Publ. Mat. **70** (2026), 27–53
DOI: 10.5565/PUBLMAT7012602

NAKAYAMA FUNCTORS ON PROPER ABELIAN SUBCATEGORIES

David Nkansah

Abstract: We construct Nakayama functors on proper abelian subcategories of triangulated categories with a Serre functor using approximation theory. This, in turn, allows for the construction of Auslander–Reiten translates. As a result, we prove that suitable proper abelian subcategories are dualising k-varieties and have enough projectives if and only if they have enough injectives. As an application, we provide a new proof of the existence of Auslander–Reiten sequences in the category of finite-dimensional modules over a finite-dimensional algebra.

2020 Mathematics Subject Classification: 18E10, 18G80, 18G25.

Key words: approximation, Auslander–Reiten translation, cover, envelope, Nakayama functor, Serre functor, triangulated category.

1. Introduction

Classical Auslander–Reiten theory was first introduced in [**5**] within the context of abelian categories. It was later extended into the triangulated realm in [**22**]. Relative versions of Auslander–Reiten theory were also introduced in both the abelian ([**7**]) and triangulated ([**26**]) cases. This paper aims to study Auslander–Reiten theory in a specific class of abelian subcategories of triangulated categories introduced in [**27**]. The class known as the *proper abelian subcategories of triangulated categories* (see Definition 1.4) generalises hearts of t-structures, and provides access to a theory concerning abelian subcategories, with possible non-zero negative extensions, of negative cluster categories.

Nakayama functors play an important role in the theory. In the abelian case, they provide an equivalence between the category of projective objects and the category of injective objects and allow for the construction of Auslander–Reiten translates. Therefore, we begin by seeking a construction of Nakayama functors in proper abelian subcategories. Our results support a change of perspective from hearts of t-structures to proper abelian subcategories, as mentioned in [**27**].

In this section, k is a field and $\mathcal{T}$ is a Krull–Schmidt Hom-finite k-linear triangulated category with suspension functor Σ.

Definition 1.1 ([**28**, Definition 2.2(i)])**.** We call a diagram $X \xrightarrow{x} Y \xrightarrow{y} Z$ in a triangulated category $\mathcal{T}$ a *short triangle* if there exists a morphism $Z \xrightarrow{z} \Sigma X$ such that the augmented diagram $X \xrightarrow{x} Y \xrightarrow{y} Z \xrightarrow{z} \Sigma X$ is a triangle in $\mathcal{T}$.

Notation 1.2. We say an exact sequence $X \xrightarrow{x} Y \xrightarrow{y} Z$ in an abelian category is left (right) exact if x is a monomorphism (y is an epimorphism). We say the sequence is short exact if it is both left exact and right exact. In particular, a projective presentation of an object X in an abelian category is a right exact sequence $P_1 \longrightarrow P_0 \longrightarrow X$, where P_0 and P_1 are projective objects in said abelian category. We use similar notation for injective copresentations.

Definition 1.3. An *additive subcategory* of an additive category is a full subcategory closed under isomorphisms, direct sums, and direct summands.

Definition 1.4 ([**28**, Definition 2.2(ii)])**.** Let $\mathcal{A}$ be an additive subcategory of a triangulated category $\mathcal{T}$. Then $\mathcal{A}$ is a *proper abelian subcategory of* $\mathcal{T}$ if it is abelian and if the following holds:

- The diagram $X \xrightarrow{x} Y \xrightarrow{y} Z$ is a short exact sequence in $\mathcal{A}$ if and only if the diagram $X \xrightarrow{x} Y \xrightarrow{y} Z$ is a short triangle in $\mathcal{T}$ whose terms X, Y, and Z all lie in $\mathcal{A}$.

Remark 1.5. Other generalisations of hearts of t-structures include *admissible abelian subcategories* [**8**, Definition 1.2.5] and *distinguished abelian subcategories* [**35**, Definition 1.1].

In the classical case, one may break down Auslander–Reiten theory into the following:

(1) The study of the collection of non-projective objects and non-injective objects. These are intimately linked to Auslander–Reiten translates and Auslander–Reiten sequences.
(2) The study of the collection of projective objects and injective objects. These are intimately linked to Nakayama functors.

Using [**26**, Theorem 3.1] (see [**18**, Theorem B] for the $(d+2)$-angulated case), (1) is well understood for proper abelian subcategories: Auslander–Reiten sequences in a proper abelian subcategory are controlled by approximation properties of the proper abelian subcategory with its ambient triangulated category. However, the aspects mentioned in (2) are not covered in [**26**] and therefore we aim to explore them in this paper. Other considerations about (1) can be found in [**31**] and [**19**] whereas other considerations about (2) can be found in [**20**].

Definition 1.6 ([**10**, p. 519])**.** A *Serre functor* $\mathcal{X} \xrightarrow{\mathbb{S}} \mathcal{X}$ on a k-linear category $\mathcal{X}$ is a k-linear autoequivalence together with isomorphisms $\mathcal{X}(X,Y) \xrightarrow{\eta_{X,Y}} \mathrm{D}\,\mathcal{X}(Y, \mathbb{S}X)$ that are natural in objects X and Y in $\mathcal{X}$. Here $(\mathsf{mod}_k)^{\mathrm{op}} \xrightarrow{\mathrm{D}} \mathsf{mod}_k$ denotes the standard k-dual functor $\mathrm{Hom}_k(-,k)$, where mod_k is the category of finite-dimensional right modules over k. In this case, we say $\mathcal{X}$ has *Serre duality*.

Let $\mathcal{A}$ be either the category of finite-dimensional (right) modules over a finite-dimensional k-algebra of finite global dimension or a hereditary Ext-finite k-linear abelian category. Consider $\mathcal{A}$ sitting inside its bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ as the heart of the canonical t-structure. It was shown in [**22**, I.4.6 and Theorem on p. 37] and [**36**, Corollary I.3.4] that a Serre functor on $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ restricts to an equivalence between projective objects in $\mathcal{A}$ and injective objects in $\mathcal{A}$ (see also [**23**]). We would like to point out two problems that may occur in a more general setting. Firstly, the Serre functor might not send a projective object in $\mathcal{A}$ to an object in $\mathcal{A}$, let alone an injective one (see Example 1.7). Secondly, the Serre functor might not induce an equivalence. Under mild assumptions, Theorem A provides a remedy to these issues by constructing a functor via approximations of a Serre functor.

Example 1.7 (Motivational example)**.** The Auslander–Reiten sequence $0 \longrightarrow {\scriptstyle 1} \longrightarrow {\scriptstyle \frac{2}{1}} \longrightarrow {\scriptstyle 2} \longrightarrow 0$ in $\mathsf{mod}\, kA_3$, where ${\scriptstyle 2}$ is the unique indecomposable non-projective non-injective kA_3-module, induces an exact fully faithful embedding of module categories $\mathsf{mod}_{kA_2} \longrightarrow \mathsf{mod}_{kA_3}$. Therefore, considering mod_{kA_3} as the heart of the canonical t-structure on the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathsf{mod}_{kA_3})$, we can identify mod_{kA_2}

as a full subcategory of $\mathcal{D}^{\mathrm{b}}(\mathsf{mod}_{kA_3})$. As the path algebra of the quiver A_3 is of finite global dimension, $\mathcal{D}^{\mathrm{b}}(\mathsf{mod}_{kA_3})$ has a Serre functor $\mathbb{S}$ given by the left derived functor of $(-) \otimes_{kA_3} \mathrm{D}(kA_3)$ (see [**22**, Theorem on p. 37]). One can check that $\mathbb{S}$ sends the image of the simple projective kA_2-module in $\mathcal{D}^{\mathrm{b}}(\mathsf{mod}_{kA_3})$ to an object that does not lie in mod_{kA_2} (identified as a full subcategory of $\mathcal{D}^{\mathrm{b}}(\mathsf{mod}_{kA_3})$). In fact, it is sent to the image of the indecomposable projective-injective object in mod_{kA_3}. If we consider the Auslander–Reiten quiver of $\mathcal{D}^{\mathrm{b}}(\mathsf{mod}_{kA_3})$, the above translates to the mapping $1 \overset{\mathbb{S}}{\longmapsto} \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ in the following diagram:

$$
\begin{tikzcd}
 & 3[-1] \arrow[dr] & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & 1[1] & \\
\cdots & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & \cdots \\
 & 1 \arrow[ur] \arrow[uurr, dotted, bend left] & & 2 \arrow[ur] & & 3 &
\end{tikzcd}
$$

Although the object $\mathbb{S}(1) = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ does not lie in mod_{kA_2}, we see that there is an irreducible morphism $\begin{smallmatrix}2\\1\end{smallmatrix} \xrightarrow{f} \begin{smallmatrix}3\\2\\1\end{smallmatrix}$, where the object $\begin{smallmatrix}2\\1\end{smallmatrix}$ does indeed lie in mod_{kA_2}. Notice that $\nu(1) = \begin{smallmatrix}2\\1\end{smallmatrix}$, where ν is the Nakayama functor on mod_{kA_2}. We will see in Subsection 5.1 that this morphism $\nu(1) \xrightarrow{f} \mathbb{S}(1)$ exhibits a strong approximation (see Definition 1.8) of $\mathbb{S}(1)$ inside of mod_{kA_2}.

We follow terminology due to [**16**, Section 1], whereas the alternative terminology in the following is due to Auslander and Smalø, who introduced the concept in [**6**].

Definition 1.8. Let $\mathcal{Y}$ be an additive subcategory of an additive category $\mathcal{X}$ and let X be an object in $\mathcal{X}$. For a morphism $Y \xrightarrow{y} X$ with Y in $\mathcal{Y}$, we have the following definitions:

- y is a *$\mathcal{Y}$-precover of X* if for each object Y' in $\mathcal{Y}$ the map

$$\mathcal{X}(Y', Y) \xrightarrow{\mathcal{X}(Y', y)} \mathcal{X}(Y', X)$$

  is an epimorphism. Diagrammatically, this means for each morphism $Y' \longrightarrow X$ there exists a morphism $Y' \longrightarrow Y$ making the following diagram commute:

$$
\begin{tikzcd}
 & Y' \arrow[d] \arrow[dl, dashed, "\exists"'] \\
Y \arrow[r, "y"] & X
\end{tikzcd}
$$

  Alternative terminology for y is a *right $\mathcal{Y}$-approximation of X*.
- y is *right minimal* if each endomorphism $Y \xrightarrow{y'} Y$ satisfying the equation $y = yy'$ is automatically an automorphism.
- y is a *$\mathcal{Y}$-cover of X* if it is both a $\mathcal{Y}$-precover of X and right minimal.

- y is a *strong $\mathcal{Y}$-cover of X* if for each object Y' in $\mathcal{Y}$ the map

$$\mathcal{X}(Y',Y) \xrightarrow{\mathcal{X}(Y',y)} \mathcal{X}(Y',X)$$

  is an isomorphism. Diagrammatically, this means for each morphism $Y' \longrightarrow X$ there exists a unique morphism $Y' \longrightarrow Y$ making the following diagram commute:

$$\begin{tikzcd} & Y' \arrow[dl, dashed, "\exists!"'] \arrow[d] \\ Y \arrow[r, "y"] & X \end{tikzcd}$$

$\mathcal{Y}$-preenvelopes (or alternatively, *left $\mathcal{Y}$-approximations*), *left minimality, $\mathcal{Y}$-envelopes*, and *strong $\mathcal{Y}$-envelopes* are defined dually.

Definition 1.9 ([**4**, p. 307])**.** Let $\mathcal{A}$ be an essentially small k-linear category and let $\mathsf{Mod}_{\mathcal{A}}$ denote the (abelian) category of k-linear functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{Mod}_k$ (see [**3**, p. 184]). We say that an object F in $\mathsf{Mod}_{\mathcal{A}}$ is *finitely presented* (see [**3**, p. 204]) if there exists a right exact sequence $\mathcal{A}(-,Y) \longrightarrow \mathcal{A}(-,X) \longrightarrow F$, with X and Y objects in $\mathcal{A}$ (alternative terminology for such an F is *coherent*; see [**1**, p. 189]). Let $\mathsf{mod}_{\mathcal{A}}$ denote the full subcategory of $\mathsf{Mod}_{\mathcal{A}}$ consisting of the finitely presented objects in $\mathsf{Mod}_{\mathcal{A}}$. The standard k-dual functors induce exact functors

$$(\mathsf{Mod}_{\mathcal{A}})^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathsf{Mod}_{\mathcal{A}^{\mathrm{op}}} \quad \text{and} \quad \mathsf{Mod}_{\mathcal{A}^{\mathrm{op}}} \xrightarrow{\mathbb{D}} (\mathsf{Mod}_{\mathcal{A}})^{\mathrm{op}}. \tag{1.1}$$

We say $\mathcal{A}$ is a *dualising k-variety* if the functors in (1.1) restrict to functors

$$(\mathsf{mod}_{\mathcal{A}})^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathsf{mod}_{\mathcal{A}^{\mathrm{op}}} \quad \text{and} \quad \mathsf{mod}_{\mathcal{A}^{\mathrm{op}}} \xrightarrow{\mathbb{D}} (\mathsf{mod}_{\mathcal{A}})^{\mathrm{op}}.$$

1.1. Main results. In this subsection, k is a field and $\mathcal{T}$ is a Krull–Schmidt Hom-finite k-linear triangulated category $\mathcal{T}$ with suspension functor Σ and with a Serre functor $\mathcal{T} \xrightarrow{\mathbb{S}} \mathcal{T}$. We also fix a full abelian subcategory $\mathcal{A}$ of $\mathcal{T}$. We let $\mathsf{Proj}\,\mathcal{A}$ denote the full subcategory of $\mathcal{A}$ consisting of the projective objects and let $\mathsf{Inj}\,\mathcal{A}$ denote the full subcategory of $\mathcal{A}$ consisting of the injective objects.

As witnessed in Example 1.7, the assignment $P \longmapsto \mathbb{S}P$ for P a projective object in $\mathcal{A}$ only defines a functor $\mathsf{Proj}\,\mathcal{A} \longrightarrow \mathcal{T}$. The idea of the next result is to correct this unwanted feature by requiring the existence of an $\mathcal{A}$-cover of $\mathbb{S}P$. It turns out, under mild assumptions (see Theorem 3.4), this requirement ensures that the assignment $P \longmapsto \mathbb{S}P \longmapsto \nu P$, where $\nu P \longrightarrow \mathbb{S}P$ is an $\mathcal{A}$-cover of $\mathbb{S}P$, is functorial and provides an equivalence $\mathsf{Proj}\,\mathcal{A} \simeq \mathsf{Inj}\,\mathcal{A}$ of additive categories.

