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PICARD GROUPS OF QUASI-FROBENIUS ALGEBRAS AND A QUESTION ON FROBENIUS STRONGLY GRADED ALGEBRAS

Sorin Dăscălescu, Constantin Năstăsescu, and Laura Năstăsescu

Dedicated to Şerban Raianu on his 70th birthday

Abstract: Our initial aim was to answer the question: does the Frobenius (symmetric) property transfer from a strongly graded algebra to its homogeneous component of trivial degree? Related to it, we investigate invertible bimodules and the Picard group of a finite-dimensional quasi-Frobenius algebra R. We compute the Picard group, the automorphism group, and the group of outer automorphisms of a 9-dimensional quasi-Frobenius algebra which is not Frobenius, constructed by Nakayama. Using these results and a semitrivial extension construction, we give an example of a symmetric strongly graded algebra whose trivial homogeneous component is not even Frobenius. We investigate associativity of isomorphisms $R^*\otimes_R R^* \simeq R$ for quasi-Frobenius algebras R, and we determine the order of the class of the invertible bimodule H^* in the Picard group of a finite-dimensional Hopf algebra H. As an application, we construct new examples of symmetric algebras.

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1. Introduction and preliminaries

A finite-dimensional algebra A over a field K is called Frobenius if $A \simeq A^*$ as left (or equivalently, as right) A-modules. If A satisfies the stronger condition that $A \simeq A^*$ as A-bimodules, then A is called a symmetric algebra. Frobenius algebras and symmetric algebras occur in algebra, geometry, topology, and quantum theory, and they have a rich representation theory, which is relevant for all these branches of mathematics. A general problem is whether a certain ring property transfers from an algebra on which a Hopf algebra (co)acts to the subalgebra of (co)invariants; of special interest is the situation where the (co)action produces a Galois extension. Particular cases of high relevance are: (1) algebras A on which a group G acts as automorphisms, and the transfer of properties to the subalgebra A^G of invariants; (2) algebras A graded by a group G, and the transfer of properties to the homogeneous component of trivial degree. In the second case, such an A is in fact a comodule algebra over the Hopf group algebra KG, and the subalgebra of coinvariants is just the component of trivial degree; moreover, the associated extension is KG-Hopf–Galois if and only if A is strongly graded.

Our main aim is to answer the following.

Question 1. *If $A = \oplus_{g\in G} A_g$ is a strongly G-graded algebra, where G is a group with neutral element e, and A is Frobenius (symmetric), does it follow that the subalgebra A_e is Frobenius (symmetric)?*

In Section 7 we answer the question in the negative, for both Frobenius and symmetric properties. There is an interesting alternative way to formulate this question for the Frobenius property. Frobenius algebras in the monoidal category of G-graded vector spaces were considered in [**5**], where they were called graded Frobenius algebras. Such objects and a shift version of them occur in non-commutative geometry, for example as Koszul duals of certain Artin–Schelter regular algebras, and also in the theory of Calabi–Yau algebras. A G-graded algebra A is graded Frobenius if $A \simeq A^*$ as graded left A-modules, where A^* is provided with a standard structure of such an object. Obviously, if A is graded Frobenius, then it is a Frobenius algebra, while the converse is not true in general. If A is strongly graded, then A is graded Frobenius if and only if A_e is Frobenius; see [**5**, Corollary 4.2]. Thus the question above can also be formulated as: if A is a strongly graded algebra which is Frobenius, is it necessarily graded Frobenius?

Question 1 cannot be reformulated in a similar way for the symmetric property. As KG is a cosovereign Hopf algebra with respect to its counit, see [**4**] for details, a concept of symmetric algebra can be defined in its category of corepresentations, i.e., in the monoidal category of G-graded vector spaces; the resulting objects are called graded symmetric algebras. As expected, A is graded symmetric if $A \simeq A^*$ as graded A-bimodules. If A is strongly graded, then A_e is symmetric whenever A is graded symmetric. However, the converse is not true; see [**5**, Remark 5.3]. This shows that Question 1 is not equivalent to asking whether a symmetric strongly graded algebra is graded symmetric, although this other question is also of interest.

The transfer of the Frobenius property from the strongly graded algebra A to A_e works well under additional conditions, for example if A is free as a left and as a right A_e-module, in particular if A is a crossed product of A_e by G; see [**6**]. If A is Frobenius, then it is left (and right) self-injective, and then so is A_e; this means that A_e is a quasi-Frobenius algebra. Thus a possible example answering Question 1 in the negative should be built on a quasi-Frobenius algebra which is not Frobenius. Moreover, by Dade's theorem, each homogeneous component of the strongly graded algebra A is an invertible A_e-bimodule, see [**13**], suggesting a study of the Picard group $\mathrm{Pic}(A_e)$ of A_e. In Section 2 we look at invertible bimodules over a finite-dimensional quasi-Frobenius algebra R. For such an R, an object of central interest is the linear dual R^* of the regular bimodule R; we show that it is an invertible R-bimodule. In the case where R is Frobenius, R^* is isomorphic to a deformation of the regular bimodule R, with the right action modified by the Nakayama automorphism ν of R with respect to a Frobenius form. It follows that the order of the class $[R^*]$ of R^* in $\mathrm{Pic}(R)$ is just the order of the class of ν in the group $\mathrm{Out}(R)$ of outer automorphisms of R. If R is not Frobenius, then R^* cannot be obtained from R by deforming the right action by an automorphism, or in other words, $[R^*]$ does not lie in the image of $\mathrm{Out}(R)$ inside $\mathrm{Pic}(R)$, and we show that it lies in the centralizer of $\mathrm{Out}(R)$. We compute the order of $[R^*]$ in $\mathrm{Pic}(R)$ for: (1) liftings of certain Hopf algebras in the braided category of Yetter–Drinfeld modules, called quantum lines, over the group Hopf algebra of a finite abelian group; (2) certain quotients of quantum planes. This order may be any positive integer, and it can also be infinite. It is known that a finite-dimensional Hopf algebra is Frobenius. In this case we prove the following.

Theorem A. *Let H be a finite-dimensional Hopf algebra with antipode S. Then the order of $[H^*]$ in $\mathrm{Pic}(H)$ is the least common multiple of the order of the class of S^2 in $\mathrm{Out}(H)$ and the order of the modular element of H^* in the group of group-like elements of H^*.*

As a particular case, one gets a well-known characterization of symmetric finite-dimensional Hopf algebras, as those unimodular Hopf algebras such that S^2 is inner.

In Section 3 we explain that if R is a finite-dimensional quasi-Frobenius algebra, and S a basic algebra of R, which is necessarily Frobenius, then $\mathrm{Pic}(R) \simeq \mathrm{Pic}(S)$, and moreover, the order of $[R^*]$ in $\mathrm{Pic}(R)$ is equal to the order of $[S^*]$ in $\mathrm{Pic}(S)$.

In Section 4 we consider an algebra of dimension 9 which is quasi-Frobenius, but not Frobenius, and we investigate its structure and determine its Picard group. This algebra was introduced by Nakayama in [**11**] in a matrix presentation; see also [**9**, Example 16.19.(5)]. We use a different presentation given in [**7**]. Let $\mathcal{R}$ be the K-algebra with basis $\mathbf{B} = \{E, X_1, X_2, Y_1, Y_2\} \cup \{F_{ij} \mid 1 \le i, j \le 2\}$, and relations

$$\begin{aligned} E^2 &= E, & F_{ij}F_{jr} &= F_{ir}, \\ EX_i &= X_i, & X_iF_{ir} &= X_r, \\ F_{ij}Y_j &= Y_i, & Y_iE &= Y_i \end{aligned}$$

for any $1 \le i, j, r \le 2$, and any other product of two elements of $\mathbf{B}$ be zero. We show that any invertible $\mathcal{R}$-bimodule is either a deformation of $\mathcal{R}$ or one of $\mathcal{R}^*$ by an automorphism of $\mathcal{R}$, and we have an exact sequence

$$1 \longrightarrow \mathrm{Inn}(\mathcal{R}) \longrightarrow \mathrm{Aut}(\mathcal{R}) \longrightarrow \mathrm{Pic}(\mathcal{R}) \longrightarrow C_2 \longrightarrow 1,$$

where C_2 is the cyclic group of order 2. If V is an $\mathcal{R}$-bimodule, and α is an automorphism of $\mathcal{R}$, we denote by ${}_1V_\alpha$ the bimodule obtained from V by changing the right action via α. We state the conclusions of this section in:

Theorem B. *The class* $[\mathcal{R}^*]$ *has order* 2 *in* $\mathrm{Pic}(\mathcal{R})$*. An invertible* $\mathcal{R}$*-bimodule is isomorphic either to* ${}_1\mathcal{R}_\alpha$ *or to* ${}_1\mathcal{R}^*_\alpha$ *for some* $\alpha \in \mathrm{Aut}(\mathcal{R})$*, and* $\mathrm{Pic}(\mathcal{R}) \simeq \mathrm{Out}(\mathcal{R}) \times C_2$.

In Section 5 we compute the automorphism group $\mathrm{Aut}(\mathcal{R})$ and the group $\mathrm{Out}(\mathcal{R})$ of outer automorphisms. For this aim, we use another presentation of $\mathcal{R}$, given in [**7**]. Thus $\mathcal{R}$ is isomorphic to the Morita ring associated with a Morita context connecting the rings K and $M_2(K)$, where the connecting bimodules are K^2 and $M_{2,1}(K) = \left[\begin{smallmatrix} K \\ K \end{smallmatrix}\right]$ with actions given by the usual matrix multiplication, and such that both Morita maps are zero. Thus $\mathcal{R}$ is isomorphic as a linear space to the matrix algebra $M_3(K)$, but its multiplication is altered by collapsing the product of the off-diagonal blocks. We prove:

Theorem C. *The automorphism group* $\mathrm{Aut}(\mathcal{R})$ *is isomorphic to a semidirect product* $(K^2 \times M_{2,1}(K)) \rtimes (K^\times \times GL_2(K))$*, and* $\mathrm{Out}(\mathcal{R}) \simeq K^\times$.

Here $K^\times$ denotes the multiplicative group associated with K. We explicitly describe the automorphisms and the outer automorphisms. Comparing to the matrix algebra $M_3(K)$, where there are no outer automorphisms, the alteration of the multiplication produces non-trivial outer automorphisms of $\mathcal{R}$. As a consequence of Theorems B and C, we see that $\mathrm{Pic}(\mathcal{R}) \simeq K^\times \times C_2$.

In Section 6 we consider an arbitrary finite-dimensional algebra R and a morphism of R-bimodules $\psi\colon R^* \otimes_R R^* \to R$ which is associative, i.e., $\psi(r^* \otimes_R s^*) \leftharpoonup t^* = r^* \rightharpoonup \psi(s^* \otimes_R t^*)$ for any $r^*, s^*, t^* \in R^*$; here $\rightharpoonup$ and $\leftharpoonup$ denote the usual left and right actions of R on R^*. Then we can form the semitrivial extension $R \rtimes_\psi R^*$, which is the cartesian product $R \times R^*$ with the usual addition, and multiplication defined by

$$(r, r^*)(s, s^*) = (rs + \psi(r^* \otimes_R s^*), (r \rightharpoonup s^*) + (r^* \leftharpoonup s))$$

for any $r, s \in R$, $r^*, s^* \in R^*$. It has a structure of a C_2-graded algebra with R as the homogeneous component of trivial degree.

Proposition A. *We have that* $R \rtimes_\psi R^*$ *is a symmetric algebra.*

If $\psi = 0$, we get a well-known construction of Tachikawa (see [**9**]); in this case R may be any finite-dimensional algebra. If ψ is an isomorphism, which implies that R^* is invertible, then $R \rtimes_\psi R^*$ is a strongly C_2-graded algebra. This suggests that in order to construct symmetric strongly graded algebras, it is natural to ask the following.

Question 2. *If R is a finite-dimensional algebra such that $R^* \otimes_R R^* \simeq R$ as R-bimodules, is it true that any isomorphism $\psi\colon R^* \otimes_R R^* \to R$ is associative?*

We address it in Section 7. We will see that if $R^* \otimes_R R^* \simeq R$, then R is necessarily quasi-Frobenius. We first answer the question if R is Frobenius, and then we derive the quasi-Frobenius case by using Morita theory to reduce to the basic algebra of R. We prove the following.

Proposition B. *Let R be a finite-dimensional algebra such that $[R^*]$ has order at most 2 in $\mathrm{Pic}(R)$. Then any isomorphism $\psi\colon R^* \otimes_R R^* \to R$ is associative.*

In particular, any isomorphism $\varphi\colon \mathcal{R}^* \otimes_{\mathcal{R}} \mathcal{R}^* \to \mathcal{R}$ resulting from Theorem B is associative, so then the strongly C_2-graded algebra $\mathcal{R} \rtimes_\varphi \mathcal{R}^*$ is symmetric, thus also Frobenius, while its component of trivial degree is not Frobenius. This answers in the negative Question 1, for both Frobenius and symmetric properties. It also answers the other question related to the symmetric property, since $\mathcal{R} \rtimes_\varphi \mathcal{R}^*$ is symmetric, but it is not graded symmetric as its homogeneous component of degree e is not symmetric.

Besides producing the large class of symmetric algebras presented above, the semitrivial extension construction is of interest in itself, at least taking into account the wealth of results of interest concerning trivial extensions, i.e., those associated to zero morphisms ψ.

