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SAITO BASES AND STANDARD BASES FOR PLANE CURVES

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Abstract: In this paper we describe how to compute a Saito basis of a cusp, a plane curve with only one Puiseux pair. Moreover, the 1-forms of the Saito basis that we obtain are characterized in terms of their divisorial orders associated to the "cuspidal" divisor of the minimal reduction of singularities of the cusp. We also introduce a new family of analytic invariants for plane curves computed in terms of Saito bases.

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1. Introduction

Let C be a plane curve in $(\mathbb{C}^2,\mathbf{0})$ and consider the $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ of holomorphic 1-forms which have the curve C invariant. In [**13**], K. Saito proved that $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ is a free module of rank 2 and a basis of $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ is called a *Saito basis* for the curve C. The objective of this paper is to compute a Saito basis for a curve C with only one Puiseux pair. Moreover, the 1-forms of the Saito basis that we compute are characterized in terms of their critical values. Let us clarify the statements.

Consider an irreducible plane curve C in $(\mathbb{C}^2,\mathbf{0})$ and let $\phi(t)$ be a primitive parametrization of C. Given $h\in\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$, we denote $\nu_C(h)=\operatorname{ord}_t(h(\phi(t)))$. Recall that the semigroup Γ of C, defined as

$$\Gamma=\{\nu_C(h) : h\in\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\},$$

is equivalent to the equisingularity data of the curve C. Given a 1-form $\omega\in\Omega^1_{\mathbb{C}^2,\mathbf{0}}$, we denote $\nu_C(\omega)=\operatorname{ord}_t(\alpha(t))+1$ with $\phi^*(\omega)=\alpha(t)\,dt$. The set of differential values Λ of the curve C, defined as

$$\Lambda=\{\nu_C(\omega) : \omega\in\Omega^1_{\mathbb{C}^2,\mathbf{0}}\},$$

is a Γ-semimodule and Λ is an analytic invariant of the curve C. There exists a basis of Λ, that is, a strictly increasing sequence $\mathcal{B}=(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_s)$ of elements of Λ, with s minimal, such that

$$\Lambda=\bigcup_{i=-1}^{s}(\lambda_i+\Gamma).$$

A set of 1-forms $(\omega_{-1},\omega_0,\omega_1,\ldots,\omega_s)$ such that $\nu_C(\omega_i)=\lambda_i$, for $-1\le i\le s$, is called a *minimal standard basis* of the curve C.

Assume now that C is a cusp, that is, an irreducible curve with a single Puiseux pair (n,m), with $2\le n<m$ and $\gcd(n,m)=1$. In this situation, the semigroup of the curve C is equal to $\Gamma=n\mathbb{Z}_{\ge0}+m\mathbb{Z}_{\ge0}$, and we say that Γ is cuspidal and that Λ is a cuspidal semimodule. Let us introduce some structural values associated to Λ.

The basis of the semimodule Λ allows us to define a chain $\Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_s = \Lambda$ with $\Lambda_i = \bigcup_{k=1}^{i}(\lambda_k + \Gamma)$, for $i = -1, 0, 1, \ldots, s$, such that $\lambda_i \notin \Lambda_{i-1}$ for $i = 0, 1, \ldots, s$. The *axes* u_i^n, u_i^m, u_i, and $\tilde{u}_i$ of Γ are defined as

$$u_i^n = \min\{\lambda_{i-1} + n\ell \in \Lambda_{i-2} : \ell \geq 1\}, \quad u_i^m = \min\{\lambda_{i-1} + m\ell \in \Lambda_{i-2} : \ell \geq 1\},$$
$$u_i = \min\{u_i^n, u_i^m\}, \qquad \tilde{u}_i = \max\{u_i^n, u_i^m\},$$

with $1 \leq i \leq s+1$, and the *critical values* t_i^n, t_i^m, t_i, $\tilde{t}_i$ are given by $t_{-1} = \lambda_{-1} = n$, $t_0 = \lambda_0 = m$, and

$$t_i^n = t_{i-1} + u_i^n - \lambda_{i-1}, \quad t_i^m = t_{i-1} + u_i^m - \lambda_{i-1},$$
$$t_i = \min\{t_i^n, t_i^m\}, \qquad \tilde{t}_i = \max\{t_i^n, t_i^m\},$$

for $1 \leq i \leq s+1$. Note that the semimodule Λ of a cusp is increasing, which means that $\lambda_i > u_i$ for $1 \leq i \leq s$ (see [**5**]). In a previous work [**4**], we have proved that these values allow us to characterize the elements of an *extended standard basis* of the curve C, that is, a set of 1-forms $\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1})$ such that $\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s$ is a minimal standard basis of C and ω_{s+1} is a 1-form with C as an invariant curve ($\nu_C(\omega_{s+1}) = \infty$) and divisorial order with respect to the cuspidal divisor equal to t_{s+1}.

Let us recall the notion of a divisorial order. Consider any sequence $\pi\colon M \to (\mathbb{C}^2, \mathbf{0})$ of point blow-ups and let $D \subset \pi^{-1}(\mathbf{0})$ be an irreducible component of the exceptional divisor. Given a point $Q \in D$, we can take coordinates (u, v) at Q such that $D = (u = 0)$ and, for any $h \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$, we can write $f \circ \pi = u^\beta \tilde{h}$, locally at Q, such that u does not divide $\tilde{h}$. We define the *divisorial order* $\nu_D(h)$ *with respect to the divisor* D by $\nu_D(h) = \beta$. Given a 1-form $\omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}$, that can be written as $\omega = a(x,y)\,dx + b(x,y)\,dy$, where (x, y) are coordinates in $(\mathbb{C}^2, \mathbf{0})$, we define the *divisorial order* $\nu_D(\omega)$ of ω *with respect to the divisor* D by $\nu_D(\omega) = \min\{\nu_D(xa), \nu_D(yb)\}$.

