we have erased the last row. Hence the unique non-trivial part in the equality $pS(\alpha) = sq$ is the equality between the last two columns of these matrices. We will distinguish between three cases: the first coefficient, the second coefficient, and finally the coefficient of index j, 1 for $j > 1$.

(i) For the first coefficient, we have

$$d_1(0,2) = -\beta_{1,2}^{-1}\beta_{0,2}\beta_{0,1}^{-1}, \text{ by Lemma 2.2.}$$

(ii) For the second coefficient, we have

$$-d_2(1,3)\beta_{0,3}\beta_{0,1}^{-1} - \beta_{2,3}^{-1}\beta_{0,3}\beta_{0,1}^{-1} = -\beta_{2,1}^{-1}, \text{ by Proposition 2.5.}$$

(iii) Finally, when $j > 1$,

$$\beta_{j,j-1}\beta_{0,j}\beta_{0,1}^{-1} - d_j(j-1,j+1)\beta_{0,j}\beta_{0,1}^{-1} - \beta_{j,j+1}^{-1}\beta_{0,j+1}\beta_{0,1}^{-1} = 0, \text{ by Proposition 2.5.}$$

To conclude, observe that as the matrix s is lower triangular with invertible elements along the diagonal it is invertible. □

Following the same proof but exchanging the roles of Λ_0 and Λ_n one can show:

Proposition 3.5. *Let $\alpha : (\Lambda_0, \dots, \Lambda_{n+1})$ be a Lagrangian path and denote by $S(\alpha)$ its Sylvester matrix. Write $\beta_{i,j}$ for $\beta_{\Lambda_i,\Lambda_j}$ and consider the maps*

$$F_{n+1} \colon \Lambda_{n+1} \longrightarrow \Lambda_1^* \oplus \cdots \oplus \Lambda_n^*$$

with matrix the transpose of the row:

$$\begin{bmatrix} \beta_{n+1,1} & \beta_{n+1,2} & \cdots & \beta_{n+1,n} \end{bmatrix}$$

and

$$U_n \colon \Lambda_0^* \longrightarrow \Lambda_1 \oplus \cdots \oplus \Lambda_n$$

with matrix the transpose of the row:

$$\begin{bmatrix} \beta_{1,0}^{-1} & 0 & \cdots & 0 \end{bmatrix}.$$

Then

(1) *The maps F_{n+1} and U_n are injective.*
(2) *The following diagram, viewed as chain maps between horizontal chain complexes, is a homotopy equivalence.*

$$\begin{tikzcd} \Lambda_{n+1} \arrow[r, "\beta_{n+1,0}"] \arrow[d, "F_{n+1}"] & \Lambda_0^* \arrow[d, "U_n"] \\ \Lambda_1^* \oplus \cdots \oplus \Lambda_n^* \arrow[r, "S(\alpha)"'] & \Lambda_1 \oplus \cdots \oplus \Lambda_n \end{tikzcd}$$

The following result is the translation in the context of general Lagrangian paths of Proposition-Définition A.2.1 in [**1**].

Corollary 3.6. *Let $\alpha : (\Lambda_0, \Lambda_1, \dots, \Lambda_n, \Lambda_{n+1})$ be a Lagrangian path and let $S(\alpha)$ denote its Sylvester matrix. The following are equivalent:*

(1) *The bilinear form $S(\alpha)$ is non-degenerate.*

(2) *The two Lagrangians Λ_0 and Λ_{n+1} are transverse.*

Proof: By Lemma 2.1(4), the map $\beta_{0,n+1}$ is invertible if and only if $L_0 \pitchfork L_{n+1}$, and in the context of Proposition 3.4 this happens if and only if $S(\alpha)$ is invertible too. $\square$

The next lemma is our fundamental tool to compute the Sylvester matrices of long Lagrangian paths.

Lemma 3.7 (Shortcut lemma)**.** *Let $\alpha : (\Lambda_0, \dots, \Lambda_{n+1})$ be a Lagrangian path with Sylvester matrix $S(\alpha)$. We suppose that there are two indices $0 \le i < j \le n+1$ such that $\Lambda_i \pitchfork \Lambda_j$. We then have two more Lagrangian paths:*

(1) *The sub-path $(\Lambda_i, \dots, \Lambda_j)$, with Sylvester matrix $S(\Lambda_i, \dots, \Lambda_j)$.*

(2) *The shortened path $(\Lambda_0, \dots, \Lambda_i, \Lambda_j, \dots, \Lambda_{n+1})$, with Sylvester matrix $S(\Lambda_0, \dots, \Lambda_i, \Lambda_j, \dots, \Lambda_{n+1})$.*

Then

$$S(\alpha) \text{ is isometric to } S(\Lambda_i, \dots, \Lambda_j) \oplus S(\Lambda_0, \dots, \Lambda_i, \Lambda_j, \dots, \Lambda_{n+1}).$$

Proof: We have three cases to consider: the case where $i = 0$, where $j = n+1$, and finally $0 < i < j < n+1$. To simplify our notations, for $0 \le s < t \le n+1$ we denote by $S(s,t)$ the matrix $S(\Lambda_s, \dots, \Lambda_t)$.

(i) $i = 0, j < n+1$.

The support of the matrix $S(0, n+1)$ is

$$\Lambda_1^* \oplus \Lambda_{j-1}^* \oplus \Lambda_j^* \oplus \cdots \oplus \Lambda_n^*$$

and its restriction to $\Lambda_1^* \oplus \cdots \oplus \Lambda_{j-1}^*$ is by construction $S(0,j)$, which is non-degenerate by Corollary 3.6. In particular,

$$S(0, n+1) = S(0,j) \perp S(0,j)^{\perp}.$$

Let us compute the orthogonal of $\Lambda_1^* \oplus \cdots \oplus \Lambda_{j-1}^*$ with respect to $S(0, n+1)$. As this matrix is trigonal, the sub-space $\Lambda_{j+1}^* \oplus \cdots \oplus \Lambda_n^*$ is in the orthogonal. Commutativity of the diagram

$$\begin{tikzcd}
\Lambda_j^* \arrow[dr, "\beta_{0,j}^{-1}"] & & \\
& \Lambda_0 \arrow[r, "\beta_{0,j+1}"] \arrow[d, "E_0"] & \Lambda_{j+1}^* \arrow[d, "{[0,\dots,0,-(\beta_{j,j+1})^{-1}]}"] \\
& \Lambda_1^* \oplus \cdots \oplus \Lambda_j^* \arrow[r, "S(0,j+1)"'] & \Lambda_1 \oplus \cdots \oplus \Lambda_j
\end{tikzcd}$$

shows that the copy of Λ_j^* included in the support of $S(0, n+1)$ via the map $E_{0,j}\beta_{0,j}^{-1}$ is also in the orthogonal of $\Lambda_1^* \oplus \Lambda_{j-1}$; indeed its image under the matrix $S(0, n+1)$ has as first $j-1$ coefficients equal to zero. Since moreover this copy of Λ_j^* is in direct sum with the preceding sub-space, for dimensional reasons the orthogonal is $\Lambda_j^* \oplus \Lambda_{j+1}^* \oplus \cdots \oplus \Lambda_n^*$. Let us now compute the restriction of $S(0, n+1)$ to this sub-space.

Observe that the inclusion of the orthogonal to $\Lambda_j^* \oplus \cdots \oplus \Lambda_n^*$ into the support $S(0,n+1)$ is given by the matrix

$$P = \begin{pmatrix} E_0\beta_{0,j}^{-1} & 0 \\ 0 & \mathrm{Id}_{n-j-1} \end{pmatrix},$$

with domain $\Lambda_j^*\oplus\cdots\oplus\Lambda_n^*$ and codomain $\Lambda_1\oplus\cdots\oplus\Lambda_n$. Let us write the matrix $S(0,n+1)$ by blocks.

$$S = \begin{pmatrix} S(0,j+1) & A_j \\ A_j^* & S(j,n+1) \end{pmatrix},$$

where

$$A_j = \begin{pmatrix} 0 & \\ \vdots & 0 \\ 0 & \\ \beta_{j,j+1}^{-1} & 0\cdots 0 \end{pmatrix}.$$

Then the matrix of $S(0,n+1)$ restricted to $\Lambda_j^*\oplus\cdots\oplus\Lambda_n^*$ is

$$P^*S(0,n+1)P = \begin{pmatrix} \beta_{0,j}^{-1*}E_0^*S(0,j+1)E_0\beta_{0,j}^{-1} & \beta_{0,j}^{-1*}E_0^*A_j \\ A_j^*E_0\beta_{0,j}^{-1} & S(j,n+1) \end{pmatrix}.$$

To compute the top left corner of the matrix, we use the commutative diagram given by Proposition 3.4:

$$\begin{tikzcd}
\Lambda_j^* \arrow[dr, "\beta_{0,j}^{-1}"] & & & & \\
& \Lambda_0 \arrow[r, "\beta_{0,j+1}"] \arrow[d, "E_0"] & \Lambda_{j+1}^* \arrow[d, "T_n"] & & \\
\Lambda_1^*\oplus\cdots\oplus\Lambda_j^* & \arrow[l, phantom] \arrow[r, "S(0,j+1)"'] & \Lambda_1\oplus\cdots\oplus\Lambda_j \arrow[r, "E_0^*"] & \Lambda_0^* \arrow[r, "\beta_{0,j}^{-1*}"] & \Lambda_j
\end{tikzcd}$$

which shows that

$$\begin{aligned}
\beta_{0,j}^{-1*}E_0^*S(0,j+1)E_0\beta_{0,j}^{-1} &= \beta_{0,j}^{-1*}E_0^*T_n\beta_{0,j+1}\beta_{0,j}^{-1} \\
&= -\beta_{j,0}^{-1}\beta_{0,j}^*(-\beta_{j,j+1}^{-1})\beta_{0,j+1}\beta_{0,j}^{-1} \\
&= -\beta_{j,0}^{-1} - \beta_{j,0}(-\beta_{j,j+1}^{-1})\beta_{0,j+1}\beta_{0,j}^{-1} \\
&= (-\beta_{j,j+1}^{-1})\beta_{0,j+1}\beta_{0,j}^{-1} \\
&= d_j(0,j+1), \text{ by Lemma 2.2.}
\end{aligned}$$