Setup 1.10. Let $\mathcal{A}$ be an extension-closed k-linear proper abelian subcategory of $\mathcal{T}$ and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. Further, assume the following:

- For each projective object P in $\mathcal{A}$ there is an $\mathcal{A}$-cover in $\mathcal{T}$ of the form

$$\nu P \xrightarrow{\alpha_P} \mathbb{S}P.$$

- For each injective object I in $\mathcal{A}$ there is an $\mathcal{A}$-envelope in $\mathcal{T}$ of the form

$$\mathbb{S}^{-1}I \xrightarrow{\beta_I} \nu^{-}I.$$

Theorem A (Theorem 3.4). *Consider Setup 1.10. Then the following hold:*

(1) *The assignment $P \mapsto \nu P$ augments to an additive functor* $\mathsf{Proj}\,\mathcal{A} \xrightarrow{\nu} \mathsf{Inj}\,\mathcal{A}$ *such that*
$$\begin{tikzcd} \nu P \arrow[r, "\alpha_P"] \arrow[d, "\nu p"'] & \mathbb{S}P \arrow[d, "\mathbb{S}p"] \\ \nu P' \arrow[r, "\alpha_{P'}"] & \mathbb{S}P' \end{tikzcd}$$
is commutative for each morphism $P \xrightarrow{p} P'$ in $\mathsf{Proj}\,\mathcal{A}$.

(2) *The assignment $I \longmapsto \nu^{-} I$ augments to an additive functor* $\mathsf{Inj}\,\mathcal{A} \xrightarrow{\nu^{-}} \mathsf{Proj}\,\mathcal{A}$ *such that*
$$\begin{tikzcd} \mathbb{S}^{-1}I \arrow[r, "\beta_I"] \arrow[d, "\mathbb{S}^{-1}i"'] & \nu^{-}I \arrow[d, "\nu^{-}i"] \\ \mathbb{S}^{-1}I' \arrow[r, "\beta_{I'}"] & \nu^{-}I' \end{tikzcd}$$
is commutative for each morphism $I \xrightarrow{i} I'$ in $\mathsf{Inj}\,\mathcal{A}$.

Moreover, the functors ν and ν^{-} are mutual quasi-inverses.

Our next main theorem shows that $\mathcal{A}$ enjoys a useful duality condition (see Definition 1.9) and if we impose that $\mathcal{A}$ is a length category, then the symmetry between the projective and injective objects in $\mathcal{A}$ becomes stronger.

Theorem B (Theorem 3.8 and 3.10). *Consider Setup 1.10. Then the following hold:*

(1) *If $\mathcal{A}$ is essentially small and has enough injectives and enough projectives, then $\mathcal{A}$ is a dualising k-variety.*

(2) *If every object in $\mathcal{A}$ is of finite length, then $\mathcal{A}$ has enough projectives if and only if $\mathcal{A}$ has enough injectives.*

Having Theorem A at our disposal, we can construct Auslander–Reiten translates τ and τ^{-} in $\mathcal{A}$ following the classical pedagogy (see the beginning of Subsection 4.1 and Definition 4.2). The fundamental properties of these mappings are given as our next main result.

Theorem C (Propositions 4.3 and 4.4). *Consider Setup 1.10 and assume each object in $\mathcal{A}$ has a projective cover and an injective envelope. Then there are Auslander–Reiten translates τ and τ^{-} on $\mathcal{A}$ and they satisfy several standard properties.*

The next result is the existence of Auslander–Reiten sequences in $\mathcal{A}$.

Theorem D (Theorem 4.7). *Consider Setup 1.10 and assume each object in $\mathcal{A}$ has a projective cover and an injective envelope. Then the following hold:*

(1) *For each indecomposable non-projective object C in $\mathcal{A}$, there exists an Auslander–Reiten sequence in $\mathcal{A}$ of the form*
$$0 \longrightarrow \tau C \longrightarrow E_C \longrightarrow C \longrightarrow 0.$$

(2) *For each indecomposable non-injective object A in $\mathcal{A}$, there exists an Auslander–Reiten sequence in $\mathcal{A}$ of the form*
$$0 \longrightarrow A \longrightarrow F_A \longrightarrow \tau^{-}A \longrightarrow 0.$$

When applying the methods presented in the main theorems above to the module category, we recover the standard Nakayama functors and Auslander–Reiten translates (see Theorem 5.4 and Proposition 5.6). As an application, we provide a new proof of the existence of Auslander–Reiten sequences in the finite-dimensional module category of a finite-dimensional algebra (see Theorem 5.7).

Theorems A, B, and D are instances of a fascinating and potentially powerful phenomenon that involves relating intrinsic properties of an abelian category with its relationship to an ambient triangulated category (see the discussion in [**13**, p. 213]). In the same paper, it was shown that the heart of a bounded t-structure of a saturated Hom-finite Krull–Schmidt k-linear triangulated category is functorially finite in said category if and only if the heart has enough projective and enough injective objects [**13**, Corollary 2.8] (see [**13**, Theorem 2.4] for a more general statement and compare with Theorem A and Theorem B(2)).

Another such example of this phenomenon can be seen in [**28**, Proposition 2.6], which says the following: given an extension-closed additive subcategory $\mathcal{A}$ of a Hom-finite Krull–Schmidt k-linear triangulated category $\mathcal{T}$ such that $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$, if each object of $\mathcal{A}$ has a $\Sigma\mathcal{A}$-envelope, then there is an exact structure $\mathcal{E}$ on $\mathcal{A}$ such that the exact category $(\mathcal{A}, \mathcal{E})$ has enough projective objects.

Remark 1.11. In [**13**, Remark 2.15], the authors mention that it would be interesting to investigate whether their result [**13**, Corollary 2.8] holds without the assumption of the triangulated category being saturated. The results in this paper assume the weaker condition that the triangulated category has a Serre functor (see [**29**, p. 3]).

1.2. Some useful results. For the reader's convenience, we record some results that will be used multiple times throughout this paper. The first result is a useful criterion for showing when a certain precover is actually a cover.

Lemma ([**32**, Lemma 2.4])**.** *Consider a non-zero morphism* $X \xrightarrow{f} Y$ *in an additive category and suppose that* $\mathrm{End}(Y)$ *is local. Then* α *is left minimal.*

The next result allows us to characterise the intrinsic property of an object being projective in a proper abelian subcategory in terms of the ambient triangulated category and its suspension functor.

Theorem ([**15**, Theorem on p. 1]; see also [**28**, Theorem 3.5])**.** *Let* $\mathcal{T}$ *be a triangulated category with suspension functor* Σ *and let* $\mathcal{A}$ *be an additive full subcategory of* $\mathcal{T}$ *that is closed under extensions and* $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$*. Then an object* P *in* $\mathcal{A}$ *is projective in* $\mathcal{A}$ *if and only if* $\mathcal{T}(P, \Sigma\mathcal{A}) = 0$*.*

The final result will be used, sometimes in conjunction with Serre duality, to show that certain Hom spaces vanish.

Lemma (Triangulated Wakamatsu's lemma [**26**, Lemma 2.1])**.** *Let* $\mathcal{T}$ *be a Krull–Schmidt* Hom*-finite* k*-linear triangulated category with suspension functor* Σ *and let* $\mathcal{A}$ *be a full subcategory of* $\mathcal{T}$ *that is closed under extensions and direct summands. Suppose that a morphism* $A \xrightarrow{\alpha} T$ *in* $\mathcal{T}$ *is an* $\mathcal{A}$*-cover of an object* T *in* $\mathcal{T}$ *and complete it to a triangle* $A \xrightarrow{\alpha} T \longrightarrow Z \longrightarrow \Sigma A$*. Then* $\mathcal{T}(\mathcal{A}, Z) = 0$*.*

1.3. Global setup. The following are taken throughout the paper:

- k is a field.
- The standard k-dual functor $(\mathsf{mod}_k)^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_k(-,k)} \mathsf{mod}_k$, where mod_k is the category of finite-dimensional right modules over k, is denoted by D.
- All subcategories are assumed to be full subcategories closed under isomorphisms.
- k-linear categories are categories enriched over the category of k-vector spaces with finite direct sums.
- For an abelian category $\mathcal{A}$, the full subcategory of $\mathcal{A}$ consisting of the projective objects is denoted by $\mathsf{Proj}\,\mathcal{A}$ and the full subcategory of $\mathcal{A}$ consisting of the injective objects is denoted by $\mathsf{Inj}\,\mathcal{A}$.
- $\mathcal{T}$ is a Krull–Schmidt Hom-finite k-linear triangulated category with suspension functor Σ and with a Serre functor $\mathcal{T} \xrightarrow{\mathbb{S}} \mathcal{T}$.

2. Lemmas

2.1. Lemmas on additive categories. Consider two objects X and X' in an additive category. We denote by $\operatorname{rad}(X, X')$ the radical morphisms between X and X' as seen in [**30**, Lemma 6]. That is, a morphism $X \xrightarrow{x} X'$ in said additive category lies in $\operatorname{rad}(X, X')$ if for each morphism $X' \xrightarrow{x'} X$ we have that $1_X - x'x$ is an automorphism. The following lemma will be used in Proposition 4.3.

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be additive categories and let $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ be a fully faithful additive functor. Then F induces an isomorphism $\operatorname{rad}(X, X') \longrightarrow \operatorname{rad}(FX, FX')$ of abelian groups.*

Proof: Let $X \xrightarrow{x} X'$ be a radical morphism in $\mathcal{X}$. We will first show that $FX \xrightarrow{Fx} FX'$ is a radical morphism in $\mathcal{Y}$. To this end, let $FX' \xrightarrow{g} FX$ be a morphism in $\mathcal{Y}$. As F is fully faithful, g is of the form Fx' for some morphism $X' \xrightarrow{x'} X$ in $\mathcal{X}$. As x is a radical morphism in $\mathcal{X}$, we have that $1_X - x'x$ is an automorphism. Therefore, $1_{FX} - gF(x)$ is an automorphism. We now have that the group isomorphism $\mathcal{X}(X, X') \longrightarrow \mathcal{Y}(FX, FX')$ induced by F restricts to an injective group homomorphism $\operatorname{rad}(X, X') \longrightarrow \operatorname{rad}(FX, FX')$. For surjectivity, we let $FX \xrightarrow{f} FX'$ be a radical morphism in $\mathcal{Y}$. Again, we may take f to be of the form Fw for some morphism $X \xrightarrow{w} X'$ in $\mathcal{X}$. It suffices to show that w is a radical morphism in $\mathcal{X}$. Let $X' \xrightarrow{w'} X$ be a morphism in $\mathcal{X}$. Then $F(1_X - w'w) = 1_{FX} - F(w')f$ is an automorphism as f is a radical morphism in $\mathcal{Y}$. As F is fully faithful, it reflects isomorphisms. Therefore, $1_X - w'w$ is an automorphism and hence w is a radical morphism in $\mathcal{X}$. $\square$

The following lemma will be a diagram trick involving a strong cover. It will be used in Lemma 2.4 and Theorem 3.4.

Lemma 2.2. *Let $\mathcal{X}$ be an additive category, let $\mathcal{Y}$ be an additive subcategory of $\mathcal{X}$, and consider the following diagram in $\mathcal{X}$:*

$$\begin{tikzcd}
Y_1 \arrow[rr, "a"] \arrow[dr, "e"'] \arrow[dd, "b"'] & & X_1 \arrow[dl, "f"] \arrow[dd, "c"] \\
& X_2 \arrow[dd, "h"] & \\
X_3 \arrow[rr, "d"] \arrow[dr, "g"'] & & Y \arrow[dl, "\alpha"] \\
& X &
\end{tikzcd}$$

with the following properties:

(1) *The objects Y and Y_1 both lie in $\mathcal{Y}$.*
(2) *All but the back-most square with vertices Y_1, X_1, X_3, and Y commute.*

If α is a strong $\mathcal{Y}$-cover of X, then the back-most square commutes.

Proof: By the commutativity of the diagram, the following equalities hold: $\alpha db = gb = he = hfa = \alpha ca$. So if α is a strong $\mathcal{Y}$-cover, then we get $db = ca$ as required. $\square$

The following lemma will be used multiple times throughout this paper, where the functor $\mathcal{X} \xrightarrow{S} \mathcal{X}$ in the statement of the lemma is taken to be the Serre functor $\mathcal{T} \xrightarrow{\mathbb{S}} \mathcal{T}$ mentioned in the global setup. It will allow us to construct additive functors through the existence of approximations.

Lemma 2.3. *Let $\mathcal{X}$ be an additive category and let $\mathcal{X} \xrightarrow{S} \mathcal{X}$ be an additive endofunctor. Let $\mathcal{Y}$ be an additive subcategory of $\mathcal{X}$ and let $\mathcal{Z}$ be an additive subcategory of $\mathcal{Y}$. Then the following hold:*

(1) *If for each object Z in $\mathcal{Z}$ there is a strong $\mathcal{Y}$-cover in $\mathcal{X}$ of the form*

$$C_Z \xrightarrow{c_Z} SZ,$$

then the assignment $Z \longmapsto C_Z$ on objects augments to an additive functor $\mathcal{Z} \xrightarrow{C_{(-)}} \mathcal{Y}$ such that the induced diagram

$$\begin{tikzcd} C_Z \arrow[r, "c_Z"] \arrow[d, "C_z"'] & SZ \arrow[d, "Sz"] \\ C_{Z'} \arrow[r, "c_{Z'}"] & SZ' \end{tikzcd}$$

is commutative for each morphism $Z \xrightarrow{z} Z'$ in $\mathcal{Z}$.

(2) *If for each object Z in $\mathcal{Z}$ there is a strong $\mathcal{Y}$-envelope in $\mathcal{X}$ of the form*

$$S^{-1}Z \longrightarrow E_Z,$$

then the assignment $Z \longmapsto E_Z$ on objects augments to an additive functor $\mathcal{Z} \longrightarrow \mathcal{Y}$ such that the induced diagram

$$\begin{tikzcd} S^{-1}Z \arrow[r] \arrow[d, "S^{-1}z"'] & E_Z \arrow[d, "E_z"] \\ S^{-1}Z' \arrow[r] & E_{Z'} \end{tikzcd}$$

is commutative for each morphism $Z \xrightarrow{z} Z'$ in $\mathcal{Z}$.

Proof: (1) Given a morphism $Z \xrightarrow{z} Z'$ between objects in $\mathcal{Z}$, we have the composition $C_Z \xrightarrow{c_Z} SZ \xrightarrow{Sz} SZ'$. As $C_{Z'} \xrightarrow{c_{Z'}} SZ'$ is a strong $\mathcal{Y}$-cover in $\mathcal{X}$, there exists a unique morphism $C_Z \xrightarrow{C_z} C_{Z'}$ such that $S(z)c_Z = c_{Z'}C_z$. The uniqueness of C_z in the assignment $z \longmapsto C_z$ on morphisms ensures functoriality.