If C is a cusp, we can consider the minimal reduction of singularities $\pi_C\colon M \to (\mathbb{C}^2, \mathbf{0})$ and we denote by D_C the "cuspidal divisor", that is, the only irreducible component of $\pi_C^{-1}(\mathbf{0})$ such that the strict transform of C intersects D_C. In this situation, if (x, y) are adapted coordinates for C, the divisorial order with respect to the divisor D_C is a monomial order since it can be computed as $\nu_{D_C}(h) = \min\{ni + mj : h_{ij} \neq 0\}$ where $h(x,y) = \sum_{i,j\geq 0} h_{ij}x^iy^j$.

Now we can state the main result of this article.

Theorem 1.1. *Let C be a curve in $(\mathbb{C}^2, \mathbf{0})$ with only one Puiseux pair and $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ be the basis of the semimodule Λ of differential values for C. There exist two 1-forms ω_{s+1}, $\tilde{\omega}_{s+1}$ having C as an invariant curve and such that*

$$\nu_{D_C}(\omega_{s+1}) = t_{s+1}, \quad \nu_{D_C}(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1},$$

where t_{s+1} and $\tilde{t}_{s+1}$ are the last critical values of Λ.

Moreover, for any pair of 1-forms as above, the set $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis for C.

The proof of the existence of the 1-forms in the theorem above is done in a constructive way.

The notion of a divisorial order with respect to a divisor allows us to introduce a new analytic invariant of any plane curve (which is not necessarily a cusp). Given a divisor D as above, we define *the pair* $(\mathfrak{s}_D(C), \tilde{\mathfrak{s}}_D(C))$ *of Saito multiplicities at* D by

$$\mathfrak{s}_D(C) = \min\{\nu_D(\omega);\ \omega \text{ belongs to a Saito basis of } C\},$$
$$\tilde{\mathfrak{s}}_D(C) = \max\{\nu_D(\omega);\ \omega \text{ belongs to a Saito basis of } C\}.$$

Then the pair $(\mathfrak{s}_D(C), \tilde{\mathfrak{s}}_D(C))$ is an analytic invariant of any plane curve C in $(\mathbb{C}^2, \mathbf{0})$. In [**6**, pp. 8–9], Y. Genzmer introduces an analytic invariant of a curve C directly related to the pair $(\mathfrak{s}_{D_1}(C), \tilde{\mathfrak{s}}_{D_1}(C))$ associated to the divisor D_1 which appears after one blow-up. More precisely, he proves that the pair of multiplicities $(\nu_0(\omega), \nu_0(\tilde{\omega}))$, where $\{\omega, \tilde{\omega}\}$ is a Saito basis for C, with $\nu_0(\omega) \leq \nu_0(\tilde{\omega})$ and such that $\nu_0(\omega) + \nu_0(\tilde{\omega})$ is maximal, is an analytic invariant for the curve C. Note that the multiplicity at the origin of a 1-form ω can also be computed as $\nu_0(\omega) = \nu_{D_1}(\omega) - 1$.

The pair $(\mathfrak{s}_{D_C}(C), \tilde{\mathfrak{s}}_{D_C}(C))$ is determined in terms of the critical values of the semimodule. More precisely,

Theorem 1.2. *Let C be a cusp in $(\mathbb{C}^2, \mathbf{0})$. Then we have*

$$(\mathfrak{s}_{D_C}(C), \tilde{\mathfrak{s}}_{D_C}(C)) = (t_{s+1}, \tilde{t}_{s+1}),$$

where t_{s+1} and $\tilde{t}_{s+1}$ are the last critical values of the semimodule of differential values of the curve C.

However, we give an example of two cusps C_1 and C_2 with the same semimodule of differential values but with $(\mathfrak{s}_{D_1}(C_1), \tilde{\mathfrak{s}}_{D_1}(C_1)) \neq (\mathfrak{s}_{D_1}(C_2), \tilde{\mathfrak{s}}_{D_1}(C_2))$.

The structure of the semimodule Λ of differential values of a plane curve plays a key role in the proofs of the results of this article. In [**15**], O. Zariski pointed out the importance of the semimodule Λ in the analytic classification of plane curves. The work of Zariski was based on the computation of the simplest possible parametrizations of irreducible plane curves. Following these ideas, C. Delorme described the structure of the semimodule of differential values for a curve with only one Puiseux pair (see [**5**]). The complete analytic classification of irreducible plane curves was given by A. Hefez and M. E. Hernandes in 2011 ([**10**], see also [**11**]). Recently, M. E. Hernandes and M. E. R. Hernandes have given the analytic classification of plane curves in the general case ([**12**]).

Some analytic invariants of a plane curve can be computed from the semimodule Λ. For instance, the Milnor $\mu(C)$ and Tjurina number $\tau(C)$ of an irreducible plane curve C can be computed as $\mu(C) - \tau(C) = \sharp(\Lambda \setminus \Gamma)$ (see [**3, 14**]). In [**9**], Y. Genzmer and M. E. Hernandes compute the difference $\mu(C) - \tau(C)$ for an irreducible plane curve C when C admits a Saito basis of a special kind called a good Saito basis. In some recent works, Y. Genzmer describes invariants associated to the Saito module to compute the generic dimension of the moduli space of a curve C (see [**6, 7, 8**]).

The article is organized as follows. In Section 2, we describe the structure of a cuspidal increasing semimodule extending some results obtained in [**4**]. In Section 3, we generalize the decomposition given by Delorme in [**5**] (see also [**4**]) of the elements of a minimal standard basis of a cusp.

The proof of the main result of the paper is given in Section 4. We introduce the notion of a *standard system* $(\mathcal{E}, \mathcal{F})$ for a cusp which is the data of an extended standard basis $\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1})$ and a family $\mathcal{F} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1})$ of 1-forms with divisorial order $\nu_{D_C}(\tilde{\omega}_k) = \tilde{t}_j$ and such that C is an invariant curve of each $\tilde{\omega}_j$, $1 \leq j \leq s+1$. Given a *standard system* $(\mathcal{E}, \mathcal{F})$, we have that the set $\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_{s+1}\}$ is a generator system of the Saito $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ (see Proposition 4.8). We show the existence of 1-forms ω_{s+1}, $\tilde{\omega}_{s+1}$ having C as an invariant curve and with divisorial values $\nu_{D_C}(\omega_{s+1}) = t_{s+1}$ and $\nu_{D_C}(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Then, given two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ as above, we prove that it is possible to construct a *special standard system* that contains ω_{s+1} and $\tilde{\omega}_{s+1}$, that is, a standard system such that the 1-forms $\tilde{\omega}_i$, $1 \leq i \leq s$, can be written in terms of ω_{s+1} and $\tilde{\omega}_{s+1}$ (see Proposition 4.9). Finally, we prove that the generator system $\mathcal{T}$ of $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ can be reduced to obtain a basis with the properties given in Theorem 1.1.