In the same way

$$\begin{aligned}
\beta_{0,j}^{-1*}E_0^*A_j &= [\beta_{j,0}^{-1}\beta_{1,0},\ldots,\beta_{j,0}^{-1}\beta_{j,0}]A_j \\
&= [\beta_{j,j+1}^{-1},0,\ldots,0]
\end{aligned}$$

and so

$$P^*S(0,n+1)P = \begin{pmatrix} d_j(0,j+1) & \beta_{j,0}^{-1}\beta_{j,0}\beta_{j,j+1}^{-1},0,\ldots,0 \\ -\beta_{j+1,j} & \\ 0 & \\ \vdots & S(j,n+1) \\ 0 & \end{pmatrix} = S(0,j,\ldots,n+1).$$

(ii) For the case where $i < n, j = n+1$ we proceed in a similar way. The same argument as before shows that

$$S(0,n+1) = S(i,n+1)^{\perp} \perp S(i,n+1),$$

and that the sub-space $\Lambda_1^* \oplus \cdots \oplus \Lambda_{i-1}^*$ is in $S(i,n+1)^{\perp}$, and the commutative diagram

$$\begin{tikzcd}
\Lambda_i^* \arrow[dr, "\beta_{n+1,i}^{-1}"] & & \\
& \Lambda_{n+1} \arrow[r, "\beta_{n+1,i-1}"] \arrow[d, "F_{n+1}"] & \Lambda_{i-1}^* \arrow[d, "{[\beta_{i,i-1}^{-1},0,\ldots,0]}"] \\
& \Lambda_i^* \oplus \cdots \oplus \Lambda_n^* \arrow[r, "S(i-1,n+1)"'] & \Lambda_i \oplus \cdots \oplus \Lambda_n
\end{tikzcd}$$

shows that the copy of Λ_i included via $F_{n+1}\beta_{n+1,i}^{-1}$ is in the orthogonal too.

Then, if we denote by

$$Q = \begin{pmatrix} \mathrm{Id}_i & 0 \\ 0 & F_{n+1}\beta_{n+1,i}^{-1} \end{pmatrix}$$

the inclusion of $\Lambda_1^* \oplus \cdots \oplus \Lambda_{i-1}^* \oplus \Lambda_i^*$ as the orthogonal to $S(i,n+1)$, we have:

$$Q^* S(0,n+1) Q = \begin{pmatrix} S(0,i) & A_{i-1}F_{n+1}\beta_{n+1,i}^{-1} \\ \beta_{i,n+1}^{-1}F_{n+1}^*A_{i-1}^* & \beta_{n+1,i}^{-1*}F_{n+1}^*S(i-1,n+1)F_{n+1}\beta_{n+1,i}^{-1} \end{pmatrix};$$

and, as before,

$$A_{i-1}F_{n+1}\beta_{n+1,i}^{-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{i-1,i} \end{pmatrix}$$

and using the above commutative diagram:

$$\begin{aligned}
Q^*S(i-1,n+1)Q &= \beta_{n+1,i}^{-1*}F_{n+1}^*S(i-1,n+1)F_{n+1}\beta_{n+1,i}^{-1} \\
&= -\beta_{i,n+1}^{-1}\beta_{i,n+1}\beta_{i,i-1}^{-1}\beta_{n+1,i-1}\beta_{n+1,i}^{-1} \\
&= \beta_{i,i-1}^{-1}\beta_{n+1,i-1}\beta_{n+1,i}^{-1} \\
&= -d_i(n+1,i-1) \\
&= d_i(i-1,n+1)
\end{aligned}$$

and hence

$$Q^*S(0,n+1)Q = \begin{pmatrix} & 0 \\ S(0,i) & \vdots \\ & 0 \\ & \beta_{i-1,i}^{-1} \\ 0\cdots 0 - \beta_{i,i-1}^{-1} & d_i(i-1,n+1) \end{pmatrix} = S(0,\ldots,i,n+1).$$

(iii) We are left the case where $0 < i < j < n+1$. Observe that if $j = i+1$, we have nothing to prove, and so assume that $i+1 < j$ and as before we start by computing $S(i,j)^{\perp}$.

Here the orthogonal is given by two sub-spaces. First, as before, the Lagrangians with index too far away from i and j are in the orthogonal:

$$(\Lambda_1^* \oplus \cdots \oplus \Lambda_{i-1}^*) \oplus (\Lambda_{j+1}^* \oplus \cdots \oplus \Lambda_n^*) \subset S(i,j)^\perp.$$

The same computation as in the first case shows also that the sub-space Λ_i^* included via the composite

$$\Lambda_i^* \xrightarrow{\beta_{j,i}^{-1}} \Lambda_j \xrightarrow{F_j} \Lambda_i^* \oplus \cdots \oplus \Lambda_{j-1}^*$$

is in the orthogonal; and as in the second case, the sub-spaces Λ_j^* included via

$$\Lambda_j^* \xrightarrow{\beta_{i,j}^{-1}} \Lambda_i \xrightarrow{E_i} \Lambda_{i+1}^* \oplus \cdots \oplus \Lambda_j^*$$

are in the orthogonal too. This orthogonal $S(i,j)^\perp$ is therefore isomorphic to $(\Lambda_1^* \oplus \cdots \oplus \Lambda_i^*) \oplus (\Lambda_j^* \oplus \cdots \oplus \Lambda_n^*)$, and the matrix of the inclusion of this sub-space into the support of $S(0,n+1)$ is given, writing blocks according to the decomposition of the source into $(\Lambda_1^* \oplus \cdots \oplus \Lambda_{i-1}^*) \oplus (\Lambda_i^* \oplus \Lambda_j^*) \oplus (\Lambda_{j+1}^* \oplus \cdots \oplus \Lambda_n^*)$, by

$$J = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

where E, written by blocks of size $2\times(j-i)$, with domain $\Lambda_i^* \oplus \Lambda_j^*$ and codomains $\Lambda_i^* \oplus \cdots \oplus \Lambda_j^*$, is

$$E = \left(\begin{array}{c|c} & 0 \\ F_j\beta_{j,i}^{-1} & \\ & E_i\beta_{i,j}^{-1} \\ 0 & \end{array}\right).$$

To compute the restriction of $S(0,n+1)$ to the $S(i,j)^\perp$, we write the matrix $S(0,n+1)$ by blocks according to the decomposition of the source as $(\Lambda_1^* \oplus \cdots \oplus \Lambda_{i-1}^*) \oplus (\Lambda_i^* \oplus \cdots \oplus \Lambda_j^*) \oplus (\Lambda_{j+1}^* \oplus \cdots \oplus \Lambda_n^*)$

$$S(0,n+1) = \begin{pmatrix} S(0,i) & A_{i-1} & 0 \\ A_{i-1}^* & S(i-1,j+1) & A_j \\ 0 & A_j^* & S(j,n+1) \end{pmatrix}.$$

The product $J^*S(0,n+1)J$ is:

$$\begin{pmatrix} S(0,i) & A_{i-1}E & 0 \\ E^*A_{i-1}^* & E^*S(i-1,j+1)E & E^*A_j \\ 0 & A_j^*E & S(j,n+1) \end{pmatrix}.$$

A first direct computation shows that

$$A_{i-1}E = \begin{pmatrix} 0 & 0 \\ -\beta_{i-1,i}^{-1} & 0 \end{pmatrix} \quad \text{and} \quad A_j^*E = \begin{pmatrix} 0 & \beta_{j+1,j}^{-1} \\ 0 & 0 \end{pmatrix}.$$

The diagrams in Propositions 3.4 and 3.5 show that

$$S(i-1,j+1)E = \begin{pmatrix} \beta_{i+1,i}^{-1}\beta_{j,i}\beta_{j,i}^{-1} & -\beta_{i,i+1}^{-1}\beta_{i,i+1}\beta_{i,j}^{-1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta_{j,j-1}^{-1}\beta_{j,j-1}\beta_{j,i}^{-1} & \beta_{j,j+1}^{-1}\beta_{i,j+1}\beta_{i,j}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \beta_{i+1,i}^{-1} & -\beta_{i,j}^{-1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta_{j,i}^{-1} & \beta_{j,j+1}^{-1}\beta_{i,j+1}\beta_{i,j}^{-1} \end{pmatrix}$$

and analogous computations to those carried out in the first two parts show that

$$E^*S(i-1,j+1)E = \begin{pmatrix} -\beta_{i,j}^{-1}F_j^*S(i-1,j)F_j\beta_{j,i}^{-1} & -\beta_{i,j}^{-1}\beta_{i,j}\beta_{i,j}^{-1} \\ -\beta_{j,i}^{-1} - \beta_{j,i}\beta_{j,j-1}^{-1}\beta_{j,j-1}\beta_{j,i}^{-1} & -\beta_{j,i}^{-1}E_i^*S(i,j+1)E_i\beta_{i,j}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} d_i(i-1,j) & -\beta_{i,j}^{-1} \\ \beta_{j,i}^{-1} & d_j(i,j+1) \end{pmatrix}.$$

To sum up, the matrix $J^*S(0,n+1)J$ is equal to

$$\left(\begin{array}{ccc|cc|ccc} & S(0,i) & & & & & 0 & \\ & & & -\beta_{i-1,j}^{-1} & 0 & & & \\ \hline 0 & 0 & \beta_{j,i-1}^{-1} & d_i(i-1,j) & -\beta_{i,j}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_{j,i}^{-1} & d_j(i,j+1) & -\beta_{j,j+1}^{-1} & 0 & 0 \\ \hline & & & 0 & \beta_{j+1,j}^{-1} & & & \\ & 0 & & 0 & 0 & & S(j+1,n+1) & \end{array}\right)$$

and we recognize $S(0\ldots ij\ldots n+1)$. $\square$

The following is an example of the flexibility in manipulating Sylvester matrices provided by the shortcut lemma.