For additivity, let $Z \xrightarrow{z_1,z_2} Z'$ be morphisms between objects in $\mathcal{Z}$. We need to show $C_{(z_1+z_2)} = C_{z_1} + C_{z_2}$. By their respective definitions and as S is an additive functor, we have the following equalities:

$$c_{Z'}C_{(z_1+z_2)} = S(z_1 + z_2)c_Z = (Sz_1 + Sz_2)c_Z = c_{Z'}(C_{z_1} + C_{z_2}).$$

Additivity follows as $c_{Z'}$ is a strong $\mathcal{Y}$-cover.

(2) is dual to (1). □

Lemma 2.3 tells us that a global choice of strong $\mathcal{Y}$-covers of the objects SZ, for each object Z in $\mathcal{Z}$, gives rise to an additive functor $\mathcal{Z} \longrightarrow \mathcal{Y}$ fitting into the commutative diagram in said lemma. The next lemma, however, uses the uniqueness of the $\mathcal{Y}$-covers to ensure that any other global choice of $\mathcal{Y}$-covers of the objects SZ would result in a canonically naturally isomorphic functor. It will be used in Theorem 5.4.

Lemma 2.4. *Let $\mathcal{X}$ be an additive category and let $\mathcal{X} \xrightarrow{S} \mathcal{X}$ be an additive endofunctor. Let $\mathcal{Y}$ be an additive subcategory of $\mathcal{X}$ and let $\mathcal{Z}$ be an additive subcategory of $\mathcal{Y}$. Then the following hold:*

(1) *Assume for each object Z in $\mathcal{Z}$ there are strong $\mathcal{Y}$-covers in $\mathcal{X}$ of the form $C_Z \xrightarrow{c_Z} SZ$ and $D_Z \xrightarrow{d_Z} SZ$. The assignments $Z \longmapsto C_Z$ and $Z \longmapsto D_Z$ augment to two functors $\mathcal{Z} \longrightarrow \mathcal{Y}$ by Lemma 2.3. These functors are naturally isomorphic.*

(2) *Assume for each object Z in $\mathcal{Z}$ there are strong $\mathcal{Y}$-envelopes in $\mathcal{X}$ of the form $S^{-1}Z \longrightarrow E_Z$ and $S^{-1}Z \longrightarrow F_Z$. The assignments $Z \longmapsto E_Z$ and $Z \longmapsto F_Z$ augment to two functors $\mathcal{Z} \longrightarrow \mathcal{Y}$ by Lemma 2.3. These functors are naturally isomorphic.*

Proof: Assume for each object Z in $\mathcal{Z}$ we have strong $\mathcal{Y}$-covers in $\mathcal{X}$ of the form $C_Z \xrightarrow{c_Z} SZ$ and $D_Z \xrightarrow{d_Z} SZ$. By Lemma 2.3, we have two functors $\mathcal{Z} \longrightarrow \mathcal{Y}$, given by the assignments $Z \longmapsto C_Z$ and $Z \longmapsto D_Z$. As C_Z is an object in $\mathcal{Y}$ and d_Z is a strong $\mathcal{Y}$-cover, there exists a unique isomorphism $C_Z \xrightarrow{\varphi_Z} D_Z$ such that $c_Z = d_Z\varphi_Z$. We now show that the collection $\{\varphi_Z \mid Z \text{ an object in } \mathcal{Z}\}$ of isomorphisms in $\mathcal{Y}$ form the components of a natural isomorphism between the functors in question. Let $Z \xrightarrow{z} Z'$ be a morphism in $\mathcal{Z}$. We obtain the diagram

$$\begin{tikzcd}
C_Z \arrow[rr, "\varphi_Z"] \arrow[dr, "c_Z"'] \arrow[dd, "C_z"'] & & D_Z \arrow[dl, "d_Z"] \arrow[dd, "D_z"] \\
& SZ \arrow[dd, "Sz"] & \\
C_{Z'} \arrow[rr, "\varphi_{Z'}"] \arrow[dr, "c_{Z'}"'] & & D_{Z'}, \arrow[dl, "d_{Z'}"] \\
& SZ' &
\end{tikzcd}$$

where all but the back-most square with vertices C_Z, $C_{Z'}$, $D_{Z'}$, and D_Z commute. By the application of Lemma 2.2, we are done. $\square$

2.2. A lemma on *k*-linear abelian subcategories. The following lemma will be used in Proposition 3.7.

Lemma 2.5. *Let $\mathcal{A}$ be a k-linear abelian subcategory of $\mathcal{T}$. Let X and P be objects in $\mathcal{A}$ with P projective. Then the following two statements are equivalent:*

(1) *There is a strong $\mathcal{A}$-cover in $\mathcal{T}$ of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$.*

(2) *There is a natural isomorphism of the form $\mathrm{D}\,\mathcal{A}(P,-) \cong \mathcal{A}(-,X)$ of functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{mod}_k$, i.e. the object X represents the functor $\mathrm{D}\,\mathcal{A}(P,-)$.*

Dually, let Y and I be objects in $\mathcal{A}$ with I injective. Then the following two statements are equivalent:

(1') *There is a strong $\mathcal{A}$-envelope in $\mathcal{T}$ of the form $\mathbb{S}^{-1}I \longrightarrow Y$.*

(2') *There is a natural isomorphism of the form $\mathrm{D}\,\mathcal{A}(-,I) \cong \mathcal{A}(Y,-)$ of functors $\mathcal{A} \longrightarrow \mathsf{mod}_k$, i.e. the object Y represents the functor $\mathrm{D}\,\mathcal{A}(-,I)$.*

Proof: (1) $\Rightarrow$ (2): Suppose there is a strong $\mathcal{A}$-cover of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$. We have a natural isomorphism $\mathcal{A}(-,X) \xrightarrow{\mathcal{T}(-,\alpha_P)|_{\mathcal{A}}} \mathcal{T}(-,\mathbb{S}P)|_{\mathcal{A}} \cong \mathrm{D}\,\mathcal{A}(P,-)$ of functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{mod}_k$, where the first whiskered composite natural transformation is a natural isomorphism as α_P is a strong $\mathcal{A}$-cover and the second natural isomorphism is given by Serre duality. Therefore, X represents the functor $\mathrm{D}\,\mathcal{A}(P,-)$.

(2) $\Rightarrow$ (1): Suppose the object X represents the functor $\mathrm{D}\,\mathcal{A}(P,-)$. Then we have a natural isomorphism $\gamma : \mathcal{A}(-,X) \cong \mathrm{D}\,\mathcal{A}(P,-) \cong \mathcal{T}(-,\mathbb{S}P)|_{\mathcal{A}}$ of functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{mod}_k$, where the second natural isomorphism is given by Serre duality. We will show that the morphism $X \xrightarrow{\gamma_X(1_X)} \mathbb{S}P$, where γ_X is the component of the natural isomorphism γ at X, is a strong $\mathcal{A}$-cover. Let $\alpha_P := \gamma_X(1_X)$. It suffices to show that for each object A in $\mathcal{A}$ the component γ_A coincides with $\mathcal{A}(A,X) \xrightarrow{\mathcal{T}(A,\alpha_P)|_{\mathcal{A}}} \mathcal{T}(A,\mathbb{S}P)|_{\mathcal{A}}$. To this end, let $A \xrightarrow{a} X$ be a morphism in $\mathcal{A}$. By naturality, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}(X,X) \arrow[r, "\gamma_X"] \arrow[d, "\mathcal{A}(a,X)"'] & \mathcal{T}(X,\mathbb{S}P)|_{\mathcal{A}} \arrow[d, "\mathcal{T}(a,\mathbb{S}P)|_{\mathcal{A}}"] \\
\mathcal{A}(A,X) \arrow[r, "\gamma_A"] & \mathcal{T}(A,\mathbb{S}P)|_{\mathcal{A}}
\end{tikzcd}$$

The equality $\gamma_A(a) = \mathcal{T}(A,\alpha_P)|_{\mathcal{A}}(a)$ is obtained by chasing the identity 1_X through this diagram.

(1') $\Rightarrow$ (2') is dual to (1) $\Rightarrow$ (2). $\square$

2.3. A lemma on abelian length categories. The following lemma will be used in Theorem 3.10.

Lemma 2.6. *Let $\mathcal{A}$ be an abelian category for which each object of $\mathcal{A}$ is of finite length. Then the following hold:*

1. *If each simple object in $\mathcal{A}$ admits a non-zero morphism to an injective object in $\mathcal{A}$, then $\mathcal{A}$ has enough injectives.*
2. *If each simple object in $\mathcal{A}$ admits a non-zero morphism from a projective object in $\mathcal{A}$, then $\mathcal{A}$ has enough projectives.*

Proof: (1) Let A be an object in $\mathcal{A}$. As A is of finite length, there exists a finite composition series $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = A$ of A. Denote the simple quotients A_{i+1}/A_i by S_{i+1} and note $A_1 = S_1$. There are short exact sequences of the form $A_i \longrightarrow A_{i+1} \longrightarrow S_{i+1}$ and by assumption there are non-zero morphisms $S_{i+1} \longrightarrow I_{i+1}$, where every I_{i+1} is an injective object in $\mathcal{A}$. In particular, these morphisms are monomorphisms in $\mathcal{A}$. By the dual of the Horseshoe lemma [**17**, Lemma 8.2.1], we inductively get monomorphisms $A_{i+1} \longrightarrow I_1 \oplus I_2 \oplus \cdots \oplus I_{i+1}$ in $\mathcal{A}$. In particular, we acquire a monomorphism $A = A_n \longrightarrow I_1 \oplus I_2 \oplus \cdots \oplus I_n$ in $\mathcal{A}$, proving our statement.

(2) is dual to (1). $\square$

2.4. A lemma on proper abelian subcategories. The following lemma will be used in Proposition 4.3, Theorem 4.7, and implicitly in Proposition 5.5.

Lemma 2.7. *Let $\mathcal{A}$ be an extension-closed k-linear proper abelian subcategory of $\mathcal{T}$ and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. Further, assume the following:*

- *For each projective object P in $\mathcal{A}$ there is a strong $\mathcal{A}$-cover in $\mathcal{T}$ of the form*
$$C_P \xrightarrow{c_P} \mathbb{S}P.$$
- *For each injective object I in $\mathcal{A}$ there is a strong $\mathcal{A}$-envelope in $\mathcal{T}$ of the form*
$$\mathbb{S}^{-1}I \longrightarrow E_I.$$

As $\mathsf{Proj}\,\mathcal{A}$ is an additive subcategory of $\mathcal{A}$, Lemma 2.3 gives rise to a functor $\mathsf{Proj}\,\mathcal{A} \longrightarrow \mathcal{A}$ given by the assignment $P \longmapsto C_P$. The following then hold:

(1) *If $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A$ is a projective presentation of A in $\mathcal{A}$, then there is an $\mathcal{A}$-precover of the form $X_A \xrightarrow{\theta} \Sigma^{-1}\mathbb{S}A$, where X_A is the kernel of the induced morphism $C_{P_1} \xrightarrow{C_{p_1}} C_{P_0}$ in $\mathcal{A}$. Moreover, if A is a non-projective object in $\mathcal{A}$, then the $\mathcal{A}$-precover θ is necessarily non-zero.*

(2) *If $B \longrightarrow I^0 \longrightarrow I^1$ is an injective copresentation of B in $\mathcal{A}$, then there is an $\mathcal{A}$-preenvelope of the form $\Sigma\mathbb{S}^{-1}B \xrightarrow{\xi} Z_B$, where Z_B is the cokernel of the induced morphism $E_{I^0} \longrightarrow E_{I^1}$ in $\mathcal{A}$. Moreover, if B is a non-injective object in $\mathcal{A}$, then the $\mathcal{A}$-preenvelope ξ is necessarily non-zero.*

Proof: (1) Our assumption allows us to use Lemma 2.3 to produce an additive functor $\mathsf{Proj}\,\mathcal{A} \longrightarrow \mathcal{A}$ given by $P \longmapsto C_P$. The projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A$ of A induces a left exact sequence $X_A \xrightarrow{x_A} C_{P_1} \xrightarrow{C_{p_1}} C_{P_0}$, where x_A is the kernel of C_{p_1}. Complete the morphism $\mathbb{S}P_1 \xrightarrow{\mathbb{S}p_1} \mathbb{S}P_0$ to a short triangle $Y \xrightarrow{y} \mathbb{S}P_1 \xrightarrow{\mathbb{S}p_1} \mathbb{S}P_0$. As $\mathcal{A}$ is a proper abelian subcategory, the short exact sequence $\Omega A \xrightarrow{\iota_1} P_0 \xrightarrow{p_0} A$, where ΩA is the syzygy of A, is a short triangle. By rotating, $\Sigma^{-1}P_0 \xrightarrow{-\Sigma p_0} \Sigma^{-1}A \longrightarrow \Omega A$ is a short triangle. Therefore, there is a short triangle $\Sigma^{-1}\mathbb{S}P_0 \xrightarrow{-\Sigma\mathbb{S}p_0} \Sigma^{-1}\mathbb{S}A \xrightarrow{w} \mathbb{S}\Omega A$. Here we used that $\mathbb{S}$ is a triangulated functor (see [**10**, Proposition 3.3] for the classical reference and see [**9**, Theorem A.4.4] for a proof which makes the natural transformation more evident). After rotating triangles, we get the following solid commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X_A \arrow[r, "x_A"] \arrow[d, dashed, "\theta_1"] & C_{P_1} \arrow[r, "C_{p_1}"] \arrow[d, "c_{P_1}"] & C_{P_0} \arrow[d, "c_{P_0}"] \\
\Sigma^{-1}\mathbb{S}P_0 \arrow[r] \arrow[d, equal] & Y \arrow[r, "y"] \arrow[d, dashed, "\theta_2"] & \mathbb{S}P_1 \arrow[r, "\mathbb{S}p_1"] \arrow[d, "\mathbb{S}\pi_1"] & \mathbb{S}P_0 \arrow[d, equal] \\
\Sigma^{-1}\mathbb{S}P_0 \arrow[r, "-\Sigma\mathbb{S}p_0"] & \Sigma^{-1}\mathbb{S}A \arrow[r, "w"] & \mathbb{S}\Omega A \arrow[r, "\mathbb{S}\iota_1"] & \mathbb{S}P_0
\end{tikzcd} \tag{2.1}$$

where the composition $\mathbb{S}P_1 \xrightarrow{\mathbb{S}\pi_1} \mathbb{S}\Omega A \xrightarrow{\mathbb{S}\iota_1} \mathbb{S}P_0$ is obtained by applying $\mathbb{S}$ to the usual factorisation $p_1\colon P_1 \xrightarrow{\pi_1} \Omega A \xrightarrow{\iota_1} P_0$ of the morphism p_1. Note that the bottom and the middle rows are both triangles. By an axiom of triangulated categories, there exists a morphism $Y \xrightarrow{\theta_2} \Sigma^{-1}\mathbb{S}A$. Since $\mathbb{S}(p_1)c_{P_1}x_A = 0$, there exists a morphism $X_A \xrightarrow{\theta_1} Y$ as y is a weak kernel of $\mathbb{S}p_1$. The morphisms θ_1 and θ_2 make the entire diagram (2.1) commute. Taking $\theta = \theta_2\theta_1$, we show that θ is indeed an $\mathcal{A}$-precover. The next steps in our proof will consist of adding many new morphisms to diagram (2.1); the reader is recommended to use diagram (2.2) below to aid their reading (the dashed arrows in diagram (2.2) will be introduced in the following argument).