The last section of the paper is devoted to introducing the analytic invariants given by the pair of Saito multiplicities at a divisor and we prove Theorem 1.2. Finally, we describe the examples which show that the Saito pairs of multiplicities with respect to the first divisor are not determined by the semimodule of differential values of the curve.

2. Structure of cuspidal semimodules

In this section we extend the description given in [**4**] of the structure of a cuspidal increasing semimodule.

Take $\Gamma \subset \mathbb{Z}_{\geq 0}$ an additive numerical semigroup, that is, Γ is a monoid generated by $\langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ with $\gcd(\overline{\beta}_0, \ldots, \overline{\beta}_g) = 1$. A set $\Lambda \subset \mathbb{Z}_{\geq 0}$ is a Γ-*semimodule* if $\gamma + \lambda \in \Lambda$ for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$. The *basis* of a Γ-semimodule Λ is the only increasing sequence

$$\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$$

satisfying that $\Lambda = \bigcup_{i=-1}^{s} (\lambda_i + \Gamma)$ and that $\lambda_j \notin \Lambda_{j-1}$, for any $j = 0, 1, \ldots, s$, where $\Lambda_i = \bigcup_{k=-1}^{i} (\lambda_k + \Gamma)$, for $i = -1, 0, 1, \ldots, s$. The basis induces the *decomposition chain* of Λ:

$$\Gamma + \lambda_{-1} = \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_s = \Lambda,$$

where each Λ_i is a semimodule with basis $\mathcal{B}_i = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_i)$, for $i = -1, 0, \ldots, s$. When $\lambda_{-1} = 0$, we say that Λ is a *normalized* semimodule. The number s above is called the *length* of the semimodule Λ.

Denote $n = \min(\Gamma \setminus \{0\})$. Given the basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$, we have that $\lambda_i \not\equiv \lambda_j \mod n$. Hence, the length s is bounded by $n - 2$.

We say that a numerical semigroup Γ is *cuspidal* if it is generated by two coprime integer numbers n, m, with $2 \leq n < m$. A Γ-semimodule Λ is *cuspidal* when Γ is cuspidal. From now on, we fix a cuspidal semigroup Γ, and we denote by $n < m$ its generators.

Any cuspidal semimodule Λ has an element $c_\Lambda \in \Lambda$ which is the minimum one satisfying the property that for every integer $k \geq c_\Lambda$ we have that $k \in \Lambda$. The element c_Λ is called the *conductor* of Λ. In the case $\Lambda = \Gamma$, the conductor takes the value $c_\Gamma = (n-1)(m-1)$. Furthermore, since $\Lambda_i \subset \Lambda_{i+1}$, then $c_{\Lambda_i} \geq c_{\Lambda_{i+1}}$. Note that if Λ is normalized, we have that $\Lambda_{-1} = \Gamma$ and in this case we have that $c_\Lambda \leq c_\Gamma$.

2.1. Axes, limits, and critical values. Let us introduce here some structural values for a cuspidal semimodule Λ with basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$.

For $1 \leq i \leq s+1$, we define the *axes* u_i^n, u_i^m, u_i, and $\tilde{u}_i$ of Λ as follows:

- $u_i^n = \min\{\lambda_{i-1} + n\ell \in \Lambda_{i-2};\ \ell \geq 1\}$. We write $u_i^n = \lambda_{i-1} + n\ell_i^n$.
- $u_i^m = \min\{\lambda_{i-1} + m\ell \in \Lambda_{i-2};\ \ell \geq 1\}$. Similarly, we put $u_i^m = \lambda_{i-1} + m\ell_i^m$.
- $u_i = \min\{u_i^n, u_i^m\}$ and $\tilde{u}_i = \max\{u_i^n, u_i^m\}$.

The numbers ℓ_i^n and ℓ_i^m are called *limits* of Λ.

Remark 2.1. If we consider the semimodule $\Lambda' = \Lambda - \lambda$, the new basis and the axes are shifted by λ and we obtain the same limits as for Λ. This is particularly interesting when $\lambda = \lambda_{-1}$ and hence Λ' is a normalized semimodule.

Remark 2.2. Let us note that $1 \leq \ell_i^m < n$ and that $1 \leq \ell_i^n < m$. To see this we can suppose that Λ is normalized and thus $c_{\Lambda_j} \leq c_\Gamma = (n-1)(m-1)$ for any $j = -1, 0, 1, \ldots, s$. Assume that $\ell_i^m \geq n$; we have

$$\lambda_{i-1} + m(\ell_i^m - 1) \geq (n-1)m \geq c_\Gamma \geq c_{\Lambda_{i-2}}.$$

Then $\lambda_{i-1} + m(\ell_i^m - 1) \in \Lambda_{i-2}$, in contradiction with the minimality of ℓ_i^m. A similar argument proves that $\ell_i^n < m$.

Remark 2.3. Notice that $u_i^n \neq u_i^m$ for each index $1 \leq i \leq s+1$. Indeed, if $u_i^n = u_i^m$, then $n\ell_i^n = m\ell_i^m$ and, since n and m are coprime, we must have $\ell_i^n = mk$ for a positive integer k and hence $\ell_i^n \geq m$, which is a contradiction.

Lemma 2.4. *Let Λ be a cuspidal semimodule of length s. Take $1 \leq i \leq s+1$. If $\lambda_{i-1} + na + mb \in \Lambda_{i-2}$ with $a, b \in \mathbb{Z}_{\geq 0}$, then either $a \geq \ell_i^n$ or $b \geq \ell_i^m$.*

Proof: (See also [**4**, Lemma 6.9].) By definition, we have

$$\lambda_{i-1} + na + mb = \lambda_r + nc + md, \quad r < i-1,$$

where c, d are nonnegative integers. We proceed by induction on $\alpha = ac + bd \geq 0$. If $\alpha = 0$, then $ac = bd = 0$. This implies that $ab = 0$, otherwise, $ab \neq 0$ and hence $c = d = 0$, that is,

$$\lambda_{i-1} + na + mb = \lambda_r,$$

which is a contradiction because $\lambda_r < \lambda_{i-1}$. Now if $a = 0$, we end by the minimality of ℓ_i^m and, similarly, if $b = 0$, we end by the minimality of ℓ_i^n.

