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GROWTH OF POWER SERIES WITH NONNEGATIVE COEFFICIENTS, AND MOMENTS OF POWER SERIES DISTRIBUTIONS

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Dedicated to the memory of Luis Báez-Duarte

Abstract: Any power series with nonnegative coefficients has an associated family of probability distributions supported on the nonnegative integers. There is a close connection between the function theoretic properties of the power series and the moments of the family of distributions. In this paper, we describe that interplay, provide simpler proofs of some known results by emphasizing the probabilistic perspective, and present some new theorems.

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1. Introduction

Every power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with nonnegative coefficients and radius of convergence $R>0$, generates a family $(X_t)_{t\in[0,R)}$ of probability distributions supported on the nonnegative integers by defining random variables X_t via

$$X_0 \equiv 0, \quad \mathbf{P}(X_t = n) = \frac{a_n t^n}{f(t)}, \quad \text{for } t \in (0, R) \text{ and } n \geq 0.$$

The systematic study of these *power series distributions* appears to have originated with Kosambi [**16**] in 1949 and Noack [**20**] in 1950, as a unifying model for several discrete distributions arising in statistics. See Section 2.2 in [**14**] and references therein.

We will restrict our study to the class $\mathcal{K}$ of *nonconstant* power series with nonnegative coefficients, positive radius of convergence, *and with* $a_0 > 0$. With this, we avoid the inconvenient case where f is a monomial, or a constant.

We will refer to the family of random variables $(X_t)_{t\in[0,R)}$ associated to f in $\mathcal{K}$ as the *Khinchin family* of f, a term and a whole framework which originates in the work of Rosenbloom [**26**]; see also [**30**].

This paper deals with the beautiful interplay between function theoretic properties of the holomorphic function given by f and certain probabilistic properties of the family (X_t) as $t \uparrow R$, in particular, the behaviour as $t \uparrow R$ of the mean $m_f(t) = \mathbf{E}(X_t)$, of the variance $\sigma_f^2(t) = \mathbf{V}(X_t)$, or, in general, of the moments $\mathbf{E}(X_t^p)$, for $p > 0$.

For instance, in Subsection 3.3 we will provide simple proofs, *of a probabilistic nature*, of some lower bounds and asymptotic lower bounds of $\sigma_f(t)$, as $t \uparrow R$, quantifying Hadamard's three-lines theorem, due to Hayman [**12**], Boĭčuk and Gol'dberg [**7**], Abi-Khuzam [**2**], and others. Later, see Theorem 5.1 and Proposition 5.3, we will show, for entire functions f in $\mathcal{K}$, how the growth of the mean $m_f(t)$ and, in general, of the moments $\mathbf{E}(X_t^p)$, with $p > 0$, and of the quotient $\sigma_f^2(t)/m_f(t)$, relate to the order of f, generalizing some results of Pólya and Szegő, and of Báez-Duarte.

We say that $f \in \mathcal{K}$ with radius of convergence $R \leq \infty$ is a *clan* (and also that the associated family $(X_t)_{t\in[0,R)}$ is a clan) if $\lim_{t\uparrow R} \sigma_f(t)/m_f(t) = 0$ or, equivalently, if $\lim_{t\uparrow R} \mathbf{E}(X_t^2)/\mathbf{E}(X_t)^2 = 1$. For clans, X_t concentrates around its mean as $t \uparrow R$, in the sense that the normalized variable $X_t/\mathbf{E}(X_t)$ tends to the constant 1 in probability. As a matter of fact, being a clan was already considered, not with this name, by Hayman in [**11**] as a property enjoyed by what are nowadays called Hayman (admissible) functions; clans also appear at least in Pólya–Szegő (see items 70 and 71 in [**25**]), and in work of Simić [**31, 32**].

Section 4 contains a detailed study of clans from a function theoretic point of view. It turns out (see Theorem 4.18) that for clans, $\lim_{t\uparrow R} \mathbf{E}(X_t^p)/\mathbf{E}(X_t)^p = 1$, for any $p > 0$. For instance, the partition function $P(z) = \prod_{k=1}^{\infty} 1/(1-z^k)$ is a clan, and for the family $(X_t)_{t>0}$ associated to P we obtain readily, for any $p > 0$, that $\mathbf{E}(X_t^p) \sim \zeta(2)^p/(1-t)^{2p}$, as $t \uparrow 1$.

For entire functions in $\mathcal{K}$, Pólya and Szegő ([**25**]) showed that entire functions with nonnegative coefficients of finite order ρ such that $\lim_{t\to\infty} \ln f(t)/t^\rho$ exists and is positive are clans; we show, more generally, see Theorem 5.9 in Section 5, that entire functions in $\mathcal{K}$ of regular growth (in a precise sense) are clans.

The entire gap series with nonnegative coefficients presented in Subsection 5.2 furnish the basic examples of entire functions of any order ρ, with $0 \leq \rho \leq +\infty$, which are not clans. A classical result of Pfluger and Pólya ([**23**]) ensures that these entire gap series have no Borel exceptional values. We show in Theorem 5.10 that entire functions in $\mathcal{K}$ with one Borel exceptional value are always clans.

Finally, some open questions are discussed in Section 6.

Notation. Generically, throughout this paper, we use $\mathbf{E}(Z)$ and $\mathbf{V}(Z)$ to denote expectation and variance of a random variable Z, and $\mathbf{P}(A)$ to denote probability of the event A (in the appropriate probability space).

For positive functions $f(t)$ and $g(t)$, defined in an interval $[0, R)$, with $0 < R \leq +\infty$, the notation $f(t) \sim g(t)$ as $t \uparrow R$ means that $\lim_{t\uparrow R} f(t)/g(t) = 1$, while $f(t) = o(g(t))$ as $t \uparrow R$ means that $\lim_{t\uparrow R} f(t)/g(t) = 0$. We write $f(t) = O(g(t))$ as $t \uparrow R$ if $f(t) \leq Cg(t)$ for a certain constant $C > 0$ and for t close enough to R; and $f(t) \asymp g(t)$ as $t \uparrow R$ if $cg(t) \leq f(t) \leq Cg(t)$ for certain constants $c, C > 0$ and for t close enough to R.

2. Khinchin families and power series distributions

Keeping the notation and definitions of [**9**], we denote by $\mathcal{K}$ the class of nonconstant power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with positive radius of convergence $R \leq \infty$, which have *nonnegative* Taylor coefficients, and with $a_0 > 0$. Thus, one coefficient of the power series f other that a_0 is also nonzero. Notice that $f(t) > 0$ for each $t \in [0, R)$. We shall resort occasionally to the fact that for f in $\mathcal{K}$ we have that

$$\max\{|f(z)| : |z| \leq t\} = \max\{|f(z)| : |z| = t\} = f(t), \quad \text{for every } t \in [0, R). \tag{2.1}$$

To any power series f in $\mathcal{K}$, we may associate a whole family of probability distributions supported on the nonnegative integers $\{0, 1, \dots\}$, indexed in the interval $[0, R)$, by specifying a family of random variables $(X_t)_{t\in[0,R)}$ with values in $\{0, 1, \dots\}$ and with probability mass functions given by

$$\mathbf{P}(X_t = n) = \frac{a_n t^n}{f(t)}, \quad \text{for each } n \geq 0 \text{ and } t \in (0, R),$$

and with $X_0 \equiv 0$. Since f has at least two coefficients which are not zero, each variable X_t, for $t \in (0, R)$, is a nonconstant random variable.

The family $(X_t)_{t\in[0,R)}$ is the *family of probability distributions associated to the power series* f. For general background on power series probability distributions, we refer the reader to Section 2.2 of [**14**].

The terminology of *Khinchin families* for $(X_t)_{t\in[0,R)}$ is used in this paper to emphasize that the focus is placed on the behaviour as $t \uparrow R$ of the variables (X_t) and of the function $f(t)$, and in the connection between the probabilistic and the function theoretic aspects.

The framework of Khinchin families arises in work of Hayman [**11**], Rosenbloom [**26**], Báez-Duarte [**4**], and others. It has been pursued recently by the authors in [**9**] and [**10**]. The main aim of that framework has been to study, in a unified manner, asymptotic formulas for the coefficients of generating power series, like the Hardy–Ramanujan asymptotic formula for the number of partitions of numbers with $P(z) = \prod_{n=1}^{\infty} 1/(1-z^n)$, the Moser–Wyman asymptotic formula for the number of partitions of sets with $B(z) = e^{e^z-1}$, and many others.

The presentation of this paper is independent of the papers above. For the general theory of Khinchin families, we refer the reader to [**9**] (and also to [**10**]). The summary of a few relevant notions follows.

2.1. Basic families. The most basic examples of Khinchin families $(X_t)_{t\in[0,R)}$ associated to power series $f \in \mathcal{K}$ are

- the *Bernoulli family*, associated to $f(z) = 1 + z$, where for each $t > 0$, the variable X_t is a Bernoulli variable with parameter $p = t/(1 + t)$;
- the *geometric family*, associated to $f(z) = 1/(1 - z)$, where for each $t \in (0, 1)$, the variable X_t follows a geometric distribution with parameter $p = 1 - t$;
- for integer $N \geq 1$, the *binomial family*, associated to $f(z) = (1 + z)^N$, where for each $t > 0$, the variable X_t follows a binomial distribution of parameters N and $p = t/(1 + t)$;
- for integer $N \geq 1$, the *negative binomial family*, associated to $f(z) = 1/(1-z)^N$, where for each $t \in (0, 1)$, the variable X_t follows a negative binomial distribution of parameters N and $p = 1 - t$;
- and the *Poisson family*, associated to $f(z) = e^z$, where for each $t > 0$, the variable X_t follows a Poisson distribution of parameter t.

We mention also the families associated to

- the (ordinary) generating function of partitions of integers: $P(z)=\prod_{n=1}^{\infty} 1/(1–z^n)$, for $|z| < 1$,
- and the (exponential) generating function of the Bell numbers (number of partitions of sets): $B(z) = e^{e^z-1}$, which is an entire power series.

2.2. Moments. Let f be a power series in $\mathcal{K}$, with radius of convergence $R > 0$, and let $(X_t)_{t\in[0,R)}$ be its associated Khinchin family. For $t \in [0, R)$, the mean and variance of X_t will be denoted by $m_f(t) = \mathbf{E}(X_t)$ and $\sigma_f^2(t) = \mathbf{V}(X_t)$, respectively. In terms of f itself, we may write

$$m_f(t) = \frac{tf'(t)}{f(t)} = t(\ln f)'(t) \quad \text{and} \quad \sigma_f^2(t) = tm_f'(t), \quad \text{for } t \in [0, R). \tag{2.2}$$

More generally, for $\beta \geq 0$ and $t \in [0, R)$, *the moment* $\mathbf{E}(X_t^\beta)$ *of exponent* β *of* X_t may be written in terms of the power series f as

$$\mathbf{E}(X_t^\beta) = \sum_{n=0}^{\infty} n^\beta \frac{a_n t^n}{f(t)}.$$

We shall be particularly interested in the comparison of the moment $\mathbf{E}(X_t^\beta)$ of exponent β with the mean $m_f(t) = \mathbf{E}(X_t)$, which acts as a sort of unit, as $t \uparrow R$.

Occasionally, we shall resort to *factorial moments* of the X_t. For integer $k \geq 0$, we denote $x^{\underline{k}} := x(x-1)\cdots(x-k+1)$, the k-th falling factorial of the number x. For $k = 0$, it is agreed that $x^{\underline{0}} = 1$. The k-th factorial moment of X_t is defined as

$$\mathbf{E}(X_t^{\underline{k}}) = \mathbf{E}(X_t(X_t-1)\cdots(X_t-k+1)),$$

with the convention that $\mathbf{E}(X_t^{\underline{0}}) = 1$. Observe that the random variable $X_t^{\underline{k}}$ given by $X_t^{\underline{k}} := X_t(X_t-1)\cdots(X_t-k+1)$ takes values in $\{0, 1, \dots\}$, and that $\mathbf{E}(X_t^{\underline{k}}) \geq 0$, for any $k \geq 0$.

In terms of the power series f itself, the k-th factorial moment is given by

$$\mathbf{E}(X_t^{\underline{k}}) = \frac{1}{f(t)} \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n t^n = \frac{t^k f^{(k)}(t)}{f(t)}, \quad \text{for } t \in [0, R). \tag{2.3}$$

Stirling numbers of the second kind $S(n, j)$ relate powers with falling factorials of a number x:

$$x^k = \sum_{j=0}^{k} S(k, j) x^{\underline{j}}, \quad \text{for } k \geq 1.$$

This translates into the following relation between moments and factorial moments (see, for instance, equation (11) of [**24**]):

$$\mathbf{E}(X_t^k) = \sum_{j=0}^{k} S(k, j) \mathbf{E}(X_t^{\underline{j}}), \quad \text{for any integer } k \geq 1 \text{ and } t \in (0, R). \tag{2.4}$$

Recall that $S(k, k) = 1$, for any $k \geq 1$.

We shall resort a couple of times to the following basic inequalities for moments of random variables.

Lemma 2.1. *Let* Y *be a positive random variable with* $\mathbf{E}(Y) = 1$.

(a) *For* $1 \leq q \leq p$,

$$1 \leq \mathbf{E}(Y^q) \leq \mathbf{E}(Y^p).$$

(b) *For* $\beta \in (0, 1]$ *and* $p > 1$,

$$1 \geq \mathbf{E}(Y^\beta) \geq \mathbf{E}(Y^p)^{-\frac{1-\beta}{p-1}}.$$

Proof: Monotonicity of moments gives

$$\mathbf{E}(Y^\beta)^{1/\beta} \leq 1 \leq \mathbf{E}(Y^q)^{1/q} \leq \mathbf{E}(Y^p)^{1/p}. \tag{2.5}$$

For part (a), we get from (2.5) that $\mathbf{E}(Y^p) \geq 1$ and thus, as $q/p \leq 1$, we learn that $\mathbf{E}(Y^p)^{q/p} \leq \mathbf{E}(Y^p)$. Raising (2.5) to the power q gives the result.

For part (b), write $1 = u\beta + (1-u)p$; explicitly, $u = (p-1)/(p-\beta) \in (0, 1]$. Now, Hölder's inequality (for $1 = u + (1-u)$) yields

$$1 = \mathbf{E}(Y) = \mathbf{E}(Y^{u\beta} \cdot Y^{(1-u)p}) \leq \mathbf{E}(Y^\beta)^u \mathbf{E}(Y^p)^{1-u},$$

and so,

$$\mathbf{E}(Y^\beta) \geq \mathbf{E}(Y^p)^{\frac{u-1}{u}} = \mathbf{E}(Y^p)^{\frac{\beta-1}{p-1}}.$$

This and (2.5) end the proof. □

Next we describe the moments and asymptotics of the moments of the most basic families.

2.2.1. Moments of geometric and negative binomial variables.

Proposition 2.2. *If $N \geq 1$ and $(X_t)_{t\in[0,1)}$ is the Khinchin family of the function $f(z) = 1/(1-z)^N$, then*

$$m_f(t) = N\frac{t}{1-t} \quad \text{and} \quad \sigma_f^2(t) = N\frac{t}{(1-t)^2}, \quad \text{for } t \in [0,1),$$

and for any $\beta \geq 0$, we have that

$$\lim_{t\uparrow 1} \frac{\mathbf{E}(X_t^\beta)}{\mathbf{E}(X_t)^\beta} = \frac{\Gamma(\beta+N)}{\Gamma(N)N^\beta}.$$

This comparison of moments is a direct consequence of the following asymptotic formula (valid for any $\beta > 0$):

$$\sum_{n=1}^\infty n^{\beta-1}t^n \sim \Gamma(\beta)\frac{1}{(1-t)^\beta}, \quad \text{as } t \uparrow 1,$$

which in turn follows from the binomial expansion and Stirling's formula.

2.2.2. Moments of Bernoulli and binomial variables. For the Khinchin family $(X_t)_{t\geq 0}$ of a *polynomial f of degree N*, we have that X_t tends in distribution to the constant N as $t \to \infty$, and also that for $\beta \geq 0$,

$$\lim_{t\to\infty} \mathbf{E}(X_t^\beta) = N^\beta = \lim_{t\to\infty} \mathbf{E}(X_t)^\beta.$$

Particular instances of this polynomial case are the Bernoulli and the binomial families, which are associated, respectively, to $f(z) = 1+z$ (of degree 1) and to $f(z) = (1+z)^N$ (of degree N).

2.2.3. Moments of Poisson variables. The Poisson family $(X_t)_{t\geq 0}$ is associated to the exponential function e^z. For $t > 0$, the variable X_t is a Poisson variable with parameter t, and thus $\mathbf{E}(X_t) = t$ and $\mathbf{V}(X_t) = t$.

For $\beta > 0$ and $t > 0$, the moment of exponent β of X_t is given by

$$\mathbf{E}(X_t^\beta) = \sum_{n=0}^\infty n^\beta \frac{t^n e^{-t}}{n!}.$$

Proposition 2.3. *For the family $(X_t)_{t\geq 0}$ associated to $f(z) = e^z$, and for any $\beta > 0$, we have*

$$\mathbf{E}(X_t^\beta) \sim t^\beta = \mathbf{E}(X_t)^\beta, \quad \text{as } t \to \infty.$$

An application of Jensen's inequality will allow us to reduce the proof of Proposition 2.3 to the case of integer exponents.