Corollary 3.8. *Let $(\Lambda_0,\Lambda_1,\ldots,\Lambda_n,\Lambda_0)$ be a Lagrangian loop and let M denote a Lagrangian that is transverse to Λ_0. Then in $W(\Bbbk)$ the following four Sylvester matrices are equal:*

(1) $S(\Lambda_0,\Lambda_1,\ldots,\Lambda_n)$.
(2) $S(\Lambda_1,\ldots,\Lambda_n,\Lambda_0)$.
(3) $S(\Lambda_0,\ldots,\Lambda_n,\Lambda_0,M)$.
(4) $S(M,\Lambda_0,\Lambda_1,\ldots,\Lambda_n,\Lambda_0)$.

Proof: We apply the shortcut lemma (Lemma 3.7) to the Lagrangians Λ_0 and Λ_n in the sequence $\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_0, M$; this gives us an isometry:

$$S(\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_0, M) = S(\Lambda_0, \Lambda_1, \ldots, \Lambda_n) \oplus S(\Lambda_0, \Lambda_n, \Lambda_0, M).$$

The matrix $S(\Lambda_0, \Lambda_n, \Lambda_0, M)$ has as support the space $\Lambda_n^* \oplus \Lambda_0^*$ and has the form

$$\begin{pmatrix} d(\Lambda_0, \Lambda_0) & \beta \\ \beta & d(M, \Lambda_0) \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \beta & d(M, \Lambda_0) \end{pmatrix}.$$

Since $\Lambda_0 \pitchfork N$, β is invertible, and this matrix represents a neutral form; this proves that (1) and (3) are equal. The equality between (3) and (2) can be shown similarly using that Λ_1 and the second copy of Λ_0 in the sequence

$$\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_0, M$$

are transverse. Finally, the same argument but using that $\Lambda_0 \pitchfork \Lambda_n$ in the sequence M, $\Lambda_0, \ldots, \Lambda_n, \Lambda_0$ shows that (4) and (1) are equal. $\square$

4. The Maslov index

4.1. Index of a Lagrangian path. For a Lagrangian path the definition of the Maslov index is straightforward.

Definition 4.1. Let $\alpha : (\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_{n+1})$ be a Lagrangian path. The class in $W(\mathbb{k})$ of the Sylvester matrix $S(\alpha)$ is the Maslov index of α; we denote it by $\mathrm{Mas}(\alpha)$, or $\mathrm{Mas}(\Lambda_0, \ldots, \Lambda_{n+1})$ if needed.

For a Lagrangian *loop*, Corollary 3.8 gives four equivalent ways to compute its Maslov index.

Lemma 4.2. *Let* $\omega : (\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_0)$ *be a Lagrangian loop based at* Λ_0*, and let* $M \pitchfork \Lambda$ *be an arbitrary Lagrangian. Then the Maslov index* $\mathrm{Mas}(\omega) \in W(\mathbb{k})$ *is defined by any of the four Sylvester matrices*

(1) $S(\Lambda_0, \Lambda_1, \ldots, \Lambda_n)$.
(2) $S(\Lambda_1, \ldots, \Lambda_n, \Lambda_0)$.
(3) $S(\Lambda_0, \ldots, \Lambda_n, \Lambda_0, M)$.
(4) $S(M, \Lambda_0, \Lambda_1, \ldots, \Lambda_n, \Lambda_0)$.

We now clarify the relation between concatenation of paths and the Maslov index.

Proposition 4.3. *Let* Λ *be a Lagrangian, let* α *be a Lagrangian path ending at* Λ*,* β *a Lagrangian path starting at* Λ*, and* ω *a Lagrangian loop based at* Λ *(here* α *and* β *could be either loops or empty). Then*

$$S(\alpha * \omega * \beta) = S(\omega) \perp S(\alpha * \beta),$$

and hence in $W(\mathbb{k})$

$$\mathrm{Mas}(\alpha * \omega * \beta) = \mathrm{Mas}(\omega) + \mathrm{Mas}(\alpha * \beta).$$

Proof: Set $\alpha = (\Lambda_0, \ldots, \Lambda_n, \Lambda)$, $\omega = (\Lambda, \Lambda_{n+2}, \ldots, \Lambda_{n+s}, \Lambda)$, and $\beta = (\Lambda, \Lambda'_{n+2}, \ldots, \Lambda'_{n+p})$. Then, in the concatenated path

$$\alpha * \omega * \beta : (\Lambda_0, \ldots, \Lambda_n, \Lambda, \Lambda_{n+2}, \ldots, \Lambda_{n+s}, \Lambda, \Lambda'_{n+2}, \ldots, \Lambda'_{n+p})$$

we observe that by hypothesis $\Lambda_{n+2} \pitchfork \Lambda$. Hence, by the shortcut lemma (Lemma 3.7),

$$\begin{aligned} S(\alpha * \omega * \beta) &= S(\Lambda_{n+2}, \ldots, \Lambda_{n+1}) \perp S(\Lambda_0, \ldots, \Lambda_{n+1}, \Lambda'_{n+2}, \ldots, \Lambda'_{n+1}, \Lambda'_{n+p}) \\ &= S(\omega) \perp S(\alpha * \beta). \end{aligned}$$

$\square$

Proposition 4.4. *Let $\alpha : (\Lambda_0, \dots, \Lambda_{n+1})$ be a Lagrangian path, let α^{-1} be its inverse path, and $\beta : (\Lambda_0, M_1, \dots, M_m, \Lambda_0)$ a loop based at Λ_0. Then in $W(\Bbbk)$*

(1) $S(\alpha * \alpha^{-1}) = 0$.
(2) $S(\alpha) = S(\beta) \perp S(\beta^{-1} * \alpha)$.
(3) $S(\alpha) = -S(\alpha^{-1})$.

Proof: (1) We argue by induction on the length n of the path α.

If $n = 0$, the path α consists of two transverse Lagrangians $L \pitchfork M$, and $\alpha * \alpha^{-1} = (LML)$, which has a zero Sylvester matrix.

If $n = 1$, the concatenated path $\alpha * \alpha^{-1}$ is (L, L_1, M, L_1, L) and its Sylvester matrix written with respect to its domain (L_1, M) and codomain (L_1^*, M^*) is

$$\begin{pmatrix} d_{M,L} & b_{L,M} \\ b_{M,L} & -d_{M,L} \end{pmatrix}.$$

As $L \pitchfork L_1$, this is a non-degenerate form. An immediate computation shows then that both L_1 and M are isotropic for this form which is therefore neutral.

For the general case, let $n \geq 1$ and choose α, a path of length $n+1$. In the middle of the loop

$$\alpha * \alpha^{-1} : (L_0, L_1, \dots, L_n, L_{n+1}, M, L_{n+1}, L_n, \dots, L_0)$$

we find a sub-path $\beta : (L_n, L_{n+1}, M, L_{n+1})$ that joins two transverse Lagrangians. The Lagrangian path $\gamma : (L_0, L_1, \dots, L_n, L_{n+1})$ has length n, and observe that

$$(L_0, L_1, \dots, L_n, L_{n+1}, L_n, \dots, L_0) = \gamma * \gamma^{-1}.$$

By the shortcut lemma (Lemma 3.7)

$$S(\alpha * \alpha^{-1}) = S(\gamma * \gamma^{-1}) \perp S(\beta * \beta^{-1}),$$

where by induction the two rightmost terms are zero in $W(\Bbbk)$.

(2) The path $\beta * \beta^{-1} * \alpha$ is given by the sequence

$$\Lambda_0, M_1, \dots, M_m, \Lambda_0, M_m, \dots, M_1, \Lambda_0, \Lambda_1, \dots, \Lambda_n, \Lambda_{n+1}.$$

As $\Lambda_0 \pitchfork \Lambda_1$ the shortcut lemma (Lemma 3.7) allows us to write

$$\begin{aligned} S(\beta * \beta^{-1} * \alpha) &= S(\Lambda_0, M_1, \dots, \Lambda_0, , \Lambda_1) \perp S(\Lambda_0, \Lambda_1, \dots, \Lambda_n, \Lambda_{n+1}) \\ &= S(\beta * \beta^{-1}) \perp S(\alpha), \text{ by Proposition 4.3} \\ &= S(\alpha), \text{ by (1)}. \end{aligned}$$

Then, using that $\Lambda_0 \pitchfork M_m$ and Proposition 4.3, we have that

$$\begin{aligned} S(\beta * \beta^{-1} * \alpha) &= S(\Lambda_0, M_1, \dots, \Lambda_0, M_m) \\ &\quad \perp S(\Lambda_0, M_m, \dots, M_1, \Lambda_0, \Lambda_1, \dots, \Lambda_{n+1}) \\ &= S(\beta) \perp S(\beta^{-1} * \alpha). \end{aligned}$$

(3) Direct from the definitions and Lemma 2.1. □

4.2. Index of a triple of Lagrangians.

Proposition 4.5. *Fix three Lagrangians Λ_0, Λ_1, Λ_2 and two Lagrangian paths α_{01} and α_{12}, where α_{ij} joins Λ_i to Λ_j. Then the class in $W(\Bbbk)$ of the bilinear symmetric form*

$$\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = S(\alpha_{01} * \alpha_{12}) \perp - S(\alpha_{01}) \perp - S(\alpha_{12})$$

is independent of the choice of paths.