Assume that $\alpha > 0$. Then $ac \neq 0$ or $bd \neq 0$. If $ac \neq 0$, let us put $a' = a - 1 \geq 0$ and $c' = c - 1 \geq 0$. We have

$$\lambda_{i-1} + na' + mb = \lambda_r + nc' + md,$$

and we finish by the induction hypothesis. We apply a similar argument if $bd \neq 0$. □

For $-1 \leq i \leq s+1$, we define inductively the *critical values* t_i^n, t_i^m, t_i, and $\tilde{t}_i$, for $-1 \leq i \leq s+1$ by putting $t_{-1} = \lambda_{-1}$ and $t_0 = \lambda_0$ and

$$\begin{aligned} t_i^n &= t_{i-1} + n\ell_i^n, & t_i^m &= t_{i-1} + m\ell_i^m, \\ t_i &= \min\{t_i^n, t_i^m\}, & \tilde{t}_i &= \max\{t_i^n, t_i^m\}, \end{aligned} \qquad 1 \leq i \leq s+1.$$

Remark 2.5. Noting that $n\ell_i^n = u_i^n - \lambda_{i-1}$ and $m\ell_i^m = u_i^m - \lambda_{i-1}$, we see that:

$$\begin{aligned} t_i^n &= t_{i-1} + (u_i^n - \lambda_{i-1}), & t_i^m &= t_{i-1} + (u_i^m - \lambda_{i-1}), \\ t_i &= t_{i-1} + (u_i - \lambda_{i-1}), & \tilde{t}_i &= t_{i-1} + (\tilde{u}_i - \lambda_{i-1}). \end{aligned}$$

Definition 2.6. We say that the cuspidal semimodule Λ is *increasing* if $\lambda_i > u_i$ for any $1 \leq i \leq s$.

Notice that if Λ is increasing, then each Λ_i is also increasing for $1 \leq i \leq s$. The notion of an increasing semimodule was introduced in [**1**].

Lemma 2.7. *Let Λ be an increasing cuspidal semimodule. For any index $1 \leq i \leq s$, we obtain $\lambda_i - \lambda_j > t_i - t_j$ for $-1 \leq j < i$.*

Proof: (See also [**4**, Lemma 7.10].) By a telescopic argument, it is enough to prove the following statements:

- $\lambda_r - \lambda_{r-1} > t_r - t_{r-1}$ for $1 \leq r \leq s$.
- $\lambda_0 - \lambda_{-1} \geq t_0 - t_{-1}$.

The second statement is straightforward since $t_{-1} = \lambda_{-1}$ and $t_0 = \lambda_0$. Let us prove that $\lambda_r - \lambda_{r-1} > t_r - t_{r-1}$ for $1 \leq r \leq s$.

The inequality $\lambda_r - \lambda_{r-1} > t_r - t_{r-1}$ is equivalent to

$$t_r = t_{r-1} + u_r - \lambda_{r-1} > t_r + u_r - \lambda_r.$$

This last inequality is equivalent to $\lambda_r > u_r$. Hence, we get the result since Λ is increasing. □

Corollary 2.8. *Let Λ be an increasing cuspidal semimodule. For any $1 \leq i \leq s$, we have*

$$u_{i+1}^n > t_{i+1}^n \quad \textit{and} \quad u_{i+1}^m > t_{i+1}^m.$$

Proof: Recalling that $t_{i+1}^n = u_{i+1}^n - (\lambda_i - t_i)$, it is enough to prove that $\lambda_i - t_i > 0$. In view of Lemma 2.7 and putting $j = -1$, we have that $\lambda_i - t_i > \lambda_{-1} - t_{-1} = 0$. □

Remark 2.9. The property of being increasing is true for cuspidal semimodules corresponding to the differential values of analytic branches of cusps (see [**5**, Lemme 12] and [**4**, Theorem 7.13]). Note that the semimodule of a branch with more than one Puiseux pair (that is, with genus greater than or equal to 2) is not increasing in general. For these reasons, we mainly consider increasing cuspidal semimodules in this paper.

2.2. Circular intervals. The circular intervals we describe here are useful for understanding the distribution of the elements of an increasing cuspidal semimodule. The notion of a circular interval was introduced in [**4**]. We are going to consider the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ as a clock with n-hours as we explain below.

Let $\varepsilon\colon \mathbb{R} \to \mathbb{S}^1$ be the map given by

$$\varepsilon(t) = \exp\left(-\frac{2\pi t\sqrt{-1}}{n}\right).$$

We define the *n-clock* $\mathbb{S}_n^1$ to be $\mathbb{S}_n^1 = \varepsilon(\mathbb{Z})$. Note that there is a bijection

$$c\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{S}_n^1$$

given by $c(k + n\mathbb{Z}) = \varepsilon(k)$. More than that, the bijection c is an isomorphism of abelian groups, where $\mathbb{S}_n^1 \subset \mathbb{C}$ has the induced multiplicative structure coming from the complex numbers $\mathbb{C}$. Note that

$$c((k + n\mathbb{Z}) + (k' + n\mathbb{Z})) = \varepsilon(k)\varepsilon(k').$$

In particular $c(k + 1 + n\mathbb{Z}) = \varepsilon(k)\varepsilon(1)$.

Notation 2.10. In order to visualize better the arithmetic of the abelian multiplicative group $\mathbb{S}_n^1$, we introduce the following notations:

$$\varepsilon(k) = k_\varepsilon, \quad \varepsilon(k)\varepsilon(k') = k_\varepsilon + k'_\varepsilon.$$

Note that there is no confusion possible with the addition in $\mathbb{C}$. For instance, we have $(-1)_\varepsilon = (n-1)_\varepsilon$, $(k+1)_\varepsilon = k_\varepsilon + 1_\varepsilon$, and $(k-1)_\varepsilon = k_\varepsilon - 1_\varepsilon = k_\varepsilon + (n-1)_\varepsilon$.