$$
\begin{tikzcd}
0 \arrow[r] & X_A \arrow[rr, "x_A"] \arrow[dd, "\theta_1"] & & C_{P_1} \arrow[r, "C_{p_1}"] \arrow[dd, "c_{P_1}"] & C_{P_0} \arrow[dd, "c_{P_0}"] \\
 & & B \arrow[ul, dashed, "\overline{b_1}"'] \arrow[ur, dashed, "b_1"] \arrow[dl, dashed, "b"] & & \\
\Sigma^{-1}\S P_0 \arrow[r] \arrow[dd, equal] & Y \arrow[rr, "y"] \arrow[dd, "\theta_2"] & & \S P_1 \arrow[r, "\S p_1"] \arrow[dd, "\S\pi_1"] & \S P_0 \arrow[dd, equal] \\
 & & B \arrow[ul, dashed, "\overline{b_3}"'] \arrow[ur, dashed, "b_3"] \arrow[dl, dashed, "b_2"'] \arrow[dr, dashed, "wb_2"] & & \\
\Sigma^{-1}\S P_0 \arrow[r, "-\Sigma\S p_0"] & \Sigma^{-1}\S A \arrow[rr, "w"] & & \S\Omega A \arrow[r, "\S\iota_1"] & \S P_0
\end{tikzcd}
\tag{2.2}
$$

First, we show that θ_1 is an $\mathcal{A}$-precover. Let $B \xrightarrow{b} Y$ be a morphism with the object B in $\mathcal{A}$. As c_{P_1} is a strong $\mathcal{A}$-precover, there exists a unique morphism $B \xrightarrow{b_1} C_{P_1}$ such that $yb = c_{P_1}b_1$. We have the following equalities: $c_{P_0}C_{p_1}b_1 = \mathbb{S}(p_1)c_{P_1}b_1 = \mathbb{S}(p_1)yb = 0$, where the last equality holds as consecutive morphisms in a triangle vanish. Then $C_{p_1}b_1 = 0$ as c_{P_0} is a strong $\mathcal{A}$-cover. As x_A is the kernel of C_{p_1}, there exists a unique morphism $B \xrightarrow{\overline{b_1}} X_A$ such that $b_1 = x_A\overline{b_1}$. We have the following equalities: $yb = c_{P_1}b_1 = c_{P_1}x_A\overline{b_1} = y\theta_1\overline{b_1}$, which gives $y(b - \theta_1\overline{b_1}) = 0$. As y is a weak kernel of $\mathbb{S}p_1$, the morphism $b - \theta_1\overline{b_1}$ factors through a morphism in the k-vector space $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0)$. But $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0) \cong \mathrm{D}\,\mathcal{T}(P_0, \Sigma B) = 0$, where the isomorphism is given by Serre duality and where the equality holds as P_0 is a projective object in $\mathcal{A}$ (see [**15**, Theorem on p. 1]). Hence, we are done.

It now suffices to show that the map $\mathcal{T}(B, Y) \xrightarrow{\mathcal{T}(B,\theta_2)} \mathcal{T}(B, \Sigma^{-1}\mathbb{S}A)$ is an epimorphism for each object B in $\mathcal{A}$. To this end, let $B \xrightarrow{b_2} \Sigma^{-1}\mathbb{S}A$ be a morphism with B in $\mathcal{A}$. By Serre duality, we have the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{T}(B, \mathbb{S}P_1) \arrow[r, "\cong"] \arrow[d, "\mathcal{T}(B,\mathbb{S}\pi_1)"'] & \mathrm{D}\,\mathcal{T}(P_1, B) \arrow[d, "\mathrm{D}\,\mathcal{T}(\pi_1,B)"] \\
\mathcal{T}(B, \mathbb{S}\Omega A) \arrow[r, "\cong"] & \mathrm{D}\,\mathcal{T}(\Omega A, B)
\end{tikzcd}
$$

As π_1 is an epimorphism, the map $\mathrm{D}\,\mathcal{T}(\pi_1, B)$ is an epimorphism and hence so is the map $\mathcal{T}(B, \mathbb{S}\pi_1)$. Therefore, there exists a morphism $B \xrightarrow{b_3} \mathbb{S}P_1$ such that $wb_2 = \mathbb{S}(\pi_1)b_3$. We have the following equalities: $\mathbb{S}(p_1)b_3 = \mathbb{S}(\iota_1)\mathbb{S}(\pi_1)b_3 = \mathbb{S}(\iota_1)wb_2 = 0$, where the last equality holds as consecutive morphisms in a triangle vanish. As y is a weak kernel of $\mathbb{S}p_1$, there exists a morphism $B \xrightarrow{\overline{b_3}} Y$ such that $b_3 = y\overline{b_3}$. We have the following equalities: $wb_2 = \mathbb{S}(\pi_1)b_3 = \mathbb{S}(\pi_1)y\overline{b_3} = w\theta_2\overline{b_3}$, which gives $w(b_2 - \theta_2\overline{b_3}) = 0$. As w is a weak kernel of $\mathbb{S}\iota_1$, the morphism $b_2 - \theta_2\overline{b_3}$ factors through a morphism in the k-vector space $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0)$. However $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0) \cong \mathrm{D}\,\mathcal{T}(P_0, \Sigma B) = 0$, where the isomorphism is given by Serre duality and where the equality holds as P_0 is a projective object in $\mathcal{A}$ (see [**15**, Theorem on p. 1]). Hence, we are done.

Suppose that A is a non-projective object in $\mathcal{A}$. By [**15**, Theorem on p. 1], there exists an object T in $\mathcal{A}$ with $\mathcal{T}(A, \Sigma T) \neq 0$. Then the k-vector space $\mathrm{D}\,\mathcal{T}(A, \Sigma T)$ is also non-zero and we have $\mathcal{T}(T, \Sigma^{-1}\mathbb{S}A) \cong \mathrm{D}\,\mathcal{T}(A, \Sigma T) \neq 0$, where the isomorphism holds by Serre duality. Therefore, we can pick a non-zero morphism $T \xrightarrow{t} \Sigma^{-1}\mathbb{S}A$. As θ is an $\mathcal{A}$-precover, the non-zero morphism t must factor through θ, resulting in θ necessarily being non-zero.

(2) is dual to (1). $\square$

3. Nakayama functors on proper abelian subcategories

3.1. Construction and fundamental properties.

Setup 3.1. We fix $\mathcal{A}$ to be an extension-closed k-linear proper abelian subcategory of $\mathcal{T}$ and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$.

Proposition 3.2. *Let $\mathcal{A}$ be a category with Setup 3.1. For P a projective object in $\mathcal{A}$ and I an injective object in $\mathcal{A}$, the following hold:*

1. *Each $\mathcal{A}$-cover in $\mathcal{T}$ of the form $C_P \xrightarrow{c_P} \mathbb{S}P$ is a strong $\mathcal{A}$-cover in $\mathcal{T}$.*
2. *Each $\mathcal{A}$-envelope in $\mathcal{T}$ of the form $\mathbb{S}^{-1}I \longrightarrow E_I$ is a strong $\mathcal{A}$-envelope in $\mathcal{T}$.*

Proof: Let $A \xrightarrow{a} C_P$ be a morphism in $\mathcal{A}$ such that $c_P a = 0$. In order to show that c_P is a strong $\mathcal{A}$-cover in $\mathcal{T}$ we need $a = 0$. Suppose a is a monomorphism in $\mathcal{A}$. As $\mathcal{A}$ is a proper abelian subcategory, there is a short triangle $A \xrightarrow{a} C_P \xrightarrow{v} B$ in $\mathcal{T}$ with B an object in $\mathcal{A}$. As $c_P a = 0$ and v is a weak cokernel of a, there exists a morphism $B \xrightarrow{b} \mathbb{S}P$ such that $c_P = bv$. As c_P is an $\mathcal{A}$-cover, there exists a morphism $B \xrightarrow{\bar{b}} C_P$ such that $b = c_P\bar{b}$. The situation is depicted in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[dr, "0"'] & C_P \arrow[r, "v"] \arrow[d, "c_P"] & B \arrow[l, dashed, bend right, "\bar{b}"'] \arrow[dl, dashed, "b"] \\
& \mathbb{S}P &
\end{tikzcd}$$

We have the following equalities: $c_P = bv = c_P\bar{b}v$. As c_P is right minimal, $\bar{b}v$ is an automorphism. As the composition of consecutive morphisms in a triangle vanishes, we have $\bar{b}va = 0$. It follows that $a = 0$.

Now suppose a is arbitrary. We choose a factorisation $a\colon A \xrightarrow{a_e} D \xrightarrow{a_m} C_P$ in $\mathcal{A}$, where a_e is an epimorphism and a_m is a monomorphism. It suffices now to show $a_m = 0$. As $\mathcal{A}$ is a proper abelian subcategory, there is a short triangle $A \xrightarrow{a_e} D \xrightarrow{d} \Sigma B$ with B an object in $\mathcal{A}$. As $c_P a_m a_e = c_P a = 0$ and d is a weak cokernel of a_e, there exists a morphism $\Sigma B \xrightarrow{b_1} \mathbb{S}P$ such that $c_P a_m = b_1 d$. The situation is depicted in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a_e"] \arrow[dr, "a"] \arrow[ddr, "0"'] & D \arrow[r, "d"] \arrow[d, "a_m"] & \Sigma B \arrow[ddl, dashed, "b_1"] \\
& C_P \arrow[d, "c_P"] & \\
& \mathbb{S}P &
\end{tikzcd}$$

The morphism b_1 lies in the k-vector space $\mathcal{T}(\Sigma B, \mathbb{S}P)$. But $\mathcal{T}(\Sigma B, \mathbb{S}P) \cong \mathrm{D}\mathcal{T}(P, \Sigma B) = 0$, where the isomorphism is given by Serre duality and where the equality holds as P is a projective objective in $\mathcal{A}$ (see [**15**, Theorem on p. 1]). Hence, $b_1 = 0$ and so

$c_P a_m = 0$. As a_m is a monomorphism in $\mathcal{A}$, the previous argument tells us that $a_m = 0$, as required.

(2) is dual to (1). □

Proposition 3.3. *Let $\mathcal{A}$ be a category with Setup 3.1. Let X be an object in $\mathcal{A}$. Then the following hold:*

(1) *If $C_X \to \mathbb{S}X$ is an $\mathcal{A}$-cover in $\mathcal{T}$, then C_X is an injective object in $\mathcal{A}$.*
(2) *If $\mathbb{S}^{-1}X \to E_X$ is an $\mathcal{A}$-envelope in $\mathcal{T}$, then E_X is a projective object in $\mathcal{A}$.*

Proof: (1) Appealing to the dual of [**15**, Theorem on p. 1], we will show that $\mathcal{T}(A, \Sigma C_X) = 0$ for each object A in $\mathcal{A}$. To this end, let A be an object in $\mathcal{A}$ and complete the morphism $C_X \longrightarrow \mathbb{S}X$ to a short triangle $C_X \longrightarrow \mathbb{S}X \longrightarrow Z$. By properties of triangulated categories, the sequence of k-vector spaces $\mathcal{T}(A, Z) \longrightarrow \mathcal{T}(A, \Sigma C_X) \longrightarrow \mathcal{T}(A, \Sigma\mathbb{S}X)$ is exact. The triangulated Wakamatsu's lemma [**26**, Lemma 2.1] ensures $\mathcal{T}(A, Z) = 0$. By Serre duality we have $\mathcal{T}(A, \Sigma\mathbb{S}X) \cong \mathrm{D}\,\mathcal{T}(X, \Sigma^{-1}A) = 0$, where the latter k-vector space vanishes since $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. It follows that $\mathcal{T}(A, \Sigma C_X) = 0$ by the exactness of the aforementioned sequence.

(2) is dual to (1). □

Theorem 3.4. *Let $\mathcal{A}$ be a category with Setup 3.1. Then the following hold:*

(1) *If for each projective object P in $\mathcal{A}$ there is an $\mathcal{A}$-cover in $\mathcal{T}$ of the form*

$$\nu P \xrightarrow{\alpha_P} \mathbb{S}P,$$

then the assignment $P \longmapsto \nu P$ augments to an additive functor $\mathsf{Proj}\,\mathcal{A} \xrightarrow{\nu} \mathsf{Inj}\,\mathcal{A}$ such that the induced diagram

$$\begin{tikzcd} \nu P \arrow[r, "\alpha_P"] \arrow[d, "\nu p"'] & \mathbb{S}P \arrow[d, "\mathbb{S}p"] \\ \nu P' \arrow[r, "\alpha_{P'}"] & \mathbb{S}P' \end{tikzcd}$$

is commutative for each morphism $P \xrightarrow{p} P'$ in $\mathsf{Proj}\,\mathcal{A}$.
(2) *If for each injective object I in $\mathcal{A}$ there is an $\mathcal{A}$-envelope in $\mathcal{T}$ of the form*

$$\mathbb{S}^{-1}I \xrightarrow{\beta_I} \nu^- I,$$

then the assignment $I \longmapsto \nu^- I$ augments to an additive functor $\mathsf{Inj}\,\mathcal{A} \xrightarrow{\nu^-} \mathsf{Proj}\,\mathcal{A}$ such that the induced diagram

$$\begin{tikzcd} \mathbb{S}^{-1}I \arrow[r, "\beta_I"] \arrow[d, "\mathbb{S}^{-1}i"'] & \nu^- I \arrow[d, "\nu^- i"] \\ \mathbb{S}^{-1}I' \arrow[r, "\beta_{I'}"] & \nu^- I' \end{tikzcd}$$

is commutative for each morphism $I \xrightarrow{i} I'$ in $\mathsf{Inj}\,\mathcal{A}$.

Moreover, if the conditions from both (1) and (2) are satisfied, then the functors ν and ν^- are mutual quasi-inverses.

Proof: (1) The existence of a functor $\mathsf{Proj}\,\mathcal{A} \longrightarrow \mathcal{A}$ given by the assignment $P \longmapsto \nu P$ is guaranteed by Lemma 2.3 and Proposition 3.2. Proposition 3.3 assures that this assignment is a well-defined additive functor $\mathsf{Proj}\,\mathcal{A} \xrightarrow{\nu} \mathsf{Inj}\,\mathcal{A}$.

(2) is dual to (1).