Lemma 2.4. *Let $0 < R \leq \infty$, and let $(U_t)_{t\in[0,R)}$ be a family of nonnegative random variables such that for some $p > 1$,*

$$\mathbf{E}(U_t^p) \sim \mathbf{E}(U_t)^p, \quad \text{as } t \uparrow R.$$

Then, for any $\beta \in (0,p]$,

$$\mathbf{E}(U_t^\beta) \sim \mathbf{E}(U_t)^\beta, \quad \text{as } t \uparrow R.$$

Proof: Write $Y_t = U_t/\mathbf{E}(U_t)$, and observe that Y_t is nonnegative and that $\mathbf{E}(Y_t) = 1$. In these terms, the hypothesis is that

$$\lim_{t\uparrow R} \mathbf{E}(Y_t^p) = 1, \quad \text{for some } p > 1,$$

and we want to prove that

$$\lim_{t\uparrow R} \mathbf{E}(Y_t^\beta) = 1, \quad \text{for all } 0 < \beta \leq p.$$

For $1 \leq \beta \leq p$, this follows from part (a) of Lemma 2.1. Part (b) of the same lemma proves the case $\beta \in (0, 1)$. □

Proof of Proposition 2.3: As $f(t) = e^t$, it follows from (2.3) that the factorial moments of X_t are given by $\mathbf{E}(X_t^{\underline{j}}) = t^j$, for any integer $j \geq 0$ and any $t > 0$. On account of (2.4), we have that, for each integer $k \geq 1$,

$$\mathbf{E}(X_t^k) = \sum_{j=0}^{k} S(k, j) t^j.$$

Since $S(k, k) = 1$, for $k \geq 1$, we obtain that for each integer $k \geq 1$,

$$\mathbf{E}(X_t^k) \sim t^k, \quad \text{as } t \to \infty.$$

Lemma 2.4 finishes the proof. □

2.2.4. Mean and variance of the partition function and of the Bell function. Beyond the most basic families, consider now the partition function $P(z)$ given by the infinite product

$$P(z) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k} = \sum_{n=0}^{\infty} p(n) z^n, \quad \text{for } |z| \leq 1.$$

The coefficient $p(n)$ is the number of partitions of the integer n.

The mean $m_P(t)$ and the variance $\sigma_P^2(t)$ are given, on account of (2.2), by

$$m_P(t) = \sum_{n=1}^{\infty} \frac{n t^n}{1 - t^n} \quad \text{and} \quad \sigma_P^2(t) = \sum_{n=1}^{\infty} \frac{n^2 t^n}{(1 - t^n)^2}, \quad \text{for } t \in [0, 1).$$

By Euler summation, one may obtain convenient asymptotic formulas describing their behaviour as $t \uparrow 1$:

$$m_P(t) \sim \frac{\zeta(2)}{(1-t)^2} \quad \text{and} \quad \sigma_P^2(t) \sim \frac{2\zeta(2)}{(1-t)^3}, \quad \text{as } t \uparrow 1. \tag{2.6}$$

See, for instance, Section 6.1 of [**9**].

We will see later on, see Subsection 4.1.1, that P is a clan. Thus Theorem 4.18 would claim, for the family (X_t) associated to the partition function P, that for any $\beta > 0$ we have that $\mathbf{E}(X_t^\beta) \sim \mathbf{E}(X_t)^\beta$, as $t \uparrow 1$, and therefore that

$$\mathbf{E}(X_t^\beta) \sim \frac{\zeta(2)^\beta}{(1-t)^{2\beta}}, \quad \text{as } t \uparrow 1,$$

for any $\beta > 0$.

The exponential generating function $B(z)$ of the Bell numbers B_n is given by

$$B(z) = e^{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad \text{for } z \in \mathbb{C}.$$

The coefficient B_n is the number of partitions of the set $\{1, \ldots, n\}$. In this case, the mean and variance admit simple formulas:

$$m_B(t) = te^t \quad \text{and} \quad \sigma_B^2(t) = t(t+1)e^t, \quad \text{for } t > 0. \tag{2.7}$$

The Bell function is also a clan: for the moments of the family (X_t) associated to the Bell function we have, for any $\beta > 0$, that

$$\mathbf{E}(X_t^\beta) \sim t^\beta e^{\beta t}, \quad \text{as } t \to \infty.$$

2.2.5. Mean and variance of some canonical products. Let $(b_k)_{k \geq 1}$ be a sequence of positive numbers increasing to ∞ in such a way that $\sum_{k=1}^{\infty} 1/b_k < +\infty$. The *canonical product* f given by

$$f(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{b_k}\right), \quad \text{for } z \in \mathbb{C}, \tag{2.8}$$

is an entire function in $\mathcal{K}$, whose set of zeros is $\{-b_k,\ k \geq 1\}$. By Hadamard's factorization theorem, these are *all* the entire functions of genus 0 with only negative (real) zeros, normalized so that $f(0) = 1$. See, for instance, Chapter 4 of [**6**].

For the mean and variance functions of f, we have, from (2.2), that

$$m_f(t) = \sum_{k=1}^{\infty} \frac{t}{t + b_k} \quad \text{and} \quad \sigma_f^2(t) = \sum_{k=1}^{\infty} \frac{b_k t}{(t + b_k)^2}, \quad \text{for } t \geq 0. \tag{2.9}$$

These canonical products provide interesting examples of the behaviour of $\sigma_f^2(t)$, as we discuss now, and they will be used in forthcoming arguments (see Subsections 3.3.1 and 5.4). Denote by

$$N(t) = \#\{k \geq 1 : b_k \leq t\}, \quad \text{for } t > 0,$$

the *counting function of zeros* of f. Thus $N(t)$ counts the number of zeros of f in the closure of the disk $\mathbb{D}(0, t)$.

The mean and variance functions of f are readily comparable.

Lemma 2.5. *For functions f as in* (2.8)*, we have*

$$\sigma_f^2(t) < m_f(t) < 2\sigma_f^2(t) + N(t), \quad \textit{for any } t > 0. \tag{2.10}$$

Proof: On the one hand, observe that

$$\frac{b_k t}{(t + b_k)^2} < \frac{t}{t + b_k}, \quad \text{for any } t > 0 \text{ and any } k \geq 1.$$

On the other hand,

$$\frac{t}{t + b_k} < 1 \quad \text{for all } t > 0, \text{ but also} \quad \frac{t}{t + b_k} < 2\frac{b_k t}{(t + b_k)^2} \quad \text{if } b_k > t > 0.$$

The statement follows from these estimates and the formulas in (2.9). □

We follow the lead of Hayman in Theorem 4 of [**12**], and show next how σ_f of these canonical products depend upon the log spacing of the b_k.

Lemma 2.6. *Consider a canonical product as in* (2.8)*, and assume further that the sequence* $(b_k)_{k\geq 1}$ *satisfies*

$$b_{k+1} \geq 2b_k, \quad \text{for any } k \geq 1. \tag{2.11}$$

For each $n \geq 2$*, we let* I_n *denote the interval*

$$I_n := \big[\sqrt{b_{n-1}b_n}, \sqrt{b_n b_{n+1}}\big].$$

Then, for any $n \geq 2$*,*

$$\begin{aligned} \sup_{t\in I_n} \sigma_f^2(t) &\leq \frac{1}{4} + 4\max\big\{\sqrt{b_n/b_{n+1}}, \sqrt{b_{n-1}/b_n}\big\}, \\ \inf_{t\in I_n} \sigma_f^2(t) &\geq \frac{1}{4}\min\big\{\sqrt{b_n/b_{n+1}}, \sqrt{b_{n-1}/b_n}\big\}. \end{aligned} \tag{2.12}$$

Proof: Consider the positive function

$$\varphi(x) = \frac{x}{(1+x)^2}, \quad \text{for } x > 0,$$

which attains a maximum value of $1/4$ at $x = 1$, and satisfies

$$\frac{x}{4} < \varphi(x) < x, \quad \text{for } x \in (0,1), \quad \text{and} \quad \frac{1}{4x} < \varphi(x) < \frac{1}{x}, \quad \text{for } x > 1. \tag{2.13}$$

In terms of φ, we may express σ_f^2 as

$$\sigma_f^2(t) = \sum_{k=1}^{\infty} \frac{b_k t}{(t+b_k)^2} = \sum_{k=1}^{\infty} \varphi(t/b_k), \quad \text{for any } t > 0.$$

Fix $n \geq 2$. If $t \leq b_{n+1}$, we have, using (2.13) and the growth condition (2.11), that

$$\sum_{k=n+1}^{\infty} \varphi(t/b_k) < \sum_{k=n+1}^{\infty} \frac{t}{b_k} < \frac{t}{b_{n+1}} \sum_{k=n+1}^{\infty} \frac{1}{2^{k-(n+1)}} = \frac{2t}{b_{n+1}}.$$

Analogously, if $t \geq b_{n-1}$,

$$\sum_{k=1}^{n-1} \varphi(t/b_k) < \sum_{k=1}^{n-1} \frac{b_k}{t} < \frac{b_{n-1}}{t} \sum_{k=1}^{n-1} \frac{1}{2^{n-1-k}} < \frac{2b_{n-1}}{t}.$$

We then have, for $n \geq 2$, that for any $t \in I_n$,

$$\sum_{k=n+1}^{\infty} \varphi(t/b_k) < 2\sqrt{b_n/b_{n+1}} \quad \text{and} \quad \sum_{k=1}^{n-1} \varphi(t/b_k) < 2\sqrt{b_{n-1}/b_n}.$$

For $n \geq 2$, it follows then that, if $t \in I_n$,

$$\varphi(t/b_n) \leq \sigma_f^2(t) \leq \varphi(t/b_n) + 4\max\big\{\sqrt{b_n/b_{n+1}}, \sqrt{b_{n-1}/b_n}\big\}.$$

The bounds in (2.12) now follow by observing that $\varphi(x)$ decreases whenever x moves away from 1. □

2.3. Derivative power series $\mathcal{D}_f$ and its family. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series in $\mathcal{K}$, with radius of convergence $R > 0$, and let $(X_t)_{t\in[0,R)}$ be its associated family. We now consider the power series $\mathcal{D}_f$ given by

$$\mathcal{D}_f(z) = zf'(z) = \sum_{n=1}^{\infty} na_n z^n, \quad \text{for } |z| < R.$$

This power series $\mathcal{D}_f$ has radius of convergence R, but it is not in $\mathcal{K}$, as $\mathcal{D}_f(0) = 0$. In any case, we denote by $(W_t)_{t\in[0,R)}$ the associated family of random variables.

Observe that for $t \in (0, R)$ and $n \geq 1$ we have that

$$\mathbf{P}(W_t = n) = \frac{na_n t^n}{tf'(t)} = \frac{1}{m_f(t)} \frac{na_n t^n}{f(t)} = \frac{n}{m_f(t)} \mathbf{P}(X_t = n).$$

Thus, for $t \in (0, R)$, we have that

$$m_{\mathcal{D}_f}(t) = \mathbf{E}(W_t) = \frac{1}{m_f(t)} \sum_{n=1}^{\infty} n^2 \mathbf{P}(X_t = n) = \frac{\mathbf{E}(X_t^2)}{m_f(t)} = \frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)}.$$

If the power series f has only two nonzero coefficients, a_0 and a_N, with $N \geq 1$, then the random variables W_t are constant, and $\mathbf{E}(X_t^2)/\mathbf{E}(X_t) = N$, for any $t \in (0, \infty)$ (in this special case, f is a polynomial and $R = +\infty$). Otherwise (if f has at least three nonzero coefficients), the quotient $\mathbf{E}(X_t^2)/\mathbf{E}(X_t)$ is monotonically increasing in the interval $(0, R)$.

In general, we have that

$$\mathbf{E}(W_t^p) = \frac{1}{m_f(t)} \mathbf{E}(X_t^{p+1}), \quad \text{for any } p > 0 \text{ and any } t \in (0, R). \tag{2.14}$$

The quotient

$$\frac{m_{\mathcal{D}_f(t)}}{m_f(t)} = \frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2} = \frac{\sigma_f^2(t)}{m_f^2(t)} + 1 \tag{2.15}$$

plays a relevant role in what follows.

3. Growth of the moments of power series distributions

Let f be a power series in $\mathcal{K}$ with radius of convergence $R > 0$ and with associated family $(X_t)_{t\in[0,R)}$.

3.1. Growth and range of the mean m_f. Since X_t is not constant for any $t \in (0, R)$, we have that $\sigma_f^2(t) > 0$, for any $t \in (0, R)$, and hence, because of (2.2), $m_f(t)$ is *strictly increasing in* $[0, R)$, though, in general, $\sigma_f(t)$ is not increasing.

We denote

$$M_f = \lim_{t\uparrow R} m_f(t).$$

As recorded in the following result, it is the case that $M_f = \infty$, other than in some exceptional instances.

Lemma 3.1 (Lemma 2.2 of [**9**]). *For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{K}$ with radius of convergence $R > 0$, we have $M_f < \infty$ in just the following two cases:*

(a) *if $R < \infty$ and $\sum_{n=0}^{\infty} na_n R^n < \infty$,*

(b) *and if $R = \infty$ and f is a polynomial.*

In the first case, we have $M_f = \big(\sum_{n=0}^{\infty} na_n R^n\big)/\big(\sum_{n=0}^{\infty} a_n R^n\big)$. For a polynomial $f \in \mathcal{K}$, we have $M_f = \deg(f)$.

As an incremental quotient, $m_f(t)$ and $\ln f(t)$ are related as in the following result.

Lemma 3.2 (Simić, [**32**]). *For $\lambda > 1$ and $t > 0$, with $\lambda t < R$, we have*

$$m_f(t)\ln\lambda \leq \ln\left(\frac{f(\lambda t)}{f(t)}\right) \leq m_f(\lambda t)\ln\lambda.$$

Lemma 3.2 follows directly from the expression (2.2) and the fact that m_f is increasing.

3.2. Relative growth of moments. We now are comparing, *in the case $M_f = +\infty$*, the growth of moments of different exponents and also the growth of the factorial moments and the moments of the X_t; this is covered, respectively, by Corollaries 3.3 and 3.4. The basic tool is the following inequality, that can be easily deduced by taking $Y_t = X_t/\mathbf{E}(X_t)$ in part (a) of Lemma 2.1: for $1 < \alpha < \beta$,

$$\frac{\mathbf{E}(X_t^\alpha)}{\mathbf{E}(X_t)^\alpha} \leq \frac{\mathbf{E}(X_t^\beta)}{\mathbf{E}(X_t)^\beta}, \quad \text{for any } t \in (0, R). \tag{3.1}$$

From (3.1), we deduce the following two corollaries.

Corollary 3.3. *Assume $M_f = +\infty$. If $1 < \alpha < \beta$, then*

$$\lim_{t\uparrow R}\frac{\mathbf{E}(X_t^\alpha)}{\mathbf{E}(X_t^\beta)} = 0.$$

Proof: From (3.1), we have that

$$\frac{\mathbf{E}(X_t^\alpha)}{\mathbf{E}(X_t^\beta)} \leq \frac{\mathbf{E}(X_t)^\alpha}{\mathbf{E}(X_t)^\beta} = m_f(t)^{\alpha-\beta}, \quad \text{for any } t \in (0, R).$$

The statement follows since $\lim_{t\uparrow R} m_f(t) = +\infty$ and $\alpha - \beta < 0$. □

Corollary 3.4. *Assume $M_f = +\infty$. For any integer $k \geq 1$, we have that*

$$\lim_{t\uparrow R}\frac{\mathbf{E}(X_t^{\underline{k}})}{\mathbf{E}(X_t^k)} = 1.$$

Proof: From Corollary 3.3, we deduce that

$$\lim_{t\uparrow R}\frac{\mathbf{E}(X_t^j)}{\mathbf{E}(X_t^k)} = 0, \quad \text{for } 0 \leq j < k.$$

The statement follows by expanding $\mathbf{E}(X_t^{\underline{k}})$ as $\mathbf{E}(X_t^k)$ plus a linear combination of the moments $\mathbf{E}(X_t^j)$ with $0 \leq j < k$. □

3.3. Growth and range of the variance $\boldsymbol{\sigma_f^2}$. Some of the results below concerning $\sigma_f^2(t)$ will depend *on the gaps among the indices* of the power series. Next we introduce some convenient notation.

Let $(n_k)_{k\geq 1}$ be the increasing sequence of indices so that $a_{n_k} \neq 0$, for each $k \geq 1$, and $a_n = 0$, if $n \notin \{n_k : k \geq 1\}$. Thus the (a_{n_k}) are the nonzero Taylor coefficients of f, and the random variables (X_t) take exactly the values (n_k). Observe that $n_1 = 0$, and that for a polynomial f, the sequence $(n_k)_{k\geq 1}$ is finite. Define $\operatorname{gap}(f)$ and $\overline{\operatorname{gap}}(f)$ as

$$\operatorname{gap}(f) = \sup_{k\geq 1}(n_{k+1} - n_k) \quad \text{and} \quad \overline{\operatorname{gap}}(f) = \limsup_{k\to\infty}(n_{k+1} - n_k).$$

It is always the case that $\operatorname{gap}(f) \geq \overline{\operatorname{gap}}(f) \geq 1$, for any $f \in \mathcal{K}$ which is not a polynomial. For polynomials, we still have $\operatorname{gap}(f) \geq 1$, and we can define $\overline{\operatorname{gap}}(f) = 0$.