Proof: We only treat the case where we change the path α_{01} by an alternate path α'_{01}; the other case is analogous, and from these two the general case follows immediately.

By construction we have a loop $\alpha_{01} * {\alpha'}_{01}^{-1}$ based at Λ_0, and by Proposition 4.4 applied to this loop and to the paths $\alpha_{01} * \alpha_{12}$ and α_{01} we get

$$\begin{aligned}
\mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_2) &= S(\alpha_{01} * \alpha_{12})\perp - S(\alpha_{01})\perp - S(\alpha_{12}) \\
&= S(\alpha'_{01} * \alpha_{01}^{-1})\perp S(\alpha'_{01} * \alpha_{01}^{-1} * \alpha_{01} * \alpha_{12}) \\
&\quad \perp - S(\alpha'_{01} * \alpha_{01}^{-1})\perp - S(\alpha'_{01} * \alpha_{01}^{-1} * \alpha_{01})\perp - S(\alpha_{12}) \\
&= S(\alpha'_{01} * \alpha_{01}^{-1})\perp - S(\alpha'_{01} * \alpha_{01}^{-1})\perp S(\alpha'_{01} * \alpha_{12}) \\
&\quad \perp S(\alpha_{01} * \alpha_{01}^{-1})\perp - S(\alpha_{01} * \alpha_{01}^{-1}) \\
&\quad \perp - S(\alpha'_{01})\perp - S(\alpha_{12}), \text{ by Proposition 4.3} \\
&= S(\alpha'_{01} * \alpha_{12})\perp - S(\alpha'_{01})\perp - S(\alpha_{12}),
\end{aligned}$$

where the last equality comes from the fact that if γ is a loop, then $S(\gamma)$ is invertible, and hence $S(\gamma)\perp - S(\gamma)$ is neutral. □

We now check that μ_{BL} satisfies the characteristic properties of the Maslov index [**3**]; in particular, point (5) settles the multiplicative ambiguity.

Theorem 4.6. *The map μ_{BL} satisfies the following properties:*

(1) *If two of the three Lagrangians coincide, then $\mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_2) = 0$.*
(2) *For any $\phi \in Sp_{2g}(\Bbbk)$, $\mu_{BL}(\phi(\Lambda_0),\phi(\Lambda_1),\phi(\Lambda_2)) = \mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_2)$.*
(3) *The Maslov index is a 2-cocycle; if Λ_0, Λ_1, Λ_2, Λ_3 are four Lagrangians, then the alternating sum*

$$\mu_{BL}(\Lambda_1,\Lambda_2,\Lambda_3) - \mu_{BL}(\Lambda_0,\Lambda_2,\Lambda_3) + \mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_3) - \mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_2)$$

is trivial.
(4) *If $\sigma \in \mathfrak{S}_3$ is a permutation of the indices 0, 1, and 2 with signature $\varepsilon(\sigma)$, then*

$$\mu_{BL}(\Lambda_{\sigma(0)},\Lambda_{\sigma(1)},\Lambda_{\sigma(2)}) = \varepsilon(\sigma)\mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_2).$$

(5) *Let μ_{KW} denote the Kashiwara–Wall index of three Lagrangians; then $2\mu_{BL} = \mu_{KW}$.*

Proof: (1) If $\Lambda_0 = \Lambda_1$, choose a Lagrangian M transverse to Λ_0, this defines a path $\alpha_{01} : (\Lambda_0 M \Lambda_0)$, and choose an arbitrary path α_{12}. As the Sylvester matrix of the loop α_{01} is zero, by the shortcut lemma (Lemma 3.7) and after regularization $\alpha_{01} * \alpha_{12} = \alpha_{12}$ and therefore

$$\begin{aligned}
\mu_{BL}(\Lambda_0,\Lambda_1,\Lambda_2) &= S(\alpha_{01} * \alpha_{12})\perp - S(\alpha_{01})\perp - S(\alpha_{12}) \\
&= S(\alpha_{12})\perp - S(\alpha_{12}) \\
&= 0.
\end{aligned}$$

When $\Lambda_1 = \Lambda_2$ the proof is as before. Finally, if $\Lambda_0 = \Lambda_2$, choose an arbitrary path α_{01} and choose as path α_{12} the path α_{01}^{-1}. Proposition 4.3 then implies that the index computed with these paths is zero.

(2) This is an immediate consequence of the equivariance properties of the Sylvester matrices; see Lemma 3.3.

(3) We choose three paths α_{01}, α_{12}, and α_{23}, where α_{ij} joins Λ_i to Λ_j, and we denote by $D(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ the alternating sum. Then,

$$\begin{aligned} D(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3) = S(\alpha_{12} * \alpha_{23}) &\perp - S(\alpha_{12}) \perp - S(\alpha_{23}) \\ &\perp - S(\alpha_{01} * \alpha_{12} * \alpha_{23}) \perp S(\alpha_{01} * \alpha_{12}) \perp S(\alpha_{23}) \\ &\perp S(\alpha_{01} * \alpha_{12} * \alpha_{23}) \perp - S(\alpha_{01}) \perp - S(\alpha_{12} * \alpha_{23}) \\ &\perp - S(\alpha_{01} * \alpha_{12}) \perp S(\alpha_{01}) \perp - S(\alpha_{12}). \end{aligned}$$

In this long orthogonal sum each bilinear form appears twice with opposite signs, and hence after regularization it is trivial.

(4) It is enough to show the statement for a transposition. Applying the cocycle result to the four Lagrangians Λ_0, Λ_1, Λ_0, Λ_2 we get

$$\begin{aligned} 0 &= \mu_{BL}(\Lambda_1, \Lambda_0, \Lambda_2) - \mu_{BL}(\Lambda_0, \Lambda_0, \Lambda_2) + \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) - \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_0) \\ &= \mu_{BL}(\Lambda_1, \Lambda_0, \Lambda_2) + \mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2). \end{aligned}$$

The same computation applied to Λ_0, Λ_1, Λ_2, Λ_0 and Λ_0, Λ_1, Λ_2, Λ_1 gives the result for the other two transpositions.

(5) Since $\Bbbk$ is a field with at least three elements by [**13**, Lemma 1.6] there exists a Lagrangian Λ' which is simultaneously transverse to the three Lagrangians Λ_0, Λ_1, Λ_2.

We choose the following paths: $\alpha_{01} : (\Lambda_0, \Lambda', \Lambda_1)$ and $\alpha_{12} : (\Lambda_1, \Lambda', \Lambda_2)$. By definition, $\alpha_{01} * \alpha_{12} = (\Lambda_0, \Lambda', \Lambda_1, \Lambda', \Lambda_2)$ and

$$\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = S(\Lambda_0, \Lambda', \Lambda_1, \Lambda', \Lambda_2) \perp - S(\Lambda_0, \Lambda', \Lambda_1) \perp - S(\Lambda_1, \Lambda', \Lambda_2).$$

By the shortcut lemma (Lemma 3.7),

$$S(\Lambda_0, \Lambda', \Lambda_1, \Lambda', \Lambda_2) = S(\Lambda_0, \Lambda', \Lambda_1, \Lambda') \perp S(\Lambda_0, \Lambda', \Lambda_2).$$

Moreover, as $\Lambda_0 \pitchfork \Lambda'$, the bilinear form $S(\Lambda_0, \Lambda', \Lambda_1, \Lambda')$ is non-degenerate. It is a form with support $\Lambda' \oplus \Lambda_1$, and with matrix:

$$\begin{pmatrix} d_{\Lambda_1,\Lambda_0} & \beta_{\Lambda_1,\Lambda'} \\ \beta_{\Lambda',\Lambda_1} & d_{\Lambda',\Lambda'} \end{pmatrix} = \begin{pmatrix} d_{\Lambda_1,\Lambda_0} & \beta_{\Lambda_1,\Lambda'} \\ \beta_{\Lambda',\Lambda_1} & 0 \end{pmatrix}.$$

As $\beta_{\Lambda_1,\Lambda'}$ is invertible, this is the matrix of a neutral form.