We now show that ν and ν^- are mutual quasi-inverses. As both ν and ν^- are additive functors, it suffices to first construct a collection of isomorphisms

$$\{I \longrightarrow \nu P \mid \text{ with } P = \nu^- I \text{ and } I \text{ an indecomposable injective object in } \mathcal{A}\}$$

natural in I, and then to construct a collection of isomorphisms

$$\{\nu^- J \longrightarrow Q \mid \text{ with } J = \nu Q \text{ and } Q \text{ an indecomposable projective object in } \mathcal{A}\}$$

natural in Q. To this end, let I be an indecomposable injective object of $\mathcal{A}$ and set $P = \nu^- I$. We have an $\mathcal{A}$-envelope $\mathbb{S}^{-1} I \xrightarrow{\beta_I} P$ which is automatically a strong $\mathcal{A}$-envelope by Proposition 3.2. As $\mathbb{S}$ is an autoequivalence, we get a strong $\mathbb{S}\mathcal{A}$-envelope $\gamma_I \colon I \cong \mathbb{S}\mathbb{S}^{-1} I \xrightarrow{\mathbb{S}\beta_I} \mathbb{S}P$. We first prove that γ_I is also a strong $\mathcal{A}$-cover.

Let A be an object in $\mathcal{A}$. By Serre duality, we have the following commutative diagram:

(3.1)
$$\begin{tikzcd}
\mathcal{T}(\mathbb{S}P, \mathbb{S}A) \arrow[r, "\cong"] \arrow[d, "\mathcal{T}(\gamma_I, \mathbb{S}A)"'] & \mathrm{D}\,\mathcal{T}(A, \mathbb{S}P) \arrow[d, "\mathrm{D}\,\mathcal{T}(A, \gamma_I)"] \\
\mathcal{T}(I, \mathbb{S}A) \arrow[r, "\cong"] & \mathrm{D}\,\mathcal{T}(A, I)
\end{tikzcd}$$

As γ_I is a strong $\mathbb{S}\mathcal{A}$-envelope, the map $\mathcal{T}(\gamma_I, \mathbb{S}A)$ is an isomorphism. By the commutativity of diagram (3.1), so too is $\mathrm{D}\,\mathcal{T}(A, \gamma_I)$ an isomorphism and therefore γ_I is a strong $\mathcal{A}$-cover as required.

Notice that the object P is projective in $\mathcal{A}$ using (2) and the definition of P. Furthermore, the $\mathcal{A}$-cover $\nu P \xrightarrow{\alpha_P} \mathbb{S}P$ is a strong $\mathcal{A}$-cover by Proposition 3.2. We now construct the morphism $I \longrightarrow \nu P$.

By the uniqueness of (strong) $\mathcal{A}$-covers, we get a (unique) isomorphism $I \xrightarrow{\overline{\gamma_I}} \nu P$ such that $\gamma_I = \alpha_P \overline{\gamma_I}$. We now show that $\overline{\gamma_I}$ is natural. Let $I \xrightarrow{i} J$ be a morphism between indecomposable injective objects of $\mathcal{A}$. We denote the morphism $\nu^- I \xrightarrow{\nu^- i} \nu^- J$ in $\mathcal{A}$ by $P \xrightarrow{p} Q$. Then we get unique morphisms $\overline{\gamma}_I$ and $\overline{\gamma}_J$ such that $\gamma_I = \alpha_P \overline{\gamma_I}$ and $\gamma_J = \alpha_Q \overline{\gamma_J}$ hold. By definition of ν^-, we have $\beta_J \mathbb{S}^{-1} i = p \beta_I$ and therefore, after applying $\mathbb{S}$ to this equation and using the natural isomorphism $\mathbb{1} \longrightarrow \mathbb{S}\mathbb{S}^{-1}$, we get $\gamma_J i = \mathbb{S}(p)\gamma_I$. We obtain the diagram:

$$\begin{tikzcd}
I \arrow[rr, "\overline{\gamma_I}"] \arrow[dr, "\gamma_I"'] \arrow[dd, "i"'] & & \nu P \arrow[dl, "\alpha_P"] \arrow[dd, "\nu p"] \\
& \mathbb{S}P \arrow[dd, "\mathbb{S}p"] & \\
J \arrow[rr, "\overline{\gamma_J}"] \arrow[dr, "\gamma_J"'] & & \nu Q \arrow[dl, "\alpha_Q"] \\
& \mathbb{S}Q &
\end{tikzcd}$$

As the right-facing square commutes by the definition of νP, we have that all but the back-most square with vertices I, J, νQ, and νP commute. By the application of Lemma 2.2, we are done. Similarly, we construct the collection of natural isomorphisms $\nu^- J \longrightarrow Q$. This proves that ν and ν^- are mutual quasi-inverses. $\square$

Remark 3.5. Because we work in a Krull–Schmidt setting, every $\mathcal{A}$-precover can be made into an $\mathcal{A}$-cover by removing direct summands from the source object of the $\mathcal{A}$-precover (see [**26**, Lemma 4.1]). As the dual situation is also true, the assumptions in Theorem 3.4 may be relaxed.

Definition 3.6. Let $\mathcal{A}$ be a category with Setup 3.1 such that the following hold:

- For each projective object P in $\mathcal{A}$ there is an $\mathcal{A}$-cover in $\mathcal{T}$ of the form
$$\nu P \xrightarrow{\alpha_P} \mathbb{S}P.$$
- For each injective object I in $\mathcal{A}$ there is an $\mathcal{A}$-envelope in $\mathcal{T}$ of the form
$$\mathbb{S}^{-1} I \xrightarrow{\beta_I} \nu^- I.$$

Then we call the functors $\operatorname{\mathsf{Proj}} \mathcal{A} \xrightarrow{\nu} \operatorname{\mathsf{Inj}} \mathcal{A}$ and $\operatorname{\mathsf{Inj}} \mathcal{A} \xrightarrow{\nu^-} \operatorname{\mathsf{Proj}} \mathcal{A}$ obtained from Theorem 3.4 the *Nakayama functors on $\mathcal{A}$ (obtained via approximations)* and we say that *$\mathcal{A}$ has a Nakayama functor.*

Proposition 3.7. *Let $\mathcal{A}$ be a category with Setup 3.1. Let X and P be objects in $\mathcal{A}$ with P projective. Then the following three statements are equivalent:*

(1) *There is an $\mathcal{A}$-cover in $\mathcal{T}$ of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$.*
(2) *There is a strong $\mathcal{A}$-cover in $\mathcal{T}$ of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$.*
(3) *There is a natural isomorphism of the form $\mathrm{D}\,\mathcal{A}(P,-) \cong \mathcal{A}(-,X)$ of functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{mod}_k$, i.e. the object X represents the functor $\mathrm{D}\,\mathcal{A}(P,-)$.*

Dually, let Y and I be objects in $\mathcal{A}$ with I injective. Then the following three statements are equivalent:

(1') *There is an $\mathcal{A}$-envelope in $\mathcal{T}$ of the form $\mathbb{S}^{-1} I \longrightarrow Y$.*
(2') *There is a strong $\mathcal{A}$-envelope in $\mathcal{T}$ of the form $\mathbb{S}^{-1} I \longrightarrow Y$.*
(3') *There is a natural isomorphism of the form $\mathrm{D}\,\mathcal{A}(-,I) \cong \mathcal{A}(Y,-)$ of functors $\mathcal{A} \longrightarrow \mathsf{mod}_k$, i.e. the object Y represents the functor $\mathrm{D}\,\mathcal{A}(-,I)$.*

Proof: (1) $\Leftrightarrow$ (2): This follows from Proposition 3.2.

(2) $\Leftrightarrow$ (3): This follows from Lemma 2.5.

(2) is dual to (1). $\square$

Theorem 3.8. *Let $\mathcal{A}$ be an essentially small category that has a Nakayama functor (see Definition 3.6). If $\mathcal{A}$ has enough projectives and enough injectives, then $\mathcal{A}$ is a dualising k-variety.*

Proof: We first show that the induced functor $(\mathsf{Mod}_{\mathcal{A}})^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathsf{Mod}_{\mathcal{A}^{\mathrm{op}}}$ restricts to a functor $(\mathsf{mod}_{\mathcal{A}})^{\mathrm{op}} \longrightarrow \mathsf{mod}_{\mathcal{A}^{\mathrm{op}}}$. To this end, let $\mathcal{A}^{\mathrm{op}} \xrightarrow{F} \mathsf{Mod}_k$ be an object in $\mathsf{mod}_{\mathcal{A}}$. Firstly, assume $F = \mathcal{A}(-,I)$ for I an injective object in $\mathcal{A}$. Then we have $\mathbb{D}(F) = \mathbb{D}\mathcal{A}(-,I) \cong \mathcal{T}(\mathbb{S}^{-1}I,-)|_{\mathcal{A}} \cong \mathcal{A}(\nu^- I,-)$, where the first isomorphism holds since it holds pointwise by Serre duality and the second isomorphism holds as β_I is a strong $\mathcal{A}$-envelope (see Proposition 3.2). Hence, $\mathbb{D}(F)$ lies in $\mathsf{mod}_{\mathcal{A}^{\mathrm{op}}}$.

Secondly, assume $F = \mathcal{A}(-,A)$ for A an object in $\mathcal{A}$. Then choose an injective copresentation $A \longrightarrow I^0 \longrightarrow I^1$ of A. As the Yoneda embedding $\mathcal{A} \longrightarrow \mathsf{Mod}_{\mathcal{A}}$ is left exact, the sequence $\mathcal{A}(-,A) \longrightarrow \mathcal{A}(-,I^0) \longrightarrow \mathcal{A}(-,I^1)$ is left exact. Using the exactness of $\mathbb{D}$, the sequence $\mathbb{D}\mathcal{A}(-,I^1) \longrightarrow \mathbb{D}\mathcal{A}(-,I^0) \longrightarrow \mathbb{D}\mathcal{A}(-,A) = \mathbb{D}(F)$ is right exact. As $\mathsf{mod}_{\mathcal{A}^{\mathrm{op}}}$ is closed under cokernels (see [**2**, Proposition on p. 41]) $\mathbb{D}(F)$ lies in $\mathsf{mod}_{\mathcal{A}^{\mathrm{op}}}$.

Lastly, assume F is arbitrary in $\mathsf{mod}_{\mathcal{A}}$. Then there is a right exact sequence of the form $\mathcal{A}(-,Y) \longrightarrow \mathcal{A}(-,X) \longrightarrow F$. As $\mathbb{D}$ is exact, the sequence $\mathbb{D}(F) \longrightarrow \mathbb{D}\mathcal{A}(-,X) \longrightarrow \mathbb{D}\mathcal{A}(-,Y)$ is left exact. As $\mathsf{mod}_{\mathcal{A}^{\mathrm{op}}}$ is closed under kernels (see [**2**, Proposition on p. 41]) $\mathbb{D}(F)$ lies in $\mathsf{mod}_{\mathcal{A}^{\mathrm{op}}}$. Moreover, the fact that the functor $\mathsf{Mod}_{\mathcal{A}^{\mathrm{op}}} \xrightarrow{\mathbb{D}} (\mathsf{Mod}_{\mathcal{A}})^{\mathrm{op}}$ restricts to a functor $\mathsf{mod}_{\mathcal{A}^{\mathrm{op}}} \xrightarrow{\mathbb{D}} (\mathsf{mod}_{\mathcal{A}})^{\mathrm{op}}$ follows by a dual argument to the one given above. □

Remark 3.9. The inclusion of points (3) and (3') in Proposition 3.7 was inspired by ongoing work by Nan Gao, Julian Külshammer, Sondre Kvamme, and Chrysostomos Psaroudakis presented at the *Homological Algebra and Representation Theory* conference (Karlovasi, Samos, 2023). They used the existence of the representations in both points (3) and (3') to define the notion of an abelian category admitting a Nakayama functor. This inspiration was instrumental in facilitating the proof of Theorem 3.8.

3.2. Proper abelian length subcategories.

Theorem 3.10. *Let $\mathcal{A}$ be a category that has a Nakayama functor (see Definition 3.6) and assume each object of $\mathcal{A}$ is of finite length. Then $\mathcal{A}$ has enough projectives if and only if $\mathcal{A}$ has enough injectives.*

Proof: ($\Rightarrow$) Assume $\mathcal{A}$ has enough projectives and let A be a simple object in $\mathcal{A}$. By assumption, there is a non-zero epimorphism $P \xrightarrow{p} A$ in $\mathcal{A}$ with P a projective object in $\mathcal{A}$. We first show that the composition $\nu P \xrightarrow{\alpha_P} \mathbb{S}P \xrightarrow{\mathbb{S}p} \mathbb{S}A$ is non-zero. Assume the contrary and complete α_P to a short triangle $\nu P \xrightarrow{\alpha_P} \mathbb{S}P \xrightarrow{g} Z$. As $\mathbb{S}(p)\alpha_P = 0$ and as g is a weak cokernel of α_P, we have that $\mathbb{S}p$ factors through a morphism in the k-vector space $\mathcal{T}(Z,\mathbb{S}A)$. But $\mathcal{T}(Z,\mathbb{S}A) \cong \mathrm{D}\,\mathcal{T}(A,Z) = 0$, where the isomorphism holds by Serre duality and the latter k-vector space vanishes by the triangulated Wakamatsu's lemma [**26**, Lemma 2.1]. Hence, we get $\mathbb{S}p = 0$ and therefore $p = 0$, a contradiction.

We now have that the composition $\nu P \xrightarrow{\alpha_P} \mathbb{S}P \xrightarrow{\mathbb{S}p} \mathbb{S}A$ is non-zero. By Serre duality, the existence of a non-zero morphism $A \xrightarrow{a} \nu P$ is ensured. Therefore, each simple object in $\mathcal{A}$ admits a non-zero morphism to an injective object in $\mathcal{A}$ (see Proposition 3.3), and by Lemma 2.6, $\mathcal{A}$ has enough injectives.

($\Leftarrow$) Dual to the previous argument. □

4. Auslander–Reiten translates on proper abelian subcategories

4.1. Construction and fundamental properties.

Setup 4.1. Throughout this section, fix a category $\mathcal{A}$ that has a Nakayama functor (see Definition 3.6) and assume that each object in $\mathcal{A}$ has a projective cover and an injective envelope.

For each indecomposable non-projective object A in $\mathcal{A}$ and for each indecomposable non-injective object B in $\mathcal{A}$ we fix

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \quad \text{and} \quad B \xrightarrow{i^0} I^0 \xrightarrow{i_1} I^1$$

to be a minimal projective presentation of A and a minimal injective copresentation of B, respectively. For each indecomposable projective object P in $\mathcal{A}$ and for each indecomposable injective object I in $\mathcal{A}$, we fix

$$0 \longrightarrow P \xrightarrow{1} P \quad \text{and} \quad I \xrightarrow{1} I \longrightarrow 0$$

to be a minimal projective presentation of P and a minimal injective copresentation of I, respectively. Furthermore, we extend to general objects of $\mathcal{A}$ by taking direct sums.