More examples answering in the negative Question 1 for the symmetric property are obtained for Frobenius algebras R such that $[R^*]$ has order 2 in $\mathrm{Pic}(R)$. We present several classes of algebras R enjoying these properties. Among them, we note that for any finite-dimensional unimodular Hopf algebra H, $[H^*]$ has order at most 2 in $\mathrm{Pic}(H)$.

We work over a field K, with multiplicative group $K^\times$. We refer to [**9**], [**10**], and [**18**] for facts related to (quasi-)Frobenius algebras and symmetric algebras, to [**13**] for results about graded rings, and to [**17**] for basic notions about Hopf algebras. We recall that if G is a group with neutral element e, an algebra A is G-graded if it has a decomposition $A = \oplus_{g\in G} A_g$ as a direct sum of linear subspaces such that $A_g A_h \subset A_{gh}$ for any $g, h \in G$; in particular, A_e is a subalgebra of A. Such an A is called strongly graded if $A_g A_h = A_{gh}$ for any $g, h \in G$.

2. Quasi-Frobenius algebras and invertible bimodules

We recall from [**2**] some basic facts concerning invertible bimodules and the Picard group. Let R be an algebra over a field K. An R-bimodule P is called invertible if it satisfies one of the following equivalent conditions: (1) There exists a bimodule Q such that $P \otimes_R Q$ and $Q \otimes_R P$ are isomorphic to R as bimodules; (2) The functor $P \otimes_R -\colon R\text{-mod} \to R\text{-mod}$ is an equivalence of categories; (3) P is a finitely generated projective generator as a left R-module, and the map $\omega\colon R \to \mathrm{End}({}_R P)$, given by $\omega(r)(p) = pr$ for any $r \in R$, $p \in P$, is a ring isomorphism.

We keep the usual convention that the multiplication in $\mathrm{End}({}_R P)$ is the inverse map composition. The set of isomorphism types of invertible R-bimodules is a group with multiplication defined by $[U] \cdot [V] = [U \otimes_R V]$, where $[U]$ denotes the class of the bimodule U with respect to the isomorphism equivalence relation. This group is called the Picard group of R, and it is denoted by $\mathrm{Pic}(R)$.

If V is an R-bimodule and α, β are elements in the group $\mathrm{Aut}(R)$ of algebra automorphisms of R, we denote by ${}_\alpha V_\beta$ the bimodule with the same underlying space as V, and left and right actions defined by $r{*}v = \alpha(r)v$ and $v{*}r = v\beta(r)$ for any $v \in V$ and $r \in R$. The following facts hold for any $\alpha, \beta, \gamma \in \mathrm{Aut}(R)$. All isomorphisms are of R-bimodules, and 1 denotes the identity morphism.

- ${}_{\gamma\alpha}R_{\gamma\beta} \simeq {}_\alpha R_\beta$, in particular ${}_\alpha R_\beta \simeq {}_1 R_{\alpha^{-1}\beta}$.
- ${}_1 R_\alpha \otimes_R {}_1 R_\beta \simeq {}_1 R_{\alpha\beta}$, thus ${}_1 R_\alpha$ is invertible, and $[{}_1 R_\alpha]^{-1} = [{}_1 R_{\alpha^{-1}}]$.
- ${}_1 R_\alpha \simeq {}_1 R_\beta$ if and only if $\alpha\beta^{-1}$ is an inner automorphism of R, i.e., there exists an invertible element $u \in R$ such that $\alpha\beta^{-1}(r) = u^{-1}ru$ for any $r \in R$. Denote by $\mathrm{Inn}(R)$ the group of inner automorphisms of R. In particular, ${}_1 R_\alpha \simeq R$ if and only if $\alpha \in \mathrm{Inn}(R)$, thus there is an exact sequence of groups $0 \to \mathrm{Inn}(R) \hookrightarrow \mathrm{Aut}(R) \to \mathrm{Pic}(R)$, the last morphism in the sequence taking α to ${}_1 R_\alpha$. The factor group $\mathrm{Aut}(R)/\mathrm{Inn}(R)$, denoted by $\mathrm{Out}(R)$, is called the group of outer automorphisms of R, and it embeds into $\mathrm{Pic}(R)$.
- ${}_\alpha V_\beta \simeq {}_\alpha R_1 \otimes_R V \otimes_R {}_1 R_\beta$.

We will also need the following.

Proposition 2.1 ([**2**, p. 73])**.** *Let U and V be invertible R-bimodules such that $U \simeq V$ as left R-modules. Then there exists $\alpha \in \mathrm{Aut}(R)$ such that $U \simeq {}_1 V_\alpha$ as R-bimodules.*

Now let V be a bimodule over the K-algebra R. Then the linear dual $V^* = \mathrm{Hom}_K(V, K)$ is an R-bimodule with actions denoted by $\rightharpoonup$ and $\leftharpoonup$, given by $(r \rightharpoonup v^*)(v) = v^*(vr)$ and $(v^* \leftharpoonup r)(v) = v^*(rv)$ for any $r \in R$, $v^* \in V^*$, $v \in V$. One can easily check that $({}_\alpha V_\beta)^* = {}_\beta(V^*)_\alpha$ for any $\alpha, \beta \in \mathrm{Aut}(R)$. If V is finite-dimensional, then $(V^*)^* \simeq V$, and this shows that two finite-dimensional bimodules V and W are isomorphic if and only if so are their duals V^* and W^*.

We are interested in a particular bimodule, namely R^*, the dual of R. Some immediate consequences of the discussion above are that for any $\alpha, \beta, \gamma \in \mathrm{Aut}(R)$:

- ${}_{\gamma\alpha}(R^*)_{\gamma\beta} \simeq {}_\alpha(R^*)_\beta$, in particular ${}_\alpha(R^*)_\beta \simeq {}_1(R^*)_{\alpha^{-1}\beta}$. Indeed, ${}_{\gamma\alpha}(R^*)_{\gamma\beta} \simeq ({}_{\gamma\beta}R_{\gamma\alpha})^* \simeq ({}_\beta R_\alpha)^* \simeq {}_\alpha(R^*)_\beta$.
- If R has finite dimension, then ${}_1(R^*)_\alpha \simeq {}_1(R^*)_\beta$ if and only if $\alpha^{-1}\beta \in \mathrm{Inn}(R)$. Indeed, ${}_1(R^*)_\alpha$ and ${}_1(R^*)_\beta$ are isomorphic if and only if so are their duals, i.e., ${}_\alpha R_1 \simeq {}_\beta R_1$, which is the same as ${}_1 R_{\alpha^{-1}} \simeq {}_1 R_{\beta^{-1}}$, i.e., $\alpha^{-1}\beta \in \mathrm{Inn}(R)$. Since $\mathrm{Inn}(R)$ is a normal subgroup of $\mathrm{Aut}(R)$, this is equivalent to $\alpha\beta^{-1} \in \mathrm{Inn}(R)$.

The following holds for any finite-dimensional algebra.

Proposition 2.2. *Let R be a finite-dimensional algebra. Then the map $\omega\colon R \to \mathrm{End}({}_R R^*)$ defined by $\omega(a)(r^*) = r^* \leftharpoonup a$ for any $r^* \in R^*$ and $a \in R$ is an isomorphism of algebras.*

Proof: Since the linear dual functor is a duality between the categories of finite-dimensional right, respectively left, R-modules, we have $R \simeq \mathrm{End}(R_R) \simeq \mathrm{End}((R_R)^*) = \mathrm{End}({}_R R^*)$. In particular R and $\mathrm{End}({}_R R^*)$ have the same dimension.

It is easy to check that ω is well defined and it is an algebra morphism. If $\omega(a) = 0$ for some a, then $r^* \leftharpoonup a = 0$ for any $r^* \in R^*$, and evaluating at 1, we get $r^*(a) = 0$. Thus a must be 0, so ω is injective, and then it is an isomorphism. $\square$

Let R be a finite-dimensional algebra. We recall that R is called quasi-Frobenius if it is injective as a left (or equivalently, right) R-module. It is known that R is quasi-Frobenius if and only if the left R-modules R and R^* have the same distinct indecomposable components (possibly occurring with different multiplicities); see [**9**, Section 16C]. Therefore a Frobenius algebra is always quasi-Frobenius.

Corollary 2.3. *Let R be a finite-dimensional algebra. Then R^* is an invertible R-bimodule if and only if R is a quasi-Frobenius algebra.*

Proof: If R^* is an invertible bimodule, then it is projective as a right R-module, so then its linear dual $(R^*)^*$ is an injective left R-module. But $(R^*)^* \simeq R$ as left R-modules, and we get that R is left self-injective.

Conversely, assume that R is quasi-Frobenius. Since R is an injective right R-module, we get that R^* is a projective left R-module. On the other hand, since the left R-modules R and R^* have the same distinct indecomposable components, we see that there is an epimorphism $(R^*)^n \to R$ for a large enough positive integer n, thus R^* is a generator as a left R-module. If we also take into account Proposition 2.2, we get that R^* is invertible. □

If R is Frobenius, then an element $\lambda \in R^*$ such that $(R \rightharpoonup \lambda) = R^*$ is called a Frobenius form on R; in this case, the map $a \mapsto (a \rightharpoonup \lambda)$ is an isomorphism of left R-modules between R and R^*, and also, the map $a \mapsto (\lambda \leftharpoonup a)$ is an isomorphism of right R-modules from R to R^*. The Nakayama automorphism of R associated with a Frobenius form λ is the map $\nu \colon R \to R$ defined such that for any $a \in R$, $\nu(a)$ is the unique element of R satisfying $(\nu(a) \rightharpoonup \lambda) = (\lambda \leftharpoonup a)$, or equivalently, $\lambda(ar) = \lambda(r\nu(a))$ for any $r \in R$; ν turns out to be an algebra automorphism. If ν and ν' are Nakayama automorphisms associated with two Frobenius forms, then there exists an invertible element $u \in R$ such that $\nu'(a) = u^{-1}\nu(a)u$ for any $a \in R$, thus ν and ν' are equal up to an inner automorphism. It follows that the class of a Nakayama automorphism in $\mathrm{Out}(R)$ does not depend on the Frobenius form; see [**9**, Section 16E], [**10**, Section 2.2], or [**18**, Chapter IV] for details.

If the quasi-Frobenius algebra R is not Frobenius, R^* is not isomorphic to any ${}_1R_\alpha$, as R^* is not isomorphic to R as left R-modules. In the Frobenius case, we have the following result; it appears in an equivalent formulation in [**18**, Proposition 3.15].

Proposition 2.4. *Let R be a Frobenius algebra. Then there exists $\nu \in \mathrm{Aut}(R)$ such that $R^* \simeq {}_1R_\nu$ as bimodules. Moreover, any such ν is the Nakayama automorphism of R associated with a Frobenius form. As a consequence, the order of $[R^*]$ in $\mathrm{Pic}(R)$ is equal to the order of the class of ν in $\mathrm{Out}(R)$.*

Proof: The first part follows directly from Proposition 2.1, since $R^* \simeq R$ as left R-modules.

Let $\gamma \colon {}_1R_\nu \to R^*$ be an isomorphism of bimodules, and let $\lambda = \gamma(1)$. Then $(R \rightharpoonup \lambda) = R^*$, so λ is a Frobenius form on R. Then for any $a, x \in R$

$$\begin{aligned}
(\lambda \leftharpoonup a)(x) &= (\gamma(1) \leftharpoonup a)(x) \\
&= \gamma(1 \cdot \nu(a))(x) \\
&= \gamma(\nu(a) \cdot 1)(x) \\
&= (\nu(a) \rightharpoonup \gamma(1))(x) \\
&= (\nu(a) \rightharpoonup \lambda)(x),
\end{aligned}$$

showing that $(\lambda \leftharpoonup a) = (\nu(a) \rightharpoonup \lambda)$, thus ν is the Nakayama automorphism associated with λ. □

Looking inside the Picard group, the previous proposition gives a new perspective on the well-known fact that a Frobenius algebra is symmetric if and only if the Nakayama automorphism is inner; see [**9**, Theorem 16.63]. Indeed, R is symmetric if and only if $R^* \simeq R$ as bimodules, i.e., ${}_1R_\nu \simeq R$, and this is equivalent to ν being inner.

The following indicates a commutation property of the class of R^* in the Picard group of R.

Proposition 2.5. *Let R be a quasi-Frobenius finite-dimensional algebra, and let $\alpha \in \mathrm{Aut}(R)$. Then $R^* \otimes_R {}_1R_\alpha \simeq {}_1R_\alpha \otimes_R R^*$ as R-bimodules. Thus the element $[R^*]$ of the Picard group $\mathrm{Pic}(R)$ lies in the centralizer of the image of $\mathrm{Out}(R)$.*

Proof: Taking into account the above considerations, we have isomorphisms of R-bimodules

$$R^* \otimes_R {}_1R_\alpha \simeq {}_1(R^*)_\alpha \simeq {}_{\alpha^{-1}}(R^*)_1 \simeq {}_{\alpha^{-1}}R_1 \otimes_R R^* \simeq {}_1R_\alpha \otimes_R R^*.$$ □

Corollary 2.6. *Let R be a Frobenius algebra. Then the class of the Nakayama automorphism of R lies in the centre of $\mathrm{Out}(R)$.*

If R is quasi-Frobenius, we are interested in the order of $[R^*]$ in the group $\mathrm{Pic}(R)$. This order is 1 if and only if R is a symmetric algebra. The following examples show that it may be any integer ≥ 2 in other quasi-Frobenius algebras, and also it can be infinite.