We divide the discussion on variance growth depending on whether R is infinite or finite.

3.3.1. Entire functions, $R = \infty$.

Lower bounds for $\sup_{t>0} \sigma_f^2(t)$ *and for* $\limsup_{t\to\infty} \sigma_f^2(t)$.

The (universal) lower bounds on σ_f that we are about to discuss originate with Hayman's results in [**12**] quantifying Hadamard's three-lines theorem for entire functions (not necessarily in $\mathcal{K}$). See also [**2**] and [**15**].

Theorem 3.5 (Boĭčuk–Gol'dberg, [**7**]). *If $f \in \mathcal{K}$ is entire, then*

$$\sup_{t>0} \sigma_f^2(t) \geq \frac{1}{4}\,\mathrm{gap}(f)^2 \quad \left(\geq \frac{1}{4}\right).$$

If, moreover, f is transcendental, then

$$\limsup_{t\to\infty} \sigma_f^2(t) \geq \frac{1}{4}\,\overline{\mathrm{gap}}(f)^2 \quad \left(\geq \frac{1}{4}\right).$$

Recall that an entire function f is termed *transcendental* if it is not a polynomial.

This result is Theorem 2 of [**7**]. See also Theorem 1 in [**1**] and Lemma 2.5 in [**22**]. The probabilistic argument below is simpler than the original proof; it uses the discreteness of the random variables X_t.

Proof of Theorem 3.5: Let $(n_k)_{k=1}^N$ be the indices of the nonzero coefficients of f, with $N \leq +\infty$.

Fix $k < N$ and take $t^\star > 0$ so that $m_f(t^\star) = (n_{k+1} + n_k)/2$, i.e., the midpoint of the interval $[n_k, n_{k+1}]$. Such $t^\star$ exists because $m_f(t)$ is a continuous (and increasing) function, $m_f(0) = 0$, and $M_f = \infty$ or $M_f = \deg(f)$ if f is a polynomial (recall Lemma 3.1).

As $X_{t^\star}$ takes the values $n_1, n_2, \ldots$, clearly $|X_{t^\star} - m_f(t^\star)| \geq \frac{1}{2}(n_{k+1} - n_k)$ with probability 1. This gives

$$\sigma_f^2(t^\star) = \mathbf{E}((X_{t^\star} - m_f(t^\star))^2) \geq \frac{1}{4}(n_{k+1} - n_k)^2.$$

The statements now follow by taking sup and lim sup in the inequality above and appealing to the definitions of gap and $\overline{\mathrm{gap}}$. □

In fact, the very same argument shows, for the centred moments, that if $f \in \mathcal{K}$ is entire, then

$$\sup_{t>0} \mathbf{E}(|X_t - m_f(t)|^p) \geq \frac{1}{2^p}\,\mathrm{gap}(f)^p, \quad \text{for any } p > 0,$$

and, moreover, that if f is transcendental, then

$$\limsup_{t\to\infty} \mathbf{E}(|X_t - m_f(t)|^p) \geq \frac{1}{2^p}\,\overline{\mathrm{gap}}(f)^p, \quad \text{for any } p > 0.$$

As for the *sharpness of the sup part* of Theorem 3.5, consider the case $f(z) = a + bz$, with $a, b > 0$, for which X_t is a Bernoulli variable with success probability $bt/(a+bt)$. In this case, one has $\sigma_f^2(t) = abt/(a+bt)^2$, which takes its maximum value of $1/4$ at $t = a/b$.

In fact, the converse is also true.

Theorem 3.6 (Abi-Khuzam, [**1**]). *For $f \in \mathcal{K}$ entire, $\sup_{t>0} \sigma_f^2(t) = 1/4$ if and only if $f(z) = a + bz$, with $a, b > 0$.*

This result is Theorem 3 of [**1**]. See also Lemma 2.5 in [**22**]. The probabilistic argument below is again simpler than the original proof.

Proof of Theorem 3.6: The 'if' part has been discussed above.

Assume that $f \in \mathcal{K}$ is entire and that $\sup_{t>0} \sigma_f^2(t) = 1/4$. Theorem 3.5 gives that $\operatorname{gap}(f) = 1$. Let a_n and a_{n+1} be any two nonzero consecutive coefficients of f, and let $t^\star$ be such that $m_f(t^\star) = n+1/2$ (observe that, in any case, $M_f \geq n+1$). We have that $|X_{t^\star} - m_f(t^\star)| \geq 1/2$. By hypothesis, $\mathbf{E}((X_{t^\star} - m_f(t^\star))^2) = \sigma_f^2(t^\star) \leq 1/4$, and thus $|X_{t^\star} - m_f(t^\star)| = 1/2$ with probability 1, which means that $X_{t^\star}$ only takes the values n and $n+1$, and thus $f(z) = a_n z^n + a_{n+1} z^{n+1}$. Since $f(0) > 0$, because $f \in \mathcal{K}$, it must be the case that $n = 0$ and $f(z) = a + bz$, with $a, b > 0$. □

For the *sharpness of the lim sup part* of Theorem 3.5, consider a canonical product h given by the infinite product

$$h(z) = \prod_{n=1}^{\infty} \Big(1 + \frac{z}{b_n}\Big),$$

where $(b_n)_{n\geq 1}$ is a sequence of positive numbers increasing to $+\infty$ so that (2.11) holds and, in fact, such that $\lim_{n\to\infty} b_{n+1}/b_n = +\infty$; in particular, $\sum_{k\geq 1} 1/b_k < +\infty$ holds. Obviously, $\operatorname{gap}(h) = 1$ and $\overline{\operatorname{gap}}(h) = 1$. It follows directly from the estimates (2.12) that $\limsup_{t\to\infty} \sigma_h^2(t) = 1/4$; this is Hayman's example in Theorem 4 of [**12**]. If h is multiplied by a polynomial p in such a way that $f = ph \in \mathcal{K}$, then it is still the case that $\limsup_{t\to\infty} \sigma_f^2(t) = 1/4$.

As it turns out, Abi-Khuzam has characterized (see Theorem 2 in [**2**] and its proof) the entire functions $f \in \mathcal{K}$ with $\limsup_{t\to\infty} \sigma_f^2(t) = 1/4$ as precisely those entire functions $f \in \mathcal{K}$ which factorize as

$$f(z) = p(z) \prod_{n=1}^{\infty} \Big(1 + \frac{z}{b_n}\Big),$$

where the b_n are as in Hayman's example, and where p is a polynomial.

Limit of $\sigma_f^2(t)$ as $t \to \infty$.

Regarding the existence and possible limits of $\sigma_f^2(t)$ as $t \to \infty$, the following holds.

(i) *Polynomials.* Polynomials $f \in \mathcal{K}$ are characterized, among the entire functions in $\mathcal{K}$, by

$$\lim_{t\to\infty} \sigma_f^2(t) = 0. \tag{3.2}$$

A direct calculation with the formulas (2.2) shows that for polynomials, (3.2) holds; in fact, $\sigma_f^2(t) = O(1/t)$, as $t \to \infty$. The converse follows, for instance, from Theorem 3.5.

(ii) *Transcendental functions.* As shown by Hilberdink in [**13**], for a transcendental entire function *it is never the case that* $\lim_{t\to\infty} \sigma_f(t)$ *exists and it is finite.*

However, there are entire functions $f \in \mathcal{K}$ for which $\limsup_{t\to\infty} \sigma_f^2(t) < +\infty$ and $\liminf_{t\to\infty} \sigma_f^2(t) > 0$. To see this, just consider a canonical product f as in (2.8), with $b_n = 2^n$, and apply (2.12).

It is also possible to have $\lim_{t\to\infty} \sigma_f(t) = \infty$, as shown, for instance, by the exponential function $f(z) = e^z$, where $\sigma_f^2(t) = t$, for $t \geq 0$.

We emphasize that for $f \in \mathcal{K}$ entire, if $\lim_{t\to\infty} \sigma_f^2(t)$ exists, then that limit is 0 (just for polynomials) or $+\infty$.

Boundedness of $\sigma_f^2(t)$.

We now discuss when, if ever,

$$\sup_{t>0} \sigma_f(t) < +\infty \tag{3.3}$$

does hold for an entire function $f \in \mathcal{K}$.

For polynomials $f \in \mathcal{K}$, (3.3) holds since, in fact, in this case, $\lim_{t\to\infty} \sigma_f(t) = 0$.

In general, *if* (3.3) *holds, then the order $\rho(f)$ of f must be zero.* (See Section 5 for details about the order of an entire function f in $\mathcal{K}$.)

To see this, observe that, since $tm'_f(t) = \sigma_f^2(t)$, integrating, we deduce that $m_f(t) = O(\ln t)$, as $t \to \infty$. A further integration, using (2.2), shows that

$$\ln f(t) = O((\ln t)^2), \quad \text{as } t \to \infty. \tag{3.4}$$

This gives that $\rho(f) = 0$; see (5.1).

Alternatively, Proposition 5.3 below shows that

$$\rho(f) \le (\sup_{t>0} \sigma_f^2(t)) \frac{1}{M_f}.$$

If f is a polynomial, then $\rho(f) = 0$; and if f is transcendental, $M_f = \infty$, and thus $\rho(f) = 0$.

However, $\rho(f) = 0$, or even the stronger condition (3.4), are not enough to ensure the boundedness of $\sigma_f^2(t)$.

Consider the canonical product

$$g(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{b_k}\right)^k,$$

where $(b_k)_{k\ge1}$ is a sequence of positive numbers increasing to ∞ satisfying (2.11) and such that $\sum_{k=1}^{\infty} k/b_k < +\infty$. For each $k \ge 1$, $-b_k$ is a zero with multiplicity k of the entire function $g \in \mathcal{K}$.

Additionally, we assume also that for some constant $H > 0$, the b_k satisfy

$$\sum_{k<n} k b_k < H b_n \quad \text{and} \quad \sum_{k>n} \frac{k}{b_k} \le \frac{H}{b_n}, \quad \text{for each } n \ge 2.$$

In this case,

$$\sigma_g^2(t) = \sum_{k=1}^{\infty} k\, \varphi\left(\frac{t}{b_k}\right), \quad \text{for any } t > 0,$$

where $\varphi(x) = x/(1+x)^2$.

Let C denote a generic positive constant. With the notations of Subsection 2.2.5 and estimating as in there, we obtain

$$n\, \varphi\left(\frac{t}{b_n}\right) \le \sigma_g^2(t) \le n + C\, n \max\left\{\sqrt{b_{n-1}/b_n}, \sqrt{b_n/b_{n+1}}\right\}, \quad \text{for } t \in I_n \text{ and } n \ge 2,$$

where $I_n = \left[\sqrt{b_{n-1}b_n}, \sqrt{b_n b_{n+1}}\right]$. It follows, in particular, since $\sigma_g^2(b_n) \ge n/4$, that $\limsup_{t\to\infty} \sigma_g^2(t) = +\infty$, and also that

$$\sup_{t\in I_n} \sigma_g^2(t) \le Cn.$$

From (2.10), we see that

$$m_g(t) \le Cn + \sum_{k\le n} k \le Cn^2, \quad \text{for any } t \in I_n \text{ and } n \ge 2. \tag{3.5}$$

Consider now the specific sequence $b_k = e^{e^k}$, $k \geq 1$, which satisfies the requirements above. In this case, (3.5) translates into

$$m_g(t) \leq C(\ln\ln t)^2, \quad \text{for } t \geq 2.$$

Integrating, this gives, for this example, that

$$\ln g(t) = O(\ln t(\ln\ln t)^2), \quad \text{as } t \to \infty,$$

which implies $\rho(g) = 0$. Notice that for any function $\Phi(t)$ slowly increasing to ∞, the sequence $(b_k)_{k\geq 1}$ can be chosen so that $\ln g(t) = O(\Phi(t)\ln t)$ just by making b_k increase fast enough. This is the best that can be expected, because if an entire function h in $\mathcal{K}$ grows as $\ln h(t) = O(\ln t)$ as $t \to \infty$, then h is a polynomial.

3.3.2. Finite radius: $R < \infty$. We now turn to functions $f \in \mathcal{K}$ with finite radius R of convergence. We have the following results on the behaviour of $\sigma_f^2(t)$. We divide the discussion according to whether M_f is finite or not.

Case $M_f = +\infty$.

In this case,

$$\sup_{t\in(0,R)} \sigma_f^2(t) = +\infty, \tag{3.6}$$

and also $\limsup_{t\uparrow R} \sigma_f^2(t) = +\infty$. To verify (3.6), assume that $\sup_{t\in(0,R)} \sigma_f^2(t) = S < +\infty$. Thus $tm_f'(t) \leq S$, for $t \in [0,R)$. Integrating between $R/2$ and $t \in (R/2,R)$, we would have that

$$m_f(t) \leq m_f(R/2) + S\ln\left(\frac{2t}{R}\right), \quad \text{for } t \in (R/2,R),$$

which implies, by letting $t \uparrow R < +\infty$, that $M_f \leq m_f(R/2) + S\ln 2 < +\infty$.

Case $M_f < +\infty$.

If $M_f < +\infty$, then $\sum_{n=0}^\infty n a_n R^n < +\infty$, see Lemma 3.1, and in fact,

$$\Sigma := \lim_{t\uparrow R} \sigma_f^2(t) = \frac{\sum_{n=0}^\infty n^2 a_n R^n}{\sum_{n=0}^\infty a_n R^n} - \left(\frac{\sum_{n=0}^\infty n a_n R^n}{\sum_{n=0}^\infty a_n R^n}\right)^2.$$

It is always the case that $\lim_{t\uparrow R} \sigma_f^2(t) > 0$, since Σ is the variance of the random variable Z that takes, for each integer $n \geq 0$, the value n with probability $a_n R^n / \left(\sum_{k=0}^\infty a_k R^k\right)$, and Z is a nonconstant variable since f is in $\mathcal{K}$.

But there is no absolute positive lower bound for $\lim_{t\uparrow R} \sigma_f^2(t)$. For $\varepsilon > 0$, the power series $f(z) = 1 + \varepsilon \sum_{n=1}^\infty z^n/n^4$ is in $\mathcal{K}$ and has radius of convergence $R = 1$. We have

$$M_f = \frac{\varepsilon\zeta(3)}{1+\varepsilon\zeta(4)} \quad \text{and} \quad \lim_{t\uparrow 1} \sigma_f^2(t) = \frac{\varepsilon\zeta(2)}{1+\varepsilon\zeta(4)} - \left(\frac{\varepsilon\zeta(3)}{1+\varepsilon\zeta(4)}\right)^2,$$

which tends to 0 as $\varepsilon \downarrow 0$.

The example $f(z) = \sum_{n=0}^\infty z^n/(1+n)^3$ shows that $\lim_{t\uparrow R} \sigma_f^2(t) = \infty$ may happen.

3.4. Growth of the quotient σ_f/m_f and gaps. Let $f \in \mathcal{K}$, not a polynomial, have radius of convergence $R > 0$. As in Subsection 3.3, we denote by $(n_k)_{k=1}^\infty$ the increasing sequence of indices of the nonzero coefficients of f. We define $\overline{G}(f)$ by

$$\overline{G}(f) = \limsup_{k\to\infty} \frac{n_{k+1}}{n_k}. \tag{3.7}$$

Clearly, $\overline{G}(f) \geq 1$. If for a given f we had $\liminf_{k\to\infty} n_{k+1}/n_k > 1$, then f would be the sum of a polynomial and a power series with Hadamard gaps; but notice that

$\overline{G}(f)$ calls for a "lim sup". Observe that the definition of $\overline{G}(f)$ involves the quotients n_{k+1}/n_k, and not the differences $n_{k+1}-n_k$ as is the case in $\operatorname{gap}(f)$ and $\overline{\operatorname{gap}}(f)$.

Theorem 3.7. *Assume that $f \in \mathcal{K}$, with radius of convergence $R>0$, is not a polynomial and that $M_f=+\infty$. Then*

$$\limsup_{t\uparrow R} \frac{\sigma_f(t)}{m_f(t)} \geq \frac{\overline{G}(f)-1}{\overline{G}(f)+1}.$$

The proof below mimics our proof of Theorem 3.5.

Proof: Since $M_f=+\infty$, we have that $m_f(t)$ is a homeomorphism from $[0,R)$ onto $[0,+\infty)$, and thus for any integer k there is $t_k \in (0,R)$ so that

$$m_f(t_k)=\frac{n_k+n_{k+1}}{2}.$$

For the random variable X_{t_k}, we have that $|X_{t_k}-m_f(t_k)| \geq (n_{k+1}-n_k)/2$ with probability 1, and thus

$$\sigma_f^2(t_k)=\mathbf{E}((X_{t_k}-m_f(t_k))^2) \geq \frac{1}{4}(n_{k+1}-n_k)^2,$$

and also

$$\frac{\sigma_f^2(t_k)}{m_f^2(t_k)} \geq \frac{(n_{k+1}-n_k)^2}{(n_{k+1}+n_k)^2}.$$

The result follows. □

Similarly, for the general centred moments of the family of functions $f \in \mathcal{K}$ as in the statement of Theorem 3.7, we have that

$$\limsup_{t\uparrow R} \frac{\mathbf{E}(|X_t-m_f(t)|^p)}{m_f(t)^p} \geq \left(\frac{\overline{G}(f)-1}{\overline{G}(f)+1}\right)^p, \quad \text{for any } p>0.$$

3.5. Zero-free region and $\boldsymbol{\sigma_f}$. A function f in $\mathcal{K}$ does not vanish on the interval $[0,R)$ and, in fact, its zeros must lie away from that segment.