Definition 4.2. Let C be an object in $\mathcal{A}$. We define the *Auslander–Reiten translate τC of C* as $\tau C = \operatorname{Ker}(\nu Q_1 \xrightarrow{\nu q_1} \nu Q_0)$, where $Q_1 \xrightarrow{q_1} Q_0 \xrightarrow{q_0} C$ is the fixed minimal projective presentation of C. Similarly, we define the *(inverse) Auslander–Reiten translate $\tau^- C$ of C* as $\tau^- C = \operatorname{Coker}(\nu^- J^0 \xrightarrow{\nu^- j^1} \nu^- J^1)$, where $C \xrightarrow{j^0} J^0 \xrightarrow{j^1} J^1$ is the minimal injective copresentation of C.

Proposition 4.3. *Let $\mathcal{A}$ be a category with Setup 4.1 and let A and A' be indecomposable objects in $\mathcal{A}$. Then the following hold:*

(1) $\tau(A \oplus A') \cong \tau A \oplus \tau A'$.
(2) *The object A is projective in $\mathcal{A} \Longleftrightarrow \tau A = 0$.*
(3) *If A is a non-projective object in $\mathcal{A}$, then τA is a non-injective object in $\mathcal{A}$.*
(4) *If A is a non-projective object in $\mathcal{A}$, then the sequence $\tau A \xrightarrow{k_A} \nu P_1 \xrightarrow{\nu p_1} \nu P_0$ is a minimal injective copresentation of τA.*
(5) *If A is a non-projective object in $\mathcal{A}$, then $\tau^- \tau A \cong A$.*
(6) *If A is a non-projective object in $\mathcal{A}$, then τA is indecomposable.*
(7) *If A and A' are non-projective objects in $\mathcal{A}$, then $A \cong A' \Longleftrightarrow \tau A \cong \tau A'$.*

Proof: (1) We have minimal projective presentations $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A$ and $P_1' \xrightarrow{p_1'} P_0' \xrightarrow{p_0'} A'$ of A and A' respectively and therefore $P_1 \oplus P_1' \xrightarrow{p_1 \oplus p_1'} P_0 \oplus P_0' \xrightarrow{p_0 \oplus p_0'} A \oplus A'$ is the fixed minimal projective presentation of $A \oplus A'$. By the application of ν, we arrive at the following solid commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tau(A \oplus A') \arrow[r] \arrow[d, dashed, "\cong"] & \nu(P_1 \oplus P_1') \arrow[r, "\nu(p_1 \oplus p_1')"] \arrow[d, "\cong"] & \nu(P_0 \oplus P_0') \arrow[d, "\cong"] \\
0 \arrow[r] & \tau A \oplus \tau A' \arrow[r] & \nu P_1 \oplus \nu P_1' \arrow[r, "\nu p_1 \oplus \nu p_1'"] & \nu P_0 \oplus \nu P_0'
\end{tikzcd}$$

where the vertical solid morphisms are the canonical isomorphisms induced by the additivity of ν (see Theorem 3.4). The dashed morphism exists by the universal property of the kernel of $\nu p_1 \oplus \nu p_1'$ and the Five lemma ensures it is an isomorphism.

(2) ($\Rightarrow$) Assume A is a projective object in $\mathcal{A}$. Then $0 \longrightarrow A \xrightarrow{1} A$ is the fixed minimal projective presentation of A. As $0 = \nu 0 \longrightarrow \nu A$ is a monomorphism, it follows that $\tau A = 0$.

($\Leftarrow$) Assume $\tau A = 0$. Then $\nu P_1 \xrightarrow{\nu p_1} \nu P_0$ is a monomorphism in $\mathcal{A}$ with νP_1 an injective object in $\mathcal{A}$. Therefore, νp_1 is a section. As ν is an equivalence, the morphism p_1 must also be a section. Hence, we have $P_0 \cong P_1 \oplus A$ and therefore A is projective.

(3) Assume that A is a non-projective object in $\mathcal{A}$. By Lemma 2.7, there is a nonzero $\mathcal{A}$-precover of the form $\tau A \xrightarrow{\theta} \Sigma^{-1} \mathbb{S} A$ (notice that τA is X_A in the statement of Lemma 2.7). Showing that τA is not an injective object in $\mathcal{A}$ amounts to finding

an object C in $\mathcal{A}$ with $\mathcal{T}(C, \Sigma\tau A) \neq 0$ (see dual of [**15**, Theorem on p. 1]). Choosing $C = A$ and using Serre duality, we have $\mathcal{T}(A, \Sigma\tau A) \cong \mathrm{D}\,\mathcal{T}(\tau A, \Sigma^{-1}\mathbb{S}A)$, where the last k-vector space is non-vanishing since the non-zero morphism θ lies in $\mathcal{T}(\tau A, \Sigma^{-1}\mathbb{S}A)$.

(4) It is clear the sequence is an injective copresentation of τA. As ν is an additive equivalence (see Theorem 3.4), Lemma 2.1 tells us that νp_1 is a radical morphism as, by [**33**, Proposition 3.10], p_1 is a radical morphism. The fact that k_A is an injective envelope follows from the dual of [**33**, Proposition 3.10]. As $\nu(p_1)k_A = 0$, the universal property of the cokernel of k_A ensures the existence of a unique morphism $\Omega^{-1}\tau A \xrightarrow{\iota_1} \nu P_0$ making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & \tau A \arrow[r, "k_A"] & \nu P_1 \arrow[r, "\nu p_1"] \arrow[d, "\pi_1"] & \nu P_0 \\
& & \Omega^{-1}\tau A \arrow[ur, dashed, "\iota_1"'] &
\end{tikzcd}$$

Here, $\Omega^{-1}\tau A$ is the cosyzygy of τA and π_1 is the cokernel of k_A. By diagram chasing, ι_1 is a monomorphism, so it suffices to show that ι_1 is an injective envelope.

To this end, we choose an injective envelope I of $\operatorname{Coker} \nu p_1$ extending the injective copresentation of τA:

$$\begin{tikzcd}
0 \arrow[r] & \tau A \arrow[r, "k_A"] & \nu P_1 \arrow[r, "\nu p_1"] \arrow[d, "\pi_1"] & \nu P_0 \arrow[r, "f"] & I \\
& & \Omega^{-1}\tau A \arrow[ur, "\iota_1"'] & &
\end{tikzcd}$$

By the dual of [**33**, Proposition 3.10], Lemma 2.1, and the fact that ν and ν^- are mutual quasi-inverses (see Theorem 3.4), it suffices to show that the morphism $g\colon P_0 \xrightarrow{\cong} \nu^-\nu P_0 \xrightarrow{\nu^- f} \nu^- I$ is a radical morphism. As $gp_1 = 0$ and p_0 is the cokernel of p_1, there exists a unique morphism $A \xrightarrow{a} \nu^- I$ such that the following diagram is commutative:

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] \arrow[dr, "g"'] & A \arrow[r] \arrow[d, "a"] & 0 \\
& & \nu^- I &
\end{tikzcd}$$

But p_0 must be a radical morphism as A is a non-projective indecomposable object. Hence, $g = ap_0$ is a radical morphism.

(5) Assume A is a non-projective object in $\mathcal{A}$. By (4), $\tau A \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0$ is a minimal injective copresentation of τA. Therefore, as minimal injective copresentations are unique up to isomorphism, we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tau A \arrow[r] & \nu P_1 \arrow[r, "\nu p_1"] \arrow[d, "\cong"] & \nu P_0 \arrow[d, "\cong"] \\
0 \arrow[r] & \tau A \arrow[r] & I^0 \arrow[r, "i^0"] & I^1
\end{tikzcd}$$

where the bottom row is our fixed injective copresentation of τA. Applying ν^-, we get the following solid commutative diagram:

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "\cong"] & P_0 \arrow[r, "p_0"] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, dashed, "\cong"] & 0 \\
\nu^-\nu P_1 \arrow[r, "\nu^-\nu p_1"] \arrow[d, "\cong"] & \nu^-\nu P_0 \arrow[r, "\nu^-\nu p_0"] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, dashed, "\cong"] & 0 \\
\nu^- I^0 \arrow[r, "\nu^- i^1"] & \nu^- I^1 \arrow[r] & \tau^-\tau A \arrow[r] & 0
\end{tikzcd}$$

where C is the cokernel of $\nu^-\nu p_1$. Then the dashed morphisms exist by the universal property of the cokernel and the Five lemma ensures that they are isomorphisms.

(6) Suppose A is a non-projective object in $\mathcal{A}$ and assume, for a contradiction, we have $\tau A \cong X \oplus Y$, for non-zero objects X and Y in $\mathcal{A}$. We first show that both X and Y are non-injective objects in $\mathcal{A}$.

For a contradiction, assume Y is a non-zero injective object in $\mathcal{A}$. As $s\colon Y \longrightarrow \tau A \xrightarrow{k_A} \nu P_1$ is a monomorphism, where $Y \longrightarrow \tau A$ is the canonical inclusion morphism, s must be a section. As $\nu(p_1)s = 0$, the object Y is isomorphic to a direct summand of νP_1 that lies in the kernel of νp_1. By applying ν^-, we get a section $\nu^- s$ with $\nu^-(\nu(p_1)s) = 0$. Similarly, the object $\nu^- Y$ is isomorphic to a direct summand of $\nu^-\nu P_1$ that lies in the kernel of $\nu^-\nu p_1$. As ν and ν^- are mutual quasi-inverses (see Theorem 3.4), there is a non-zero direct summand of P_1 contained in the kernel of p_1. Using that the projective cover $P_1 \longrightarrow \Omega A$ is unique up to isomorphism, we get a contradiction to the dual of [**34**, Corollary 1.4].

Now, we have $A \cong \tau^-\tau A \cong \tau^- X \oplus \tau^- Y$ by (5) and the dual of (1). By the dual of (2), both objects $\tau^- X$ and $\tau^- Y$ are non-zero, a contradiction as A is indecomposable.

(7) ($\Rightarrow$) Assuming $A \cong A'$, the result follows by the uniqueness of minimal projective presentations, the universal property of kernels and the Five lemma.

($\Leftarrow$) Conversely, assume $\tau A \cong \tau A'$. By (5), we have $A \cong \tau^-\tau A$ and $A' \cong \tau^-\tau A'$. Then by (6) and (3), τA and $\tau A'$ are indecomposable non-injective objects in $\mathcal{A}$. Therefore, the dual argument of the "($\Rightarrow$)" implication of (7) gives us $\tau^-\tau A \cong \tau^-\tau A'$. □

We state the dual version of the above result for completeness.

Proposition 4.4. *Let $\mathcal{A}$ be a category with Setup 4.1 and let B and B' be indecomposable objects in $\mathcal{A}$. Then the following hold:*

(1) $\tau^-(B \oplus B') \cong \tau^- B \oplus \tau^- B'$.
(2) *The object B is injective in $\mathcal{A} \Longleftrightarrow \tau^- B = 0$.*
(3) *If B is a non-injective object in $\mathcal{A}$, then $\tau^- B$ is a non-projective object in $\mathcal{A}$.*
(4) *If B is a non-injective object in $\mathcal{A}$, then the sequence $\nu^- I^0 \longrightarrow \nu^- I^1 \longrightarrow \tau^- B$ is a minimal projective presentation of $\tau^- B$.*
(5) *If B is a non-injective object in $\mathcal{A}$, then $\tau\tau^- B \cong B$.*
(6) *If B is a non-injective object in $\mathcal{A}$, then $\tau^- B$ is indecomposable.*
(7) *If B and B' are non-injective objects in $\mathcal{A}$, then $B \cong B' \Longleftrightarrow \tau^- B \cong \tau^- B'$.*

Corollary 4.5. *Let $\mathcal{A}$ be a category with Setup 4.1. There is a bijective correspondence between the following:*

(1) *Isomorphism classes of indecomposable non-projective objects A in $\mathcal{A}$.*
(2) *Isomorphism classes of indecomposable non-injective objects B in $\mathcal{A}$.*

The bijective correspondence is given by the assignments $A \longmapsto \tau A$ and $B \longmapsto \tau^{-}B$.

4.2. The existence of Auslander–Reiten sequences in proper abelian subcategories.

Remark 4.6. In [**26**, Theorem 3.1], the triangulated categories are assumed to be essentially small. This assumption can be dropped due to [**25**, Proposition 5.15].

Theorem 4.7. *Let $\mathcal{A}$ be a category with Setup 4.1. Then the following hold:*

(1) *For each indecomposable non-projective object C in $\mathcal{A}$, there exists an Auslander–Reiten sequence in $\mathcal{A}$ of the form*
$$0 \longrightarrow \tau C \longrightarrow E_C \longrightarrow C \longrightarrow 0.$$
(2) *For each indecomposable non-injective object A in $\mathcal{A}$, there exists an Auslander–Reiten sequence in $\mathcal{A}$ of the form*
$$0 \longrightarrow A \longrightarrow F_A \longrightarrow \tau^{-}A \longrightarrow 0.$$

Proof: (1) By Lemma 2.7, there exists a non-zero $\mathcal{A}$-precover of the form $\tau C \longrightarrow \Sigma^{-1}\mathbb{S}C$. But τC is an indecomposable object in $\mathcal{A}$ by Proposition 4.3 and hence by the dual of [**32**, Lemma 2.4] $\tau C \longrightarrow \Sigma^{-1}\mathbb{S}C$ is an $\mathcal{A}$-cover. The existence of the required Auslander–Reiten sequence is ensured by [**26**, Theorem 3.1] (see also Remark 4.6).

(2) is dual to (1). □

5. Application to the module category of a finite-dimensional algebra

In this section, we recover the standard Nakayama functors associated with the category of finite-dimensional modules over a finite-dimensional algebra and give a new proof of the existence of Auslander–Reiten sequences in such a category. The following setup applies throughout this section.

Setup 5.1. Let A be a finite-dimensional k-algebra and let mod_A denote the category of finite-dimensional right modules over A. We also let proj_A denote the full subcategory of finitely generated projective A-modules and let inj_A denote the full subcategory of finitely generated injective A-modules.

Remark 5.2. In [**36**, Theorem I.2.4], the ground field is assumed to be algebraically closed. This assumption can be dropped due to [**25**, Theorem 3.6].