For the first example, we recall that if H is a finite-dimensional Hopf algebra, then a left integral on H is an element $\lambda \in H^*$ such that $h^*\lambda = h^*(1)\lambda$ for any $h^* \in H^*$; the multiplication of H^* is given by the convolution product. Any finite-dimensional Hopf algebra H is a Frobenius algebra, and a non-zero left integral λ on H is a Frobenius form; see [**10**, Theorem 12.5].

Example 2.7. Let C be a finite abelian group, and let $C^* = \mathrm{Hom}(C, K^\times)$ be its character group. We consider certain Hopf algebras in the braided category of Yetter–Drinfeld modules over the group Hopf algebra KC, called quantum lines, and their liftings, obtained by a bosonization construction. We obtain some finite-dimensional pointed Hopf algebras with coradical KC; see [**1**], [**3**]. There are two classes of such objects.

- Hopf algebras of the type $H_1(C, n, c, c^*)$, where $n \geq 2$ is an integer, $c \in C$, and $c^* \in C^*$, such that $c^n \neq 1$, $(c^*)^n = 1$, and $c^*(c)$ is a primitive nth root of unity. It is generated as an algebra by the Hopf subalgebra KC and a $(1, c)$-skew-primitive element x, i.e., the comultiplication works as $\Delta(x) = c \otimes x + x \otimes 1$ on x, subject to relations $x^n = c^n - 1$ and $xg = c^*(g)gx$ for any $g \in C$. Note that the required conditions show that c^* has order n.
- Hopf algebras of the type $H_2(C, n, c, c^*)$, where $n \geq 2$ is an integer, $c \in C$, and $c^* \in C^*$, such that $c^*(c)$ is a primitive nth root of unity. It is generated as an algebra by the Hopf subalgebra KC and a $(1, c)$-skew-primitive element x, subject to relations $x^n = 0$ and $xg = c^*(g)gx$ for any $g \in C$. In this case, the order of c^*, which we denote by m, is a multiple of n.

If H is any of $H_1(C, n, c, c^*)$ or $H_2(C, n, c, c^*)$, a linear basis of H is $\mathcal{B} = \{gx^j \mid g \in C,\ 0 \leq j \leq n-1\}$, thus the dimension of H is $n|C|$, and the linear map $\lambda \in H^*$ such that $\lambda(c^{1-n}x^{n-1}) = 1$ and λ takes any other element of $\mathcal{B}$ to 0 is a left integral on H; see [**3**, Proposition 1.17].

If $g \in C$, then

$$(\lambda \leftharpoonup g)(g^{-1}c^{1-n}x^{n-1}) = \lambda(c^{1-n}x^{n-1}) = 1$$

and $\lambda \leftharpoonup g$ takes any other element of $\mathcal{B}$ to 0, while

$$(g \rightharpoonup \lambda)(g^{-1}c^{1-n}x^{n-1}) = \lambda(g^{-1}c^{1-n}x^{n-1}g) = c^*(g)^{n-1}\lambda(c^{1-n}x^{n-1}) = c^*(g)^{n-1}$$

and $g \rightharpoonup \lambda$ takes any other element of $\mathcal{B}$ to 0. These show that $(g \rightharpoonup \lambda) = c^*(g)^{n-1}(\lambda \leftharpoonup g)$, so the Nakayama automorphism ν associated with the Frobenius form λ satisfies $\nu(g) = c^*(g)^{1-n}g$.

On the other hand, if we denote $\xi = c^*(c)$, we have

$$(x \rightharpoonup \lambda)(c^{1-n}x^{n-2}) = \lambda(c^{1-n}x^{n-1}) = 1$$

and $x \rightharpoonup \lambda$ takes any other element of $\mathcal{B}$ to 0, while

$$(\lambda \leftharpoonup x)(c^{1-n}x^{n-2}) = \lambda(xc^{1-n}x^{n-2}) = \xi^{1-n}\lambda(c^{1-n}x^{n-1}) = \xi$$

and $\lambda \leftharpoonup x$ takes any other element of $\mathcal{B}$ to 0. Thus we get $\nu(x) = \xi x$.

Denote the order of c^* by m; we noticed that $m = n$ in the case of $H_1(C, n, c, c^*)$, and $m = dn$ for some positive integer d in the case of $H_2(C, n, c, c^*)$. If j is a positive integer, then $\nu^j = 1$ if and only if $\xi^j = 1$ and $c^*(g)^{j(1-n)} = 1$ for any $g \in C$. If the latter condition is satisfied, then $(c^*)^{j(1-n)} = 1$, or equivalently, $m|j(1-n)$, hence $n|j(1-n)$, and then $n|j$, so the condition $\xi^j = 1$ is automatically satisfied. Thus the order of ν is the least positive integer j such that $m|j(1-n)$. For any such j we have $n|j$, so $j = bn$ for some integer b. Then $m|j(1-n)$ is equivalent to $d|b(n-1)$, and also to $\frac{d}{(d,n-1)}|b \cdot \frac{n-1}{(d,n-1)}$. Since $\frac{d}{(d,n-1)}$ and $\frac{n-1}{(d,n-1)}$ are relatively prime, the latter condition is equivalent to $\frac{d}{(d,n-1)}|b$. We conclude that the least such b is $\frac{d}{(d,n-1)}$, and the order of ν is

$$j = bn = \frac{dn}{(d, n-1)} = \frac{m}{\left(\frac{m}{n}, n-1\right)}.$$

This shows that for $H_1(C, n, c, c^*)$, where $m = n$, the order of ν is necessarily n, while for $H_2(C, n, c, c^*)$, the order may be larger than n, depending on the value of m.

Now we show that for any $1 \leq j < \frac{m}{(\frac{m}{n}, n-1)}$, ν^j is not an inner automorphism. Indeed, if it were, then there would exist an invertible u such that $\nu^j(r) = u^{-1}ru$ for any r in the Hopf algebra (which is either $H_1(C, n, c, c^*)$ or $H_2(C, n, c, c^*)$). In particular, for any $g \in C$, $c^*(g)^{j(1-n)}g = u^{-1}gu$. Applying the counit ε, one gets $c^*(g)^{j(1-n)} = 1$ for any $g \in C$, so $(c^*)^{j(1-n)} = 1$. Hence $m|j(1-n)$, and we have seen above that this implies that j must be at least $\frac{m}{(\frac{m}{n}, n-1)}$, a contradiction.

We conclude that if A is a Hopf algebra of type $H_1(C, n, c, c^*)$ or $H_2(C, n, c, c^*)$, then the order of the Nakayama automorphism ν of A in the group of algebra automorphisms of A, as well as the order of the class of ν in $\mathrm{Out}(A)$ (which is the same as the order of $[A^*]$ in $\mathrm{Pic}(A)$) is $\frac{m}{(\frac{m}{n}, n-1)}$, where m is the order of c^* in C^*. In the case of $H_1(C, n, c, c^*)$, where $m = n$, this order is just n.

A particular case is when $C = C_n = \langle c \rangle$ is the cyclic group of order $n \geq 2$. Then for any linear character $c^* \in C^*$ such that $c^*(c)$ is a primitive nth root of unity, $H_1(C, n, c, c^*)$ is a Taft Hopf algebra. For such algebras, the order of the Nakayama automorphism associated with a left integral as a Frobenius form is computed in [**18**, Example 5.9, p. 614].

Example 2.8. Let q be a non-zero element of a field K, and let $K_q[X,Y]$ be the quantum plane, which is the K-algebra generated by X and Y, subject to the relation $YX = qXY$. Let $R_q = K_q[X,Y]/(X^2, Y^2)$, which has dimension 4, and a basis $\mathcal{B} = \{1, x, y, xy\}$, where x, y denote the classes of X, Y in R. We have $x^2 = y^2 = 0$ and $yx = qxy$. Denote by $\mathcal{B}^* = \{1^*, x^*, y^*, (xy)^*\}$ the basis of R_q^* dual to $\mathcal{B}$. Then

$$1 \rightharpoonup (xy)^* = (xy)^*, \quad x \rightharpoonup (xy)^* = qy^*, \quad y \rightharpoonup (xy)^* = x^*, \quad (xy) \rightharpoonup (xy)^* = 1^*,$$

showing that the linear map from R_q to R_q^* which takes r to $r \rightharpoonup (xy)^*$ is an isomorphism. Thus R_q is a Frobenius algebra and $\lambda = (xy)^*$ is a Frobenius form on R_q. Now since $(xy)^* \leftharpoonup x = y^*$ and $(xy)^* \leftharpoonup y = qx^*$, the Nakayama automorphism associated with λ is $\nu \in \mathrm{Aut}(R_q)$, given by $\nu(x) = q^{-1}x$, $\nu(y) = qy$. Then it is clear that the order of ν in the automorphism group of R_q is n if q is a primitive nth root of unity in K, and it is infinite when no non-trivial power of q is 1. This fact was observed in [**18**, Example 10.7, p. 417] by using periodic modules with respect to actions of the syzygy and Auslander–Reiten operators.

We show that if t is a positive integer such that $q^t \neq 1$, then ν^t is not even an inner automorphism. Indeed, if it were, then $\nu^t(x) = u^{-1}xu$, or $ux = q^t xu$ for some invertible $u \in R_q$. If we write $u = a1 + bx + cy + dxy$ with $a, b, c, d \in K$, this means that $ax + qcxy = q^t(ax + cxy)$, showing that $a = 0$. But then u cannot be invertible, since $xyu = 0$, a contradiction.

In conclusion, if q is not a root of unity, ν has infinite order in $\mathrm{Out}(R_q)$, and so does $[R_q^*]$ in $\mathrm{Pic}(R_q)$, while if q is a primitive nth root of unity, then $[R_q^*]$ has order n in $\mathrm{Pic}(R_q)$.

We end this example with the remark that Nakayama and Nesbitt constructed in [**12**, p. 665] a class of examples of Frobenius algebras which are not symmetric, presented in a matrix form. More precisely, in the presentation of [**9**, Example 16.66], for any non-zero elements $u, v \in K$, let $A_{u,v}$ be the subalgebra of $M_4(K)$ consisting of all matrices of the type

$$\begin{bmatrix} a & b & c & d \\ 0 & a & 0 & uc \\ 0 & 0 & a & vb \\ 0 & 0 & 0 & a \end{bmatrix},$$

where $a, b, c, d \in K$. Then $A_{u,v}$ is Frobenius for any $u, v \in K^\times$, and it is symmetric if and only if $u = v$. The algebra $A_{u,v}$ has a basis consisting of the elements

$$I_4, \quad x = E_{12} + vE_{34}, \quad y = E_{13} + uE_{24}, \quad z = E_{14},$$

where E_{ij} denote the usual matrix units in $M_4(K)$, and they satisfy the relations

$$x^2 = 0, \quad y^2 = 0, \quad xy = uz, \quad yx = vz.$$

These show that in fact $A_{u,v}$ is isomorphic to the quotient $R_{u^{-1}v}$ of the quantum plane.

If H is a finite-dimensional Hopf algebra, let $t \in H$ be a non-zero left integral in H, i.e., $ht = \varepsilon(h)t$ for any $h \in H$, where ε is the counit of H. As the space of left integrals is one-dimensional and th is a left integral for any $h \in H$, there is a linear map $\mathcal{G}\colon H \to K$ such that $th = \mathcal{G}(h)t$ for any $h \in H$. In fact, $\mathcal{G}$ is an algebra morphism, thus an element of the group $G(H^*)$ of group-like elements of H^*. We call $\mathcal{G}$ the distinguished group-like element of H^*, and also the right modular element of H^*.