Finally,

$$\mu_{BL}(\Lambda_0, \Lambda_1, \Lambda_2) = S(\Lambda_0, \Lambda', \Lambda_2) \perp - S(\Lambda_0, \Lambda', \Lambda_1) \perp - S(\Lambda_1, \Lambda', \Lambda_2)$$

is the regularization of a bilinear form with support $\Lambda' \oplus \Lambda' \oplus \Lambda'$ and with matrix

$$\begin{pmatrix} d_{\Lambda_2,\Lambda_0} & 0 & 0 \\ 0 & d_{\Lambda_1,\Lambda_0} & 0 \\ 0 & 0 & d_{\Lambda_1,\Lambda_2} \end{pmatrix}.$$

For an explicit computation that shows that this is indeed half the Kashiwara–Wall index we refer the reader to [**15**, Proposition 7.8.3]. $\square$

5. Triviality mod I^2 of Maslov's cocycle

We will now focus on the 2-cocycle of the symplectic group associated to the Maslov index. As in Subsection 2.2, we have a preferred Lagrangian L in the symplectic space $H(L) = L \oplus L^*$, and by point (3) in Theorem 4.6, the map

$$\begin{aligned} \mu\colon Sp_{2g}(\Bbbk) \times Sp_{2g}(\Bbbk) &\longrightarrow W(\Bbbk) \\ (A, B) &\longmapsto \mu_{BL}(L, AL, ABL) \end{aligned}$$

is a 2-cocycle on the symplectic group, and hence determines a group extension

$$0 \longrightarrow W(\Bbbk) \longrightarrow \Gamma \longrightarrow Sp_{2g}(\Bbbk) \longrightarrow 1.$$

In this section we identify this extension as the push-out of a canonical extension defined from the stabilizers of the Lagrangians L and L^* along a map which resembles the Maslov index of a Lagrangian path. We then show from the properties of the Maslov index of a loop that the reduction mod I^2 of the above extension splits uniquely, and compute explicitly the splitting. For $\Bbbk = \mathbb{R}$ this computation was carried out by Turaev in [**18**], but his methods do not seem to generalize to an arbitrary field.

5.1. Sturm sequences and Sylvester matrices. In Section 2 we introduced the stabilizers of L and L^* for the canonical action of the symplectic group $\mathcal{L}_g$

$$\mathcal{S}_L = \left\{ \left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right) \mid q\colon L^* \longrightarrow L \text{ symmetric} \right\}$$

and

$$\mathcal{S}_{L^*} = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right) \mid q\colon L \longrightarrow L^* \text{ symmetric} \right\}.$$

Elements in these sub-groups will be called *elementary matrices* and will be denoted by $E(q)$. To distinguish the identity matrix in both sub-groups we will write

$$\underline{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{S}_L \quad \text{and} \quad \overline{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{S}_{L^*}.$$

The free product $\mathcal{S}_L * \mathcal{S}_{L^*}$ comes with an obvious evaluation morphism $E\colon \mathcal{S}_L * \mathcal{S}_{L^*} \to Sp_{2g}(\Bbbk)$, and since $\Bbbk$ is a field, the map E is surjective. Indeed, it is well known that over a field the symplectic group is generated by elementary matrices of the shape $E(q)$ as above and by matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix},$$

where $\alpha \in GL_g(\Bbbk)$. But over a field (cf. [**5**, Theorem 66]) every invertible matrix is the product of two symmetric matrices, and if $\alpha = p^{-1}q$ is such a factorization, then the following two equalities in $Sp_{2g}(\Bbbk)$,

$$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} 1 & -q^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} 0 & -q^{-1} \\ q & 0 \end{pmatrix} = m(q)$$

and

$$m(-p)m(q) = \begin{pmatrix} p^{-1}q & 0 \\ 0 & {}^t(p^{-1}q)^{-1} \end{pmatrix}$$

show that the elementary matrices suffice to generate the symplectic group.

In the symplectic space $H(L)$, we have two preferred Lagrangians L and L^* and we will borrow the following notation convention from [**1**, Section 2.2]: L_n will denote the Lagrangian L if the integer n is even and the Lagrangian L^* if it is odd. By definition, a reduced word in the free product $\mathcal{S}_L * \mathcal{S}_{L^*}$ is a sequence of symmetric linear maps $\underline{q} : (q_m, q_{m+1}, \dots, q_n)$ where $q_j\colon L_j \to L_{j+1}$. Such a sequence is said to be of *type* (m,n) and is called a *Sturm sequence*. Barge–Lannes ([**1**]) associate to a Sturm sequence as before a Sylvester matrix with support $L_m \oplus L_{m+1} \oplus \cdots \oplus L_n$ by the rule:

$$S(\underline{q}) = \begin{pmatrix} (-1)^m q_m & 1 & 0 & \cdots & 0 \\ 1 & (-1)^{m+1} q_{m+1} & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & (-1)^{n-1} q_{n-1} & 1 \\ 0 & \cdots & 0 & 1 & (-1)^n q_n \end{pmatrix}.$$

The sign that appears in this definition comes from our convention in defining the affine structure on the sets of Lagrangians transverse to L_0 or L_1; see Subsection 2.2. Let us now link the Sylvester matrix associated to a Lagrangian path and that associated to a Sturm sequence.

Proposition 5.1 ([**1**, Proposition C.1])**.** *Let m and n be two integers with $m \leq n$ and let $\Lambda_{m-1}, \Lambda_m, \ldots, \Lambda_n$ be a finite sequence of Lagrangians of the symplectic space $H(L)$ such that $\Lambda_{m-1} = L_{m-1}$. The following conditions are equivalent:*

(1) $\Lambda_{k-1} \pitchfork \Lambda_k$ *for* $m \leq k \leq n$.

(2) *There exists a Sturm sequence $(q_m, q_{m+1}, \ldots, q_n)$ on L of type (m,n) such that*

$$\Lambda_k = E(q_m, q_{m+1}, \ldots, q_k) \cdot L_k \text{ for } m \leq k \leq n.$$

Moreover, if condition (1) is satisfied, then the Sturm sequence that appears in (2) is unique.

Proof: The implication (2) $\Rightarrow$ (1) is clear. Conversely, we consider a Lagrangian path $\alpha : (L_{m-1}, \Lambda_m, \ldots, \Lambda_n, \Lambda_n)$. By definition $\Lambda_m \in \mathcal{L}_{g,\pitchfork L_{m-1}}$ and hence there exists a unique $q_{m-1} \in \mathcal{S}_{L_{m-1}}$ such that $E(q_m)L_m = \Lambda_m$. Assume by induction that we have constructed a unique Sturm sequence $(q_m, \ldots, q_k)$ for $k \geq n-1$ such that for all $s \leq k$ $\Lambda_s = E(q_m, q_{m-1}, \ldots, q_s)L_s$. Let $\Lambda'_{s+1} = E(q_m, q_{m-1}, \ldots, q_s)^{-1}\Lambda_{s+1}$. By construction $\Lambda'_{s+1} \pitchfork L_s$ and hence there exists a unique $q_{s+1} \in \mathcal{S}_{L_s}$ such that $\Lambda'_{s+1} = E(q_{s+1}L_{s+1})$, and

$$\begin{aligned} \Lambda_{s+1} &= E(q_m, q_{m-1}, \ldots, q_s)\Lambda'_{s+1} \\ &= E(q_m, q_{m-1}, \ldots, q_s)E(q_{s+1})L_{s+1} \\ &= E(q_m, q_{m-1}, \ldots, q_s, q_{s+1})L_{s+1}. \end{aligned}$$

□

Remark 5.2. As the symplectic group acts transitively on $\mathcal{L}_g$, given any Lagrangian path, by pushing it by a suitable element in $Sp_{2g}(\Bbbk)$ we may assume that its initial Lagrangian is either L or L^*, hence Proposition 5.1 appears as a sort of normal form for the Sylvester matrix associated to the path.

Let $K = \ker(E\colon \mathcal{S}_L * \mathcal{S}_{L^*} \to Sp_{2g}(\Bbbk))$. By definition we have a short exact sequence of groups

$$1 \longrightarrow K \longrightarrow \mathcal{S}_L * \mathcal{S}_{L^*} \longrightarrow Sp_{2g}(\Bbbk) \longrightarrow 1.$$

5.2. Four natural functions on K. Let (m,n) denote one of the four couples $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. Up to possibly adding to a word $w \in \mathcal{S}_L * \mathcal{S}_{L^*}$ either $\overline{0}$ or $\underline{0}$ at the beginning or at the end, we may assume that w is of the type (m,n), i.e. it can be identified with a Sturm sequence of type (m,n). Let us define

$$\begin{aligned} f_{m,n}\colon \mathcal{S}_L * \mathcal{S}_{L^*} &\longrightarrow W(\Bbbk) \\ w &\longmapsto S(w), \text{ where } w \text{ is of type } (m,n). \end{aligned}$$

Proposition 5.3. *The four functions f_{00}, f_{01}, f_{10}, and f_{11} are all well defined.*

Proof: We only treat the case of the function f_{00}; the other three cases can be treated in a similar way. It is enough to show that the value of f_{00} on w is independent of the choice of the representative of type 00 of w. There are only two types of ambiguity in the choice of the representative:

(i) Given a representative of type 00 we may add to it the word $\underline{0}\overline{0}$ at the beginning (resp. $\overline{0}\underline{0}$ at the end).

(ii) Inside a representative of type 00 we may find a sub-word of the form $a\overline{0}b$ with $a, b \in \mathcal{S}_L$ (resp. $a\underline{0}b$ with $a, b \in \mathcal{S}_{L^*}$).