The following result shows how the variance function of f determines a specific zero-free region for $f \in \mathcal{K}$ containing the interval $[0,R)$.

Proposition 3.8. *Let $f \in \mathcal{K}$ have radius of convergence $R>0$. If for some $t \in [0,R)$ and some $\theta \in [-\pi,\pi]$ we have $f(te^{\imath\theta})=0$, then*

$$|\theta| \cdot \sigma_f(t) \geq \frac{\pi}{2}.$$

Thus, $f \in \mathcal{K}$ does not vanish in the region

$$\Omega_f=\left\{z=te^{\imath\theta} : t \in [0,R) \text{ and } |\theta| < \frac{\pi}{2\sigma_f(t)}\right\}.$$

For the proof, we may use the following lemma.

Lemma 3.9 (Sakovič, [**29**])**.** *Let Y be a random variable and let $\theta \in \mathbb{R}$. If $\mathbf{E}(e^{\imath\theta Y})=0$, then*

$$\theta^2\mathbf{V}(Y) \geq \frac{\pi^2}{4}.$$

The bound on Lemma 3.9 is sharp. Simply consider the random variable Z which takes values ± 1 with probability $1/2$; then $\mathbf{V}(Z)=1$ and $\mathbf{E}(e^{\imath\theta Z})=\cos\theta$, which vanishes at $\pi/2$. (Actually, equality in Lemma 3.9 only happens for this simple symmetric random variable Z.)

The result of Sakovič appeared in [**29**]. As presented in the more accessible reference [**27**], Lemma 3.9 follows most ingeniously as follows.

Proof of Lemma 3.9 following Rossberg [**27**]*:* Consider the function

$$\varphi(t)=t^2-1+\frac{4}{\pi}\cos\frac{\pi t}{2},$$

which happens to be positive for all $t\in\mathbb{R}$, except for $t=\pm 1$, where $\varphi(t)=0$. (A misprinted sign in the definition of φ in [**27**] has been corrected.) Assume that $\mathbf{E}(e^{\imath\theta Y})=0$. Consider $W=Y-\mathbf{E}(Y)$, so that $\mathbf{E}(e^{\imath\theta W})=0$, and thus $\Re\mathbf{E}(e^{\imath\theta W})=\mathbf{E}(\cos(\theta W))=0$. This yields

$$0\le \mathbf{E}\Big(\varphi\Big(\frac{2\theta}{\pi}W\Big)\Big)=\theta^2\frac{4}{\pi^2}\mathbf{E}(W^2)-1=\theta^2\frac{4}{\pi^2}\mathbf{V}(Y)-1. \qquad \square$$

A more direct proof of Lemma 3.9, but with a weaker constant, appears, for instance, in Proposition 7.8 of [**3**] (see also Lemma 2.3 of [**9**]).

Proof of Proposition 3.8: It follows from Lemma 3.9 and from observing that if $f(te^{\imath\theta})=0$, then $\mathbf{E}(e^{\imath\theta X_t})=f(te^{\imath\theta})/f(t)=0$. $\square$

Alternatively, to verify Proposition 3.8, we may use Lemma 1 of [**1**], which gives that for $f\in\mathcal{K}$,

$$f(t)^2-|f(te^{\imath\theta})|^2\le 4\sin^2(\theta/2)f(t)^2\sigma_f^2(t),\quad \text{for } t\in(0,R) \text{ and } \theta\in[-\pi,\pi].$$

If $f(te^{\imath\theta})=0$, then $1\le 2|\sin(\theta/2)|\sigma_f(t)$, and thus $|\theta|\sigma_f(t)\ge 1$. This gives a weaker result with the constant $\pi/2$ replaced by 1. Lemma 1 of [**1**] is stated for entire functions, but it is valid for general $f\in\mathcal{K}$.

Remark 3.10 (On Boĭchuk–Gol'dberg's Theorem 3.5)*.* From Sakovič's Lemma 3.9, we may deduce Boĭchuk–Gol'dberg's Theorem 3.5 in the weaker form, that is,

$$\sup_{t>0}\sigma_f^2(t)\ge 1/4,$$

for any entire function $f\in\mathcal{K}$. To see this, we may assume that $\sup_{t>0}\sigma_f^2(t)<+\infty$. As discussed in Subsection 3.3.1, this yields that the entire function f is of order 0, and Hadamard's factorization theorem gives that f is an infinite canonical product or a polynomial (nonconstant and not a monomial, since $f\in\mathcal{K}$). (This is the starting point of Hayman's proof of Theorem 3 of [**12**].) In any case, f vanishes at some $z_0\neq 0$. Write $z_0=r_0e^{\imath\theta_0}$, with $r_0>0$ and $|\theta_0|\le\pi$. Lemma 3.9 gives that

$$\pi\sigma_f(r_0)\ge\frac{\pi}{2},$$

and thus, that $\sigma_f(r_0)\ge 1/2$, and, in particular, that $\sup_{t>0}\sigma_f^2(t)\ge 1/4$.

This same reasoning also gives that if f is not a polynomial, then

$$\limsup_{t\to\infty}\sigma_f^2(t)\ge 1/4.$$

4. Clans

In previous sections, we have compared the growth of $\mathbf{E}(X_t^\beta)$ with that of $\mathbf{E}(X_t)^\beta$, as $t \uparrow R$, for the basic examples of Khinchin families. Motivated by Hayman [**11**] (see Remark 4.19), next we introduce a particular kind of Khinchin families which we shall call *clans*, for which $\mathbf{E}(X_t^2) \sim \mathbf{E}(X_t)^2$ as $t \uparrow R$. Concretely,

Definition 4.1. Let f in $\mathcal{K}$ have radius of convergence $R \leq \infty$. We say that f is a *clan* (and also that the associated family $(X_t)_{t\in[0,R)}$ is a clan) if

$$\lim_{t\uparrow R} \frac{\sigma_f(t)}{m_f(t)} = 0. \tag{4.1}$$

For a clan, the normalized variables $Y_t = X_t/\mathbf{E}(X_t)$, for $t \in (0, R)$, converge in probability to the constant 1 as $t \uparrow R$, since its variance $\mathbf{V}(Y_t) = \sigma_f^2(t)/m_f(t)^2$ converges to 0 as $t \uparrow R$.

This clan condition is equivalent to

$$\lim_{t\uparrow R} \frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2} = 1.$$

In terms of just the mean m_f, being a clan, see (2.15), is equivalent to

$$\lim_{t\uparrow R} \frac{t m_f'(t)}{m_f(t)^2} = 0,$$

while in terms of the derivative power series $\mathcal{D}_f$, the condition for being a clan becomes

$$\lim_{t\uparrow R} \frac{m_{\mathcal{D}_f}(t)}{m_f(t)} = 1.$$

Alternatively, if we define

$$L_f(t) := \frac{f(t)f''(t)}{f'(t)^2}, \quad \text{for } t \in (0, R), \tag{4.2}$$

and since

$$\frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2} = \frac{1}{m_f(t)} + L_f(t), \tag{4.3}$$

we have that f is a clan if and only if

$$\lim_{t\uparrow R} L_f(t) = 1 - 1/M_f. \tag{4.4}$$

4.1. Some examples.

4.1.1. Examples of clans. The Poisson family associated to $f(z) = e^z$ is a clan, since in this case $m_f(t) = t$ and $\sigma_f^2(t) = t$, and the radius of convergence is $R = \infty$.

The Bernoulli and binomial families are also clans. In fact, all polynomials f are clans, since for them $\sigma_f(t) \to 0$, while $m_f(t) \to \deg(f)$, as $t \to \infty$. Further, we have the following.

Lemma 4.2. *Let $f \in \mathcal{K}$ be a clan. Then $M_f < +\infty$ if and only if f is a polynomial.*

Proof: Polynomials in $\mathcal{K}$ are clans and have M_f finite; in fact, M_f coincides with its degree.

To show the converse, let $f(z) = \sum_{n=0}^\infty a_n z^n$ be in $\mathcal{K}$, not a polynomial, and with $M_f < \infty$. Then, due to Lemma 3.1, we have that $R < \infty$ and $\sum_{n=1}^\infty n a_n R^n < +\infty$.

Then since

$$\frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2} = \frac{\left(\sum_{n=0}^{\infty} n^2 a_n t^n\right)\left(\sum_{n=0}^{\infty} a_n t^n\right)}{\left(\sum_{n=0}^{\infty} n a_n t^n\right)^2}, \quad \text{for } t \in (0, R),$$

and since f is a clan, taking the limit as $t \uparrow R$, we obtain that

$$\frac{\left(\sum_{n=0}^{\infty} n^2 a_n R^n\right)\left(\sum_{n=0}^{\infty} a_n R^n\right)}{\left(\sum_{n=0}^{\infty} n a_n R^n\right)^2} = 1.$$

If we now set $b_n = a_n R^n / \left(\sum_{j=0}^{\infty} a_j R^j\right)$, for each $n \geq 0$, which satisfy $\sum_{n=0}^{\infty} b_n = 1$, then the identity above becomes

$$\sum_{n=0}^{\infty} n^2 b_n = \left(\sum_{n=0}^{\infty} n b_n\right)^2.$$

This means that $b_n = 0$, for each $n \geq 0$ except for one value of n, which would imply that f is a monomial: a contradiction. □

The partition function $P(z) = \prod_{k=1}^{\infty} 1/(1-z^k)$ and the Bell function $B(z) = e^{e^z - 1}$ are also clans. This follows immediately from (2.6) and (2.7).

In Subsection 4.1.3, we exhibit an ample class of functions, that includes the generating function of the partitions and its variants, which are clans.

4.1.2. Clans and L_f. The characterization of a clan given in (4.4) using the function L_f immediately gives the following.

Lemma 4.3. *Let f be a power series in $\mathcal{K}$ with radius of convergence $R \leq +\infty$ and such that $M_f = +\infty$. Then f is a clan if and only if*

$$\lim_{t \uparrow R} L_f(t) = 1. \tag{4.5}$$

In particular, if f is an entire transcendental function *in $\mathcal{K}$, then f is a clan if and only if* (4.5) *holds.*

Recall, from Lemma 3.1, that entire transcendental functions in $\mathcal{K}$ have $M_f = \infty$. For any transcendental entire function, it is always the case that

$$\liminf_{t \to \infty} L_f(t) \geq 1. \tag{4.6}$$

This follows from Lemma 3.1 and the general identity (4.3). This has been pointed out by Simić on page 682 of [**31**]. But, in fact, we have the following.

Lemma 4.4. *For any transcendental entire function f in $\mathcal{K}$,*

$$\liminf_{t \to \infty} L_f(t) = 1. \tag{4.7}$$

Proof: Since f is trascendental, we have that $M_f = +\infty$. Let c be such that $m_f(c) = 1$. From the identity

$$L_f(s) = \left(s\left(1 - \frac{1}{m_f(s)}\right)\right)', \quad \text{for } s > 0,$$

we deduce, since $M_f = +\infty$, that

$$\int_c^t L_f(s)\, ds = t\left(1 - \frac{1}{m_f(t)}\right), \quad \text{for } t > c,$$

and so, that

$$\lim_{t \to \infty} \frac{1}{t} \int_c^t L_f(s)\, ds = 1.$$

This implies, given (4.6), that (4.7) holds. □

4.1.3. Hayman class, strong Gaussianity, and clans. Let f be in $\mathcal{K}$ with radius of convergence $R>0$, and let $(X_t)_{t\in[0,R)}$ be the associated family. Write $\breve{X}_t$ for the normalized random variable

$$\breve{X}_t=\frac{X_t-m_f(t)}{\sigma_f(t)},\quad \text{for } t\in(0,R).$$

The characteristic function of $\breve{X}_t$ is

$$\mathbf{E}(e^{\imath\theta\breve{X}_t})=\mathbf{E}(e^{\imath\theta X_t/\sigma_f(t)})e^{-\imath\theta m_f(t)/\sigma_f(t)},\quad \text{for } t\in(0,R) \text{ and } \theta\in\mathbb{R}.$$

As introduced by Báez-Duarte in [**4**], a power series $f\in\mathcal{K}$ and its family $(X_t)_{t\in[0,R)}$ are termed *strongly Gaussian* if

$$\lim_{t\uparrow R}\sigma_f(t)=+\infty,\quad \text{and}\quad \lim_{t\uparrow R}\int_{|\theta|<\pi\sigma_f(t)}|\mathbf{E}(e^{\imath\theta\breve{X}_t})-e^{-\theta^2/2}|\,d\theta=0.$$

Every strongly Gaussian function is Gaussian, that is, its normalized Khinchin family, $(\breve{X}_t)_{t\in[0,R)}$, converges in distribution, as $t\uparrow R$, to the standard normal variable or, equivalently,

$$\lim_{t\uparrow R}\mathbf{E}(e^{\imath\theta\breve{X}_t})=e^{-\theta^2/2},\quad \text{for each } \theta\in\mathbb{R}.$$

See [**9**] for definitions and proofs. The main interest of strong Gaussianity is that if a power series $f(z)=\sum_{n\geq 0}a_nz^n$ in $\mathcal{K}$ is strongly Gaussian, then

$$a_n\sim\frac{f(t_n)}{\sqrt{2\pi}t_n^n\sigma_f(t_n)},\quad \text{as } n\to\infty, \tag{4.8}$$

where t_n is the unique value such that $m_f(t_n)=n$.

All (*Hayman*) admissible functions, in the terminology of [**11**] (see Definition on pages 68–69) or functions in the *Hayman class*, are strongly Gaussian. See, for instance, Theorem 3.8 in [**9**].

The exponential $f(z)=e^z$ is strongly Gaussian. In [**9**] and [**10**], some criteria are given to check when a power series in $\mathcal{K}$ is strongly Gaussian, and these criteria are applied to find asymptotic estimations on the growth of the coefficients of generating functions of combinatorial interest.

For admissible functions, Hayman proved (but the proof also works for strongly Gaussian functions) the following central limit theorem (see [**11**] and [**9**] for more details).

Theorem 4.5 (Hayman's local central limit theorem)**.** *If $f(z)=\sum_{n=0}^\infty a_nz^n$ in $\mathcal{K}$ is strongly Gaussian, then*

$$\lim_{t\uparrow R}\left(\sup_{n\in\mathbb{Z}}\left|\frac{a_nt^n}{f(t)}\sqrt{2\pi}\sigma_f(t)-e^{-(n-m_f(t))^2/(2\sigma_f^2(t))}\right|\right)=0,$$

where for $n<0$ it is understood that $a_n=0$.

As a corollary of this theorem, we obtain the following.

Corollary 4.6. *If $f\in\mathcal{K}$ is strongly Gaussian, then f is a clan.*

Proof: Restricting the supremum of Theorem 4.5 to $n=-1$, we get

$$\lim_{t\uparrow R}\exp\left(-\frac{(m_f(t)+1)^2}{2\sigma_f(t)^2}\right)=0.$$

Since $\lim_{t\uparrow R}\sigma_f(t)=+\infty$, because f is strongly Gaussian, we deduce that f is a clan. □

This corollary shows that a large collection of functions ranging from $f(z) = e^z$ to the generating functions of partitions $\prod_{j\geq 1}(1-z^j)^{-1}$, $|z|<1$, are clans; see [**9**] and [**10**] for these and other interesting examples.

Notice that there are clans in $\mathcal{K}$ which are not strongly Gaussian. For instance, if $g \in \mathcal{K}$ is strongly Gaussian, then $f(z) = g(z^N)$, with $N \in \mathbb{N}$, is a clan, but it is not strongly Gaussian since $a_n \equiv 0$ when n is not a multiple of N, and hence (4.8) cannot hold.

4.1.4. Some power series that are not clans. The geometric and negative binomial families *are not clans*, since for $f(z) = 1/(1-z)^N$, see Proposition 2.2, we have that $\lim_{t\uparrow 1} \sigma_f(t)/m_f(t) = 1/\sqrt{N}$.

In fact, for each $\alpha > 0$, the function $f(z) = 1/(1-z)^\alpha$, which is in $\mathcal{K}$, is not a clan, because $\lim_{t\uparrow 1} \sigma_f(t)/m_f(t) = 1/\sqrt{\alpha}$.

Even further, for the function $f(z) = 1 + \ln(1/(1-z))$, which is in $\mathcal{K}$, it holds that $\lim_{t\uparrow 1} \sigma_f(t)/m_f(t) = +\infty$.

Many other examples of power series in $\mathcal{K}$ which are not clans are provided by the following immediate corollary of Theorem 3.7.

Corollary 4.7. *Let f be in $\mathcal{K}$. If $M_f = +\infty$ and $\overline{G}(f) > 1$, then f is* not *a clan.*

For instance, power series with radius of convergence $R = 1$, like $1 + \sum_{k=1}^\infty z^{2^k}$, or entire power series like $1 + \sum_{k=1}^\infty z^{2^k}/(2^k)!$, are in $\mathcal{K}$ but they are not clans.