Theorem 2.9. *Let H be a finite-dimensional Hopf algebra with antipode S and counit ε, and let $\mathcal{G}$ be the modular element in H^*. If n is a positive integer, then $[H^*]^n = 1$ in $\mathrm{Pic}(H)$ if and only if S^{2n} is inner and $\mathcal{G}^n = \varepsilon$. As a consequence, the order of $[H^*]$ in $\mathrm{Pic}(H)$ is the least common multiple of the order of the class of S^2 in $\mathrm{Out}(H)$ and the order of $\mathcal{G}$ in $G(H^*)$.*

Proof: Let λ be a non-zero left integral on H, which is a Frobenius form on H, and let ν be the associated Nakayama automorphism. By [**16**, Theorem 3(a)], in the reformulation of [**10**, Proposition 12.8], $\nu(h) = \sum \mathcal{G}(h_2)S^2(h_1)$ for any $h \in H$. Let $\ell_{\mathcal{G}} \colon H \to H$ be the linear map defined by $\ell_{\mathcal{G}}(h) = \mathcal{G} \rightharpoonup h = \sum \mathcal{G}(h_2)h_1$. We have $\nu = S^2\ell_{\mathcal{G}}$. We note that $\mathcal{G}S^2 = \mathcal{G}$. Indeed, it is clear that $\mathcal{G}S = S^*(\mathcal{G}) = \mathcal{G}^{-1}$, since the dual map S^* of S is the antipode of the dual Hopf algebra H^*, and it takes a group-like element to its inverse. Now we have

$$\begin{aligned}
(\ell_{\mathcal{G}}S^2)(h) &= \mathcal{G} \rightharpoonup S^2(h)\\
&= \sum \mathcal{G}(S^2(h_2))S^2(h_1)\\
&= \sum \mathcal{G}(h_2)S^2(h_1)\\
&= S^2(\mathcal{G} \rightharpoonup h)\\
&= (S^2\ell_{\mathcal{G}})(h),
\end{aligned}$$

showing that $\ell_{\mathcal{G}}S^2 = S^2\ell_{\mathcal{G}}$. Since $\rightharpoonup$ is a left action, we have $(\ell_{\mathcal{G}})^n = \ell_{\mathcal{G}^n}$ for any positive integer n, and it follows that $\nu^n = S^{2n}\ell_{\mathcal{G}^n}$. Now if $\ell_{\mathcal{G}^n} = \varepsilon$ and S^{2n} is inner, then $\nu^n = S^{2n}$ is inner, so $[H^*]^n = [{}_1H_\nu]^n = [{}_1H_{\nu^n}] = 1$ in $\mathrm{Pic}(H)$. Conversely, if $[H^*]^n = 1$, then ν^n is inner. Let $\nu^n(h) = u^{-1}hu$ for some invertible $u \in H$. Then $S^{2n}(\ell_{\mathcal{G}^n}(h)) = u^{-1}hu$ for any $h \in H$, and applying ε and using that $\varepsilon S = \varepsilon$, we obtain $\varepsilon(\ell_{\mathcal{G}^n}(h)) = \varepsilon(u^{-1})\varepsilon(h)\varepsilon(u) = \varepsilon(h)$. As $\varepsilon(\ell_{\mathcal{G}^n}(h)) = \varepsilon\big(\sum \mathcal{G}^n(h_2)h_1\big) = \mathcal{G}^n(h)$, we get $\mathcal{G}^n = \varepsilon$. Consequently, $\nu^n = S^{2n}$, so S^{2n} is inner. $\square$

We note that in the particular case where $n = 1$, the previous theorem says that a finite-dimensional Hopf algebra H is a symmetric algebra if and only if $\mathcal{G} = \varepsilon$, i.e., H is unimodular, and S^2 is inner. This is a result of [**14**]; see also [**10**, Theorem 12.9].

3. Picard groups of quasi-Frobenius algebras

We start with some general considerations. Let R and S be two Morita-equivalent K-algebras. Let $(R, S, {}_RP_S, {}_SQ_R, P \otimes_S Q \xrightarrow{f} R, Q \otimes_R P \xrightarrow{g} S)$ be a strict Morita context connecting R and S, i.e., P and Q are bimodules as the indices indicate, f is an isomorphism of R-bimodules, g is an isomorphism of S-bimodules, and denoting $f(p \otimes_S q) = [p, q]$ and $g(q \otimes_R p) = (q, p)$, the conditions $[p, q]p' = p(q, p')$ and $(q, p)q' = q[p, q']$ hold for any $p, p' \in P$, $q, q' \in Q$.

As explained in [**20**, pp. 301–302], a K-linear monoidal equivalence $F = Q \otimes_R (-) \otimes_R P$ is induced between the monoidal categories of R-bimodules and S-bimodules, with quasi-inverse $G = P \otimes_S (-) \otimes_S Q$.

Proposition 3.1. *The mapping $[M] \mapsto [F(M)]$ is an isomorphism between the groups $\mathrm{Pic}(R)$ and $\mathrm{Pic}(S)$. In particular, the order of $[M]$ in $\mathrm{Pic}(R)$ is equal to the order of $[Q \otimes_R M \otimes_R P]$ in $\mathrm{Pic}(S)$.*

Proof: Since F is a monoidal equivalence, we see that if M is an invertible R-bimodule, then $F(M)$ is an invertible S-bimodule, and moreover, the mapping $[M] \mapsto [F(M)]$ is a group morphism. Its inverse takes $[N]$ to $[G(N)]$ for any invertible S-bimodule N. $\square$

Proposition 3.2. *Assume that R and S are Morita-equivalent finite-dimensional K-algebras, and let F be the monoidal equivalence between the categories of R-bimodules and S-bimodules described above. Then there is an isomorphism of S-bimodules $F(R^*) \simeq S^*$. In particular, the order of $[R^*]$ in $\mathrm{Pic}(R)$ is equal to the order of $[S^*]$ in $\mathrm{Pic}(S)$.*

Proof: We first list some basic facts. We denote by ${}_A\mathcal{M}_B$ the category of left A, right B-bimodules, Hom_{A-} means morphisms of left A-modules, while Hom_{-A} means morphisms of right A-modules. Item (i) below is the tensor-Hom adjunction, and (ii) is the duality property, where the structure of the involved objects is enriched to bimodules. Item (iii) is basic Morita theory, and so is (iv), with the mention that the isomorphism is also of S-bimodules.

(i) We have that $(M \otimes_B N)^* \simeq \mathrm{Hom}_{-B}(M, N^*)$ in ${}_C\mathcal{M}_A$ whenever $M \in {}_A\mathcal{M}_B$ and $N \in {}_B\mathcal{M}_C$ are finite-dimensional.

(ii) We have that $\mathrm{Hom}_{A-}(U, V) \simeq \mathrm{Hom}_{-A}(V^*, U^*)$ in ${}_B\mathcal{M}_C$ whenever $U \in {}_A\mathcal{M}_B$ and $V \in {}_A\mathcal{M}_C$ are finite-dimensional.

(iii) $\mathrm{Hom}_{R-}(P, R) \simeq Q$ in ${}_S\mathcal{M}_R$.

(iv) $\mathrm{Hom}_{-R}(Q, Q) \simeq S$ in ${}_S\mathcal{M}_S$.

Using these, we have the following isomorphisms of S-bimodules:

$$\begin{aligned}
(Q \otimes_R R^* \otimes_R P)^* &\simeq \mathrm{Hom}_{-R}(Q, (R^* \otimes_R P)^*) && (\text{by (i) for } M = {}_SQ_R, \\
& && \qquad N = {}_R(R^* \otimes_R P)_S) \\
&\simeq \mathrm{Hom}_{-R}(Q, \mathrm{Hom}_{-R}(R^*, P^*)) && (\text{by (i) for } M = {}_RR^*_R,\ N = {}_RP_S) \\
&\simeq \mathrm{Hom}_{-R}(Q, \mathrm{Hom}_{R-}(P, R)) && (\text{by (ii) for } U = {}_RP_S,\ V = {}_RR_R) \\
&\simeq \mathrm{Hom}_{-R}(Q, Q) && (\text{by (iii)}) \\
&\simeq S && (\text{by (iv)}).
\end{aligned}$$

Taking duals, we find that $F(R^*) = Q \otimes_R R^* \otimes_R P \simeq S^*$ as S-bimodules. □

Remark 3.3. One of the referees indicated to us an alternative way of proving this by using basic results on the Nakayama functor. Thus if we denote by $T = Q \otimes_R (-)$ the equivalence functor between the categories of left R-modules and left S-modules, and by $L = P \otimes_S (-)$ its quasi-inverse, we have isomorphisms

$$F(R^*) \otimes_S (-) \simeq T \circ \mathrm{N}^{\mathrm{r}}_{R\text{-mod}} \circ L \simeq T \circ L \circ \mathrm{N}^{\mathrm{r}}_{S\text{-mod}} \simeq S^* \otimes_S (-)$$

in the category of right exact linear functors from S-mod to R-mod (see [**8**, Lemma 3.15 and Theorem 3.18]), where N^{r} denotes the right exact Nakayama functor introduced in [**8**]. It follows that $F(R^*)$ and S^* are isomorphic.

Now if R is a finite-dimensional quasi-Frobenius algebra, we consider a basic algebra S of R. As explained in [**18**, p. 172], S can be constructed as follows. Take a complete system of orthogonal primitive idempotents in R, and let e be the sum of a system of representatives of the isomorphism types of the idempotents in this system. Then $S = eRe$ is a basic algebra of R. It is Frobenius and Morita-equivalent to R; see [**18**, Theorem 6.16, p. 173, and Corollary 3.11, p. 351]. A basic algebra of R is not uniquely determined, but it is unique up to an isomorphism. As a consequence of the above discussion, we obtain:

Corollary 3.4. *If R is a finite-dimensional quasi-Frobenius algebra, and S is a basic algebra of R, then $\mathrm{Pic}(R) \simeq \mathrm{Pic}(S)$ and the order of $[R^*]$ in $\mathrm{Pic}(R)$ is equal to the order of the class of the Nakayama automorphism of S (with respect to some Frobenius form) in $\mathrm{Out}(S)$.*

4. The structure of $\mathcal{R}$ and $\mathcal{R}^*$, and the Picard group of $\mathcal{R}$

Let $\mathcal{R}$ be the K-algebra presented in the introduction. It has basis

$$\mathbf{B} = \{E, X_1, X_2, Y_1, Y_2\} \cup \{F_{ij} \mid 1 \le i, j \le 2\},$$

and relations

$$\begin{aligned} E^2 &= E, & F_{ij}F_{jr} &= F_{ir}, \\ EX_i &= X_i, & X_iF_{ir} &= X_r, \\ F_{ij}Y_j &= Y_i, & Y_iE &= Y_i \end{aligned}$$

for any $1 \le i, j, r \le 2$, and any other product of two elements of $\mathbf{B}$ is zero.

Let

$$\begin{aligned} \mathcal{V}_1 &= \langle X_1, F_{11}, F_{21}\rangle, \\ \mathcal{V}_1' &= \langle X_2, F_{12}, F_{22}\rangle, \\ \mathcal{V}_2 &= \langle Y_1, Y_2, E\rangle. \end{aligned}$$

Then $\mathcal{R} = \mathcal{V}_1 \oplus \mathcal{V}_1' \oplus \mathcal{V}_2$ is a decomposition of $\mathcal{R}$ into a direct sum of indecomposable left $\mathcal{R}$-modules, and $\mathcal{V}_1 \simeq \mathcal{V}_1' \not\simeq \mathcal{V}_2$. Indeed, right multiplication by F_{12} is an isomorphism from $\mathcal{V}_1$ to $\mathcal{V}_1'$, with inverse the right multiplication by F_{21}, while $\mathcal{V}_1$ and $\mathcal{V}_2$ are not isomorphic since they have different annihilators.

Similarly, a decomposition of $\mathcal{R}$ into a direct sum of indecomposable right $\mathcal{R}$-modules is $\mathcal{R} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_2'$, with $\mathcal{U}_2 \simeq \mathcal{U}_2' \not\simeq \mathcal{U}_1$, where

$$\begin{aligned} \mathcal{U}_1 &= \langle E, X_1, X_2\rangle, \\ \mathcal{U}_2 &= \langle F_{11}, F_{12}, Y_1\rangle, \\ \mathcal{U}_2' &= \langle F_{21}, F_{22}, Y_2\rangle. \end{aligned}$$

The quotient algebra of $\mathcal{R}$ by the nilpotent ideal $\langle X_1, X_2, Y_1, Y_2\rangle$ is isomorphic to $K \times M_2(K)$, so the Jacobson radical $J(\mathcal{R}) = \langle X_1, X_2, Y_1, Y_2\rangle$. Then $\mathcal{U}_1 J(\mathcal{R}) = \langle X_1, X_2\rangle$ and $\mathcal{U}_2 J(\mathcal{R}) = \langle Y_1\rangle$, so the isomorphism types of simple right $\mathcal{R}$-modules are $S_1 = \mathcal{U}_1/\mathcal{U}_1 J(\mathcal{R}) \simeq \langle Y_1\rangle = \mathrm{soc}(\mathcal{U}_2)$ and $S_2 = \mathcal{U}_2/\mathcal{U}_2 J(\mathcal{R}) \simeq \langle X_1, X_2\rangle = \mathrm{soc}(\mathcal{U}_1)$, where $\mathrm{soc}(U)$ denotes the socle of the module U. These show that the multiplicity $\mu(S_i, \mathcal{U}_j)$ of the simple module S_i in a composition series of $\mathcal{U}_j$ is 1 for any $1 \le i, j \le 2$. Clearly, $\mathrm{End}_{\mathcal{R}}(S_i) \simeq K$ for each i.

Now we look at $\mathcal{R}^* = \mathrm{Hom}_K(\mathcal{R}, K)$, with the $\mathcal{R}$-bimodule structure induced by that of $\mathcal{R}$; we denote by $\rightharpoonup$ and $\leftharpoonup$ the left and right actions of $\mathcal{R}$ on $\mathcal{R}^*$. Denote by $\mathbf{B}^* = \{E^*, F_{ij}^*, X_i^*, Y_j^* \mid 1 \le i, j \le 2\}$ the basis of $\mathcal{R}^*$ dual to $\mathbf{B}$. On basis elements, the left action of $\mathcal{R}$ on$\mathcal{R}^*$ is

$$\begin{array}{llll} E \rightharpoonup E^* = E^*, & E \rightharpoonup F_{ij}^* = 0, & E \rightharpoonup X_i^* = 0, & E \longrightarrow Y_i^* = Y_i^*, \\ F_{ij} \rightharpoonup E^* = 0, & F_{ij} \rightharpoonup F_{rp}^* = \delta_{jp} F_{ri}^*, & F_{ij} \rightharpoonup X_r^* = \delta_{jr} X_i^*, & F_{ij} \rightharpoonup Y_r^* = 0, \\ X_i \rightharpoonup E^* = 0, & X_i \rightharpoonup F_{rj}^* = 0, & X_i \rightharpoonup X_j^* = \delta_{ij} E^*, & X_i \rightharpoonup Y_j^* = 0, \\ Y_i \rightharpoonup E^* = 0, & Y_i \rightharpoonup F_{rj}^* = 0, & Y_i \rightharpoonup X_j^* = 0, & Y_i \rightharpoonup Y_j^* = F_{ji}^*, \end{array}$$

for any $1 \le i, j, r, p \le 2$.