(i) Let $w : (q_2, q_3, \ldots, q_{2n})$ be a Sturm sequence of type 00. We will only show the equality $f_{00}(\underline{0}\overline{0}w) = f_{00}(w)$; the other case is similar. Observe that

$$S(\underline{0}\overline{0}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a non-degenerate and neutral form. The Lagrangian path associated to $\underline{0}\overline{0}w$ is

$$(L_0, L_1, E(\underline{0})L_0, E(\underline{0}\overline{0})L_1, E(\underline{0}\overline{0}q_2)L_0, \ldots, E(\underline{0}\overline{0}w)L_1).$$

Since

$$L_1 = \begin{pmatrix} 1 & 0 \\ q_2 & 0 \end{pmatrix} L_1 \pitchfork \begin{pmatrix} 1 & 0 \\ q_2 & 0 \end{pmatrix} L_0 = E(\underline{0}\overline{0}q_2)L_0$$

the shortcut lemma (Lemma 3.7) tells us that

$$(**) \qquad S(\underline{0}\overline{0}w) = S(\alpha) \perp S(\beta),$$

where α is the Lagrangian path

$$(L_1, E(\underline{0})L_0, E(\underline{0}\overline{0})L_1, E(\underline{0}\overline{0}q_2)L_0)$$

and β is the Lagrangian path

$$(L_0, L_1, E(\underline{0}\overline{0}q_2)L_0, \ldots, E(\underline{0}\overline{0}w)L_1).$$

Since $E(\underline{0}) = E(\underline{0}\overline{0}) = \mathrm{Id}$ the path α is simply

$$(L_1, L_0, E(\overline{0})L_1, E(\overline{0}q_2)L_0),$$

which is associated to the Sturm sequence $(\overline{0}, q_2)$ and whose Sylvester matrix is neutral

$$S(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & q_2 \end{pmatrix}.$$

In β one recognizes the Lagrangian path associated to the sequence w and equality $(**)$ in $W(\Bbbk)$ states that

$$S(\underline{0}\overline{0}q_2 \cdots q_{2n}) = S(q_2 \cdots q_{2n}).$$

(ii) Once more we only treat the first of the two sub-cases. The Sturm sequence associated to the representative of type 00 is:

$$(q_0, q_1, \ldots, q_{2r}, \overline{0}, q_{2r+2}, \ldots, q_{2n}),$$

and its associated Lagrangian path is

$$(L_0, L_1, E(q_0)L_0, \ldots, E(q_0 \cdots q_{2r-1})L_1, E(q_0 \cdots q_{2r})L_0, E(q_0 \cdots q_{2r}\overline{0})L_1,$$
$$E(q_0 \cdots q_{2r}\overline{0}q_{2r+2})L_0, \ldots, E(q_0 \cdots q_{2r}\overline{0}q_{2r+2} \cdots q_{2n})L_0).$$

Simplifying the evaluations, taking into account that E is a group homomorphism and that $E(\overline{0}) = \mathrm{Id}$, this path is exactly

$$(L_0, L_1, E(q_0)L_0, \ldots, \mathbf{E(q_0 \cdots q_{2r-1})L_1}, E(q_0 \cdots q_{2r})L_0, E(q_0 \cdots q_{2r})L_1,$$
$$\mathbf{E(q_0 \cdots (q_{2r} + q_{2r+2}))L_0}, \ldots, E(q_0 \cdots (q_{2r} + q_{2r+2}) \cdots q_{2n})L_0).^1$$

Now, the two Lagrangians in boldface characters are mutually transverse, for

$$L_1 = \begin{pmatrix} 1 & 0 \\ q_{2r} + q_{2r+2} & 1 \end{pmatrix} L_1 = E(q_{2r} + q_{2r+2})L_1 \pitchfork E(q_{2r} + q_{2r+2})L_0$$

and hence $E(q_0 \cdots q_{2r-1})L_1 \pitchfork E(q_0 \cdots (q_{2r} + q_{2r+2}))L_0$.

We apply the shortcut lemma (Lemma 3.7) to the two Lagrangians and observe that the Sylvester matrix of the initial sequence is therefore isometric to the direct sum of the two Sylvester matrices of the Lagrangian path

$$(L_0, L_1, E(q_0)L_0, \ldots, E(q_0 \cdots q_{2r-1})L_1,$$
$$E(q_0 \cdots (q_{2r} + q_{2r+2}))L_0, \ldots, E(q_0 \cdots (q_{2r} + q_{2r+2}) \cdots q_{2n})L_0),$$

that is associated to the Sturm sequence

$$(q_0, q_1, \ldots, q_{2r} + q_{2r+2}, \ldots, q_{2n}),$$

and of the path

$$(E(q_0 \cdots q_{2r-1})L_1, E(q_0 \cdots q_{2r})L_0, E(q_0 \cdots q_{2r})L_1, E(q_0 \cdots (q_{2r} + q_{2r+2}))L_0).$$

Since $E(q_{2r})L_1 = L_1$, this last path is the image by $E(q_0 \cdots q_{2r})$ of the path

$$(L_1, L_0, L_1, E(q_{2r+2})L_0),$$

whose Sylvester matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & q_{2r+2} \end{pmatrix}$$

and is clearly neutral. □

Given a sub-word k of w which represents an element in the kernel K, the function f_{00} behaves almost as a group homomorphism would do.

Proposition 5.4. *Let $\underline{k} = k_0, \ldots, k_{2k+1} \in K$ be a word of type* 01 *and let*

$$\underline{w} = (q_0, \ldots, q_{2r-1}, k_0, \ldots, k_{2\ell+1}, q_{2\ell+2}, \ldots, q_{2n})$$

be an arbitrary word in $\mathcal{S}_L * \mathcal{S}_{L^*}$ of type 00 *that contains $\underline{k}$ as a sub-word. Then*

$$f_{00}(\underline{w}) = f_{01}(\underline{k}) + f_{00}(q_0, \ldots, q_{2r-1}, q_{2\ell+2}, \ldots, q_{2n}).$$

Proof: We write down the Lagrangian path associated to the Sturm sequence $\underline{w}$:

$$\big(L_0, L_1, E(q_0)L_0, \ldots, E(q_0 \cdots q_{2r-1})L_1,$$
$$E(q_0 \cdots q_{2r-1}k_0)L_0, \ldots, E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1})L_1,$$
$$E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0, \ldots, E(\underline{w})L_0\big).$$

Since the sequence $\underline{k}$ represents an element in K, the kernel of the evaluation map, we know that $E(\underline{k}) = \mathrm{Id}$. In the Lagrangian path above, the last Lagrangian on the second line and the first Lagrangian on the third line are by definition transverse:

$$E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1})L_1 \pitchfork E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0.$$

But $E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1}) = E(q_0 \cdots q_{2r-1})$, hence

$$E(q_0 \cdots q_{2r-1})L_1 \pitchfork E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0.$$

[1] If $r = 0$, by convention $E(q_0 \cdots q_{2r-1}) = \mathrm{Id}$.

We can now apply the shortcut lemma (Lemma 3.7) to the last Lagrangian on the first line and to the first Lagrangian on the third line; this tells us that $f_{00}(\underline{w})$ is the sum of the classes of the Sylvester matrices associated to the following two Lagrangian paths.

Firstly we have the path

$$(L_0, L_1, E(q_0)L_0, \dots, E(q_0 \cdots q_{2r-1})L_1,\\ E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0, \dots, E(\underline{w})L_0).$$

Since $E(\underline{k}) = \mathrm{Id}$, this is nothing other than the Lagrangian path

$$(L_0, L_1, E(q_0)L_0, \dots, E(q_0 \cdots q_{2r-1})L_1,\\ E(q_0 \cdots q_{2r-1}q_{2\ell+2})L_0, \dots, E(q_0 \cdots q_{2r-1}q_{2\ell+2} \cdots q_{2n})L_0),$$

whose class is by definition $f_{00}(q_0, \dots, q_{2r-1}, q_{2\ell+2}, \dots, q_{2n})$.

Secondly, we have the path

$$(E(q_0 \cdots q_{2r-1})L_1, E(q_0 \cdots q_{2r-1}k_0)L_0, \dots,\\ E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1})L_1, E(q_0 \cdots q_{2r-1}k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0).$$

By definition of the action of the symplectic group on Lagrangian paths, this path is the image by $E(q_0 \cdots q_{2r-1})$ of the Lagrangian path

$$(L_1, E(k_0)L_0, \dots, E(k_0 \cdots k_{2\ell+1})L_1, E(k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0).$$

Here again, as $E(k_0 \cdots k_{2\ell+1}) = \mathrm{Id}$, $E(k_0 \cdots k_{2\ell+1})L_1 = L_1$, and we recognize a Lagrangian loop with an extra term to its right:

$$E(k_0 \cdots k_{2\ell+1}q_{2\ell+2})L_0 = E(q_{2\ell+2})L_0.$$

By Corollary 3.8, the Witt class of the Sylvester matrix associated to this path coincides with that of the Sylvester matrix associated to the path

$$(L_0, L_1, E(k_0)L_0, \dots, E(k_0 \cdots k_{2\ell+1})L_1),$$

which by definition is $f_{01}(\underline{k})$. $\square$

Remark 5.5. If $\underline{k} \in K$ is a word of type 01, by applying Corollary 3.8 as in the last part of the above argument one can show that

$$f_{00}(\underline{k0}) = f_{01}(\underline{k}) + f_{00}(\underline{0}) = f_{01}(\underline{k}),$$

because in $W(\Bbbk)$, $f_{00}(\underline{0}) = 0$.

In particular all four functions f_{00}, f_{11}, f_{01}, and f_{01} coincide on K.