Let us consider two points $P, Q \in \mathbb{S}_n^1$. There are $\alpha \in \mathbb{Z}$ and an integer number β with $0 \leq \beta \leq n-1$ such that $P = \varepsilon(\alpha)$ and $Q = \varepsilon(\alpha + \beta)$. This number β, with $0 \leq \beta \leq n-1$, does not depend on the chosen α such that $P = \varepsilon(\alpha)$ and we call it the *separation $S(P,Q)$ from P to Q*. That is, if $P = \varepsilon(\alpha)$, we have that $Q = \varepsilon(\alpha + S(P,Q))$. We have that $S(P,P) = 0$ and that

$$S(P,Q) + S(Q,P) = n, \quad \text{if } Q \neq P.$$

We define the *circular interval* $\langle P, Q\rangle$ to be

$$\langle P, Q\rangle = \{\varepsilon(\alpha + k);\ k = 0, 1, \ldots, S(P, Q)\} \subset \mathbb{S}^1_n.$$

Note that if $P \neq Q$, we get

$$\langle P, Q\rangle \cup \langle Q, P\rangle = \mathbb{S}^1_n, \quad \langle P, Q\rangle \cap \langle Q, P\rangle = \{P, Q\}.$$

Remark 2.11. Given three points $P, Q, R \in \mathbb{S}^1_n$ with $P \neq Q$ and such that

$$R \in \langle P, Q\rangle,$$

we have that $S(P, Q) = S(P, R) + S(R, Q) \leq n - 1$.

Consider a list $B = (z_{-1}, z_0, z_1, \ldots, z_s)$ of two by two distinct points $z_j \in \mathbb{S}^1_n$, with $s \geq 0$. For any index $0 \leq i \leq s$, we define the *i-left bound* $b^\ell_i(B)$ *and the i-right bound* $b^r_i(B)$ *of* B to be integer numbers with

$$-1 \leq b^\ell_i(B), b^r_i(B) \leq i - 1$$

and such that the following holds:

- If $k = b^\ell_i(B)$, then $S(z_k, z_i) \leq S(z_q, z_i)$, for any $-1 \leq q \leq i - 1$.
- If $\tilde{k} = b^r_i(B)$, then $S(z_i, z_{\tilde{k}}) \leq S(z_i, z_q)$, for any $-1 \leq q \leq i - 1$.

Remark 2.12. Denote $k = b^\ell_i(B)$ and $\tilde{k} = b^r_i(B)$. The bounds are the integer numbers k, $\tilde{k}$ with $-1 \leq k, \tilde{k} \leq i - 1$ defined by the two following properties:

(a) $z_i \in \langle z_k, z_{\tilde{k}}\rangle$.
(b) If $z_j \in \langle z_k, z_{\tilde{k}}\rangle$ with $-1 \leq j \leq i$, then $j \in \{i, k, \tilde{k}\}$.

2.3. Circular intervals in a cuspidal semimodule. Let us recall that Γ is the semigroup generated by n, m, with $2 \leq n < m$, and n, m are without common factors.

Let $\rho\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the quotient map, which we also denote by $\rho(k) = \overline{k}$. Since $(n, m) = 1$, the class $\overline{m}$ is a unit in $\mathbb{Z}/n\mathbb{Z}$, thus, we have a ring isomorphism

$$\xi\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}, \quad \xi(\overline{k}) = \frac{\overline{k}}{\overline{m}}.$$

Throughout this paper, we consider the map $\zeta\colon \mathbb{Z} \to \mathbb{S}^1_n$ defined by $\zeta(k) = (c \circ \xi \circ \rho)(k)$. Let us note that $\zeta(k + an) = \zeta(k)$ and that $\zeta(mk) = \varepsilon(k) = k_\varepsilon$.

Consider the intervals $I_q = \{nq, nq + 1, \ldots, nq + n - 1\} \subset \mathbb{Z}$, with $q \in \mathbb{Z}$. For a set $S \subset \mathbb{Z}$, we define the *q-level set* $R_q(S)$ by

$$R_q(S) = \zeta(S \cap I_q) \subset \mathbb{S}^1_n.$$

Remark 2.13. If $S \subset \mathbb{Z}$ satisfies the property that $n + S \cap I_{q-1} \subset S \cap I_q$, we have that $R_{q-1}(S) \subset R_q(S)$. This is the case of cuspidal semimodules.

Let us consider a cuspidal semimodule Λ of length $s \geq 0$ with basis

$$\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s).$$

We see the basis $\mathcal{B}$ in the clock $\mathbb{S}^1_n$ as $B = \zeta(\mathcal{B}) = (z_{-1}, z_0, z_1, \ldots, z_s)$, where we have that $z_j = \zeta(\lambda_j)$, for $j = -1, 0, 1, \ldots, s$.

Note that $z_i \neq z_j$ if $i \neq j$. Indeed, saying that $z_i = z_j$ means that $\lambda_i - \lambda_j \in n\mathbb{Z}$, which is not possible in view of the definition of a basis.

Take an index $1 \leq i \leq s + 1$. We define the *tops* q^n_i and q^m_i of Λ by the property that $u^n_i \in I_{q^n_i}$ and $u^m_i \in I_{q^m_i}$. We also define the *tops* q_i and $\tilde{q}_i$ to be such that $u_i \in I_{q_i}$ and $\tilde{u}_i \in I_{\tilde{q}_i}$. Recall that

$$\{u^n_i, u^m_i\} = \{u_i, \tilde{u}_i\}.$$

As a consequence, we have that $\{q^n_i, q^m_i\} = \{q_i, \tilde{q}_i\}$. Note that $q_i \leq \tilde{q}_i$, since $u_i < \tilde{u}_i$.

We also need to consider the integers v_i that indicate the first levels $R_{v_i}(\Lambda)$ such that $z_i \in R_{v_i}(\Lambda)$. In other words, each v_i is defined by the property that $\lambda_i \in I_{v_i}$, for $i = -1, 0, 1, \ldots, s$.