We will identify $\mathcal{U}_1^*$ with $\langle E^*, X_1^*, X_2^*\rangle$ inside $\mathcal{R}^*$, and similarly for the duals of $\mathcal{U}_2$, $\mathcal{U}_2'$, $\mathcal{V}_1$, $\mathcal{V}_1'$, $\mathcal{V}_2$.

Lemma 4.1. *We have $\mathcal{U}_1^* \simeq \mathcal{V}_1$ and $\mathcal{U}_2^* \simeq \mathcal{V}_2$ as left $\mathcal{R}$-modules. Consequently, $\mathcal{V}_1^* \simeq \mathcal{U}_1$ and $\mathcal{V}_2^* \simeq \mathcal{U}_2$ as right $\mathcal{R}$-modules, $\mathcal{R}^* \simeq \mathcal{V}_1 \oplus \mathcal{V}_2^2$ as left $\mathcal{R}$-modules, and $\mathcal{R}^* \simeq \mathcal{U}_1^2 \oplus \mathcal{U}_2$ as right $\mathcal{R}$-modules.*

Proof: It follows from the action table above that the linear map taking X_1 to E^*, F_{11} to X_1^*, and F_{21} to X_2^* is an isomorphism of left $\mathcal{R}$-modules from $\mathcal{V}_1$ to $\mathcal{U}_1^*$. Also, the mapping $Y_1 \mapsto F_{11}^*$, $Y_2 \mapsto F_{12}^*$, $E \mapsto Y_1^*$ defines an isomorphism $\mathcal{V}_2 \simeq \mathcal{U}_2^*$. □

The proof of the following corollary was suggested by one of the referees. Our initial proof was more computational.

Corollary 4.2. *We have* $\dim_K(\mathcal{U}_i \otimes_{\mathcal{R}} \mathcal{V}_j) = 1$ *for any* $1 \leq i, j \leq 2$.

Proof: By the tensor-Hom adjunction and taking into account Lemma 4.1, we have linear isomorphisms

$$(\mathcal{U}_i \otimes_{\mathcal{R}} \mathcal{V}_j)^* = \operatorname{Hom}_K(\mathcal{U}_i \otimes_{\mathcal{R}} \mathcal{V}_j, K) \simeq \operatorname{Hom}_{\mathcal{R}}(\mathcal{U}_i, \mathcal{V}_j^*) \simeq \operatorname{Hom}_{\mathcal{R}}(\mathcal{U}_i, \mathcal{U}_j)$$

for any i, j. Now the result follows since $\mathcal{U}_i$ is a projective cover of S_i, and we have by [**10**, Proposition 2.8] that

$$\dim_K(\operatorname{Hom}_{\mathcal{R}}(\mathcal{U}_i, \mathcal{U}_j)) = \mu(S_i, \mathcal{U}_j) \dim_K(\operatorname{End}_{\mathcal{R}}(S_i)) = 1. \qquad \square$$

Remark 4.3. The only non-zero tensor monomials formed with elements of $\mathbf{B}$ in tensor products of the form $\mathcal{U}_i \otimes_{\mathcal{R}} \mathcal{V}_j$ are: $E \otimes_{\mathcal{R}} X_1 = X_1 \otimes_{\mathcal{R}} F_{11} = X_2 \otimes_{\mathcal{R}} F_{21}$ in $\mathcal{U}_1 \otimes_{\mathcal{R}} \mathcal{V}_1$, $F_{11} \otimes_{\mathcal{R}} F_{11} = F_{12} \otimes_{\mathcal{R}} F_{21}$ in $\mathcal{U}_2 \otimes_{\mathcal{R}} \mathcal{V}_1$, $Y_1 \otimes_{\mathcal{R}} E = F_{11} \otimes_{\mathcal{R}} Y_1 = F_{12} \otimes_{\mathcal{R}} Y_2$ in $\mathcal{U}_2 \otimes_{\mathcal{R}} \mathcal{V}_2$, and $E \otimes_{\mathcal{R}} E$ in $\mathcal{U}_1 \otimes_{\mathcal{R}} \mathcal{V}_2$. Indeed, it is straightforward to check that any other such tensor monomial in some $\mathcal{U}_i \otimes_{\mathcal{R}} \mathcal{V}_j$ is zero. Then the above mentioned tensor monomials must be non-zero since each $\mathcal{U}_i \otimes_{\mathcal{R}} \mathcal{V}_j$ has dimension 1.

Now let $\mathcal{S} = e\mathcal{R}e$ be a basic algebra of $\mathcal{R}$, where $e = E + F_{11}$, so then $\mathcal{S} = \langle E, F_{11}, X_1, Y_1 \rangle$.

Proposition 4.4. *The order of the class of a Nakayama automorphism of* $\mathcal{S}$ *in* $\operatorname{Out}(\mathcal{S})$ *is* 2.

Proof: For simplicity of notation, we denote just for this proof $F_{11} = F$, $X_1 = X$, $Y_1 = Y$. Also denote by $\mathcal{B}^* = \{E^*, F^*, X^*, Y^*\}$ the dual basis of $\mathcal{B} = \{E, F, X, Y\}$ in $\mathcal{S}^*$. A direct computation shows that the only non-zero elements of the form $b \rightarrow b^*$, where $b \in \mathcal{B}$, $b^* \in \mathcal{B}^*$, and $\rightarrow$ denotes the left action of $\mathcal{S}$ on $\mathcal{S}^*$, are

$$E \longrightarrow E^* = E^*, \quad F \longrightarrow F^* = F^*, \quad F \longrightarrow X^* = X^*,$$
$$X \longrightarrow X^* = E^*, \quad E \longrightarrow Y^* = Y^*, \quad Y \longrightarrow Y^* = F^*,$$

while the only non-zero elements of the form $b^* \leftarrow b$, with $b \in \mathcal{B}$, $b^* \in \mathcal{B}^*$, are

$$E^* \longleftarrow E = E^*, \quad F^* \longleftarrow F = F^*, \quad X^* \longleftarrow E = X^*,$$
$$X^* \longleftarrow X = F^*, \quad Y^* \longleftarrow F = Y^*, \quad Y^* \longleftarrow Y = E^*.$$

Using these relations, we see that $\lambda = X^* + Y^*$ is a Frobenius form of $\mathcal{S}$, and the corresponding Nakayama automorphism is $\nu \colon \mathcal{S} \to \mathcal{S}$, given by

$$\nu(E) = F, \quad \nu(F) = E, \quad \nu(X) = Y, \quad \nu(Y) = X,$$

thus $\nu^2 = \mathrm{Id}$. On the other hand, ν is not an inner automorphism, otherwise $\mathcal{S}$ would be a symmetric algebra, and then so would be $\mathcal{R}$, as it is Morita-equivalent to $\mathcal{S}$; this is a contradiction. Alternatively, a direct simple argument can be given to show that ν is not inner. Indeed, if $u \in \mathcal{S}$ were invertible such that $u^{-1}Eu = F$, then writing $u = \alpha E + \beta F + \gamma X + \delta Y$ for some scalars α, β, γ, δ, we would derive from $Eu = uF$ that $\alpha = \beta = 0$, and then $u^2 = (\gamma X + \delta Y)^2 = 0$, so u could not be invertible. □

Corollary 4.5. *The order of* $\mathcal{R}^*$ *in* $\operatorname{Pic}(\mathcal{R})$ *is* 2, *thus there is an isomorphism of* $\mathcal{R}$*-bimodules* $\varphi \colon \mathcal{R}^* \otimes_{\mathcal{R}} \mathcal{R}^* \to \mathcal{R}$.

Proof: It follows from Corollary 3.4 and the previous proposition. □

Remark 4.6. It is possible to obtain a direct computational proof of the previous corollary, and an explicit isomorphism $\varphi\colon \mathcal{R}^*\otimes_{\mathcal{R}}\mathcal{R}^*\to\mathcal{R}$, by taking into account Lemma 4.1, Remark 4.3, the table with the left $\mathcal{R}$-action on $\mathcal{R}^*$, and a similar one with the right action. One can obtain a basis of $\mathcal{R}^*\otimes_{\mathcal{R}}\mathcal{R}^*$ consisting of the elements

$$\mathcal{E}=Y_1^*\otimes_{\mathcal{R}}X_1^*=Y_2^*\otimes_{\mathcal{R}}X_2^*,\quad \mathcal{F}_{ij}=X_i^*\otimes_{\mathcal{R}}Y_j^*,\quad \mathcal{X}_i=E^*\otimes_{\mathcal{R}}Y_i^*,\quad \mathcal{Y}_i=F_{1i}^*\otimes_{\mathcal{R}}X_1^*,$$

where $1\le i,j\le 2$.

Moreover, the linear map $\varphi\colon \mathcal{R}^*\otimes_{\mathcal{R}}\mathcal{R}^*\to\mathcal{R}$ given by $\varphi(\mathcal{E})=E$, $\varphi(\mathcal{F}_{ij})=F_{ij}$, $\varphi(\mathcal{X}_i)=X_i$, $\varphi(\mathcal{Y}_i)=Y_i$ for any $1\le i,j\le 2$, is an isomorphism of $\mathcal{R}$-bimodules.

We aim to compute the Picard group of $\mathcal{R}$. One possibility is to compute first the Picard group of the Frobenius algebra $\mathcal{S}$, and then to use Corollary 3.4. As determining the invertible bimodules and the group of exterior automorphisms of $\mathcal{S}$ is of comparable difficulty to determining those of $\mathcal{R}$, we prefer to look directly at the Picard group of $\mathcal{R}$. This will also make the description more explicit.

Lemma 4.7. *Let P be an invertible $\mathcal{R}$-bimodule. Then P is isomorphic either to $\mathcal{V}_1\oplus\mathcal{V}_2^2$ or to $\mathcal{V}_1^2\oplus\mathcal{V}_2$ as a left $\mathcal{R}$-module.*

Proof: Let Q be a bimodule such that $[Q]=[P]^{-1}$ in $\mathrm{Pic}(R)$. Since P is a finitely generated projective left module over the finite-dimensional algebra $\mathcal{R}$, it is isomorphic to a finite direct sum of principal indecomposable left $\mathcal{R}$-modules, say $P\simeq\mathcal{V}_1^a\oplus\mathcal{V}_2^b$ for some non-negative integers a, b. But P is a generator as a left $\mathcal{R}$-module, so $\mathcal{R}$ is a direct summand in the left $\mathcal{R}$-module P^m for some positive integer m. Thus by the Krull–Schmidt theorem, both a and b are positive. Similarly, $Q\simeq\mathcal{U}_1^c\oplus\mathcal{U}_2^d$ as right $\mathcal{R}$-modules for some integers $c,d>0$. Now there are linear isomorphisms

$$\mathcal{R}\simeq Q\otimes_{\mathcal{R}}P\simeq(\mathcal{U}_1\otimes_{\mathcal{R}}\mathcal{V}_1)^{ca}\oplus(\mathcal{U}_1\otimes_{\mathcal{R}}\mathcal{V}_2)^{cb}\oplus(\mathcal{U}_2\otimes_{\mathcal{R}}\mathcal{V}_1)^{da}\oplus(\mathcal{U}_2\otimes_{\mathcal{R}}\mathcal{V}_2)^{db}.$$

Counting dimensions and using Corollary 4.2, we see that $(c+d)(a+b)=9$. As $a,b,c,d>0$, we must have $c+d=a+b=3$, so then either $a=1$ and $b=2$, or $a=2$ and $b=1$. □

Theorem 4.8. *Any invertible $\mathcal{R}$-bimodule is isomorphic either to ${}_1\mathcal{R}_\alpha$ or to ${}_1\mathcal{R}_\alpha^*$ for some $\alpha\in\mathrm{Aut}(\mathcal{R})$. As a consequence, $\mathrm{Pic}(\mathcal{R})\simeq\mathrm{Out}(\mathcal{R})\times C_2$, where C_2 is the cyclic group of order 2.*

Proof: We know that a bimodule of type ${}_1\mathcal{R}_\alpha$, with $\alpha\in\mathrm{Aut}(\mathcal{R})$, is invertible; the inverse of $[{}_1\mathcal{R}_\alpha]$ in $\mathrm{Pic}(\mathcal{R})$ is $[{}_1\mathcal{R}_{\alpha^{-1}}]$. Moreover, $[{}_1\mathcal{R}_\alpha]\cdot[{}_1\mathcal{R}_\beta]=[{}_1\mathcal{R}_{\alpha\beta}]$, and $[{}_1\mathcal{R}_\alpha]$ depends only on the class of α modulo $\mathrm{Inn}(\mathcal{R})$.