Observe that for $f(z) = 1/(1-z)$, which is not a clan, we have $M_f = +\infty$ and $\overline{G}(f) = 1$.

4.2. Some basic properties of clans. We now register a few properties of power series in $\mathcal{K}$ that are clans, that is, functions in $\mathcal{K}$, with radius of convergence $R > 0$, and satisfying the limit condition given in (4.1).

- It follows from Chebyshev's inequality that if $(X_t)_{t\in[0,R)}$ is a clan, then for any $\varepsilon > 0$,
$$\lim_{t\uparrow R} \mathbf{P}\left(\left|\frac{X_t}{\mathbf{E}(X_t)} - 1\right| > \varepsilon\right) = 0,$$
and thus that $X_t/\mathbf{E}(X_t)$ converges in probability to the constant 1 as $t \uparrow R$: the random variable X_t concentrates about its mean $m_f(t)$ as $t \uparrow R$.
- By Lemma 2.4, we have that if f is a clan, then
$$\lim_{t\uparrow R} \frac{\mathbf{E}(X_t^p)}{\mathbf{E}(X_t)^p} = 1, \quad \text{for any } p \in (0, 2].$$
Theorem 4.18 below will show that if f is a clan, then this limit result actually holds for any $p > 0$.
- If f is a clan (with at least three nonzero coefficients), then $\mathcal{D}_f = zf'(z)$ is also a clan. This will be proved right after Theorem 4.18. (The condition of three nonzero coefficients excludes the case in which the variables associated to $\mathcal{D}_f$ are constant.)
- If g is a clan, then for any integer $N \geq 1$, $f(z) = g(z^N)$ is also a clan, since for $t \in [0, R^{1/N})$, $m_f(t) = N m_g(t^N)$ and $\sigma_f(t) = N\sigma_g(t^N)$, where m_g, σ_g^2, and m_f, σ_f^2 denote the mean and variance functions of the Khinchin families of g and f.
- If f and g are clans with the same radius of convergence $R > 0$, then their product $h \equiv fg$ is also a clan. For we have $m_h = m_f + m_g$ and $\sigma_h^2 = \sigma_f^2 + \sigma_g^2 = o(m_f^2 + m_g^2)$ as $t \uparrow R$, and thus $\sigma_h = o(m_h)$ as $t \uparrow R$. In particular, if f is a clan and $N \geq 1$ is an integer, then f^N is also a clan.

- Finally, if f and g are entire functions which are clans, then the composition $f\circ g$ is a clan. This is clear if both f and g are polynomials. Otherwise, this follows from the identity
$$L_{f\circ g}(t)=L_f(g(t))+\frac{L_g(t)}{m_f(g(t))},\quad \text{for any } t>0,$$
and by combining Lemma 4.3 and the fact that, for a polynomial h in $\mathcal{K}$ of degree N, we have that $\lim_{t\to\infty} L_h(t)=1-1/N$, while $\lim_{t\to\infty} m_h(t)=N$.
In particular, if g is an entire function which is a clan, then e^g is a clan.

4.3. Weak clans. We say that $f\in\mathcal{K}$ is a *weak clan* if
$$\liminf_{t\uparrow R}\frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2}=1,\quad \text{or, equivalently, if}\quad \liminf_{t\uparrow R}\frac{\sigma_f(t)}{m_f(t)}=0.$$
Of course, clans are weak clans.

Proposition 4.8. *Every* entire *function f in $\mathcal{K}$ is a weak clan.*

As a consequence of this result, for entire functions $f\in\mathcal{K}$ which are not clans, the quotient $\sigma_f(t)/m_f(t)$ must oscillate and has no limit as $t\to\infty$.

In contrast, power series $f\in\mathcal{K}$ with finite radius of convergence *need not be weak clans*. Recall, for instance, the examples $f(z)=1/(1-z)$ and $f(z)=1+\ln(1/(1-z))$, for which $\lim_{t\uparrow 1}\sigma_f(t)/m_f(t)$ is 1 and $+\infty$, respectively.

Proof of Proposition 4.8: Polynomials are clans. For a transcendental entire function $f\in\mathcal{K}$ we have, because of Lemma 4.2, that $M_f=+\infty$. That f is a weak clan then follows from combining (4.7) with (4.3). □

Remark 4.9. In fact, more is true. The following argument is based on Rosenbloom's proof in [**26**] of the Wiman–Valiron theorem.

For a transcendental entire function f, we are going to show that, for every $\eta>0$ and $\varepsilon>0$, there exists a set $G_\varepsilon\subset(0,R)$ of logarithmic measure not exceeding ε, such that

$$\lim_{\substack{t\to\infty\\ t\notin G_\varepsilon}}\frac{\sigma_f^2(t)}{m_f(t)^{1+\eta}}=0, \tag{4.9}$$

which, in particular, implies that
$$\liminf_{t\to\infty}\frac{\sigma_f^2(t)}{m_f(t)^{1+\eta}}=0.$$

For $a>0$, consider $H_a=\{x\ge a:\sigma_f^2(x)\ge m_f(x)^{1+\eta/2}\}$. For $x\in H_a$, we have that
$$\frac{m_f'(x)}{m_f(x)^{1+\eta/2}}\ge\frac{1}{x},$$
and thus, using that $M_f=+\infty$, we deduce that
$$\int_{H_a}\frac{dx}{x}\le\frac{2}{\eta m_f(a)^{\eta/2}}.$$
Let $a=a(\varepsilon)$ be such that $2/m_f(a)^{\eta/2}\le\eta\varepsilon$. Then, for $G_\varepsilon=H_{a(\varepsilon)}$, we have that G_ε has logarithmic measure at most ε, and for $t\notin G_\varepsilon$, we conclude that
$$\sigma_f^2(t)\le m_f(t)^{1+\eta/2}.$$
Since $M_f=+\infty$, this gives (4.9).

4.4. On the mean function of a clan. Hayman showed in Lemmas 2 and 3 of [**11**] that the mean $m_f(t)$ of functions $f \in \mathcal{K}$ in the Hayman class cannot grow too slowly. Recall, from Subsection 4.1.3, that the Hayman class is a subclass of the class of strongly Gaussian functions, and thus, by Corollary 4.6, functions in the Hayman class are clans. Next, building upon Hayman's approach, we present a characterization of clans in terms of m_f alone.

Observe that as a first-order approximation we have that

$$m_f(t+t/m_f(t)) \approx m_f(t) + m_f'(t)\frac{t}{m_f(t)} = m_f(t) + \frac{\sigma_f^2(t)}{m_f(t)},$$

and thus that

$$\frac{m_f(t+t/m_f(t))}{m_f(t)} \approx 1 + \frac{\sigma_f^2(t)}{m_f(t)^2}.$$

This suggests that clans, for which $\lim_{t\uparrow R} \sigma_f(t)/m_f(t) = 0$, may be characterized in terms of the behaviour of the quotient $m_f(t+t/m_f(t))/m_f(t)$ as $t \uparrow R$. This is the content of Theorem 4.10.

Also, as a second-order approximation we have, using (2.2), that

$$\begin{aligned}\ln f(t+t/m_f(t)) - \ln f(t) &\approx (\ln f)'(t)\frac{t}{m_f(t)} + \frac{1}{2}(\ln f)''(t)\frac{t^2}{m_f(t)^2}\\ &= 1 + \frac{1}{2}\frac{\sigma_f^2(t)}{m_f(t)^2} - \frac{1}{2}\frac{1}{m_f(t)},\end{aligned} \tag{4.10}$$

which in turn suggests that clans, at least when $M_f = +\infty$, may be characterized in terms of the behaviour of the difference $\ln f(t+t/m_f(t)) - \ln f(t)$, as $t \uparrow R$. This is the content of Theorem 4.16.

Theorem 4.10. *Let $f \in \mathcal{K}$ have radius of convergence $0 < R \le \infty$.*

If $R = \infty$, the power series f is a clan if and only if

$$\lim_{t\uparrow R} \frac{m_f(t+t/m_f(t))}{m_f(t)} = 1. \tag{4.11}$$

If $R < \infty$, the power series f is a clan if and only if (4.11) *holds, and besides,*

$$\lim_{t\uparrow R} (R-t)m_f(t) = \infty. \tag{4.12}$$

Remark 4.11. The classical Borel lemma, see for instance Chapter 9 of [**28**], claims that for any function $\mu(t)$ continuous and increasing in $[T,\infty)$ for some T, and if $a > 1$, there exists an exceptional set E of logarithmic measure at most $a/(a-1)$ so that

$$\mu(t+t/\mu(t)) \le a\mu(t), \quad \text{for any } t \in [T,+\infty) \setminus E.$$

Proof of Theorem 4.10: If f is a polynomial, then $\lim_{t\to\infty} m_f(t) = \deg(f)$, and thus (4.11) holds. We may assume thus that f is not a polynomial.

For the *direct part*, we assume that f is a clan and not a polynomial, and thus that $M_f = +\infty$. Consider $t \in (0,R)$, and denote by $\Delta(t)$ the supremum

$$\Delta(t) = \sup_{s\in[t,R)} \frac{\sigma_f^2(s)}{m_f(s)^2}.$$

Since f is a clan, $\lim_{t\uparrow R} \Delta(t) = 0$. Now, take $0 < t \le r \le s < R$. Since $\sigma_f^2(r) = r m_f'(r)$, we have that

$$\frac{m_f'(r)}{m_f(r)^2} \le \frac{\Delta(t)}{r}.$$

After integration in the interval (t, s) and using that $\ln y \le y-1$, for any $y > 1$, we get that

$$\frac{1}{m_f(t)} - \frac{1}{m_f(s)} \le \Delta(t) \ln \frac{s}{t} \le \frac{\Delta(t)}{t}(s-t). \tag{4.13}$$

If $R < \infty$, by taking the limit as $s \uparrow R$, and using that $M_f = +\infty$, we get that

$$\frac{1}{m_f(t)} \le \frac{\Delta(t)}{t}(R-t), \quad \text{for } t \in [0, R), \tag{4.14}$$

and since $\lim_{t\uparrow R} \Delta(t) = 0$, we deduce that (4.12) holds.

If R is finite, from (4.14) and because $\lim_{t\uparrow R} \Delta(t) = 0$, there exists $T \in (0, R)$ such that $t + t/m_f(t) < R$ for every $t \in (T, R)$; this, of course, is also true if $R = +\infty$. In (4.13), take $t \in (T, R)$ and $s = t + t/m_f(t) < R$ and multiply by $m_f(t) > 0$ on both sides to get

$$0 \le 1 - \frac{m_f(t)}{m_f(t + t/m_f(t))} \le \Delta(t).$$

Then (4.11) follows since $\lim_{t\uparrow R} \Delta(t) = 0$.

Next, the *converse part* of the statement.

Notice first that we have that $M_f = +\infty$. For $R = +\infty$, this follows since f is not a polynomial. For $R < +\infty$, this follows from the assumption (4.12).

Observe that, no matter whether R is finite or not, we always have that $t + t/m_f(t) < R$ for $t \in [T, R)$, for appropriate $T \in (0, R)$, and thus that $m_f(t+t/m_f(t))$ is well defined.

Denote $\lambda(t) = 1+1/m_f(t)$. Consider the power series $\mathcal{D}_f$. From Lemma 3.2 applied to $\mathcal{D}_f$, we have that

$$m_{\mathcal{D}_f}(t) \ln \lambda(t) < \ln \frac{\mathcal{D}_f(\lambda(t)t)}{\mathcal{D}_f(t)}, \quad \text{for } t \in [T, R). \tag{4.15}$$

Since $\mathcal{D}_f(t) = m_f(t)f(t)$, for $t \in [0, R)$, appealing again to Lemma 3.2, but now to f itself, we have that, for $t \in [T, R)$,

$$\ln \frac{\mathcal{D}_f(\lambda(t)t)}{\mathcal{D}_f(t)} = \ln \frac{m_f(\lambda(t)t)}{m_f(t)} + \ln \frac{f(\lambda(t)t)}{f(t)} \le \ln \frac{m_f(\lambda(t)t)}{m_f(t)} + m_f(\lambda(t)t) \ln \lambda(t). \tag{4.16}$$

Combining inequalities (4.15) and (4.16), dividing by $m_f(t) \ln \lambda(t)$, and using equation (2.15), we obtain that

$$1 \le \frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2} = \frac{m_{\mathcal{D}_f}(t)}{m_f(t)} \le \frac{\ln \frac{m_f(\lambda(t)t)}{m_f(t)}}{m_f(t) \ln \lambda(t)} + \frac{m_f(\lambda(t)t)}{m_f(t)}, \quad \text{for } t \in [T, R).$$

Since $\lim_{t\uparrow R} m_f(\lambda(t)t)/m_f(t)=1$, by (4.11), and $\lim_{t\uparrow R} m_f(t) \ln \lambda(t)=1$, because $\lim_{t\uparrow R} m_f(t) = \infty$, we deduce that $\lim_{t\uparrow R} \mathbf{E}(X_t^2)/\mathbf{E}(X_t)^2 = 1$, and thus that f is a clan. □

4.5. On the quotient $f(\lambda t)/f(t)$. Let f be in $\mathcal{K}$, with radius of convergence $R \le +\infty$. For $\lambda > 1$, we consider the quotient $f(\lambda t)/f(t)$. Bounds for this quotient appeared already in Lemma 3.2 of Simić.

Lemma 4.12. *If f is a power series in $\mathcal{K}$ with radius of convergence $R \le +\infty$, then for $\lambda > 1$, the function $f(\lambda t)/f(t)$ is increasing for $t \in (0, R/\lambda)$.*

Proof: Fix $\lambda > 1$. Consider the function $g(t) = \ln f(\lambda t) - \ln f(t)$, for $t \in (0, R/\lambda)$. The derivative of g multiplied by t is $m_f(\lambda t) - m_f(t)$, which is always positive, since m_f is strictly increasing in $(0, R)$. □

Lemma 4.13. *Let f be an entire power series in $\mathcal{K}$ and let $\lambda > 1$. If f is a polynomial of degree d, then $f(\lambda t)/f(t)$ is bounded for $t \in (0,\infty)$; in fact,* $\lim_{t\to\infty} f(\lambda t)/f(t) = \lambda^d$. *If f is transcendental, then* $\lim_{t\to\infty} f(\lambda t)/f(t) = +\infty$.

Proof: By Lemma 4.12, we have that $\lim_{t\to\infty} f(\lambda t)/f(t) = \sup_{t>0} f(\lambda t)/f(t) := H$, which could be $+\infty$.

If $H < +\infty$, we have, in particular, that $f(\lambda^n) \le H^n f(1)$, for each $n \ge 1$ and thus that f is a polynomial of degree at most $\ln H/\ln\lambda$. For if a_k is the k-th coefficient of f, then, by Cauchy estimates of coefficients, we have that $0 \le a_k \le f(\lambda^n)/\lambda^{n\,k} \le f(1)H^n/\lambda^{n\,k}$, for any $n \ge 1$, and so $a_k = 0$ for any $k > \ln H/\ln\lambda$. Conversely, if $f \in \mathcal{K}$ is a polynomial of degree d, then $\lim_{t\to\infty} f(\lambda t)/f(t) = \lambda^d$. □

Regarding Lemma 4.13, see [**25**, item 24], and compare with Lemma 3.2.

Lemma 4.14. *Let f be an entire power series in $\mathcal{K}$. Assume that for a continuous function $\lambda(t)$ defined in $(0,+\infty)$ with values in $(1,+\infty)$ we have that $f(\lambda(t)t)/f(t)$ is bounded for $t \in (0,+\infty)$. Then*

$$\lambda(t) = 1 + O(1/m_f(t)), \quad \text{as } t \to \infty.$$

Proof: Assume first that f is not a polynomial, and thus that $M_f = +\infty$. Let $J > 0$ be such that $f(\lambda(t)t)/f(t) \le e^J$, for every $t > 0$. Lemma 3.2 gives us that

$$m_f(t)\ln\lambda(t) \le J, \quad \text{for } t > 0. \tag{4.17}$$

Define $\delta(t) := \lambda(t) - 1$. Because of (4.17), we have that

$$m_f(t)\delta(t) \le (e^{J/m_f(t)} - 1)m_f(t), \quad \text{for } t > 0.$$

And so

$$\limsup_{t\to\infty} m_f(t)\delta(t) \le \limsup_{t\to\infty} (e^{J/m_f(t)} - 1)m_f(t) = J.$$

The equality on the right holds because $M_f = +\infty$.

For a polynomial f, if $f(\lambda(t)t)/f(t)$ is bounded for $t \in (0,+\infty)$, then it follows immediately that $\lambda(t)$ must be bounded. This is consistent with the conclusion of the lemma, since $\lim_{t\to\infty} m_f(t) = \deg(f)$. □

The discussion above leads us to consider most naturally the case $\lambda(t) = 1 + 1/m_f(t)$.