By [**36**, Proposition I.1.4] and [**36**, Proposition I.2.3] (see also Remark 5.2), we can associate to the Serre functor $\mathbb{S}$ and the collection of natural isomorphisms
$$\{\mathcal{T}(X,Y) \xrightarrow{\eta_{X,Y}} \mathrm{D}\,\mathcal{T}(Y,\mathbb{S}X) \mid \text{for } X \text{ and } Y \text{objects in } \mathcal{T}\}$$
a collection of linear forms
$$\{\mathcal{T}(Z,\mathbb{S}Z) \xrightarrow{\eta_Z} k \;\big|\; \text{ for } Z \text{ an object in } \mathcal{T} \text{ and } \eta_Z \coloneqq \eta_{Z,Z}(1_Z)\}$$
satisfying $\eta_Z(h_Z)\neq 0$ for each connecting homomorphism h_Z in an Auslander–Reiten triangle of the form $\Sigma^{-1}\mathbb{S}Z \longrightarrow Y \longrightarrow Z \xrightarrow{h_Z} \mathbb{S}Z$. Then for each morphism $X \xrightarrow{f} Y$ we have $\eta_{X,Y}(f)(-) = \eta_X(-\circ f)$ as linear forms on $\mathcal{T}(Y,\mathbb{S}X)$. It will also be useful to note

that, for X and Y objects in $\mathcal{T}$, the morphism $\mathcal{T}(X, \mathbb{S}Y) \xrightarrow{\text{ev}} \mathrm{D}^2\,\mathcal{T}(X, \mathbb{S}Y) \xrightarrow{\mathrm{D}(\eta_{Y,X})} \mathrm{D}\,\mathcal{T}(Y, X)$ is given by $g \longmapsto \eta_Y(g \circ -)$, where ev is the evaluation map isomorphism.

Remark 5.3. Similarly to Serre duality, the standard Nakayama functor $\mathrm{N}(-) = - \otimes_A \mathrm{D}(A)$ on mod_A exhibits a duality as follows. For each finitely generated A-module M and for each finitely generated projective A-module P, we have the following composition of natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}_A(P, M) \longrightarrow \mathrm{D}(P \otimes_A \mathrm{D}(M)) &\longrightarrow \mathrm{D}(P \otimes_A \mathrm{Hom}_A(M, \mathrm{D}(A))) \\ &\longrightarrow \mathrm{D}(\mathrm{Hom}_A(M, \mathrm{N}(P)).\end{aligned}$$

Here, the first isomorphism uses the evaluation isomorphism $M \cong \mathrm{D}^2(M)$ and the tensor-Hom adjunction, the second isomorphism uses the canonical isomorphism $M \cong M \otimes_A A$ and the tensor-Hom adjunction and the third isomorphism follows by the tensor evaluation isomorphism (a reference for these isomorphisms can be found in [**12**, Section I.1]). We will denote this composition by

$$\mathrm{Hom}_A(P, M) \xrightarrow{\varepsilon_{P,M}} \mathrm{D}\,\mathrm{Hom}_A(M, \mathrm{N}(P)).$$

The isomorphism $\varepsilon_{P.M}$ is natural in both P and M. We will refer to this as *Nakayama duality.*

5.1. Recovering the standard Nakayama functors on the module category.

Theorem 5.4. *Let mod_A be a k-linear subcategory of $\mathcal{T}$. Let $\mathrm{N}(-) = - \otimes_A \mathrm{D}(A)$ and let $\mathrm{N}^-(-) = \mathrm{Hom}_A(\mathrm{D}(A), -)$ be the standard Nakayama functors on mod_A. Then the following hold:*

(1) *For each indecomposable finitely generated projective A-module P, there is a strong mod_A-cover of the form $\mathrm{N}(P) \xrightarrow{\alpha_P} \mathbb{S}P$ such that the diagram*

$$\begin{tikzcd} \mathrm{N}(P) \arrow[r, "\alpha_P"] \arrow[d, "\mathrm{N}(p)"'] & \mathbb{S}P \arrow[d, "\mathbb{S}p"] \\ \mathrm{N}(Q) \arrow[r, "\alpha_Q"] & \mathbb{S}Q \end{tikzcd}$$

is commutative for each A-module homomorphism $P \xrightarrow{p} Q$ between indecomposable finitely generated projective A-modules. Moreover, the system of strong mod_A-covers $\{\mathrm{N}(P) \xrightarrow{\alpha_P} \mathbb{S}P\}$ given above gives rise to a functor $\mathsf{proj}_A \xrightarrow{\nu} \mathsf{inj}_A$ by Theorem 3.4. The restricted Nakayama functor $\mathsf{proj} \xrightarrow{\mathrm{N}} \mathsf{inj}_A$ is naturally isomorphic to ν.

(2) *For each indecomposable finitely generated injective A-module I, there is a strong mod_A-envelope of the form $\mathbb{S}^{-1}I \longrightarrow \mathrm{N}^-(I)$ such that the diagram*

$$\begin{tikzcd} \mathbb{S}^{-1}I \arrow[r] \arrow[d, "\mathbb{S}^{-1}i"'] & \mathrm{N}^-(I) \arrow[d, "\mathrm{N}^-(i)"] \\ \mathbb{S}^{-1}J \arrow[r] & \mathrm{N}^-(J) \end{tikzcd}$$

is commutative for each A-module homomorphism $I \xrightarrow{i} J$ between indecomposable finitely generated injective A-modules. Moreover, the system of strong mod_A-envelopes $\{\mathbb{S}^{-1}I \longrightarrow \mathrm{N}^{-1}(I)\}$ given above gives rise to a functor $\mathsf{inj}_A \xrightarrow{\nu^-} \mathsf{proj}_A$ by Theorem 3.4. The restricted functor $\mathsf{inj}_A \xrightarrow{\mathrm{N}^-} \mathsf{proj}_A$ is naturally isomorphic to ν^-.

Proof: (1) *Construction of α_P.* Using Remark 5.3, we have a natural isomorphism:

$$\gamma_P^{(-)}\colon \operatorname{Hom}_A(-,\mathrm{N}(P)) \xrightarrow{\mathrm{D}(\varepsilon_{P,-})\circ \mathrm{ev}} \mathrm{D}\operatorname{Hom}_A(P,-) \xrightarrow{(\mathrm{D}(\eta_{P,-})\circ \mathrm{ev})^{-1}} \mathcal{T}(-,\mathbb{S}P).$$

Let $\alpha_P \coloneqq \gamma_P^{\mathrm{N}(P)}(1_{\mathrm{N}(P)})$, where $\gamma_P^{\mathrm{N}(P)}$ is the component of the natural isomorphism γ_P at $\mathrm{N}(P)$.

α_P is a strong mod_A-cover. By definition, the morphism α_P is a strong mod_A-cover if and only if for each finitely generated A-module M, the k-linear map

$$\operatorname{Hom}_A(M,\mathrm{N}(P)) \xrightarrow{\mathcal{T}(M,\alpha_P)} \mathcal{T}(M,\mathbb{S}P)$$

is an isomorphism. As $\gamma_P^{(-)}$ is a natural isomorphism, it therefore suffices to show, for each finitely generated A-module M, that the invertible component γ_P^M coincides with $\operatorname{Hom}_A(M,\mathrm{N}(P)) \xrightarrow{\mathcal{T}(M,\alpha_P)} \mathcal{T}(M,\mathbb{S}P)$. To this end, let $M \xrightarrow{m} \mathrm{N}(P)$ be an A-module homomorphism. By naturality, the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_A(\mathrm{N}(P),\mathrm{N}(P)) \arrow[r, "\gamma_P^{\mathrm{N}(P)}"] \arrow[d, "\operatorname{Hom}_A(m,\mathrm{N}(P))"'] & \mathcal{T}(\mathrm{N}(P),\mathbb{S}P) \arrow[d, "\mathcal{T}(m,\mathbb{S}P)"] \\
\operatorname{Hom}_A(M,\mathrm{N}(P)) \arrow[r, "\gamma_P^M"] & \mathcal{T}(M,\mathbb{S}P)
\end{tikzcd}$$

The equality $\gamma_P^M(m) = \mathcal{T}(M,\alpha_P)(m)$ is obtained by chasing the identity $1_{\mathrm{N}(P)}$ through this diagram.

The commutativity of the induced diagram. Let $P \xrightarrow{p} Q$ be a morphism between indecomposable finitely generated projective A-modules. We want to show that the diagram

$$\begin{tikzcd}
\mathrm{N}(P) \arrow[r, "\alpha_P"] \arrow[d, "\mathrm{N}(p)"'] & \mathbb{S}P \arrow[d, "\mathbb{S}p"] \\
\mathrm{N}(Q) \arrow[r, "\alpha_Q"] & \mathbb{S}Q
\end{tikzcd} \tag{5.1}$$

is commutative. Using the naturality of γ in both components, the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_A(\mathrm{N}(P),\mathrm{N}(P)) \arrow[r, "\cong"] \arrow[d, "\operatorname{Hom}_A(\mathrm{N}(P),\mathrm{N}(p))"'] & \mathrm{D}\operatorname{Hom}_A(P,\mathrm{N}(P)) \arrow[r, "\cong"] \arrow[d] & \mathcal{T}(\mathrm{N}(P),\mathbb{S}P) \arrow[d, "\mathcal{T}(\mathrm{N}(P),\mathbb{S}p)"] \\
\operatorname{Hom}_A(\mathrm{N}(P),\mathrm{N}(Q)) \arrow[r, "\cong"] & \mathrm{D}\operatorname{Hom}_A(Q,\mathrm{N}(P)) \arrow[r, "\cong"] & \mathcal{T}(\mathrm{N}(P),\mathbb{S}Q) \\
\operatorname{Hom}_A(\mathrm{N}(Q),\mathrm{N}(Q)) \arrow[r, "\cong"] \arrow[u, "\operatorname{Hom}_A(\mathrm{N}(p),\mathrm{N}(Q))"] & \mathrm{D}\operatorname{Hom}_A(Q,\mathrm{N}(Q)) \arrow[r, "\cong"] \arrow[u] & \mathcal{T}(\mathrm{N}(Q),\mathbb{S}Q) \arrow[u, "\mathcal{T}(\mathrm{N}(p),\mathbb{S}Q)"']
\end{tikzcd} \tag{5.2}$$

Commutativity of (5.1) is equivalent to the validity of the equation

$$\mathcal{T}(\mathrm{N}(P),\mathbb{S}p)(\alpha_P) = \mathcal{T}(\mathrm{N}(p),\mathbb{S}Q)(\alpha_Q)$$

and this, by commutativity of (5.2), is equivalent to the validity of the equation

$$\operatorname{Hom}_A(\mathrm{N}(P),\mathrm{N}(p))(1_{\mathrm{N}(P)}) = \operatorname{Hom}_A(\mathrm{N}(p),\mathrm{N}(Q))(1_{\mathrm{N}(Q)}),$$

which indeed holds.

The construction of ν is naturally isomorphic to N. As any finitely generated projective A-module P is a finite direct sum of indecomposable finitely generated projective A-modules and as the finite direct sum of strong covers is again a strong cover, we have strong mod_A-covers of the form $\mathrm{N}(P) \longrightarrow \mathbb{S}P$, where P is a finitely generated projective A-module. Theorem 3.4 therefore gives rise to a functor $\mathsf{proj}_A \xrightarrow{\nu} \mathsf{inj}_A$. The fact that the functors ν and the restricted Nakayama functor $\mathsf{proj}_A \xrightarrow{\mathrm{N}} \mathsf{inj}_A$ are naturally isomorphic follows from Lemma 2.4.

(2) This follows by a similar dual argument to (1). □

5.2. Recovering the existence of Auslander–Reiten sequences in module categories.

Proposition 5.5. *Let mod_A be an extension-closed proper abelian k-linear subcategory of $\mathcal{T}$ and assume $\mathcal{T}(\mathsf{mod}_A, \Sigma^{-1}\mathsf{mod}_A) = 0$. Let $\mathrm{N}(-) = - \otimes_A \mathrm{D}(A)$ and $\mathrm{N}^-(-) = \mathrm{Hom}_A(D(A), -)$ be the standard Nakayama functors on mod_A and let t and t^- denote the standard Auslander–Reiten translates in mod_A. Then the following hold:*

1. *If $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M$ is a projective presentation of a non-projective indecomposable finitely generated A-module M, then there is a mod_A-cover of the form $\mathrm{t}(M) \longrightarrow \Sigma^{-1}\mathbb{S}M$ in $\mathcal{T}$, where $\mathrm{t}(M) \longrightarrow \mathrm{N}(P_1)$ is the kernel of the induced A-module homomorphism $\mathrm{N}(P_1) \longrightarrow \mathrm{N}(P_0)$ in mod_A.*
2. *If $L \longrightarrow I^0 \longrightarrow I^1$ is an injective copresentation of a non-injective indecomposable finitely generated A-module L, then there is a mod_A-envelope of the form $\Sigma\mathbb{S}^{-1}L \longrightarrow \mathrm{t}^-(L)$ in $\mathcal{T}$, where $\mathrm{N}^-(I^1) \longrightarrow \mathrm{t}^-(L)$ is the cokernel of the induced A-module homomorphism $\mathrm{N}(I^0) \longrightarrow \mathrm{N}(I^1)$ in mod_A.*

Proof: (1) By Theorem 5.4 and Lemma 2.7, there is a non-zero mod_A-precover of the form $\tau M \xrightarrow{\theta} \Sigma^{-1}\mathbb{S}M$, where $\tau M \longrightarrow \nu P_1$ is the kernel of the induced morphism $\nu P_1 \xrightarrow{\nu p_1} \nu P_0$ in mod_A. We have left exact sequences $\tau M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0$ and $\mathrm{t}(M) \longrightarrow \mathrm{N}(P_1) \xrightarrow{\mathrm{N}(p_1)} \mathrm{N}(P_0)$. By the equivalence $\nu \cong \mathrm{N}$ in Theorem 5.4 and by the universal property of the kernel of $\mathrm{N}(p_1)$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r] \arrow[d, dashed, "f"] & \nu P_1 \arrow[r, "\nu p_1"] \arrow[d, dashed, "\cong"] & \nu P_1 \arrow[d, dashed, "\cong"] \\
0 \arrow[r] & \mathrm{t}(M) \arrow[r] & \mathrm{N}(P_1) \arrow[r, "\mathrm{N}(p_1)"] & \mathrm{N}(P_0)
\end{tikzcd}$$

where the right two vertical A-module homomorphisms are isomorphisms. The Five lemma ensures that the leftmost A-module homomorphism is also an isomorphism. Hence, $\mathrm{t}(M) \xrightarrow{f^{-1}} \tau M \xrightarrow{\theta} \Sigma^{-1}\mathbb{S}M$ is a mod_A-precover. Finally, the minimality of the non-zero mod_A-precover θf^{-1} follows as $\mathrm{t}(M)$ is indecomposable (see the dual of [**32**, Lemma 2.4]).

(2) is dual to (1). □

Proposition 5.6. *Let mod_A be an extension-closed proper abelian k-linear subcategory of $\mathcal{T}$ and assume $\mathcal{T}(\mathsf{mod}_A, \Sigma^{-1}\mathsf{mod}_A) = 0$. Let t and t^- denote the standard Auslander–Reiten translates in mod_A. Then the following hold:*

1. *For each indecomposable finitely generated non-projective A-module M, there is an Auslander–Reiten sequence in mod_A of the form*

$$0 \longrightarrow \mathrm{t}(M) \longrightarrow E_M \longrightarrow M \longrightarrow 0.$$

(2) *For each indecomposable finitely generated non-injective A-module L, there is an Auslander–Reiten sequence in mod_A of the form*

$$0 \longrightarrow L \longrightarrow F_L \longrightarrow \mathrm{t}^{-}(L) \longrightarrow 0.$$

Proof: (1) By Proposition 5.5, there exists a mod_A-cover of the form $\mathrm{t}(M) \longrightarrow \Sigma^{-1}\mathbb{S}M$. The existence of the required Auslander–Reiten sequence is ensured by [**26**, Theorem 3.1] (see Remark 4.6).