By Corollary 2.3, $\mathcal{R}^*$ is an invertible $\mathcal{R}$-bimodule, and then so is $\mathcal{R}^*\otimes_{\mathcal{R}}{}_1\mathcal{R}_\alpha\simeq{}_1\mathcal{R}_\alpha^*$. Since $\mathcal{R}^*\otimes_{\mathcal{R}}{}_1\mathcal{R}_\alpha\simeq{}_1\mathcal{R}_\alpha\otimes_{\mathcal{R}}\mathcal{R}^*$ by Proposition 2.5, and $\mathcal{R}^*\otimes_{\mathcal{R}}\mathcal{R}^*\simeq\mathcal{R}$ by Corollary 4.5, we get that the subset $\mathcal{P}$ of $\mathrm{Pic}(\mathcal{R})$ consisting of all ${}_1\mathcal{R}_\alpha$ and ${}_1\mathcal{R}_\alpha^*$, with $\alpha\in\mathrm{Aut}(\mathcal{R})$, is a subgroup isomorphic to $\mathrm{Out}(\mathcal{R})\times C_2$; an isomorphism between $\mathcal{P}$ and $\mathrm{Out}(\mathcal{R})\times C_2$ takes ${}_1\mathcal{R}_\alpha$ to $(\hat\alpha,e)$, and ${}_1\mathcal{R}_\alpha^*$ to $(\hat\alpha,c)$, where $C_2=\langle c\rangle$, e is the neutral element of C_2, and $\hat\alpha$ is the class of α in $\mathrm{Out}(\mathcal{R})$.

Let P be an invertible $\mathcal{R}$-bimodule. By Lemma 4.7 we see that as a left $\mathcal{R}$-module, P is isomorphic either to $\mathcal{R}$ or to $\mathcal{R}^*$. Now Proposition 2.1 shows that either $P\simeq{}_1\mathcal{R}_\alpha$ or $P\simeq{}_1\mathcal{R}_\alpha^*$ as $\mathcal{R}$-bimodules for some $\alpha\in\mathrm{Aut}(\mathcal{R})$. We conclude that $\mathrm{Pic}(\mathcal{R})=\mathcal{P}$, which ends the proof. □

5. Automorphisms of $\mathcal{R}$

The aim of this section is to compute the automorphism group and the group of outer automorphisms of $\mathcal{R}$. We will use a presentation of $\mathcal{R}$ given in [**7**, Remark 4.1], where it is explained that $\mathcal{R}$ is isomorphic to the Morita ring $\left[\begin{smallmatrix} K & X \\ Y & M_2(K) \end{smallmatrix}\right]$ associated with the Morita context connecting the rings K and $M_2(K)$, by the bimodules $X = K^2$ and $Y = M_{2,1}(K)$, with all actions given by the usual matrix multiplication, such that both Morita maps are zero. The multiplication of this Morita ring is given by

$$\begin{bmatrix} \alpha & x \\ y & f \end{bmatrix} \begin{bmatrix} \alpha' & x' \\ y' & f' \end{bmatrix} = \begin{bmatrix} \alpha\alpha' & \alpha x' + xf' \\ \alpha' y + fy' & ff' \end{bmatrix}$$

for any $\alpha, \alpha' \in K$, $f, f' \in M_2(K)$, $x, x' \in X$, and $y, y' \in Y$. This Morita ring and $M_3(K)$ coincide as K-vector spaces, but they have different multiplications. An algebra isomorphism between $\mathcal{R}$ and $\left[\begin{smallmatrix} K & X \\ Y & M_2(K) \end{smallmatrix}\right]$ takes E, $(X_i)_{1\le i\le 2}$, $(Y_i)_{1\le i\le 2}$, $(F_{ij})_{1\le i,j\le 2}$ to the elements in the Morita ring corresponding to the "matrix units" in K, X, Y, $M_2(K)$. Throughout this section, we will identify $\mathcal{R}$ with $\left[\begin{smallmatrix} K & X \\ Y & M_2(K) \end{smallmatrix}\right]$.

The multiplicative group $K^\times \times GL_2(K)$ acts on the additive group $K^2 \times M_{2,1}(K)$ by

$$(\lambda, P) \cdot (x_1, y_1) = (\lambda x_1 P^{-1}, Py_1)$$

for any $\lambda \in K^\times$, $P \in GL_2(K)$, $x_1 \in K^2$, $y_1 \in M_{2,1}(K)$, so we can form a semidirect product $(K^2 \times M_{2,1}(K)) \rtimes (K^\times \times GL_2(K))$.

For any $x_1 \in K^2$, $y_1 \in M_{2,1}(K)$, $\lambda \in K^\times$, $P \in GL_2(K)$ define $\varphi_{x_1,y_1,\lambda,P}\colon \mathcal{R} \to \mathcal{R}$ by

$$\varphi_{x_1,y_1,\lambda,P}\left(\begin{bmatrix} \alpha & x \\ y & f \end{bmatrix}\right) = \begin{bmatrix} \alpha & \alpha x_1 + \lambda x P^{-1} - x_1 PfP^{-1} \\ \alpha y_1 + Py - PfP^{-1}y_1 & PfP^{-1} \end{bmatrix}.$$

Theorem 5.1. *We have that $\varphi_{x_1,y_1,\lambda,P}$ is an algebra automorphism of $\mathcal{R}$ for any $x_1 \in K^2$, $y_1 \in M_{2,1}(K)$, $\lambda \in K^\times$, $P \in GL_2(K)$, and that $\Phi\colon (K^2 \times M_{2,1}(K)) \rtimes (K^\times \times GL_2(K)) \to \mathrm{Aut}(\mathcal{R})$, given by $\Phi(x_1, y_1, \lambda, P) = \varphi_{x_1,y_1,\lambda,P}$, is an isomorphism of groups. An automorphism $\varphi_{x_1,y_1,\lambda,P}$ of $\mathcal{R}$ is inner if and only if $\lambda = 1$. As a consequence, $\mathrm{Out}(\mathcal{R}) \simeq K^\times$.*

Proof: Let $\varphi \in \mathrm{Aut}(\mathcal{R})$. Since the Jacobson radical of $\mathcal{R}$ is $J(\mathcal{R}) = \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right]$, φ induces an automorphism $\tilde{\varphi}$ of the algebra $\mathcal{R}/J(\mathcal{R}) \simeq K \times M_2(K)$, thus $\tilde{\varphi}$ acts as identity on the first position, and as an inner automorphism associated to some $P \in GL_2(K)$ on the second one. Lifting to $\mathcal{R}$, we see that $\varphi\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1 & x_1 \\ y_1 & 0 \end{smallmatrix}\right]$ for some $x_1 \in K^2$ and $y_1 \in M_{2,1}(K)$, and $\varphi\left(\left[\begin{smallmatrix} cc0 & 0 \\ 0 & f \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & \mu(f) \\ \omega(f) & PfP^{-1} \end{smallmatrix}\right]$ for some linear maps $\mu\colon M_2(K) \to X$ and $\omega\colon M_2(K) \to Y$.

On the other hand, since $\varphi(J(\mathcal{R})) \subset J(\mathcal{R})$, $\varphi\left(\left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right]\right) \in \left[\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right]$, so then

$$\varphi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \in \begin{bmatrix} 1 & x_1 \\ y_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \subset \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

This shows that $\varphi\left(\left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & \theta(x) \\ 0 & 0 \end{smallmatrix}\right]$ for a linear map $\theta\colon X \to X$; thus $\theta(x) = xA$ for any $x \in X$, where $A \in M_2(K)$.

Similarly we see that $\varphi\left(\left[\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & 0 \\ By & 0 \end{smallmatrix}\right]$ for any $y \in Y$, where $B \in M_2(K)$. Thus we obtain that φ must be of the form

$$\varphi\left(\begin{bmatrix} \alpha & x \\ y & f \end{bmatrix}\right) = \begin{bmatrix} \alpha & \alpha x_1 + xA + \mu(f) \\ \alpha y_1 + By + \omega(f) & PfP^{-1} \end{bmatrix}. \tag{1}$$

By equating the corresponding entries, we see that the matrices $\varphi\left(\left[\begin{smallmatrix}\alpha & x\\ y & f\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha' & x'\\ y' & f'\end{smallmatrix}\right]\right)$ and $\varphi\left(\left[\begin{smallmatrix}\alpha\alpha' & \alpha x'+xf'\\ y\alpha'+fy' & ff'\end{smallmatrix}\right]\right)$ are equal if and only if the equations

$$\alpha\mu(f') + \alpha x_1 Pf'P^{-1} + xAPf'P^{-1} + \mu(f)Pf'P^{-1} = xf'A + \mu(ff') \tag{2}$$

and

$$\alpha'\omega(f) + \alpha' PfP^{-1}y_1 + PfP^{-1}By' + PfP^{-1}\omega(f') = Bfy' + \omega(ff') \tag{3}$$

are satisfied for any $\alpha, \alpha' \in K$, $x, x' \in K^2$, $y, y' \in M_{2,1}(K)$, $f, f' \in M_2(K)$. If in equation (2) we take $f = 0$, we get $\alpha(\mu(f') + x_1Pf'P^{-1}) + xAPf'P^{-1} - xf'A = 0$. As this holds for any $\alpha \in K$, we must have

$$\mu(f') + x_1Pf'P^{-1} = 0 \tag{4}$$

and $x(APf'P^{-1} - f'A) = 0$. As x runs through K^2, we get $APf'P^{-1} - f'A = 0$, showing that $APf' = f'AP$ for any f', so $AP \in KI_2$, or equivalently,

$$A \in KP^{-1}. \tag{5}$$

On the other hand, it is clear that if equations (4) and (5) hold, then (2) is satisfied.

In a similar way, we see that (3) is true if and only if

$$\omega(f) = -PfP^{-1}y_1$$

and

$$B \in KP.$$

These show that a map φ of the form given in (1) is a ring morphism if and only if

$$\mu(f) = -x_1PfP^{-1}, \quad \omega(f) = -PfP^{-1}y_1, \quad A \in KP^{-1}, \quad B \in KP.$$

Thus take $A = \lambda P^{-1}$ and $B = \rho P$, with $\lambda, \rho \in K$; in fact, in order for φ to be injective one needs $\lambda, \rho \in K^\times$. For any $x_1 \in K^2$, $y_1 \in M_{2,1}(K)$, $\lambda, \rho \in K^\times$, $P \in GL_2(K)$, denote by $\psi_{x_1,y_1,\lambda,\rho,P}\colon \mathcal{R} \to \mathcal{R}$ the map defined by

$$\psi_{x_1,y_1,\lambda,\rho,P}\left(\begin{bmatrix}\alpha & x\\ y & f\end{bmatrix}\right) = \begin{bmatrix}\alpha & \alpha x_1 + \lambda xP^{-1} - x_1PfP^{-1}\\ \alpha y_1 + \rho Py - PfP^{-1}y_1 & PfP^{-1}\end{bmatrix}.$$

The considerations above show that $\psi_{x_1,y_1,\lambda,\rho,P}$ is an algebra endomorphism of $\mathcal{R}$. As it is clearly injective, it is in fact an automorphism of $\mathcal{R}$. We showed that any automorphism of $\mathcal{R}$ is one such $\psi_{x_1,y_1,\lambda,\rho,P}$.

A straightforward computation shows that

$$\psi_{x_1',y_1',\lambda',\rho',P'}\psi_{x_1,y_1,\lambda,\rho,P} = \psi_{x_1'+\lambda' x_1(P')^{-1},\, y_1'+\rho' P' y_1,\, \lambda'\lambda,\, \rho'\rho,\, P'P}. \tag{6}$$

Consider the additive group $A = K^2 \times M_{2,1}(K)$ and the multiplicative group $B = K^\times \times K^\times \times GL_2(K)$. Then B acts on A by $(\lambda, \rho, P)\cdot(x_1, y_1) = (\lambda x_1P^{-1}, \rho Py_1)$, and (6) shows that

$$\Psi\colon A \rtimes B \longrightarrow \operatorname{Aut}(\mathcal{R}), \quad \Psi(x_1, y_1, \lambda, \rho, P) = \psi_{x_1,y_1,\lambda,\rho,P}$$

is a group morphism. We have also seen that Ψ is surjective. Now $\psi_{x_1,y_1,\lambda,\rho,P}$ is the identity morphism if and only if

$$PfP^{-1} = f, \quad \alpha x_1 + \lambda xP^{-1} - x_1PfP^{-1} = x, \quad \alpha y_1 + \rho Py - PfP^{-1}y_1 = y$$

for any $\alpha \in K$, $x \in K^2$, $y \in M_{2,1}(K)$, $f \in M_2(K)$. If we take $\alpha = 1$, $x = 0$, $f = 0$ in the second relation, we get $x_1 = 0$. Hence $\lambda xP^{-1} = x$ for any x, so $P = \lambda I_2$. Similarly, the third relation shows that $y_1 = 0$ and $P = \rho^{-1}I_2$. Therefore $\mathrm{Ker}(\Psi) = 0 \times B_0$, where $B_0 = \{(\lambda, \lambda^{-1}, \lambda I_2) \mid \lambda \in K^\times\}$. As B_0 acts trivially on A, the action of B induces an action of the factor group $\frac{B}{B_0}$ on A, and then $\mathrm{Aut}(\mathcal{R}) \simeq \frac{A \rtimes B}{0 \rtimes B_0} \simeq A \rtimes \frac{B}{B_0}$. Denoting by $\bar{b}$ the class of some $b \in B$ modulo B_0, we see that

$$\overline{(\lambda, \rho, P)} = \overline{(\rho^{-1}, \rho, \rho I_2)(\lambda\rho, 1, \rho^{-1}P)} = \overline{(\lambda\rho^{-1}, 1, \rho^{-1}P)},$$

so there is a group isomorphism $\Gamma\colon K^\times \times GL_2(K) \to \frac{B}{B_0}$ taking (λ, P) to $\overline{(\lambda, 1, P)}$. Γ induces an action of $K^\times \times GL_2(K)$ on A, given by

$$(\lambda, P) \cdot (x_1, y_1) = (\lambda, 1, P) \cdot (x_1, y_1) = (\lambda x_1 P^{-1}, P y_1).$$

We obtain a composition of group isomorphisms

$$\Phi\colon A \rtimes (K^\times \times GL_2(K)) \longrightarrow A \rtimes \frac{B}{B_0} \longrightarrow \mathrm{Aut}(\mathcal{R})$$

given by $\Phi(x_1, y_1, \lambda, P) = \psi_{x_1,y_1,\lambda,1,P}$. Now we denote $\psi_{x_1,y_1,\lambda,1,P} = \varphi_{x_1,y_1,\lambda,P}$ and the first part of the statement is proved.