Lemma 4.15. *Let $f \in \mathcal{K}$ with radius of convergence $R \le \infty$. Assume that*

$$\text{there exists } T \in (0,R) \text{ such that } t + t/m_f(t) < R, \text{ for any } t \in [T,R). \tag{4.18}$$

Then

$$\frac{f(t + t/m_f(t))}{f(t)} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\mathbf{E}(X_t^k)}{\mathbf{E}(X_t)^k}, \quad \text{for any } t \in (T,R) \tag{4.19}$$

and, in particular,

$$\frac{f(t + t/m_f(t))}{f(t)} \ge \frac{1}{k!}\frac{\mathbf{E}(X_t^k)}{\mathbf{E}(X_t)^k}, \quad \text{for any } k \ge 0. \tag{4.20}$$

The condition (4.18) in Lemma 4.15 is satisfied whenever f is a clan. To see this, observe first that if $R = +\infty$, then (4.18) is obvious. Now, if $R < +\infty$, and f is a clan, then (4.12) of Theorem 4.10 gives, in particular, that $(R-t)m_f(t) > R > t$, for any $t \in [T,R)$, for some $T \in (0,R)$, and thus that $t + t/m_f(t) < R$, for $t \in [T,R)$, which is (4.18).

Proof of Lemma 4.15: Fix $t \in [T, R)$. The radius of convergence of the Taylor expansion of f around t exceeds $t/m_f(t)$, and this gives that

$$f(t + t/m_f(t)) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{t^k f^{(k)}(t)}{m_f(t)^k}.$$

Dividing by $f(t)$ and appealing to the expression (2.3) of the factorial moments of the (X_t) in terms of f, we obtain (4.19). The inequalities in (4.20) follow since all the summands in (4.19) are nonnegative. □

In the next result, as suggested by the second-order approximation (4.10), clans are characterized as in Theorem 4.10, but involving $\ln f(t)$ instead of $m_f(t)$.

Theorem 4.16. *Let $f \in \mathcal{K}$ with radius of convergence $R \leq \infty$. Assume that $M_f = +\infty$. Then f is a clan if and only if condition* (4.18) *holds and*

$$\lim_{t \uparrow R} \ln\left(\frac{f(t + t/m_f(t))}{f(t)}\right) = 1. \tag{4.21}$$

Concerning the hypothesis $M_f = +\infty$ of Theorem 4.16, observe that for any polynomial $f \in \mathcal{K}$ of degree N, we have that

$$\lim_{t \to \infty} \frac{f(t + t/m_f(t))}{f(t)} = \left(1 + \frac{1}{N}\right)^N < e.$$

Proof: Assume first that f is a clan.

We have observed after the statement of Lemma 4.15 that condition (4.18) is satisfied by clans. To verify (4.21), denote $\lambda(t) := 1 + 1/m_f(t)$, for $t \in (0, R)$. Observe that since $\lim_{t \uparrow R} m_f(t) = M_f = +\infty$, we have that

$$\lim_{t \uparrow R} m_f(t) \ln \lambda(t) = 1. \tag{4.22}$$

From Lemma 3.2, we have, for $T \in (0, R)$ as in condition (4.18), that

$$m_f(t) \ln \lambda(t) \leq \ln \frac{f(\lambda(t)t)}{f(t)} \leq \frac{m_f(\lambda(t)t)}{m_f(t)} (m_f(t) \ln \lambda(t)), \quad \text{for any } t \in [T, R).$$

Using (4.22) and (4.11) of Theorem 4.10, the limit (4.21) follows.

For the converse implication, assuming now that (4.18) and (4.21) hold, we will verify that f is a clan. It is enough to show that

$$\limsup_{t \uparrow R} \frac{\mathbf{E}(X_t^2)}{\mathbf{E}(X_t)^2} \leq 1,$$

or, because of Corollary 3.4 and the hypothesis $M_f = +\infty$, that

$$\limsup_{t \uparrow R} \frac{\mathbf{E}(X_t^{\underline{2}})}{\mathbf{E}(X_t)^2} \leq 1. \tag{4.23}$$

Using hypothesis (4.18) and Lemma 4.15 we obtain that

$$\frac{f(t + t/m_f(t))}{f(t)} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathbf{E}(X_t^{\underline{k}})}{\mathbf{E}(X_t)^k}, \quad \text{for } t \in [T, R).$$

Fix an integer $N \geq 3$. Since the summands above are all nonnegative, we may bound

$$\frac{f(t + t/m_f(t))}{f(t)} \geq \sum_{k=0}^{N} \frac{1}{k!} \frac{\mathbf{E}(X_t^{\underline{k}})}{\mathbf{E}(X_t)^k}, \quad \text{for } t \in [T, R).$$

We split the sum on the right separating the summands corresponding to $k \le 2$ and those with $3 \le k \le N$:

$$\frac{f(t+t/m_f(t))}{f(t)} \ge 1 + 1 + \frac{1}{2}\frac{\mathbf{E}(X_{\frac{2}{t}})}{\mathbf{E}(X_t)^2} + \sum_{k=3}^{N} \frac{1}{k!}\frac{\mathbf{E}(X_{\frac{k}{t}})}{\mathbf{E}(X_t)^k}.$$

For $3 \le k \le N$, we have that

$$\liminf_{t\uparrow R} \frac{\mathbf{E}(X_{\frac{k}{t}})}{\mathbf{E}(X_t)^k} = \liminf_{t\uparrow R} \frac{\mathbf{E}(X_{\frac{k}{t}})}{\mathbf{E}(X_t^k)}\frac{\mathbf{E}(X_t^k)}{\mathbf{E}(X_t)^k} \ge 1, \tag{4.24}$$

since Jensen's inequality gives that $\mathbf{E}(X_t^k) \ge \mathbf{E}(X_t)^k$ and $\lim_{t\uparrow R} \mathbf{E}(X_{\frac{k}{t}})/\mathbf{E}(X_t^k) = 1$, because of Corollary 3.4 and the hypothesis $M_f = +\infty$.

Now fix $\tau \in (0,1)$. Because of (4.24), there exists $S = S(N,\tau) \in [T,R)$ so that $\mathbf{E}(X_{\frac{k}{t}})/\mathbf{E}(X_t)^k \ge \tau$, for any $t \in [S,R)$ and any $3 \le k \le N$. Thus, we have that

$$\frac{f(t+t/m_f(t))}{f(t)} \ge 1 + 1 + \frac{1}{2}\frac{\mathbf{E}(X_{\frac{2}{t}})}{\mathbf{E}(X_t)^2} + \tau\sum_{k=3}^{N} \frac{1}{k!}, \quad \text{for } t \in [S,R).$$

From this inequality and the hypothesis (4.21), we deduce that

$$e \ge 1 + 1 + \frac{1}{2}\limsup_{t\uparrow R}\frac{\mathbf{E}(X_{\frac{2}{t}})}{\mathbf{E}(X_t)^2} + \tau\sum_{k=3}^{N} \frac{1}{k!}.$$

Now letting $\tau \uparrow 1$, and then $N \to \infty$, we obtain that

$$e \ge 1 + 1 + \frac{1}{2}\limsup_{t\uparrow R}\frac{\mathbf{E}(X_{\frac{2}{t}})}{\mathbf{E}(X_t)^2} + \left(e - 1 - 1 - \frac{1}{2}\right),$$

and conclude that (4.23) holds. □

In terms of the moment-generating function of X_t, clans are characterized as follows.

Corollary 4.17. *Let $f \in \mathcal{K}$ with radius of convergence $R \le \infty$. Assume that $M_f = +\infty$. Then f is a clan if and only if condition* (4.18) *holds and*

$$\lim_{t\uparrow R} \mathbf{E}(e^{(X_t - m_f(t))\nu(t)}) = 1, \tag{4.25}$$

where $\nu_f(t) := \ln(1 + 1/m_f(t))$ for $t > 0$.

Proof: The results follows from the expression

$$\mathbf{E}(e^{(X_t - m_f(t))\nu(t)}) = \frac{f(t+t/m_f(t))}{f(t)}\left(1 + \frac{1}{m_f(t)}\right)^{-m_f(t)}$$

and Theorem 4.16. □

Observe that, for polynomials in $\mathcal{K}$, (4.25) holds.

4.6. Moments of clans. Our next result shows that for a clan f, any moment, not just the second one, is asymptotically equivalent, as $t \uparrow R$, to the corresponding power of $m_f(t)$.

Theorem 4.18. *If $f \in \mathcal{K}$ with radius of convergence $R \le \infty$ is a clan with associated family $(X_t)_{t\in[0,R)}$, then*

$$\lim_{t\uparrow R} \frac{\mathbf{E}(X_t^p)}{\mathbf{E}(X_t)^p} = 1, \quad \textit{for every } p > 0. \tag{4.26}$$

As a consequence of Theorem 4.18, we can now prove that, as anticipated in Subsection 4.2, if f is a clan (with at least three nonzero coefficients), then $\mathcal{D}_f = zf'(z)$ is also a clan. Let $(X_t)_{t\in[0,R)}$ and $(W_t)_{t\in[0,R)}$ denote, respectively, the Khinchin families of f and of $\mathcal{D}_f$.

As observed in (2.14), the moments of the families (W_t) and (X_t) are related by

$$\mathbf{E}(W_t^p) = \frac{1}{m_f(t)}\mathbf{E}(X_t^{p+1}), \quad \text{for any } p > 0 \text{ and any } t \in (0, R).$$

Thus,

$$\frac{\mathbf{E}(W_t^2)}{\mathbf{E}(W_t)^2} = \frac{\mathbf{E}(X_t^3)}{m_f(t)}\frac{m_f(t)^2}{\mathbf{E}(X_t^2)^2} = \frac{\mathbf{E}(X_t^3)}{m_f(t)^3}\left(\frac{m_f(t)^2}{\mathbf{E}(X_t^2)}\right)^2.$$

Since f is a clan, both fractions on the far right tend towards 1 as $t \uparrow R$ and, consequently, $\mathcal{D}_f$ is, as claimed, also a clan.

As another consequence, which we have also anticipated, observe that since the partition function $P(z) = \prod_{k=1}^{\infty} 1/(1-z^k)$ is a clan, Theorem 4.18 and (2.6) give for the moments of its associated family (X_t) that, for any $p > 0$,

$$\mathbf{E}(X_t^p) \sim \mathbf{E}(X_t)^p \sim \frac{\zeta(2)^p}{(1-t)^{2p}}, \quad \text{as } t \uparrow 1.$$

Remark 4.19. Hayman, in Theorem III of [**11**], shows that the successive derivatives of Hayman (admissible) functions satisfy $t^k f^{(k)}(t)/f(t) \sim m_f(t)^k$, as $t \uparrow R$, which is equivalent to the conclusion of Theorem 4.18, i.e., $\lim_{t\uparrow R} \mathbf{E}(X_t^k)/\mathbf{E}(X_t)^k = 1$ for $k \geq 1$ integer. Our probabilistic proof below shows that this conclusion is valid under the simple and more general notion of clan.

Proof of Theorem 4.18: Due to Lemma 2.4, it is enough to prove (4.26) for any integer $k \geq 1$.

If f is a polynomial of degree N, we have that $\lim_{t\to\infty} \mathbf{E}(X_t^k) = N^k$ for any integer $k \geq 1$. In particular,

$$\lim_{t\to\infty} \frac{\mathbf{E}(X_t^k)}{\mathbf{E}(X_t)^k} = \frac{N^k}{N^k} = 1.$$

We assume now that f is not a polynomial and, consequently, that $M_f = +\infty$.

We first check that

$$\limsup_{t\uparrow R} \frac{\mathbf{E}(X_t^k)}{\mathbf{E}(X_t)^k} \leq ek!. \tag{4.27}$$

Corollary 3.4, using that $M_f = +\infty$, gives that the inequality (4.27) is equivalent to

$$\limsup_{t\uparrow R} \frac{\mathbf{E}(X_t^{\underline{k}})}{\mathbf{E}(X_t)^k} \leq ek!,$$

which follows from (4.20) and Theorem 4.16.

Denote $V_t := X_t/\mathbf{E}(X_t)$, for $t \in (0, R)$, so that, for any integer $k \geq 1$ and any $t \in (0, R)$,

$$\frac{\mathbf{E}(X_t^k)}{\mathbf{E}(X_t)^k} = \mathbf{E}(V_t^k).$$

We aim to show that, for a clan, $\lim_{t\uparrow R} \mathbf{E}(V_t^k) = 1$ for any integer $k \geq 1$. For $k = 1$, we have that $\mathbf{E}(V_t) = 1$, for any $t \in (0, R)$, and the case $k = 2$ is just the definition of clan.

By (4.27), for any integer $k \geq 1$, the moments of V_t satisfy that

$$\limsup_{t\uparrow R} \mathbf{E}(V_t^{2k}) \leq e(2k)!. \tag{4.28}$$

Fix an integer $k \geq 3$. Consider a constant $\omega > 0$ and apply the Jensen, Cauchy–Schwarz, and Chebyshev inequalities:

$$1 = \mathbf{E}(V_t)^k \leq \mathbf{E}(V_t^k) = \mathbf{E}(V_t^k \mathbf{1}_{\{|V_t-1|>\omega\}}) + \mathbf{E}(V_t^k \mathbf{1}_{\{|V_t-1|\leq\omega\}})$$
$$\leq \mathbf{E}(V_t^{2k})^{1/2}\mathbf{P}(|V_t-1|>\omega)^{1/2} + (1+\omega)^k \leq \mathbf{E}(V_t^{2k})^{1/2}\frac{\sigma_f(t)}{m_f(t)}\frac{1}{\omega} + (1+\omega)^k,$$

where by $\mathbf{1}_A$ we denote the indicator function of the event A. Since f is a clan, $\lim_{t\uparrow R}\sigma_f(t)/m_f(t) = 0$, and this and the bound in (4.28) combine to imply that

$$1 \leq \limsup_{t\uparrow R}\mathbf{E}(V_t^k) \leq (1+\omega)^k,$$

for any $\omega > 0$. Therefore, $\limsup_{t\uparrow R}\mathbf{E}(V_t^k) = 1$. Now since $k \geq 1$, we have that $\mathbf{E}(V_t^k) \geq \mathbf{E}(V_t)^k = 1$ for any $t \in (0,R)$, and we get, as desired, that

$$\lim_{t\uparrow R}\mathbf{E}(V_t^k) = 1, \quad \text{for every integer } k \geq 1. \qquad \square$$

Remark 4.20. Theorem 4.18 does not hold for general families or sequences of random variables, i.e., if $(V_n)_{n\geq 1}$ is a sequence of nonnegative random variables, then the condition $\lim_{n\to\infty}\mathbf{E}(V_n^2)/\mathbf{E}(V_n)^2 = 1$ does not imply that $\lim_{n\to\infty}\mathbf{E}(V_n^p)/\mathbf{E}(V_n)^p = 1$ for $p > 2$ (although this would be the case for $p < 2$, because of Lemma 2.4).

Indeed, for $n \geq 1$, define V_n taking the value $\sqrt{n}/\ln(n+1)$ with probability $1/(n+1)$ and the value $\frac{1}{n}((n+1) - \sqrt{n}/\ln(n+1))$ with probability $n/(n+1)$.

For this sequence of random variables, we have that $\mathbf{E}(V_n) = 1$ for any $n \geq 1$ and $\lim_{n\to\infty}\mathbf{E}(V_n^p) = 1$, if $p \leq 2$, but $\lim_{n\to\infty}\mathbf{E}(V_n^p) = +\infty$, if $p > 2$.

5. Order of entire functions and power series distributions

In this section, we deal with entire functions f in $\mathcal{K}$. In particular, we shall be interested in the relation between the order (of growth) of f and the growth of the mean $m_f(t) = \mathbf{E}(X_t)$, of the variance $\sigma_f^2(t) = \mathbf{V}(X_t)$, and of moments $\mathbf{E}(X_t^p)$, with $p > 0$, of the associated family of probability distributions.

Recall that the order $\rho(f)$ of an entire function f in $\mathcal{K}$ is given by

$$\rho(f) := \limsup_{t\to\infty}\frac{\ln\ln\max_{|z|=t}\{|f(z)|\}}{\ln t} = \limsup_{t\to\infty}\frac{\ln\ln f(t)}{\ln t}, \tag{5.1}$$

where we have used (2.1) in the second expression. On the other hand, for any entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, Hadamard's formula (see Theorem 2.2.2 in [**6**]) gives $\rho(f)$ in terms of the coefficients of f:

$$\rho(f) = \limsup_{n\to\infty}\frac{n\ln n}{\ln(1/|a_n|)}. \tag{5.2}$$

5.1. The order of f entire and the moments $\mathbf{E}(X_t^p)$.

We can express the order of an entire function $f \in \mathcal{K}$ in terms of the mean $m_f(t)$ and, in fact, of any moment $\mathbf{E}(X_t^p)$, as follows.

Theorem 5.1. *Let $f \in \mathcal{K}$ be an entire function of order $\rho(f) \leq +\infty$. Then*

$$\limsup_{t\to\infty}\frac{\ln[\mathbf{E}(X_t^p)^{1/p}]}{\ln t} = \rho(f), \quad \textit{for any } p \geq 1. \tag{5.3}$$

The case $p = 1$ of Theorem 5.1, i.e., $\limsup_{t\to\infty}\ln m_f(t)/\ln t = \rho(f)$, appears in item 52 on page 9 of [**25**].

Proof: We abbreviate and write

$$\Lambda_p := \limsup_{t\to\infty} \frac{\ln[\mathbf{E}(X_t^p)^{1/p}]}{\ln t}, \quad \text{for } p > 0.$$

Observe that $\Lambda_p \le \Lambda_q$ if $0 < p \le q$, by Jensen's inequality.