The following statements concern the properties of being circular intervals for the levels of Λ and some derived properties of the conductor.

Lemma 2.14 ([**4**, Lemma A.3])**.** *Given $\mu \in \mathbb{Z}$, the set $R_q(\mu+\Gamma)$ is a circular interval for all $q \in \mathbb{Z}$ (up to considering the empty set as a circular interval).*

Proposition 2.15 ([**4**, Proposition A.5])**.** *Assume that Λ is normalized (that is, $\lambda_{-1} = 0$) and that $R_q(\Lambda_{s-1})$ is a circular interval for anyq $\geq v_s$. We have*

(a) $\langle 0_\varepsilon, z_s - 1_\varepsilon \rangle \subset R_q(\Lambda_{s-1})$*, for* $q \geq q^n_{s+1} - 1$.
(b) $\langle z_s, (n-1)_\varepsilon \rangle \subset R_q(\Lambda)$*, for* $q \geq q^m_{s+1} - 1$.

In particular, we obtain that $R_q(\Lambda) = \mathbb{S}^1_n$ for any $q \geq \tilde{q}_{s+1} - 1$.

Proposition 2.16 ([**4**, Proposition A.6])**.** *Assume that Λ is normalized and increasing. Then $R_q(\Lambda)$ is a circular interval for any $q \geq q_{s+1}$.*

Corollary 2.17. *Assume that Λ is normalized and increasing. Then $\tilde{u}_{s+1} \geq c_\Lambda + n$, where c_Λ is the conductor of Λ.*

Proof: (See also [**4**, Corollary A.7].) First, let us show that $R_q(\Lambda_{s-1})$ is a circular interval for $q \geq v_s$.

If $s = 0$, we have that $\Lambda_{s-1} = \Lambda_{-1} = \Gamma$, and we apply Lemma 2.14 by taking $\mu = 0$. Assume now that $s \geq 1$. By Proposition 2.16, we know that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq q_s$. Moreover, we have that $\lambda_s > u_s$ since Λ is an increasing semimodule. This implies that $v_s \geq q_s$, hence we get that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v_s$, as desired.

We end the proof as follows. By Proposition 2.15, we have that $R_q(\Lambda) = \mathbb{S}^1_n$ for any $q \geq \tilde{q}_{s+1} - 1$. This implies that we have $k \in \Lambda$ for any $k \geq n\tilde{q}_{s+1} - n$, and hence $n\tilde{q}_{s+1} - n \geq c_\Lambda$. Finally, by definition of the tops, we have $\tilde{u}_{s+1} \geq n\tilde{q}_{s+1}$ and we get the result. □

Remark 2.18. Notice that Propositions 2.15 and 2.16 and Corollary 2.17 are also true for increasing cuspidal semimodules such that λ_{-1} is a multiple nk of n. Indeed, in this case, we obtain the desired statements by applying the propositions to $\Lambda - nk$.

2.4. Distribution of the elements of the basis. Throughout this section, we consider a cuspidal semimodule Λ of length $s \geq 0$ with basis $\mathcal{B}$, that we read in the clock $\mathbb{S}^1_n$ as $B = \zeta(\mathcal{B})$ as in the previous section. We are going to describe a pattern for the distribution of the points z_i in

$$B = (z_{-1}, z_0, z_1, \ldots, z_s)$$

by computing the bounds $b^\ell_i(B)$ and $b^r_i(B)$ of B in terms of the axes u^n_{i+1} and u^m_{i+1}.

Lemma 2.19. *For any $0 \leq i \leq s$, there are unique integer numbers k^n_i and k^m_i such that*

(a) $-1 \leq k^n_i, k^m_i \leq i - 1$.
(b) *There is* $b_{i+1} \geq 0$ *such that* $u^n_{i+1} = \lambda_i + n\ell^n_{i+1} = \lambda_{k^n_i} + mb_{i+1}$.
(c) *There is* $a_{i+1} \geq 0$ *such that* $u^m_{i+1} = \lambda_i + m\ell^m_{i+1} = \lambda_{k^m_i} + na_{i+1}$.

Proof: The existence of k_i^n and k_i^m comes from the definition of axes and limits. Let us show their uniqueness. Assume that there is another $k \neq k_i^n$ with $-1 \leq k \leq i-1$ and a natural number b such that

$$u_{i+1}^n = \lambda_i + n\ell_{i+1}^n = \lambda_{k_i^n} + mb_{i+1} = \lambda_k + mb.$$

Then either $\lambda_k \in (\lambda_{k_i^n} + \Gamma)$ or $\lambda_{k_i^n} \in (\lambda_k + \Gamma)$, in contradiction with the definition of a basis. The uniqueness of k_i^m is shown in the same way. □

The numbers b_{i+1} and a_{i+1} are the *colimits* of Λ.

Notation 2.20. We denote by k_i and $\tilde{k}_i$ the following numbers:

$$k_i = \begin{cases} k_i^n, & \text{if } u_{i+1} = u_{i+1}^n, \\ k_i^m, & \text{if } u_{i+1} = u_{i+1}^m; \end{cases} \quad \tilde{k}_i = \begin{cases} k_i^n, & \text{if } \tilde{u}_{i+1} = u_{i+1}^n, \\ k_i^m, & \text{if } \tilde{u}_{i+1} = u_{i+1}^m. \end{cases}$$

Remark 2.21. Note that $1 \leq b_{i+1} < n$. To see this, it is enough to consider the case of a normalized Λ. Indeed, if $b_{i+1} \geq n$, we have

$$\lambda_i + n(\ell_{i+1}^n - 1) = \lambda_{k_i^n} + mb_{i+1} - n \geq (m-1)n \geq c_\Gamma \geq c_{\Lambda_{i-1}}.$$

Thus $\lambda_i + n(\ell_{i+1}^n - 1) \in \Lambda_{i-1}$, in contradiction with the minimality of ℓ_{i+1}^n. Now, as a consequence, we have that the separation $S(z_{k_i^n}, z_i)$ is given by $S(z_{k_i^n}, z_i) = b_{i+1}$. Recalling that $1 \leq \ell_{i+1}^m < n$, see Remark 2.2, we have that the separation $S(z_i, z_{k_i^m})$ is given by $S(z_i, z_{k_i^m}) = \ell_{i+1}^m$.