A direct computation shows that an element $\left[\begin{smallmatrix} \beta & z \\ g & m \end{smallmatrix}\right]$ of $\mathcal{R}$ is invertible if and only if $\beta \neq 0$ and $m \in GL_2(K)$, and in this case its inverse is $\left[\begin{smallmatrix} \beta^{-1} & -\beta^{-1}zm^{-1} \\ -\beta^{-1}m^{-1}g & m^{-1} \end{smallmatrix}\right]$, and the associated inner automorphism of $\mathcal{R}$ takes $\left[\begin{smallmatrix} \alpha & x \\ y & f \end{smallmatrix}\right]$ to

$$\begin{bmatrix} \alpha & \alpha\beta^{-1}z + \beta^{-1}xm - \beta^{-1}zm^{-1}fm \\ -\alpha m^{-1}g + \beta m^{-1}y + m^{-1}fg & m^{-1}fm \end{bmatrix},$$

so it is just $\psi_{\beta^{-1}z,-m^{-1}g,\beta^{-1},\beta,m^{-1}}$. Hence $\varphi_{x_1,y_1,\lambda,P} = \psi_{x_1,y_1,\lambda,1,P}$ is inner if and only if $\psi_{x_1,y_1,\lambda,1,P} = \psi_{\beta^{-1}z,-m^{-1}g,\beta^{-1},\beta,m^{-1}}$ for some $\beta \in K^\times$, $z \in K^2$, $g \in M_{2,1}(K)$, $m \in GL_2(K)$, and taking into account the description of the kernel of Ψ, this equality is equivalent to $(x_1, y_1, \lambda, 1, P) = (\beta^{-1}z, -m^{-1}g, \beta^{-1}, \beta, m^{-1})(0, 0, \rho, \rho^{-1}, \rho I_2) = (\beta^{-1}z, -m^{-1}g, \beta^{-1}\rho, \beta\rho^{-1}, \rho m^{-1})$ for some $\rho \in K^\times$. Equating the corresponding positions, we get $1 = \beta\rho^{-1}$, so $\rho = \beta$, and then $\lambda = \beta^{-1}\rho = 1$, $z = \beta x_1 = \rho x_1$, $m = \rho P^{-1}$, and $g = -my_1 = -\rho P^{-1}y_1$. We conclude that $\varphi_{x_1,y_1,\lambda,P}$ is inner if and only if $\lambda = 1$, and in this case, by making the choice $\rho = 1$, $\varphi_{x_1,y_1,1,P}$ is the inner automorphism associated with the invertible element $\left[\begin{smallmatrix} 1 & x_1 \\ -P^{-1}y_1 & P^{-1} \end{smallmatrix}\right]$.

We got that $\mathrm{Inn}(\mathcal{R}) = \Phi(A \rtimes (1 \times GL_2(K))$, so then

$$\mathrm{Out}(\mathcal{R}) = \frac{\mathrm{Aut}(\mathcal{R})}{\mathrm{Inn}(\mathcal{R})} \simeq \frac{A \rtimes (K^\times \times GL_2(K))}{A \rtimes (1 \times GL_2(K))} \simeq K^\times.$$

Finally, we note that the outer automorphism corresponding to $\lambda \in K^\times$ through the isomorphism $\mathrm{Out}(\mathcal{R}) \simeq K^\times$ is (the class of) $\varphi_{0,0,\lambda,I_2}$. □

6. Semitrivial extensions

Let R be a finite-dimensional K-algebra, and consider the R-bimodule R^* with actions denoted by $\rightharpoonup$ and $\leftharpoonup$. Let $\psi\colon R^* \otimes_R R^* \to R$ be a morphism of R-bimodules, and denote $\psi(r^* \otimes_R s^*)$ by $[r^*, s^*]$ for any $r^*, s^* \in R^*$. We say that ψ is associative if $[r^*, s^*] \rightharpoonup t^* = r^* \leftharpoonup [s^*, t^*]$ for any $r^*, s^*, t^* \in R^*$; in other words, we have a Morita context $(R, R, R^*, R^*, \psi, \psi)$ connecting the rings R and R, with both bimodules being R^*, and both Morita maps equal to ψ. It follows from Morita theory that if ψ is associative and surjective, then it is an isomorphism of R-bimodules.

If $\psi\colon R^* \otimes_R R^* \to R$ is an associative morphism of R-bimodules, we consider the semitrivial extension $R \rtimes_\psi R^*$, which is the cartesian product $R \times R^*$ with the usual addition, and multiplication defined by

$$(r, r^*)(s, s^*) = (rs + [r^*, s^*], (r \rightharpoonup s^*) + (r^* \leftharpoonup s))$$

for any $r, s \in R$, $r^*, s^* \in R^*$. Then $R \rtimes_\psi R^*$ is an algebra with identity element $(1, 0)$; this construction was introduced in [**15**]. Moreover, it is a C_2-graded algebra, where $C_2 = \langle c \rangle$ is a cyclic group of order 2, with homogeneous components $(R \rtimes_\psi R^*)_e = R \times 0$ and $(R \rtimes_\psi R^*)_c = 0 \times R^*$; here e denotes the neutral element of C_2. It is a strongly graded algebra if and only if ψ is surjective, thus an isomorphism.

Proposition 6.1. *Let R be a finite-dimensional algebra and let $\psi\colon R^* \otimes_R R^* \to R$ be an associative morphism of R-bimodules. Then $R \rtimes_\psi R^*$ is a symmetric algebra.*

Proof: If we evaluate both sides of $[r^*, s^*] \rightharpoonup t^* = r^* \leftharpoonup [s^*, t^*]$ at 1, we get

$$t^*([r^*, s^*]) = r^*([s^*, t^*]) \text{ for any } r^*, s^*, t^* \in R^*. \tag{7}$$

Denote $A = R \rtimes_\psi R^*$ and define

$$\Phi\colon A \longrightarrow A^*,\ (\Phi(r, r^*))(s, s^*) = r^*(s) + s^*(r) \text{ for any } r, s \in R,\ r^*, s^* \in R^*.$$

If $\Phi(r, r^*) = 0$, then $r^*(s) = (\Phi(r, r^*))(s, 0) = 0$ for any $s \in R$, so $r^* = 0$, and $s^*(r) = (\Phi(r, r^*))(0, s^*) = 0$ for any $s^* \in R^*$, so $r = 0$. This shows that Φ is injective, thus a linear isomorphism. Moreover, if $(x, x^*), (r, r^*), (s, s^*) \in A$, then

$$\begin{aligned}
(\Phi((x, x^*)(r, r^*)))(s, s^*) &= (\Phi(xr + [x^*, r^*], (x \rightharpoonup r^*) + (x^* \leftharpoonup r)))(s, s^*) \\
&= (x \rightharpoonup r^*)(s) + (x^* \leftharpoonup r)(s) + s^*(xr + [x^*, r^*]) \\
&= r^*(sx) + x^*(rs) + s^*(xr) + s^*([x^*, r^*]) \\
&= r^*(sx) + x^*(rs) + s^*(xr) + r^*([s^*, x^*]) \quad \text{(by (7))} \\
&= (s \rightharpoonup x^* + s^* \leftharpoonup x)(r) + r^*(sx + [s^*, x^*]) \\
&= (\Phi(r, r^*))(sx + [s^*, x^*], s \rightharpoonup x^* + s^* \leftharpoonup x) \\
&= (\Phi(r, r^*))((s, s^*)(x, x^*)) \\
&= ((x, x^*) \rightharpoonup \Phi(r, r^*))(s, s^*),
\end{aligned}$$

showing that Φ is a morphism of left A-modules, and

$$\begin{aligned}
(\Phi(x, x^*) \leftharpoonup (r, r^*))(s, s^*) &= (\Phi(x, x^*))((r, r^*)(s, s^*)) \\
&= (\Phi(x, x^*))(rs + [r^*, s^*], (r \rightharpoonup s^*) + (r^* \leftharpoonup s)) \\
&= x^*(rs) + x^*([r^*, s^*]) + s^*(xr) + r^*(sx) \\
&= x^*(rs) + s^*([x^*, r^*]) + s^*(xr) + r^*(sx) \quad \text{(by (7))} \\
&= (\Phi((x, x^*)(r, r^*)))(s, s^*) \quad \text{(by the computations above)},
\end{aligned}$$

so Φ is also a morphism of right A-modules. We conclude that Φ is an isomorphism of A-bimodules. □

We first mention two particular cases of interest.

The first one is for an arbitrary finite-dimensional algebra R and the zero morphism $\psi\colon R^* \otimes_R R^* \to R$. The associated semitrivial extension, called in fact the trivial extension, is $R \times R^*$, with the multiplication given by $(r, r^*)(s, s^*) = (rs, (r \rightharpoonup s^*) + (r^* \leftharpoonup s))$ for any $r, s \in R$, $r^*, s^* \in R^*$. This is just the example of Tachikawa of a symmetric algebra constructed from R; see [**9**, Example 16.60].

The second one is for a symmetric finite-dimensional algebra R. As $R^* \simeq R$ as bimodules, a semitrivial extension $R \rtimes_\psi R^*$ is isomorphic to $R \times R$ with multiplication $(r, a)(s, b) = (rs + \gamma(a \otimes_R b), rb + as)$, where $\gamma\colon R \otimes_R R \to R$ is a morphism of R-bimodules. As such a γ is of the form $\gamma(a \otimes_R b) = zab$ for any $a, b \in R$, where z is an element in the centre of R, any semitrivial extension of this kind is isomorphic to the algebra $A_z = R \times R$ for some $z \in \mathrm{Cen}(R)$, whose multiplication is given by $(r, a)(s, b) = (rs + zab, rb + as)$.

7. Order 2 elements in Picard groups and associative isomorphisms

In order to construct semitrivial extensions that are strongly C_2-graded, we consider finite-dimensional algebras R such that $R^* \otimes_R R^* \simeq R$, and we are interested in the associativity of isomorphisms $R^* \otimes_R R^* \to R$. We have seen in Corollary 2.3 that such an R is necessarily quasi-Frobenius, it is clear that $[R^*]$ has order at most 2 in $\mathrm{Pic}(R)$, and we addressed Question 2 in the introduction, asking whether any such isomorphism is associative.

The following shows that the answer to the question depends only on the algebra, and not on a particular choice of the isomorphism.

Proposition 7.1. *If R is a finite-dimensional algebra such that $R^* \otimes_R R^* \simeq R$ as bimodules and there exists an associative isomorphism $R^* \otimes_R R^* \to R$, then any other such isomorphism is associative.*

Proof: Let $\psi, \psi'\colon R^* \otimes_R R^* \to R$ be isomorphisms of bimodules, and assume that ψ is associative. Then $\psi'\psi^{-1}$ is an automorphism of the bimodule R, so it is the multiplication by a central invertible element c. Therefore $\psi'(y) = c\psi(y)$ for any $y \in R^* \otimes_R R^*$. We note that $c \rightharpoonup r^* = r^* \leftharpoonup c$ for any $r^* \in R^*$, since $(c \rightharpoonup r^*)(a) = r^*(ac) = r^*(ca) = (r^* \leftharpoonup c)(a)$ for any $a \in R$.

Now for any $r^*, s^*, t^* \in R^*$

$$\begin{aligned}
\psi'(r^* \otimes_R s^*) \rightharpoonup t^* &= (c\psi(r^* \otimes_R s^*)) \rightharpoonup t^* \\
&= c \rightharpoonup (\psi(r^* \otimes_R s^*) \rightharpoonup t^*) \\
&= c \rightharpoonup (r^* \leftharpoonup \psi(s^* \otimes_R t^*)) \\
&= (c \rightharpoonup r^*) \leftharpoonup \psi(s^* \otimes_R t^*) \\
&= (r^* \leftharpoonup c) \leftharpoonup \psi(s^* \otimes_R t^*) \\
&= r^* \leftharpoonup (c\psi(s^* \otimes_R t^*)) \\
&= r^* \leftharpoonup \psi'(s^* \otimes_R t^*),
\end{aligned}$$

showing that ψ' is associative as well. $\square$

The following answers in the positive our question in the Frobenius case.