First we show that

$$\Lambda_p \le \rho(f), \quad \text{for any } p > 0.$$

Fix $p > 0$. The inequality holds trivially if $\rho(f) = +\infty$, so we may assume $\rho(f) < +\infty$. Let $\omega > \rho(f)$ and take $\tau = (\omega + \rho(f))/2$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, for $z \in \mathbb{C}$, then Hadamard's formula (5.2) gives $N = N_\tau > 0$ such that

$$a_n \le \frac{1}{n^{n/\tau}}, \quad \text{if } n \ge N.$$

For t such that $t^\tau > N$, we have

$$\begin{aligned} f(t)\mathbf{E}(X_t^p) &= \sum_{n=1}^{\infty} n^p a_n t^n = \sum_{n\le t^\omega} n^p a_n t^n + \sum_{n>t^\omega} n^p a_n t^n \\ &\le t^{p\omega} f(t) + \sum_{n\ge 1} n^p \frac{1}{n^{n/\tau}} n^{n/\omega} = t^{p\omega} f(t) + C, \end{aligned}$$

where $C = C(\omega, \rho(f), p) < +\infty$. Thus,

$$\mathbf{E}(X_t^p) \le t^{p\omega} + C/f(t), \quad \text{if } t^\tau > N,$$

and so $\Lambda_p \le \omega$, for every $\omega > \rho(f)$, which implies, as desired, that $\Lambda_p \le \rho(f)$ for $p > 0$ fixed above.

To finish the proof, it is enough to show that $\rho(f) \le \Lambda_1$, because $\Lambda_1 \le \Lambda_p$ for $p \ge 1$, and this, combined with the fact that $\Lambda_p \le \rho(f)$, for any $p > 0$, would give (5.3).

We may assume that $\Lambda_1 < +\infty$, since otherwise there is nothing to prove. We observe first that, for any $\omega > \Lambda_1$, there exists $T = T_\omega$ such that

$$m_f(t) = \mathbf{E}(X_t) \le t^\omega, \quad \text{for } t \ge T,$$

which, recall (2.2), can be written in terms of f as

$$\frac{f'(t)}{f(t)} < t^{\omega-1}, \quad \text{for } t \ge T.$$

Upon integration, the above inequality gives that

$$\ln f(t) - \ln f(T) \le \frac{1}{\omega}(t^\omega - T^\omega), \quad \text{if } t \ge T,$$

which implies, by the very definition (5.1) of order, that $\rho(f) \le \omega$. From this, we deduce, as desired, that $\rho(f) \le \Lambda_1$. □

Remark 5.2. We mention that, if $\rho(f)$ is finite, then (5.3) holds also for $p \in (0,1)$.

To see this, fix $p \in (0,1)$. We just need to show that $\rho(f) \le \Lambda_p$. We are going to interpolate Λ_1 between Λ_p and Λ_2.

Let $u = 1/(2-p) \in (1/2, 1)$, so that

$$1 = pu + 2(1-u).$$

By Hölder's inequality, we have that

$$\mathbf{E}(X_t) = \mathbf{E}(X_t^{pu} X_t^{2(1-u)}) \le \mathbf{E}(X_t^p)^u \mathbf{E}(X_t^2)^{1-u}, \quad \text{for any } t > 0,$$

and thus,

$$\Lambda_1 \le pu\,\Lambda_p + 2(1-u)\Lambda_2.$$

By (5.3), $\Lambda_1 = \Lambda_2 = \rho(f) < +\infty$, so

$$\rho(f) \le pu\,\Lambda_p + 2(1-u)\rho(f).$$

This, substituting the value of u, gives that $\rho(f) \le \Lambda_p$.

5.2. Entire gap series, order, and clans. Recall, from Definition 4.1, that an entire function f in $\mathcal{K}$ is a clan if

$$\lim_{t\to\infty} \frac{\sigma_f(t)}{m_f(t)} = 0.$$

We shall now exhibit examples of entire functions in $\mathcal{K}$ of *any given order* ρ, $0 \le \rho \le \infty$, *which are not clans.* These (counter)examples will be used in forthcoming discussions.

Fix $0 < \rho < +\infty$.

Consider the increasing sequence of integers given by $n_1 = 0$, $n_2 = 1$, and $n_{k+1} = kn_k$, for any $k \ge 2$, and let

$$f(z) = 1 + \sum_{k=2}^{\infty} \frac{1}{n_k^{n_k/\rho}} z^{n_k}.$$

The function f is entire and belongs to $\mathcal{K}$. Hadamard's formula (5.2) gives $\rho(f) = \rho$.

Recall the gap parameter $\overline{G}(f)$ given in (3.7), and observe that, in this case, $\overline{G}(f) = \limsup_{k\to\infty} k = +\infty$, so from Theorem 3.7 we deduce that

$$\limsup_{t\to\infty} \frac{\sigma_f(t)}{m_f(t)} \ge 1 \tag{5.4}$$

holds, and, in particular, that f is not a clan.

With the same specification of the sequence n_k, the power series g and h in $\mathcal{K}$ given by

$$g(z) = 1 + \sum_{k=2}^{\infty} \frac{1}{n_k^{n_k^2}} z^{n_k} \quad \text{and} \quad h(z) = 1 + \sum_{k=2}^{\infty} \frac{1}{n_k^{n_k/\sqrt{\ln n_k}}} z^{n_k}$$

are entire, of respective orders $\rho(g) = 0$ and $\rho(h) = +\infty$, and are such that (5.4) holds, and so, in particular, they are not clans.

The examples above of entire power series which are not clans are based on the seminal examples of Borel [**8**], see also [**37**] of Whittaker, of entire functions whose lower order does not coincide with the order.

5.3. The order of f entire and the quotient $\sigma_f^2(t)/m_f(t)$. The next result compares the order of the entire function with the quotient $\sigma_f^2(t)/m_f(t)$ as $t \to \infty$.

Proposition 5.3. *Let $f \in \mathcal{K}$ be an entire function of order $\rho(f) \le \infty$. Then*

$$\liminf_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)} \le \liminf_{t\to\infty} \frac{\ln m_f(t)}{\ln t} \le \liminf_{t\to\infty} \frac{\ln\ln f(t)}{\ln t}$$

$$\le \limsup_{t\to\infty} \frac{\ln\ln f(t)}{\ln t} = \limsup_{t\to\infty} \frac{\ln m_f(t)}{\ln t} = \rho(f) \le \limsup_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)}.$$

The equality statements in the middle of the comparisons of Proposition 5.3 are the very definition of order and the case $p = 1$ of Theorem 5.1. On the other hand, Báez-Duarte shows in Proposition 7.7 of [**3**] that

$$\liminf_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)} \le \rho(f) \le \limsup_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)}.$$

Proof of Proposition 5.3: We will check first that

$$\limsup_{t\to\infty} \frac{\ln\ln f(t)}{\ln t} \le \limsup_{t\to\infty} \frac{\ln m_f(t)}{\ln t} \le \limsup_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)}. \tag{5.5}$$

Of course, we already know that the first two lim sup coincide with the order $\rho(f)$.

For the inequality on the right of (5.5), let us denote

$$L = \limsup_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)}$$

and assume that $L < +\infty$, since otherwise there is nothing to prove. Recall that

$$\frac{\sigma_f^2(t)}{m_f(t)} = \frac{t m_f'(t)}{m_f(t)}.$$

Take $\omega > L$. Then there exists $T > 0$ such that

$$\frac{t m_f'(t)}{m_f(t)} \le \omega, \quad \text{for any } t \ge T,$$

and, by integration, for $t > T$,

$$\ln m_f(t) - \ln m_f(T) \le \omega(\ln t - \ln T),$$

and thus

$$\limsup_{t\to\infty} \frac{\ln m_f(t)}{\ln t} \le \omega, \quad \text{and consequently,} \quad \limsup_{t\to\infty} \frac{\ln m_f(t)}{\ln t} \le L.$$

The proof of the inequality on the left of (5.5),

$$\limsup_{t\to\infty} \frac{\ln\ln f(t)}{\ln t} \le \limsup_{t\to\infty} \frac{\ln m_f(t)}{\ln t},$$

follows as above using that

$$t(\ln f(t))' = m_f(t).$$

The proof of the inequalities for the lim inf,

$$\liminf_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)} \le \liminf_{t\to\infty} \frac{\ln m_f(t)}{\ln t} \le \liminf_{t\to\infty} \frac{\ln\ln f(t)}{\ln t},$$

is analogous. □

Concerning Proposition 5.3, a few observations are in order.

(i) The three lim inf in the statement, in general, do not give the order of f, since the third one is the lower order of f, which, for instance, and again by Borel [**8**], does not have to coincide with the order.

(ii) As for the third lim sup in the second line: as pointed out by Báez-Duarte (see [**3**, p. 100]), in general, the order $\rho(f)$ is not given by $\limsup_{t\to\infty} \sigma_f^2(t)/m_f(t)$, as proposed by Kosambi in Lemma 4 of [**17**]. The example of Baéz-Duarte is the canonical product f given by

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z^{n^2}}{n^{4n^2}}\right), \tag{5.6}$$

which is an entire function in $\mathcal{K}$. Borel's theorem (see, for instance, Theorem 2.6.5 in [**6**]) tells us that the order $\rho(f)$ of any canonical product coincides with the exponent of convergence of its zeros. The zeros of f have exponent of convergence $3/4$, and thus f has order $\rho(f) = 3/4$. From the case $p = 1$ of Theorem 5.1, we see that f satisfies that, say, $m_f(t) \le C\,t^{7/8}$, for some $C > 0$ and every $t \ge 1$. As $f(n^4 e^{i\pi/n^2}) = 0$, we obtain from Lemma 3.9 that

$$\sigma_f^2(n^4) \ge \frac{1}{4} n^4, \quad \text{for any } n \ge 1,$$

and thus that

$$\frac{\sigma_f^2(n^4)}{m_f(n^4)} \ge \frac{1}{4C} n^{1/2}, \quad \text{for any } n \ge 1.$$

So, for this particular function f, we have that $\limsup_{t\to\infty} \sigma_f^2(t)/m_f(t) = +\infty$, but $\liminf_{t\to\infty} \sigma_f^2(t)/m_f(t) \le \rho(f) = 3/4$.

Alternatively, and more generally, recall that, for any $\rho \in [0, +\infty]$, we have exhibited in Subsection 5.2 an entire transcendental function $f \in \mathcal{K}$, with order $\rho(f) = \rho$, and such that $\limsup_{t\to\infty} \sigma_f(t)/m_f(t) \ge 1$, and thus such that $\limsup_{t\to\infty} \sigma_f^2(t)/m_f(t) = +\infty$. Also, $\liminf_{t\to\infty} \sigma_f^2(t)/m_f(t) \le \rho(f) < +\infty$, because of Proposition 5.3.

(iii) On the other hand, as a consequence of Proposition 5.3, for an entire function $f \in \mathcal{K}$, we have that *if the limit* $\lim_{t\to\infty} \sigma_f^2(t)/m_f(t)$ *exists* (including the possibility of being ∞), then the order $\rho(f)$ of f is precisely

$$\rho(f) = \lim_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)}. \tag{5.7}$$

In Báez-Duarte's example (5.6), the limit of $\sigma_f^2(t)/m_f(t)$ as $t \to \infty$ does not exist, and (5.7) does not hold. In general, (5.7) does not hold for any entire function f in $\mathcal{K}$ for which the lower order does not coincide with the order.

For the class of entire functions in $\mathcal{K}$ of genus 0 which we are to discuss in Subsection 5.4 below, the identity (5.7) holds (see the comments after Proposition 5.4).

The identity (5.7) also holds for the class of nonvanishing entire functions in $\mathcal{K}$ of finite order. To see this, let $f(z) = e^{g(z)} \in \mathcal{K}$, where g is entire (not necessarily in $\mathcal{K}$). We may assume that $g(t) \in \mathbb{R}$ for $t > 0$, and further that, for some $T > 0$, $g(t) > 0$ for $t \ge T$. Assume that f has finite order. Hadamard's factorization theorem gives that g is a polynomial, say of degree N. Thus $\rho(f) = N$, by (5.1). If the leading coefficient of the polynomial g is b, then

$$m_f(t) = t f'(t)/f(t) = t g'(t) \sim bN\,t^N, \qquad \text{as } t \to \infty,$$

$$\sigma_f^2(t) = t m_f'(t) = t g'(t) + t^2 g''(t) \sim bN^2\,t^N, \quad \text{as } t \to \infty.$$

Therefore, (5.7) holds.

5.4. Entire functions of genus 0 with negative zeros. We consider again the entire transcendental functions f in $\mathcal{K}$ of genus 0 (and thus of order ≤ 1) whose zeros are all real and negative which we have considered in Subsection 2.2.5. Recall that, if normalized so that $f(0) = 1$, they are the canonical products of the form

$$f(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{b_j}\right), \quad \text{for } z \in \mathbb{C}, \tag{5.8}$$

and $(b_j)_{j\ge 1}$ is an increasing sequence of positive real numbers with $\sum_{j=1}^{\infty} 1/b_j < \infty$. The zeros $-b_1, -b_2, -b_3, \ldots$ of f all lie on the negative real axis.

We keep the notation $N(t)$ for the counting function of the zeros of f:

$$N(t) = \#\{j \ge 1 : b_j \le t\}, \quad \text{for } t > 0,$$

which is a nondecreasing function such that $N(t) \to \infty$ as $t \to \infty$.

Recall also that Borel's theorem tells us that the order $\rho(f)$ of the canonical product coincides with the exponent of convergence of its zeros, which, in turn (see Theorem 2.5.8 in [**6**]), is given by

$$\limsup_{t\to\infty} \frac{\ln N(t)}{\ln t} = \rho(f).$$

The function $\ln f(t)$, for $t \in (0, +\infty)$, may be expressed in terms of the counting function $N(t)$; concretely, by integration by parts, one obtains

$$\ln f(t) = \sum_{j=1}^{\infty} \ln\left(1 + \frac{t}{b_j}\right) = \int_0^{\infty} \frac{tN(x)}{x(x+t)}\, dx = \int_0^{\infty} \frac{N(ty)}{y(y+1)}\, dy. \tag{5.9}$$

This representation (5.9) of $\ln f(t)$ in terms of the counting function $N(t)$ comes from Valiron; see [**36**].

From precise asymptotic information on the counting function $N(t)$ of f, one may obtain asymptotic information on the mean and variance function of the family associated to f.

Proposition 5.4. *Let f be an entire function of genus 0 with only negative zeros, and given by (5.8). Assume that for $\rho \in (0,1)$ we have*

$$N(t) \sim Ct^{\rho}, \quad \text{as } t \to \infty. \tag{5.10}$$

Then

(a) $\ln f(t) \sim \dfrac{C\pi}{\sin(\pi\rho)}\, t^{\rho}$, *as* $t \to \infty$,

(b) $m_f(t) \sim \dfrac{C\pi\rho}{\sin(\pi\rho)}\, t^{\rho}$, *as* $t \to \infty$,

(c) $\sigma_f^2(t) \sim \dfrac{C\pi\rho^2}{\sin(\pi\rho)}\, t^{\rho}$, *as* $t \to \infty$.

Conversely, if (a), (b), or (c) holds, then (5.10) holds.

It follows that for entire functions of genus 0 given by formula (5.8), and whenever $N(t) \sim Ct^{\rho}$ as $t \to \infty$, with $\rho \in (0,1)$, then

$$\lim_{t\to\infty} \frac{\sigma_f^2(t)}{m_f(t)} = \rho,$$

as was pointed out (but left unproved) by Báez-Duarte in Proposition 7.9 of [**3**], and also that equality holds in Proposition 5.3. Furthermore, it follows that the entire functions of Proposition 5.4 are clans; but see Proposition 5.7 for a more general statement.

Proof: That (c) $\Rightarrow$ (b) $\Rightarrow$ (a) follows immediately by integration, since $m_f(t) = t(\ln f)'(t)$ and $\sigma_f^2(t) = tm_f'(t)$; recall the formulas (2.2).

That (a) implies (5.10) is a classical Tauberian theorem; see [**36**] and [**33, 34**].