Example 2.22. Take the semimodule $\Lambda = \Gamma \setminus \{0\}$. The basis of Λ is $\mathcal{B} = (n, m)$. Note that $\lambda_{-1} = n$ and $\lambda_0 = m$, and thus we have $\Lambda_{-1} = n + \Gamma$ and $\Lambda_0 = \Lambda = \Gamma \setminus \{0\}$.

The limit ℓ_1^n is the smallest positive integer such that

$$m + n\ell_1^n = \lambda_0 + n\ell_1^n \in \Lambda_{-1} = n + \Gamma.$$

After solving the equation $m + n\ell_1^n = n + mb_1$, we obtain that $\ell_1^n = 1 = b_1$. Moreover, we have $u_1^n = n + m = t_1^n$.

In the same way, in order to compute ℓ_1^m, we solve $m + m\ell_1^m = n + na_1$, obtaining $\ell_1^m = n - 1$ and $a_1 = m - 1$. Therefore, $u_1^m = nm = t_1^m$.

We conclude that $u_1 = u_1^n = n + m$, $\tilde{u}_1 = u_1^m = nm$, $t_1 = t_1^n$, and $\tilde{t}_1 = t_1^m$. As expected, we have that $k_0^n = k_0^m = -1$, which are the 0-bounds of the list

$$B = (0_\varepsilon, 1_\varepsilon) = (z_{-1}, z_0)$$

(note that $\zeta(m) = 1_\varepsilon$).

Any cuspidal semimodule Λ with basis $(n, m, \dots)$ has the same first axes, first critical values, first limits, first colimits, and 0-bounds as the ones computed above, since their computation depends only on $\Lambda_0 = \Gamma \setminus \{0\}$.

Lemma 2.23. *Consider* $0 \leq i \leq s$ *and take integer numbers* $-1 \leq k, k' \leq i-1$ *with* $k \neq k'$. *Assume that we have the following equalities:*

$$\lambda_i + ne = \lambda_k + mb; \quad \lambda_i + ne' = \lambda_{k'} + mb', \tag{1}$$

where $e, e' \in \mathbb{Z}$ *and* $0 \leq b, b' < n$. *Then we have that* $e < e'$ *if and only if* $b < b'$.

Proof: Equations (1) lead us to

$$\lambda_k = \lambda_{k'} + n(e - e') + m(b' - b),$$
$$\lambda_{k'} = \lambda_k + n(e' - e) + m(b - b').$$

Note that $\lambda_k \notin \lambda_{k'} + \Gamma$ and $\lambda_{k'} \notin \lambda_k + \Gamma$ since λ_k and $\lambda_{k'}$ are different elements of the basis of Λ. We conclude that $b < b'$ if and only if $e < e'$. □

Proposition 2.24. *Consider $0 \leq i \leq s$ and take integer numbers $-1 \leq k, k' \leq i-1$, with $k \neq k'$. We have*

(a) *Assume that $\lambda_i + ne = \lambda_k + mb$, $\lambda_i + ne' = \lambda_{k'} + mb'$, where $e, e' \in \mathbb{Z}$ and $0 \leq b, b' < n$. Then $e < e' \Leftrightarrow \lambda_i + ne < \lambda_i + ne' \Leftrightarrow S(z_k, z_i) < S(z_{k'}, z_i)$. In particular, taking $k = k_i^n$, we have $S(z_{k_i^n}, z_i) < S(z_{k'}, z_i)$.*

(b) *Assume that $\lambda_i + mf = \lambda_k + na$, $\lambda_i + mf' = \lambda_{k'} + na'$, where $a, a' \in \mathbb{Z}$ and $0 \leq f, f' < n$. Then $f < f' \Leftrightarrow \lambda_i + mf < \lambda_i + mf' \Leftrightarrow S(z_i, z_k) < S(z_i, z_{k'})$. In particular, taking $k = k_i^m$, we have $S(z_i, z_{k_i^m}) < S(z_i, z_{k'})$.*

Proof: Notice that $S(z_i, z_k) = f$ and $S(z_i, z_{k'}) = f'$; this proves the second statement. For the first statement, we apply Lemma 2.23, by noting that $S(z_k, z_i) = b$ and $S(z_{k'}, z_i) = b'$. □

Corollary 2.25. *We have that $k_i^n = b_i^\ell(B)$ and $k_i^m = b_i^r(B)$ for $0 \leq i \leq s$.*

Remark 2.26. Take an integer number $\lambda \in \mathbb{Z}$. Then $\mathcal{B}' = \lambda + \mathcal{B}$ is the basis of $\Lambda' = \lambda + \Lambda$ and $B' = \zeta(\mathcal{B}') = B + \lambda_\varepsilon$. Thus, the bounds of B' are the same ones as the bounds of B. Anyway, the axes for Λ' are the ones of Λ shifted by λ; this implies also that bounds, limits, and colimits coincide for both semimodules.

For the particular case when the semimodule Λ is increasing, we can give a more accurate description of the bounds, as shown in the next proposition:

Proposition 2.27. *Assume that Λ is increasing and take $1 \leq i \leq s$. We have*

(a) *If $u_i = u_i^n$, then $k_i^n = i-1$ and $k_i^m = k_{i-1}^m$.*

(b) *If $u_i = u_i^m$, then $k_i^n = k_{i-1}^n$ and $k_i^m = i-1$.*

Proof: In view of Remark 2.26, it is enough to consider the normalized case $\lambda_{-1} = 0$. Let us do the proof of (a); the proof of (b) is similar and we do not explicate it. Thus, we take the assumption that $u_i = u_i^n$.

First, let us suppose that $i = 1$. By considering the bounds in the list (z_{-1}, z_0, z_1), we deduce that $k_0^n = k_0^m = -1$ and either $z_1 \in \langle z_{-1}, z_0 \rangle$ or $z_1 \in \langle z_0, z_{-1} \rangle$. Let us show that we actually have that $z_1 \in \langle z_0, z_{-1} \rangle$; this gives $k_1^n = 0$ and $k_1^m = -1$ as desired.