Lemma 7.2. *Let R be a Frobenius algebra such that $R^* \otimes_R R^* \simeq R$ as bimodules. Then any isomorphism $\psi\colon R^* \otimes_R R^* \to R$ is associative.*

Proof: Let $\lambda \in R^*$ be a Frobenius form and let ν be the Nakayama automorphism associated with λ. We have seen in Proposition 2.4 that $\theta\colon {}_1R_\nu \to R^*$, $\theta(r) = r \rightharpoonup \lambda$, is a bimodule isomorphism. Then $R^* \otimes_R R^* \simeq {}_1R_\nu \otimes_R {}_1R_\nu \simeq {}_1R_{\nu^2}$, so ${}_1R_{\nu^2} \simeq R$, which shows that ν^2 is inner; let $\nu^2(r) = u^{-1}ru$ for any $r \in R$, where u is an invertible element of R. Now for any $a \in R$

$$\begin{aligned} \lambda(au) &= \lambda(u\nu(a)) && \text{(since } \nu \text{ is the Nakayama automorphism)} \\ &= \lambda(u\nu^2(\nu^{-1}(a))) \\ &= \lambda(uu^{-1}\nu^{-1}(a)u) && \text{(since } \nu^2 \text{ is inner)} \\ &= \lambda(\nu^{-1}(a)u) \\ &= \lambda(ua) && \text{(since } \nu \text{ is the Nakayama automorphism),} \end{aligned}$$

showing that $\lambda(au) = \lambda(ua)$, or equivalently, $u \rightharpoonup \lambda = \lambda \leftharpoonup u$. Therefore $\theta(u) = u \rightharpoonup \lambda = \lambda \leftharpoonup u = \nu(u) \rightharpoonup \lambda = \theta(\nu(u))$, so $\nu(u) = u$, since θ is injective.

It is easy to check that $\delta\colon {}_1R_\nu \otimes_R {}_1R_\nu \to {}_1R_{\nu^2}$, $\delta(r \otimes_R s) = r\nu(s)$ for any $r, s \in R$, and $\omega\colon {}_1R_{\nu^2} \to R$, $\omega(r) = ru^{-1}$ for any $r \in R$ are both bimodule isomorphisms. Composing them, we obtain an isomorphism $F = \omega\delta\colon {}_1R_\nu \otimes_R {}_1R_\nu \to R$, $F(r \otimes_R s) = r\nu(s)u^{-1}$. Denoting by $*$ the right action of R on ${}_1R_\nu$, we see that for any $r, s, t \in R$

$$\begin{aligned} r * F(s \otimes_R t) &= r * (s\nu(t)u^{-1}) \\ &= r\nu(s\nu(t)u^{-1}) \\ &= r\nu(s)u^{-1}tu\nu(u^{-1}) && \text{(since } \nu^2(t) = u^{-1}tu) \\ &= r\nu(s)u^{-1}t && \text{(since } \nu(u) = u) \\ &= F(r \otimes_R s)t. \end{aligned}$$

Since ${}_1R_\nu \simeq R^*$, F induces an associative bimodule isomorphism $F'\colon R^* \otimes_R R^* \to R$, and then any such isomorphism is associative by Proposition 7.1. □

In the initial version of the paper, we were able to answer Question 2 only in the Frobenius case. One of the referees indicated to us how the quasi-Frobenius case can be derived from the Frobenius one. Some of the steps of the approach were explained in Section 3, and we show below how the conclusion can be reached. We keep the notation at the beginning of Section 3, where R and S are two Morita-equivalent algebras, $(R, S, P, Q, [,], (,))$ is a strict Morita context connecting them, $F = Q \otimes_R (-) \otimes_R P$ is the induced monoidal equivalence between their categories of bimodules, and G is its quasi-inverse. Denote by $\theta_M\colon F(M) \otimes_S F(M) \to F(M \otimes_R M)$,

$$\theta_M(q_1 \otimes_R m_1 \otimes_R p_1 \otimes_S q_2 \otimes_R m_2 \otimes_R p_2) = q_1 \otimes_R m_1[p_1, q_2] \otimes_R m_2 \otimes_R p_2,$$

and $\mu\colon F(R) \to S$, $\mu(q \otimes_R r \otimes_R p) = (qr, p)$, the isomorphisms of S-bimodules associated with this monoidal equivalence.

If M is an R-bimodule and $\psi\colon M \otimes_R M \to R$ is an R-bimodule isomorphism, consider the isomorphism $\tilde{\psi} = \mu F(\psi)\theta_M\colon F(M) \otimes_S F(M) \to S$.

Proposition 7.3. (i) *The mapping $\psi \mapsto \tilde{\psi}$ is a bijective correspondence between the isomorphisms of R-bimodules $M \otimes_R M \xrightarrow{\sim} R$ and the isomorphisms of S-bimodules $F(M) \otimes_S F(M) \xrightarrow{\sim} S$.*

(ii) *If $\psi\colon M \otimes_R M \to R$ is an associative isomorphism of R-bimodules, then $\tilde{\psi}$ is associative.*

Proof: (i) follows immediately since F is full and faithful. For (ii), let $z_i = q_i \otimes_R m_i \otimes_R p_i \in F(M)$ for $1 \leq i \leq 3$. Then $\tilde{\psi}(z_1 \otimes_S z_2) = (q_1\psi(m_1[p_1, q_2] \otimes_R m_2), p_2)$, so

$$\begin{aligned}
\tilde{\psi}(z_1 \otimes_S z_2)z_3 &= (q_1\psi(m_1[p_1, q_2] \otimes_R m_2), p_2)q_3 \otimes_R m_3 \otimes_R p_3 \\
&= q_1\psi(m_1[p_1, q_2] \otimes_R m_2)[p_2, q_3] \otimes_R m_3 \otimes_R p_3 \\
&= q_1 \otimes_R \psi(m_1 \otimes_R [p_1, q_2]m_2)[p_2, q_3]m_3 \otimes_R p_3 \\
&= q_1 \otimes_R m_1\psi([p_1, q_2]m_2[p_2, q_3] \otimes_R m_3) \otimes_R p_3 \\
&= q_1 \otimes_R m_1 \otimes_R [p_1, q_2]\psi(m_2[p_2, q_3] \otimes_R m_3)p_3 \\
&= q_1 \otimes_R m_1 \otimes_R p_1(q_2\psi(m_2[p_2, q_3] \otimes_R m_3), p_3) \\
&= z_1\tilde{\psi}(z_2 \otimes_S z_3).
\end{aligned}$$

An alternative proof can be done with a categorical approach, using the fact that F is a monoidal equivalence and showing the commutativity of some diagrams. □

Corollary 7.4. *Let M be an R-bimodule. Then any isomorphism of R-bimodules $M \otimes_R M \xrightarrow{\sim} R$ is associative if and only if any isomorphism of S-bimodules $F(M) \otimes_S F(M) \xrightarrow{\sim} S$ is associative.*

Proof: The only if part follows directly from Proposition 7.3, while the if part follows by applying the direct implication for the quasi-inverse G of F and the S-bimodule $F(M)$. □

Now we can answer Question 2 in general.

Proposition 7.5. *Let R be a finite-dimensional algebra such that $R^* \otimes_R R^* \simeq R$ as bimodules. Then any isomorphism $\psi\colon R^* \otimes_R R^* \to R$ is associative.*

Proof: We have seen that R is necessarily quasi-Frobenius. Let S be a basic algebra of R. Then S is Frobenius. We know that $F(R^*) \simeq S^*$ by Proposition 3.2, and that $S^* \otimes_S S^* \simeq S$ by Proposition 7.3(i) (or alternatively, by Corollary 3.4). By Lemma 7.2, any isomorphism $S^* \otimes_S S^* \xrightarrow{\sim} S$ is associative. Corollary 7.4 shows now that any isomorphism $R^* \otimes_R R^* \xrightarrow{\sim} R$ is associative. □

As a consequence, we can construct semitrivial extensions that are strongly graded algebras from any algebra whose dual has order at most 2 in the Picard group.

Proposition 7.6. *Let R be a finite-dimensional algebra such that there exists an isomorphism of R-bimodules $\psi\colon R^* \otimes_R R^* \to R$. Then $A = R \rtimes_\psi R^*$ is a symmetric algebra and a strongly C_2-graded algebra with grading given by $A_e = R \rtimes 0$ and $A_c = 0 \rtimes R^*$.*

In the particular case where $R = \mathcal{R}$ is Nakayama's 9-dimensional algebra, and $\varphi\colon \mathcal{R}^* \otimes_{\mathcal{R}} \mathcal{R}^* \to \mathcal{R}$ is an isomorphism of bimodules, for example the one described in Remark 4.6, we obtain an example answering in the negative our initial Question 1, for both the symmetric property and the Frobenius property.

Corollary 7.7. *We have that $\mathcal{R} \rtimes_\varphi \mathcal{R}^*$ is a symmetric strongly C_2-graded algebra whose homogeneous component of trivial degree is not Frobenius.*

This example also answers a question posed by the referee of our paper [**6**]. It was proved in [**6**, Proposition 2.1] that if B is a subalgebra of a Frobenius algebra A, such that A is free as a left B-module and also as a right B-module, then B is Frobenius, too. The question was whether the conclusion remains valid if we only suppose that A is projective as a left B-module and as a right B-module. The example constructed in Corollary 7.7 shows that the answer is negative. Indeed, A is even symmetric, and it is projective as a left A_e-module and as a right A_e-module, although A_e is not a Frobenius algebra.

As another consequence of Proposition 7.6, we obtain a class of examples of strongly graded algebras that are symmetric as algebras, while their homogeneous component of trivial degree is Frobenius, but not symmetric. Indeed, we can take a Frobenius algebra R such that the order of $[R^*]$ in $\mathrm{Pic}(R)$ is 2; in other words, the Nakayama automorphism ν with respect to a Frobenius form is not inner, but ν^2 is inner. Then there is an isomorphism of R-bimodules $\psi\colon R^* \otimes_R R^* \to R$, and by Lemma 7.2, it is associative. Hence we can form the semitrivial extension $R \rtimes_\psi R^*$, a strongly C_2-graded algebra which is symmetric, and its homogeneous component of trivial degree is isomorphic to R, which is Frobenius, but not symmetric.

We have several classes of examples of Frobenius algebras R such that $[R^*]$ has order 2 in $\mathrm{Pic}(R)$:

- A first class follows from Example 2.7. For $R = H_1(C, n, c, c^*)$, the order of $[R^*]$ is 2 if and only if $n = 2$. Thus we obtain such an R if we have a finite abelian group C, an element $c \in C$ with $c^2 \neq 1$, and a linear character $c^* \in C^*$ such that $(c^*)^2 = 1$ and $c^*(c) = -1$. A particular family of such examples is when we take $C = \langle c \rangle \simeq C_{2r}$, where $r \geq 2$, and $c^* \in C^*$ defined by $c^*(c) = -1$, obtaining a Hopf algebra of dimension $4r$, generated by the group-like element c and the $(1, c)$-skew-primitive element x, subject to relations $c^{2r} = 1$, $x^2 = c^2 - 1$, $xc = -cx$.
- A second class follows from Example 2.7, too. For $R = H_2(C, n, c, c^*)$, the order of $[R^*]$ is 2 if and only if $\frac{m}{(\frac{m}{n}, n-1)} = 2$, where m is the order of c^*. It is easy to check that this happens if and only if $m = n = 2$. Thus we need a finite abelian group C, a character $c^* \in C^*$ such that $(c^*)^2 = 1$, and an element $c \in C$ such that $c^*(c) = -1$ (in particular, the order of c must be even). A particular family of such examples is when we take $C = \langle c \rangle \simeq C_{2r}$, where $r \geq 1$, and $c^* \in C^*$ defined by $c^*(c) = -1$, obtaining a Hopf algebra of dimension $4r$, generated by the group-like element c and the $(1, c)$-skew-primitive element x, subject to relations $c^{2r} = 1$, $x^2 = 0$, $xc = -cx$. For $r = 1$ this is just Sweedler's 4-dimensional Hopf algebra.
- Another example is $R_{-1} = K_{-1}[X, Y]/(X^2, Y^2)$ from Example 2.8 for $q = -1$.
- Let H be a unimodular finite-dimensional Hopf algebra, i.e., the spaces of left integrals and right integrals coincide in H; equivalently, the unimodular element $\mathcal{G}$ is trivial. By Radford's formula, see [**10**, Theorem 12.10] or [**17**, Theorem 10.5.6], $S^4(h) = a^{-1}ha$ for any $h \in H$, where a is the modular element of H^* regarded inside H via the isomorphism $H \simeq H^{**}$. Thus S^4 is inner, and then the order of S^2 in $\mathrm{Out}(H)$ is either 1 or 2. By Theorem 2.9, in the first case $[H^*]$ has order 1 in $\mathrm{Pic}(H)$, and H is symmetric, while in the second case, $[H^*]$ has order 2 in $\mathrm{Pic}(H)$. We conclude that a class of Frobenius algebras such as we are looking for is the family of all unimodular finite-dimensional Hopf algebras that are not symmetric. A class of such objects was explicitly constructed in [**19**].

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Sorin Dăscălescu
University of Bucharest, Faculty of Mathematics and Computer Science, Str. Academiei 14, Bucharest 1, RO-010014, Romania
E-mail address: sdascal@fmi.unibuc.ro
ORCID: 0000-0002-0496-9543

Constantin Năstăsescu
Institute of Mathematics of the Romanian Academy, PO-Box 1-764, RO-014700, Bucharest, Romania
E-mail address: Constantin_nastasescu@yahoo.com
ORCID: 0009-0007-4645-0508

Laura Năstăsescu
Max Planck Institut für Mathematik, Vivatsgasee 7, 53111 Bonn, Germany
Institute of Mathematics of the Romanian Academy, PO-Box 1-764, RO-014700, Bucharest, Romania
E-mail address: lauranastasescu@gmail.com
ORCID: 0009-0001-4855-6900

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