The proof of the direct part of Proposition 5.4 follows from the representation (5.9) and the following standard identity for the Euler Beta function:

$$\int_0^{\infty} \frac{y^{\eta}}{(1+y)^2}\, dy = \mathrm{Beta}(1+\eta, 1-\eta) = \frac{\pi\eta}{\sin(\pi\eta)}, \quad \text{for any } \eta \in [0,1). \tag{5.11}$$

The representation (5.9) gives that

$$\ln f(t) = t^\rho \int_0^\infty \frac{y^\rho}{y(y+1)} \frac{N(ty)}{(ty)^\rho}\, dy.$$

Since $N(ty)/(ty)^\rho \to C$ as $t \to \infty$ and $y^\rho/(y(y+1))$ is integrable in $[0,\infty)$,

$$\lim_{t\to\infty} \frac{\ln f(t)}{t^\rho} = C \int_0^\infty \frac{y^\rho}{y(y+1)}\, dy.$$

Integrating by parts, using that $\rho < 1$ and (5.11), we obtain that

$$\int_0^\infty \frac{y^\rho}{y(y+1)}\, dy = \frac{1}{\rho}\int_0^\infty \frac{y^\rho}{(y+1)^2}\, dy = \frac{\pi}{\sin(\pi\rho)}.$$

That is,

$$\ln f(t) \sim \frac{C\pi}{\sin(\pi\rho)} t^\rho, \quad \text{as } t \to \infty.$$

For the mean and the variance, we have the representations

$$m_f(t) = t^\rho \int_0^\infty \frac{y^\rho}{(y+1)^2} \frac{N(ty)}{(ty)^\rho}\, dy \quad \text{and} \quad \sigma_f^2(t) = t^\rho \int_0^\infty \frac{y^\rho (y-1)}{(y+1)^3} \frac{N(ty)}{(ty)^\rho}\, dy.$$

Arguing as above, (b) and (c) follow from (5.10). □

Remark 5.5. For general canonical products f of genus $p \geq 0$ with only negative zeros, there is a representation of $\ln f(t)$ analogous to (5.9), which is the case $p = 0$ (see, for instance, Theorem 7.2.1 in [**5**]):

$$\ln f(t) = (-1)^p \int_0^\infty \frac{(t/x)^{p+1}}{1 + t/x} N(x) \frac{dx}{x}.$$

This expression means in particular that for p odd, the canonical product f is bounded by 1, for $t \in (0,\infty)$, and thus shows that f is not in $\mathcal{K}$. Observe, in any case, that for a primary factor $E_p(z)$, which for $|z| < 1$ is $E_p(z) = \exp\big(-\sum_{j=p+1}^\infty z^j/j\big)$, we have that its $(2p+2)$-coefficient is $-p/(2(p+1)^2)$, and therefore $E_p(-z/a)$ with $a > 0$, which vanishes at $-a$, is not in $\mathcal{K}$, if $p \geq 1$; in general, thus, canonical products of nonzero genus with only negative zeros are not in $\mathcal{K}$.

Remark 5.6. For entire functions f of genus 0 and only negative zeros, if the number of zeros is *comparable* to a power t^ρ with $\rho \in (0,1)$ ($N(t) \asymp t^\rho$, as $t \to \infty$), so are the mean and variance of its Khinchin family. This follows most directly from the representation (5.9).

Entire functions of genus 0 with only negative zeros are always clans.

Proposition 5.7. *Every entire function f in $\mathcal{K}$ defined by* (5.8) *with $\sum_{j\geq 1} 1/b_j < +\infty$ is a clan.*

Proof: Assume that f is not a polynomial. For f given by (5.8), we have, recalling (2.10), that

$$\sigma_f^2(t) < m_f(t),$$

and thus $\sigma_f^2(t)/m_f^2(t) \leq 1/m_f(t)$. Since $m_f(t) \to \infty$ as $t \to \infty$, we obtain that f is a clan. □

5.5. Entire functions, proximate orders, and clans. For $\rho \geq 0$, a *proximate ρ-order* $\rho(t)$ is a continuously differentiable function defined in $(0, +\infty)$ and such that

$$\lim_{t\to\infty} \rho(t) = \rho \quad \text{and} \quad \lim_{t\to\infty} \rho'(t) t \ln t = 0.$$

Traditionally, proximate orders are allowed to have a discrete set of points where they are not differentiable, but have both one-sided derivatives at those points; see Section 7.4 in [**5**].

If for a proximate ρ-order $\rho(t)$ we write $V(t) = t^{\rho(t)}$, for $t > 0$, then (see, for instance, Lemma 5 in Section 12, Chapter I, of [**18**]) for every $\lambda > 0$ we have that

$$\lim_{t\to\infty} \frac{V(\lambda t)}{V(t)} = \lambda^{\rho}. \tag{5.12}$$

In other terms, the function $V(t)/t^{\rho}$ is a *slowly varying* function.

Theorem 5.8 (Valiron's proximate theorem for $\mathcal{K}$)**.** *If f is an entire function in $\mathcal{K}$ of finite order $\rho \geq 0$, then there is a proximate ρ-order $\rho(t)$ such that*

$$\limsup_{t\to\infty} \frac{\ln f(t)}{t^{\rho(t)}} = 1.$$

See Theorem 7.4.2 in [**5**]; the smoothness which we require in our definition of proximate ρ-order $\rho(t)$ is provided by Proposition 7.4.1 and Theorem 1.8.2 (the smooth variation theorem) in [**5**].

We have the following.

Theorem 5.9. *Let f be an entire function in $\mathcal{K}$ of finite order $\rho > 0$, let $\rho(t)$ be a proximate ρ-order, and let $\tau > 0$. Then*

$$\lim_{t\to\infty} \frac{\ln f(t)}{t^{\rho(t)}} = \tau \tag{5.13}$$

if and only if

$$\lim_{t\to\infty} \frac{m_f(t)}{t^{\rho(t)}} = \tau\rho. \tag{5.14}$$

If either (5.13) *or* (5.14) *holds, then f is a clan.*

Observe that in both (5.13) and (5.14) a 'lim' is assumed, and not a 'lim sup' as in Valiron's Theorem 5.8, which encompasses all finite-order entire functions. Condition (5.13) of Theorem 5.9 concerns entire functions that are said to have *regular growth.*

Comparing with Valiron's theorem, the limit τ instead of 1 amounts to no extra generality, since replacing $\rho(t)$ with $\rho^{\star}(t) = \rho(t) + \ln\tau/\ln t$, we have that $\rho^{\star}(t)$ is also a proximate ρ-order and

$$\lim_{t\to\infty} \frac{\ln f(t)}{t^{\rho^{\star}(t)}} = 1.$$

That (5.14) implies that f is a clan is due to Simić, [**32**]. In [**32**], see also [**31**], it is claimed, in the terminology of the present paper, that any entire function of *finite order* in $\mathcal{K}$ is a clan, which is not the case; see, for instance, Subsection 5.2. The error in the argument originates in a misprint in the statement of Theorem 2.3.11 on page 81 of [**5**]: the lim sup appearing in that statement should be a lim inf (which is what is actually proved in [**5**]). See also page 101 of [**19**] for a similar warning. The argument of [**32**] shows precisely that (5.14) implies that f is a clan.

As for the implication (5.13) $\Rightarrow$ (5.14), compare with Lemma 3.1 in [**21**].

For constant proximate ρ-order, ($\rho(t) = \rho$, for $t > 0$) that (5.13) implies (5.14) and then that f is a clan is due to Pólya and Szegö with an argument involving some delicate estimates: combine items 70 and 71, on page 12, of their [**25**].

Proof: Fix $\lambda > 1$. Lemma 3.2 gives us that

$$m_f(t)\ln\lambda \le \ln f(\lambda t) - \ln f(t) \le m_f(\lambda t)\ln\lambda, \quad \text{for } t > 0,$$

and thus dividing by $V(t) = t^{\rho(t)}$,

$$\text{(5.15)} \qquad \frac{m_f(t)}{V(t)} \le \frac{1}{\ln\lambda}\left[\frac{\ln f(\lambda t)}{V(\lambda t)}\frac{V(\lambda t)}{V(t)} - \frac{\ln f(t)}{V(t)}\right] \le \frac{m_f(\lambda t)}{V(\lambda t)}\frac{V(\lambda t)}{V(t)}, \quad \text{for } t > 0.$$

We first prove that (5.13) ⇒ (5.14). From (5.12), (5.13), and (5.15), and letting $t \to \infty$, we deduce that

$$\limsup_{t\to\infty} \frac{m_f(t)}{t^{\rho(t)}} \le \tau\frac{\lambda^\rho - 1}{\ln\lambda} \le \lambda^\rho \liminf_{t\to\infty}\frac{m_f(t)}{t^{\rho(t)}}.$$

Letting $\lambda \downarrow 1$, equation (5.14) follows.

We now prove that (5.14) ⇒ (5.13). Assume first that

$$\text{(5.16)} \qquad \limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} < +\infty.$$

If (5.16) holds, then from the first inequality of (5.15), and using (5.14), we deduce that

$$\tau\rho\ln\lambda + \liminf_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \le \lambda^\rho \liminf_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}},$$

while the second inequality of (5.15) gives

$$\lambda^\rho \limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \le \limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} + \tau\rho\lambda^\rho\ln\lambda.$$

Writing the two inequalities above as

$$\tau\rho \le \frac{\lambda^\rho - 1}{\ln\lambda}\liminf_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \quad \text{and} \quad \frac{\lambda^\rho - 1}{\ln\lambda}\limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \le \tau\rho\lambda^\rho,$$

and by letting $\lambda \downarrow 1$, we get that

$$\tau \le \liminf_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \quad \text{and} \quad \limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \le \tau,$$

so (5.13) follows.

To show that (5.16) holds, we first observe that

$$\frac{d}{ds}s^{\rho(s)} = (s\ln s\cdot\rho'(s) + \rho(s))s^{\rho(s)-1}, \quad \text{for } s > 0,$$

and from the defining properties of the proximate orders, we deduce that, for appropriately large $A > 0$,

$$\text{(5.17)} \qquad \frac{d}{ds}s^{\rho(s)} \ge \frac{\rho}{2}s^{\rho(s)-1}, \quad \text{for each } s \ge A.$$

From (5.14), by incrementing A if necessary, we deduce that

$$\frac{d}{ds}\ln f(s) = \frac{m_f(s)}{s} \le 2\tau\rho\, s^{\rho(s)-1} \le 4\tau\frac{d}{ds}s^{\rho(s)}, \quad \text{for } s \ge A,$$

where (5.17) was used in the last inequality. We thus have that

$$\ln f(t) \le \ln f(A) + 4\tau\, t^{\rho(t)} - 4\tau A^{\rho(A)}, \quad \text{for } t \ge A,$$

from which we obtain that

$$\limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}} \le 4\tau,$$

as wanted.

This argument shows in fact that

$$\frac{1}{\rho}\liminf_{t\to\infty}\frac{m_f(t)}{t^{\rho(t)}}\le\liminf_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}}\le\limsup_{t\to\infty}\frac{\ln f(t)}{t^{\rho(t)}}\le\frac{1}{\rho}\limsup_{t\to\infty}\frac{m_f(t)}{t^{\rho(t)}}.$$

For another proof of this last chain of inequalities, see, for instance, Theorem 4 in [**35**].

Finally, we prove that (5.14) $\Rightarrow$ f is a clan. From (5.12) and (5.14), and taking into account that $\rho>0$, we deduce that

$$\lim_{t\to\infty}\frac{m_f(\lambda t)}{m_f(t)}=\lambda^\rho,\quad\text{for any }\lambda>0, \tag{5.18}$$

and thus that the mean m_f is a regularly varying function; see Section 1.4 of [**5**].

To show that f is a clan, we may assume that f is not a polynomial, and thus that $\lim_{t\to\infty}m_f(t)=+\infty$.

Now, (5.18) implies that for any function $\lambda(t)$ such that $\lambda(t)>1$ and such that $\lim_{t\to\infty}\lambda(t)=1$, we have that

$$\lim_{t\to\infty}\frac{m_f(\lambda(t)\,t)}{m_f(t)}=1. \tag{5.19}$$

To see this, fix $\varepsilon>0$. Then we have that $1<\lambda(t)\le 1+\varepsilon$, for $t\ge t_\varepsilon$. Therefore,

$$1\le\frac{m_f(\lambda(t)\,t)}{m_f(t)}\le\frac{m_f((1+\varepsilon)t)}{m_f(t)},\quad\text{for }t\ge t_\varepsilon.$$

Thus, $\limsup_{t\to\infty}m_f(\lambda(t)\,t)/m_f(t)\le(1+\varepsilon)^\rho$, and thus (5.19) holds.

Applying (5.19) with $\lambda(t)=1+1/m_f(t)$ and appealing to Theorem 4.10, we conclude that f is a clan. □

5.6. Exceptional values and clans. The entire gap series of Subsection 5.2, which are our basic examples of entire functions in $\mathcal{K}$ which are not clans, have no Borel exceptional values. This follows, for instance, from a classical result of Pfluger and Pólya [**23**]. Next we show that, in general, entire functions which are not clans have no Borel exceptional values.

Recall that, by definition, a is a Borel exceptional value of an entire function f of finite order if the exponent of convergence of the a-values of f (i.e., the zeros of $f(z)-a=0$) is strictly smaller than the order of f; a theorem of Borel claims that a nonconstant entire function of finite order can have at most one Borel exceptional value.

Theorem 5.10. *If the entire function $f\in\mathcal{K}$ has finite order and has one Borel exceptional value, then f is a clan.*

Proof: Let ρ be the order of f. Assume that $a\in\mathbb{C}$ is the Borel exceptional value of f. Denote by s the exponent of convergence of the zeros of $f(z)-a$. Thus $s<\rho$, since a is a Borel exceptional value of f.

Let $f(z)=a+P(z)e^{Q(z)}$ be the Hadamard factorization of $f-a$, where P is the canonical product formed with the zeros of $f-a$, and Q is a polynomial of degree d and leading coefficient $c\neq 0$. The order of P is s, and thus the order ρ of f must be the integer d.

Now,

$$|f(t)-a|=|P(t)|e^{\Re Q(t)}\quad\text{and}\quad\ln|f(t)-a|=\ln|P(t)|+\Re Q(t),\quad\text{for }t>0.$$

Take $s' \in (s, d)$. For a certain t' depending on s', we have for $t \geq t'$ that $\ln|P(z)| \leq |z|^{s'}$, if $|z| = t$. Besides,

$$\frac{\Re Q(t)}{t^d} = \Re c + O\left(\frac{1}{t}\right), \quad \text{as } t \to \infty.$$

We conclude that

$$\lim_{t\to\infty} \frac{\ln|f(t) - a|}{t^d} = \Re c,$$

and therefore that

$$\lim_{t\to\infty} \frac{\ln f(t)}{t^d} = \Re c.$$

Observe that if $\Re c = 0$, then $\Re Q(t) = O(t^{d-1})$ as $t \to \infty$, and that would mean that

$$\limsup_{t\to\infty} \frac{\ln f(t)}{t^h} = 0,$$

for some h such that $(s <)h < d$, and thus, in particular, that f would be of order at most h, which is not the case.

Thus $\Re c > 0$, and condition (5.13) of Theorem 5.9 holds, so f is a clan. □

For an entire function, not necessarily of finite order, a Picard exceptional value is a value that is taken just a finite number of times. For Picard exceptional values and clans, we have the following result, which came out in a conversation of one of the authors with Walter Bergweiler.

Proposition 5.11. *If $f = Pe^g$ is in $\mathcal{K}$, where P is a polynomial and g is an entire function in $\mathcal{K}$* of finite order*, then f is a clan.*

The value 0 is Picard exceptional for $f = Pe^g$. It is not assumed that P is in $\mathcal{K}$, but it is assumed that g is in $\mathcal{K}$. Observe also that the assumption is that g is of finite order; if e^g were of finite order, that f is a clan would follow from Theorem 5.10.

Proof: The entire function f is transcendental, since $g \in \mathcal{K}$ is not a constant. From Lemma 4.3, we have that f is a clan if and only if $\lim_{t\to\infty} L_f(t) = 1$. To show that this limit holds, we verify first that g satisfies

$$\lim_{t\to\infty} \frac{g''(t)}{g'(t)^2} = 0. \tag{5.20}$$

Condition (5.20) clearly holds if g is a polynomial.

Assume thus that g is not a polynomial. From the case $p = 1$ of Theorem 5.1 applied to the derivative g', which is also of finite order, it follows that for some finite constant $S > 0$ and radius $R_1 > 0$ we have that

$$\frac{g''(t)}{g'(t)} \leq t^S, \quad \text{for } t > R_1.$$

Besides, since g' is not a polynomial, we have, for some radius R_2, that $g'(t) > t^{S+1}$, for $t > R_2$. And thus (5.20) holds.

Next, a calculation, recall (4.2), gives that

$$L_f(t) = \left(\frac{P''(t)}{P(t)} \frac{1}{g'(t)^2} + 2\frac{P'(t)}{P(t)} \frac{1}{g'(t)} + \frac{g''(t)}{g'(t)^2} + 1\right) \Big/ \left(\frac{P'(t)}{P(t)} \frac{1}{g'(t)} + 1\right)^2, \quad \text{for } t > 0.$$

Since P is a polynomial, we have that $P'(t)/P(t)$ and $P''(t)/P(t)$ tend to 0 as $t \to \infty$. Besides, since $g \in \mathcal{K}$, we have that $\lim_{t\to\infty} g'(t) = +\infty$. Using now (5.20), it is deduced that $\lim_{t\to\infty} L_f(t) = 1$. □

6. Some questions

Question 1. If $f \in \mathcal{K}$ has radius of convergence $R \leq \infty$ and its Khinchin family $(X_t)_{t\in[0,R)}$ satisfies for *some* value of $p \geq 2$ that

$$\lim_{t\uparrow R} \frac{\mathbf{E}(X_t^p)}{\mathbf{E}(X_t)^p} = 1, \tag{6.1}$$

then f is a clan. This follows directly from Lemma 2.4. Assume that (6.1) is satisfied for *some* value $p \in (1,2)$. Is this enough to deduce that f is a clan?

Question 2. Is it the case that, for every entire function f in $\mathcal{K}$,

$$\limsup_{t\to\infty} \frac{\sigma_f(t)}{m_f(t)} \leq 1?$$

Question 3. If g is an entire function in $\mathcal{K}$ (not necessarily of finite order), it is natural to ask whether $f = e^g$ is always a clan or not. This is the case if g is a clan or if g has finite order, as we have seen in Subsection 4.2 and in Proposition 5.11.

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