(2) is dual to (1). □

The following were defined in [**24**, pp. 351–352] and [**22**, Chapter II.2.1] (see also [**21**]): let $\widehat{A}$ denote the repetitive algebra of A, let $\mathsf{mod}_{\widehat{A}}$ denote the module category of finitely generated modules over $\widehat{A}$ and let $\underline{\mathsf{mod}}_{\widehat{A}}$ denote the stable module category of $\widehat{A}$. We now recover the existence of Auslander–Reiten sequences in mod_A.

Theorem 5.7. *Let M be an indecomposable finitely generated non-projective A-module and let L be an indecomposable finitely generated non-injective A-module. Let* t *and* t^{-} *denote the standard Auslander–Reiten translates in mod_A and let $\mathrm{N}^{-}(-) = \mathrm{Hom}_A(D(A), -)$ be the standard Nakayama functor on mod_A. Then the following hold:*

(1) *There is an Auslander–Reiten sequence in mod_A of the form*

$$0 \longrightarrow \mathrm{t}(M) \longrightarrow E_M \longrightarrow M \longrightarrow 0.$$

(2) *There is an Auslander–Reiten sequence in mod_A of the form*

$$0 \longrightarrow L \longrightarrow F_L \longrightarrow \mathrm{t}^{-}(L) \longrightarrow 0.$$

Proof: (1) and (2): By [**22**, Theorem on p. 16 and Lemma on p. 62], the category $\underline{\mathsf{mod}}_{\widehat{A}}$ is a triangulated category. We denote the suspension functor of $\underline{\mathsf{mod}}_{\widehat{A}}$ by Σ. By [**22**, Proposition on p. 67], mod_A is the heart of a t-structure on $\underline{\mathsf{mod}}_{\widehat{A}}$ and hence by [**8**, Theorem 1.3.6 on p. 31], mod_A is an extension-closed k-linear proper abelian subcategory of $\underline{\mathsf{mod}}_{\widehat{A}}$. By [**22**, Lemma on p. 63], we have isomorphisms $\underline{\mathrm{Hom}}_{\widehat{A}}(M, \Sigma^i L) \cong \mathrm{Ext}^i_A(M, L)$ for all i in $\mathbb{Z}$ and therefore $\underline{\mathrm{Hom}}_{\widehat{A}}(\mathsf{mod}_A, \Sigma^{-1}\,\mathsf{mod}_A) = 0$. It was known in [**24**, Lemma on p. 354], that $\mathsf{mod}_{\widehat{A}}$ has Auslander–Reiten sequences. Later, in [**22**], it was stated that $\underline{\mathsf{mod}}_{\widehat{A}}$ has the so-called Auslander–Reiten triangles (one can use [**25**, Proposition 5.11], for example). Then by [**36**, Theorem I.2.4] (see also Remark 5.2), the triangulated category $\underline{\mathsf{mod}}_{\widehat{A}}$ has a Serre functor. As $\mathsf{mod}_{\widehat{A}}$ is Hom-finite, so is $\underline{\mathsf{mod}}_{\widehat{A}}$. To show that $\underline{\mathsf{mod}}_{\widehat{A}}$ is Krull–Schmidt, it suffices to show that each idempotent in $\underline{\mathsf{mod}}_{\widehat{A}}$ splits (see [**37**, p. 52], [**22**, p. 26], [**11**, Corollary A.2], [**33**, Section 4], and [**38**, Theorem 6.1]).

Let us first show that each idempotent in $\mathsf{mod}_{\widehat{A}}$ splits. Following [**22**, p. 60], we let $M = (M_i, m_i)_{i\in\mathbb{Z}}$ be a finitely generated $\widehat{A}$-module. That is, M_i is a finitely generated A-module with finitely many M_i being non-zero and $M_i \xrightarrow{m_i} \mathrm{N}^{-}(M_{i+1})$ is an A-module homomorphism such that $\mathrm{N}^{-}(m_{i+1})m_i = 0$. Let $M \xrightarrow{e=(e_i)_{i\in\mathbb{Z}}} M$ be an $\widehat{A}$-module endomorphism. That is, $M_i \xrightarrow{e_i} M_i$ is an A-module homomorphism such that

$$\begin{tikzcd} M_i \arrow[r, "m_i"] \arrow[d, "e_i"'] & \mathrm{N}^{-}(M_{i+1}) \arrow[d, "\mathrm{N}^{-}(e_{i+1})"] \\ M_i \arrow[r, "m_i"] & \mathrm{N}^{-}(M_{i+1}) \end{tikzcd}$$

commutes for each i in $\mathbb{Z}$. Assume that e is an idempotent in $\mathrm{End}_{\widehat{A}}(M)$. Then for each i in $\mathbb{Z}$, the A-module homomorphism $M_i \xrightarrow{e_i} M_i$ is an idempotent in $\mathrm{End}_A(M_i)$.

Hence, for each i in $\mathbb{Z}$, there exists a factorisation $e_i\colon M_i \xrightarrow{r_i} X_i \xrightarrow{s_i} M_i$ of e_i such that r_i and s_i are A-module homomorphisms satisfying the equation $1_{X_i} = r_i s_i$. We define an A-module homomorphism $X_i \xrightarrow{x_i} \mathrm{N}^-(X_{i+1})$ as the following composition:

$$x_i\colon X_i \xrightarrow{\quad s_i \quad} M_i \xrightarrow{\quad m_i \quad} \mathrm{N}^-(M_{i+1}) \xrightarrow{\mathrm{N}^-(r_{i+1})} \mathrm{N}^-(X_{i+1}).$$

It is not hard to check that the pair $(X_i, x_i)_{i\in\mathbb{Z}}$ defines a finitely generated $\widehat{A}$-module and that the sequences $r = (r_i)_{i\in\mathbb{Z}}$ and $s = (s_i)_{i\in\mathbb{Z}}$ define $\widehat{A}$-module homomorphisms such that $e = sr$ and $1_M = rs$. Hence, e splits. By [**14**, Proposition 5.9], the stable module category $\underline{\mathsf{mod}}_{\widehat{A}}$ has split idempotents and therefore is Krull–Schmidt. Both parts of the theorem now follow from Proposition 5.6. □

Acknowledgements. I am grateful to Peter Jørgensen for his invaluable supervision. I deeply appreciate our insightful discussions, which have significantly enhanced my understanding of algebra. I am also immensely thankful to everyone at the (extended) Aarhus Homological Algebra Group. Special thanks to David Pauksztello for his useful feedback on the first draft of this article and to the kind referees for their comments and suggestions. Everyone mentioned above has helped improve the readability of this article and deepened my understanding of the topic. This work was supported by the Independent Research Fund Denmark (grant no. 1026-00050B).

References

[1] M. Auslander, Coherent functors, in: *Proceedings of the Conference on Categorical Algebra* (La Jolla, Calif., 1965), Springer-Verlag New York, Inc., New York, 1966, pp. 189–231. DOI: 10.1007/978-3-642-99902-4_8.

[2] M. Auslander, *Representation Dimension of Artin Algebras*, With the assistance of Bernice Auslander, Queen Mary College Mathematics Notes, Queen Mary College, University of London, 1971.

[3] M. Auslander, Representation theory of Artin algebras I, *Comm. Algebra* **1(3)** (1974), 177–268. DOI: 10.1080/00927877408548230.

[4] M. Auslander and I. Reiten, Stable equivalence of dualizing R-varieties, *Advances in Math.* **12(3)** (1974), 306–366. DOI: 10.1016/S0001-8708(74)80007-1.

[5] M. Auslander and I. Reiten, Representation theory of Artin algebras III. Almost split sequences, *Comm. Algebra* **3(3)** (1975), 239–294. DOI: 10.1080/00927877508822046.

[6] M. Auslander and S. O. Smalø, Preprojective modules over Artin algebras, *J. Algebra* **66(1)** (1980), 61–122. DOI: 10.1016/0021-8693(80)90113-1.

[7] M. Auslander and S. O. Smalø, Almost split sequences in subcategories, *J. Algebra* **69(2)** (1981), 426–454. DOI: 10.1016/0021-8693(81)90214-3.

[8] A. A. Beĭlinson, J. Bernstein, and. P. Deligne, Faisceaux pervers, in: *Analyse et topologie sur les espaces singuliers (I)* (Luminy, 1981), Astérisque **100**, Société Mathématique de France, Paris, 1982, pp. 5–171.

[9] R. Bocklandt, Graded Calabi Yau algebras of dimension 3, *J. Pure Appl. Algebra* **212(1)** (2008), 14–32. DOI: 10.1016/j.jpaa.2007.03.009.

[10] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and mutations (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **53(6)** (1989), 1183–1205, 1337; translation in: *Math. USSR-Izv.* **35(3)** (1990), 519–541. DOI: 10.1070/IM1990v035n03ABEH000716.

[11] X.-W. Chen, Y. Ye, and P. Zhang, Algebras of derived dimension zero, *Comm. Algebra* **36(1)** (2008), 1–10. DOI: 10.1080/00927870701649184.

[12] L. W. Christensen, H.-B. Foxby, and H. Holm, *Derived Category Methods in Commutative Algebra*, Springer Monogr. Math., Springer, Cham, 2024. DOI: 10.1007/978-3-031-77453-9.

[13] R. Coelho Simões, D. Pauksztello, and D. Ploog, Functorially finite hearts, simple-minded systems in negative cluster categories, and noncrossing partitions, *Compos. Math.* **158(1)** (2022), 211–243. DOI: 10.1112/s0010437x21007648.

[14] M. Cortés-Izurdiaga, S. Crivei, and M. Saorín, Reflective and coreflective subcategories, *J. Pure Appl. Algebra* **227(5)** (2023), Paper no. 107267, 43 pp. DOI: 10.1016/j.jpaa.2022.107267.

[15] M. J. Dyer, Exact subcategories of triangulated categories, Unpublished manuscript (2005). https://www3.nd.edu/ dyer/papers/extri.pdf.

[16] E. E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39(3)** (1981), 189–209. DOI: 10.1007/BF02760849.
[17] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Volume 1, Second revised and extended edition, De Gruyter Exp. Math. **30**, Walter de Gruyter GmbH & Co. KG, Berlin, 2011. DOI: 10.1515/9783110215212.
[18] F. Fedele, Auslander–Reiten $(d+2)$-angles in subcategories and a $(d+2)$-angulated generalisation of a theorem by Brüning, *J. Pure Appl. Algebra* **223(8)** (2019), 3554–3580. DOI: 10.1016/j.jpaa.2018.11.017.
[19] F. Fedele, d-Auslander–Reiten sequences in subcategories, *Proc. Edinb. Math. Soc. (2)* **63(2)** (2020), 342–373. DOI: 10.1017/s0013091519000312.
[20] F. Fedele, Almost split morphisms in subcategories of triangulated categories, *J. Algebra Appl.* **21(12)** (2022), Paper no. 2250239, 27 pp. DOI: 10.1142/S0219498822502395.
[21] D. Happel, On the derived category of a finite-dimensional algebra, *Comment. Math. Helv.* **62(3)** (1987), 339–389. DOI: 10.1007/BF02564452.
[22] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc. Lecture Note Ser. **119**, Cambridge University Press, Cambridge, 1988. DOI: 10.1017/CBO9780511629228.
[23] D. Happel, Auslander–Reiten triangles in derived categories of finite-dimensional algebras, *Proc. Amer. Math. Soc.* **112(3)** (1991), 641–648. DOI: 10.2307/2048684.
[24] D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras, *Proc. London Math. Soc. (3)* **46(2)** (1983), 347–364. DOI: 10.1112/plms/s3-46.2.347.
[25] O. Iyama, H. Nakaoka, and Y. Palu, Auslander–Reiten theory in extriangulated categories, *Trans. Amer. Math. Soc. Ser. B* **11** (2024), 248–305. DOI: 10.1090/btran/159.
[26] P. Jørgensen, Auslander–Reiten triangles in subcategories, *J. K-Theory* **3(3)** (2009), 583–601. DOI: 10.1017/is008007021jkt056.
[27] P. Jørgensen, Proper abelian subcategories of triangulated categories and their tilting theory, Preprint (2021). arXiv:2109.01395.
[28] P. Jørgensen, Abelian subcategories of triangulated categories induced by simple minded systems, *Math. Z.* **301(1)** (2022), 565–592. DOI: 10.1007/s00209-021-02913-5.
[29] Y. Kawamata, Derived categories and birational geometry, in: *Algebraic Geometry—Seattle 2005*, Proc. Sympos. Pure Math. **80**, Part 2, American Mathematical Society, Providence, RI, 2009, pp. 655–665. DOI: 10.1090/pspum/080.2/2483950.
[30] G. M. Kelly, On the radical of a category, *J. Austral. Math. Soc.* **4(3)** (1964), 299–307. DOI: 10.1017/S1446788700024071.
[31] M. Kleiner, Approximations and almost split sequences in homologically finite subcategories, *J. Algebra* **198(1)** (1997), 135–163. DOI: 10.1006/jabr.1997.7159.
[32] H. Krause, Auslander–Reiten theory via Brown representability, Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part IV, *K-Theory* **20(4)** (2000), 331–344. DOI: 10.1023/A:1026571214620.
[33] H. Krause, Krull–Schmidt categories and projective covers, *Expo. Math.* **33(4)** (2015), 535–549. DOI: 10.1016/j.exmath.2015.10.001.
[34] H. Krause and M. Saorín, On minimal approximations of modules, in: *Trends in the Representation Theory of Finite Dimensional Algebras* (Seattle, WA, 1997), Contemp. Math. **229**, American Mathematical Society, Providence, RI, 1998, pp. 227–236. DOI: 10.1090/conm/229/03321.
[35] M. Linckelmann, On abelian subcategories of triangulated categories, *Math. Z.* **306(1)** (2024), Paper no. 7, 43 pp. DOI: 10.1007/s00209-023-03396-2.
[36] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, *J. Amer. Math. Soc.* **15(2)** (2002), 295–366. DOI: 10.1090/S0894-0347-02-00387-9.
[37] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. **1099**, Springer-Verlag, Berlin, 1984. DOI: 10.1007/BFb0072870.
[38] A. Shah, Krull–Remak–Schmidt decompositions in Hom-finite additive categories, *Expo. Math.* **41(1)** (2023), 220–237. DOI: 10.1016/j.exmath.2022.12.003.

Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark
E-mail address: david.nkansah@math.au.dk
ORCID: 0000-0002-2342-8654

Received on January 11, 2024.
Accepted on September 6, 2024.