This leads us to the key observation:

Lemma 5.6. *The function $f_{01}\colon K \to W(\Bbbk)$ is a group homomorphism invariant under the conjugation action of $\mathcal{S}_L * \mathcal{S}_{L^*}$; it takes values in I^2, the square of the fundamental ideal.*

Proof: Let $\underline{k}$ and $\underline{\ell}$ denote two elements in K, and fix for each of them a representative of type 01, respectively $\underline{k}_{01}$ and $\underline{\ell}_{01}$. Compute

$$\begin{aligned} f_{01}(\underline{k}_{01}\underline{\ell}_{01}) &= f_{00}(\underline{k}_{01}\underline{\ell}_{01}\underline{0}), && \text{by Remark 5.5}\\ &= f_{01}(\underline{k}_{01}) + f_{00}(\underline{\ell}_{01}\underline{0}), && \text{by Proposition 5.4}\\ &= f_{01}(\underline{k}_{01}) + f_{01}(\underline{\ell}_{01}), && \text{by Remark 5.5.} \end{aligned}$$

To show invariance under conjugation, we fix a word $w_{00} \in \mathcal{S}_L * \mathcal{S}_{L^*}$ of type 00. Then $w_{00}\overline{0}k_{01}w_{00}^{-1}\overline{0}$ is a representative of type 01 of the conjugate of $\underline{k}$ by $\underline{w}$, and as before:

$$\begin{aligned} f_{01}(w_{00}\overline{0}k_{01}w_{00}^{-1}\overline{0}) &= f_{00}(w_{00}\overline{0}k_{01}w_{00}^{-1}\overline{0}\underline{0}), && \text{by Remark 5.5} \\ &= f_{01}(k_{01}) + f_{00}(w_{00}\overline{0}w_{00}^{-1}\overline{0}\underline{0}), && \text{by Proposition 5.4.} \end{aligned}$$

But $w_{00}\overline{0}w_{00}^{-1}\overline{0}\underline{0}$ is a representative of type 00 of $\underline{0}$, hence

$$f_{00}(w_{00}\overline{0}w_{00}^{-1}\overline{0}\underline{0}) = f_{00}(\underline{0}) = 0.$$

We finally have to show that for all $\underline{k} \in K$ we have $f_{01}(\underline{k}) \in I^2$. Given a representative $k_{01} = k_0, k_1, \ldots, k_{2r+1}$ of type 01 of $\underline{k}$, observe that the Sylvester matrix $S(k_{01})$ has as support the direct sum of $r+1$ copies of the pair $L \oplus L^*$; in particular, this vector space has even dimension and hence the class of $S(k_{01})$ in $W(\Bbbk)$ lies in $I = \ker(W(\Bbbk) \to \mathbb{Z}/2)$.

We now need to compute the discriminant of $S(k_{01})$. By definition of K,

$$E(k_0 k_1 \cdots k_{2r+1}) = \mathrm{Id} \in GL_g(\Bbbk) \subset Sp_{2g}(\Bbbk).$$

Now, if in a Sturm sequence of even length one multiplies the associated elementary matrices in the symplectic group, and as a result one obtains a block-diagonal matrix

$$\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \in GL_g(\Bbbk) \subset Sp_{2g}(\Bbbk),$$

then [**1**, Scholie 5.5.6, p. 111] shows that the determinant of a is equal to the discriminant of the Sylvester matrix associated to the initial Sturm sequence. In our present case this gives us, as wanted,

$$\mathrm{dis}(S(k_{01})) = \det \mathrm{Id} = 1. \qquad \square$$

5.3. A cocycle for the fundamental extension. Let us push out our original exact sequence

$$1 \longrightarrow K \longrightarrow \mathcal{S}_L * \mathcal{S}_{L^*} \longrightarrow Sp_{2g}(\Bbbk) \longrightarrow 1$$

along the composite morphism $f_{01}\colon K \to I^2 \hookrightarrow W(\Bbbk)$:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "f_{01}"] & \mathcal{S}_L * \mathcal{S}_{L^*} \arrow[r] \arrow[d] \arrow[dl, "f_{00}"] & Sp_{2g}(\Bbbk) \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & W(\Bbbk) \arrow[r] & \Gamma \arrow[r] & Sp_{2g}(\Bbbk) \arrow[r] & 1 \end{tikzcd}$$

We caution the reader that the diagonal arrow labeled f_{00} is of course not a group homomorphism. Invariance of f_{01} with respect to the conjugation action implies that the bottom extension is a central extension. Finally, commutativity of the upper left triangle allows us to write an explicit 2-cocycle for the bottom central extension:

Definition-Proposition 5.7. *Let $x, y \in Sp_{2g}(\Bbbk)$ and choose arbitrary type 00 lifts of these elements, respectively denoted $\tilde{x}$ and $\tilde{y}$, in $\mathcal{S}_L * \mathcal{S}_{L^*}$. Then the function*

$$\mu(x, y) = f_{00}(\tilde{x}\overline{0}\tilde{y}) - f_{00}(\tilde{x}) - f_{00}(\tilde{y})$$

is independent of the choice of the liftings and defines a 2-cocycle for the extension

$$0 \longrightarrow W(\Bbbk) \longrightarrow \Gamma \longrightarrow Sp_{2g}(\Bbbk) \longrightarrow 1.$$

Proof: Recall that Γ as a set is a quotient of the set $W(\Bbbk) \times (\mathcal{S}_L * \mathcal{S}_{L^*})$. One checks from the definitions that the function which on a couple $(w, \widetilde{x})$ takes the value

$$r(w, \widetilde{x}) = w + f_{00}(\widetilde{x})$$

is in fact a retraction of the extension. A direct and classical computation shows then that the 2-cocycle associated to this retraction is indeed μ. $\square$

We now identify the above cocyle with the Maslov index of a triple of Lagrangians introduced in Theorem 4.6.

Proposition 5.8. *For any $x, y \in Sp_{2g}(\Bbbk)$ we have:*

$$\mu(x, y) = \mu_{BL}(x^{-1}L, L, yL).$$

Proof: By definition $f_{00}(\tilde{x})$ is the Witt class of the Sylvester matrix associated to a precise path from L_0 to $E(\tilde{x})L_0$, say $\alpha_{\tilde{x}}$, and f_{00} is the Sylvester matrix associated to a precise path from L_0 to $E(\tilde{y})L_0$, say $\alpha_{\tilde{y}}$. The problem is that $f_{00}(\tilde{x}\overline{0}\tilde{y})$ is not exactly the Witt class of the Sylvester matrix associated to the concatenation $\alpha_{\tilde{x}} * \alpha_{\tilde{y}}$; instead it is the class of the concatenation $\alpha_{\tilde{x}} * E(\tilde{x}\overline{0})\alpha_{\tilde{y}}$. By the equivariance of the Sylvester matrix of a Lagrangian path, Lemma 3.3, $f_{00}(\tilde{x}\overline{0}\tilde{y})$ is also the class of the Lagrangian path $E(\tilde{x})^{-1}(\alpha_{\tilde{x}} * E(\tilde{x}\overline{0})\alpha_{\tilde{y}} = (E(\tilde{x})^{-1}(\alpha_{\tilde{x}})) * \alpha_{\tilde{y}}$, and the same argument shows that $f_{00}(\tilde{x})$ is the class of the path $E(\tilde{x})^{-1}\alpha_{\tilde{x}}$, a path from $E(\tilde{x})^{-1}L$ to L. So the definition of $\mu(x, y)$ is precisely that of $\mu_{BL}(x^{-1}L, L, yL)$. $\square$

From the general properties of the Maslov index of a triple of Lagrangians, we recovered the following well-known properties of the Maslov cocycle.

Proposition 5.9. *The value of μ at (x, y) only depends on the three Lagrangians L, $x^{-1}L$, and yL. For any $\phi \in Sp_{2g}(\Bbbk)$, the value of $\mu(x, y) \in W(\Bbbk)$ only depends on the triple of Lagrangians $(\phi x^{-1}L, \phi L, \phi yL)$. More precisely it only depends on the cosets determined by the elements x^{-1} and y in $Sp_{2g}(\Bbbk)/\operatorname{Stab}(L)$. In particular if two of the three Lagrangians L, xL, and xyL coincide, then $\mu(x, y) = 0$.*

5.4. Triviality of μ modulo I^2 and computations. Let us now consider the mod I^2 reduction of the central extension defined by the 2-cocycle μ:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "f_{01}"] & \mathcal{S}_L * \mathcal{S}_{L^*} \arrow[r] \arrow[d] & Sp_{2g}(\Bbbk) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & W(\Bbbk) \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & Sp_{2g}(\Bbbk) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & W(\Bbbk)/I^2 \arrow[r] & \overline{\Gamma} \arrow[r] & Sp_{2g}(\Bbbk) \arrow[r] & 1
\end{tikzcd}$$

As $f_{01} \colon K \to W(\Bbbk)$ factors through I^2, the bottom extension trivially splits. Moreover, according to Proposition 5.4, the function $f_{00} \colon \mathcal{S}_L * \mathcal{S}_{L^*} \to W(\Bbbk) \bmod I^2$ factors through $Sp_{2g}(\Bbbk)$, and the shape of μ given in Definition-Proposition 5.7 tells us that:

Theorem 5.10. *The function $\Phi \colon Sp_{2g}(\Bbbk) \to W(\Bbbk)/I^2$ that associates to $x \in Sp_{2g}(\Bbbk)$ the element $f_{00}(x) \bmod I^2$ is the unique function on $Sp_{2g}(\Bbbk)$ that satisfies the equation*

$$\forall x, y \in Sp_{2g}(\Bbbk) \quad \Phi(xy) - \Phi(x) - \Phi(y) = \mu(x, y) \bmod I^2.$$

Proof: Only the unicity part requires a further argument. By construction any two functions satisfying the equation stated in the theorem differ by a group homomorphism $Sp_{2g}(\Bbbk) \to W(\Bbbk)/I^2$, and the unicity statement is an elementary consequence of the fact that $Sp_{2g}(\Bbbk)$ is perfect, unless $g = 2$ and $\Bbbk = \mathbb{F}_3$. In this singular case we have

$Sp_2(\mathbb{F}_3) = SL_2(\mathbb{F}_3)$; the abelianization of this group is $\mathbb{Z}/3$ and is induced by the exceptional map $PSL_2(\mathbb{F}_3) \simeq \mathfrak{A}_4 \to \mathbb{Z}/3$ [**14**, p. 78]. But as $W(\mathbb{F}_3) \simeq W(\mathbb{F}_3)/I^2 \simeq \mathbb{Z}/4$ [**12**, Lemma 1.5, p. 87], there are no non-trivial homomorphisms $Sp_2(\mathbb{F}_3) \to W(\mathbb{F}_3)/I^2$ and unicity follows. □

We conclude with an observation on the function Φ that is both elementary and new. Recall that for any $a \in \Bbbk^*$ then $\langle 1, -a\rangle$ denotes the associated Pfister form, and that the stabilizer of a Lagrangian L, $\mathrm{Stab}_L \subset Sp_{2g}(\Bbbk)$, is made of those matrices that according to the decomposition $\Bbbk^{2g} = L \oplus L^*$ are of the form

$$\begin{pmatrix} x & u \\ 0 & {}^t x^{-1} \end{pmatrix},$$

where $x \in GL_g(\Bbbk)$ and u is a symmetric bilinear form on L^*. In particular there is a split short exact sequence:

$$1 \longrightarrow \mathcal{S}_{L^*} \longrightarrow \mathrm{Stab}_L \longrightarrow GL_g(\Bbbk) \longrightarrow 1.$$

A section is given by the function $h\colon x \mapsto h(x) = \left(\begin{smallmatrix} x & 0 \\ 0 & {}^t x^{-1} \end{smallmatrix}\right)$.

Proposition 5.11. *The restriction of Φ to the sub-group Stab_L is a morphism with values in the fundamental ideal I. It factors through the canonical projection $\mathrm{Stab}_L \to GL_g(\Bbbk)$; more precisely:*

$$\Phi\left(\begin{pmatrix} x & u \\ 0 & {}^t x^{-1} \end{pmatrix}\right) = \langle 1, -\det x\rangle \in I/I^2 = \Bbbk^\times/(\Bbbk^\times)^2.$$

Proof: That $\Phi|_{\mathrm{Stab}_L}$ is a morphism is a direct consequence of the properties of the cocycle μ; see Proposition 5.9. The relation among symplectic matrices

$$\begin{pmatrix} 1 & u^t x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & {}^t x^{-1} \end{pmatrix} = \begin{pmatrix} x & u \\ 0 & {}^t x^{-1} \end{pmatrix}$$

shows that to compute Φ it is enough to consider the cases where $u = 0$ on the one hand and $x = \mathrm{Id}$ on the other.

(i) Let us start with $u \neq 0$ and $x = \mathrm{Id}$. Our matrix is therefore an element in $\mathcal{S}_{L^*}$, and we can use the Sturm sequence $\underline{0}, u, \underline{0}$ to compute the value of Φ. The associated Sylvester matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -u & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

An immediate computation shows that this quadratic form, whose support is $L \oplus L^* \oplus L$, has as kernel L, embedded as the elements of the form $(x, 0, -x)$. Its regularization therefore has as support $L \oplus L^* \oplus L/L \simeq L \oplus L^*$. The regularized matrix of $S(\underline{0}u\underline{0})$ is then

$$\begin{pmatrix} 0 & 1 \\ 1 & -u \end{pmatrix},$$

which is neutral and therefore $\Phi(u) = 0$.

(ii) Now suppose $u = 0$. Recall that since we work over a field, given $x \in GL_g(\Bbbk)$, there exist two symmetric forms p and q such that $x = p^{-1}q$.

Let us define

$$m(q) = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & -q^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}.$$

Then

$$m(-p)m(q) = \begin{pmatrix} p^{-1}q & 0 \\ 0 & {}^t(p^{-1}q)^{-1} \end{pmatrix}.$$

Let us denote by r the type 00 Sturm sequence

$$\underline{-p},\quad \overline{p^{-1}},\quad \underline{-p+q},\quad \overline{-q^{-1}},\quad \underline{q},$$

where we underline elements in $\mathcal{S}_{L^*}$ and overline elements in $\mathcal{S}_L$. To compute $\Phi(h(x))$ we use the extended sequence $r\overline{0}\underline{0}$. Since x stabilizes L, and $r\overline{0}$ is a preimage of x, Proposition 5.9 shows that

$$\begin{aligned}\Phi(h(x)) &= f_{01}(r\overline{0}) + f_{00}(\underline{0}) \bmod I^2\\ &= f_{01}(r\overline{0}) \bmod I^2.\end{aligned}$$

Since x also stabilizes L^*, $E(r\overline{0})L^* = L^* \pitchfork L$, and the Sylvester matrix of $r\overline{0}$ is non-degenerate. The support of this quadratic form is $L \oplus L^* \oplus L \oplus L^* \oplus L \oplus L^*$, which has even dimension $6g$, and hence $\Phi(h(x)) \in I$.

We now have to compute the discriminant of $S(r\overline{0})$. As in the proof of Lemma 5.6, the evaluation of the components of the Sturm sequence $r\overline{0}$ gives by construction

$$E(r\overline{0}) = h(x),$$

and as $r\overline{0}$ has an even number of elements, Scholie 5.2.6 in [**1**] tells us exactly that

$$\operatorname{disc}(S(r\overline{0})) = \det(x^{-1}) = \det(x) \bmod (\Bbbk^\times)^2. \qquad \square$$

Let us now consider the group $\mathrm{Stab}_{L\oplus L^*}$ of those elements that stabilize the *decomposition* $L\oplus L^*$ of $H(L)$. The elements of this group fall into two disjoint families:

(a) Those elements of the form

$$h(x) = \begin{pmatrix} x & 0\\ 0 & {}^t x^{-1}\end{pmatrix}$$

with $x \in GL_g(\Bbbk)$, that stabilize both Lagrangians separately.

(b) Those elements of the form

$$m(y) = \begin{pmatrix} 0 & -{}^t y^{-1}\\ y & 0\end{pmatrix}$$

with $y \in GL_g(\Bbbk)$, that swap the two Lagrangians L and L^*.

These two types of elements together clearly generate the group $\mathrm{Stab}_{L\oplus L^*}$.

Proposition 5.12. *The restriction of the function Φ to the sub-group $\mathrm{Stab}_{L\oplus L^*}$ is a group homomorphism. Moreover,*

$$\Phi\left(\begin{pmatrix} 0 & -{}^t y^{-1}\\ y & 0\end{pmatrix}\right)$$

is the Witt class of the pair $((-1)^{\frac{g(g-1)}{2}}\det(y), 3g) \in (\Bbbk^\times/(\Bbbk^\times)^2, \mathbb{Z}/4)$ via the morphism F defined at the end of Subsection 2.1.

Proof: It is enough to show that $\forall v, w \in GL_g(\Bbbk)$, $\Phi(m(v)m(w)) = \Phi(m(v)) + \Phi(m(w))$.

We apply the relation that characterizes Φ to $m(v)$ and $m(v^{-1})m(w) = h(v^{-1}w)$; by Proposition 5.9

$$0 = \mu(m(v), m(v^{-1})m(w)) = \Phi(m(w)) - \Phi(m(v)) - \Phi(m(v^{-1})m(w)),$$

which means that

$$\Phi(m(w)) - \Phi(m(v)) = \Phi(m(v^{-1})m(w)).$$

As this relation is satisfied for arbitrary v and w, it is enough to show that $\Phi(m(v)) = -\Phi(m(v^{-1}))$. For this we compute $\Phi(m(v))$ from relation $(*)$. A representative of $m(v)$ is given by the Sturm sequence $v : \underline{v}, \overline{-v^{-1}}, \underline{v}$, whose associated Sylvester matrix is

$$S(v) = \begin{pmatrix} v & 1 & 0 \\ 1 & v^{-1} & 1 \\ 0 & 1 & v \end{pmatrix}.$$

Since $E(v)L_0 = m(v)L_0 = L_1 \pitchfork L_0$, the Sylvester matrix $S(v)$ is non-degenerate, and its rank modulo 4 equals g. To compute the discriminant of $S(v)$, we calculate directly

$$\begin{vmatrix} v & 1 & 0 \\ 1 & v^{-1} & 1 \\ 0 & 1 & v \end{vmatrix} = \begin{vmatrix} v & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & v \end{vmatrix} = \begin{vmatrix} v & 1 & 0 \\ 0 & -v^{-1} & 1 \\ 0 & 0 & v \end{vmatrix} = (-1)^g \det(v)$$

and hence

$$\operatorname{disc}(S(v)) = (-1)^{\frac{3g(3g-1)}{2}}(-1)^g \det(v) = (-1)^{\frac{g(g-1)}{2}} \det(v). \qquad \square$$

More generally, Propositions 5.9 and 5.12 show that the value of Φ on the matrices on the left-hand side of the following equalities in the symplectic group only depends on v and is computed using the two preceding propositions

$$\begin{pmatrix} v & x \\ -{}^tx^{-1} & 0 \end{pmatrix} = \begin{pmatrix} x & v \\ 0 & -{}^tx^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -{}^tx^{-1} \\ x & v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & v \\ 0 & -{}^tx^{-1} \end{pmatrix}.$$

Again by Propositions 5.9 and 5.11 we can compute Φ on the matrix

$$\begin{pmatrix} x & 0 \\ qx & -{}^tx^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & {}^tx^{-1} \end{pmatrix}.$$

More precisely:

$$\Phi\left(\begin{pmatrix} x & 0 \\ v & {}^tx^{-1} \end{pmatrix}\right) = \langle 1, -\det x \rangle + [vx^{-1}],$$

where $[vx^{-1}]$ stands for the Witt class of the regularization of the quadratic form vx^{-1} with support L, and which is again symmetric.

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