Since $\Lambda_{-1} = \Gamma$, we have that $R_q(\Lambda_{-1})$ is a circular interval for $q \geq 0$, due to Lemma 2.14. Recall that $u_1^n = u_1 \in I_{q_1}$. Noting that $z_{-1} = 0_\varepsilon$ and applying Proposition 2.15, we have

$$\langle z_{-1}, z_0 - 1_\varepsilon \rangle \subset R_q(\Lambda_{-1}), \quad q \geq q_1^n - 1 = q_1 - 1.$$

On the other hand, we have that $z_0 \in R_q(\Lambda_0)$ for any $q \geq q_1$ since $\lambda_0 < u_1$ and hence $v_0 \leq q_1$. Thus, we have $\langle z_{-1}, z_0 \rangle \subset R_{q_1}(\Lambda_0)$. Note that $\lambda_1 > u_1$, since Λ is increasing; this implies that $z_1 \notin R_{q_1}(\Lambda_0)$ and thus we necessarily have that $z_1 \in \langle z_0, z_{-1} \rangle$.

Now, assume that $i > 1$. Our first step is to show that $z_i \in \langle z_{k_{i-1}^n}, z_{k_{i-1}^m} \rangle$. By Proposition 2.16, we have that $R_q(\Lambda_{i-2})$ is a circular interval for $q \geq q_{i-1}$. Since $z_{k_{i-1}^n}$ and $z_{k_{i-1}^m}$ belong to $R_{q_{i-1}}(\Lambda_{i-2})$, we have

$$\text{either } \langle z_{k_{i-1}^n}, z_{k_{i-1}^m} \rangle \subset R_{q_{i-1}}(\Lambda_{i-2}) \quad \text{or} \quad \langle z_{k_{i-1}^m}, z_{k_{i-1}^n} \rangle \subset R_{q_{i-1}}(\Lambda_{i-2}).$$

Noting that $z_{i-1} \notin R_{q_{i-1}}(\Lambda_{i-2})$ and $z_{i-1} \in \langle z_{k_{i-1}^n}, z_{k_{i-1}^m} \rangle$, we conclude

$$\langle z_{k_{i-1}^m}, z_{k_{i-1}^n} \rangle \subset R_{q_{i-1}}(\Lambda_{i-2}).$$

Noting also that $z_i \notin R_{q_{i-1}}(\Lambda_{i-2})$, we obtain that $z_i \in \langle z_{k_{i-1}^n}, z_{k_{i-1}^m} \rangle$, as desired.

Thus, we have $z_i, z_{i-1} \in \langle z_{k^n_{i-1}}, z_{k^m_{i-1}} \rangle$ and hence there are two possibilities: either $z_i \in \langle z_{k^n_{i-1}}, z_{i-1} \rangle$ or $z_i \in \langle z_{i-1}, z_{k^m_{i-1}} \rangle$. Let us show that $z_i \in \langle z_{i-1}, z_{k^m_{i-1}} \rangle$ holds, which ends the proof. By Proposition 2.15, we have

$$\langle 0_\varepsilon, z_{i-1} - 1_\varepsilon \rangle \subset R_{q_i - 1}(\Lambda_{i-2}) \subset R_{q_i-1}(\Lambda_{i-1}).$$

Since $z_{i-1} \in R_{q_i}(\Lambda_{i-1})$, we have that $\langle 0_\varepsilon, z_{i-1} \rangle \subset R_{q_i}(\Lambda_{i-1})$. Recalling that $k^n_{i-1} = b^\ell_{i-1}(B)$, we necessarily have that $z_{k^n_{i-1}} \in \langle 0_\varepsilon, z_{i-1} \rangle$, noting that $0_\varepsilon = z_{-1}$. Hence, we have

$$\langle z_{k^n_{i-1}}, z_{i-1} \rangle \subset R_{q_i}(\Lambda_{i-1}).$$

Since $z_i \notin R_{q_i}(\Lambda_{i-1})$, we obtain that $z_i \in \langle z_{i-1}, z_{k^m_{i-1}} \rangle$ as desired. □

Remark 2.28. Note that Proposition 2.27 implies the following statements:

(a) If $k^n_i = i - 1$, then $u_i = u^n_i$.

(b) If $k^m_i = i - 1$, then $u_i = u^m_i$.

Indeed, we have that $i - 1 \in \{k^n_i, k^m_i\}$; if $k^n_i = i - 1$, then necessarily $k^m_i \neq i - 1$ (note that $i \geq 1$) and we are in the situation of the first statement of Proposition 2.27. A similar argument applies when $k^m_i = i - 1$.

2.5. Relations between parameters. Let Λ be an increasing cuspidal semimodule with basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ and let us denote

$$B = \zeta(\mathcal{B}) = (z_{-1}, z_0, z_1, \ldots, z_s).$$

In this section we describe inductive features of axes, limits, and colimits of Λ.

Lemma 2.29. *Take $1 \leq k < i \leq s + 1$. We have:*

(a) *The axes and the critical values u_i, t_i satisfy that $u_i > u_k$ and $t_i > t_k$.*

(b) *The axes and the critical values $\tilde{u}_i$, $\tilde{t}_i$ satisfy that $\tilde{u}_i < \tilde{u}_k$ and $\tilde{t}_i < \tilde{t}_k$.*

Proof: It is enough to consider the case $k = i - 1$.

Let us prove property (a). By definition of the axes, we have that $u_i > \lambda_{i-1}$. Since the semimodule is increasing, we have that $\lambda_{i-1} > u_{i-1}$. We get that $u_i > u_{i-1}$. Moreover, by Remark 2.5, we see that

$$t_i = t_{i-1} + (u_i - \lambda_{i-1}) > t_{i-1}.$$

This ends the proof of property (a).

Let us prove property (b). We do it for the case that $\tilde{u}_i = u^n_i = \lambda_{i-1} + n\ell^n_i$; the proof for the case $\tilde{u}_i = u^m_i$ is done in a similar way. By Proposition 2.27, there are two cases: either $k^n_{i-1} = i - 2$ or $k^m_{i-1} = i - 2$. We shall see that $\tilde{u}_i < \tilde{u}_{i-1}$ and that $\tilde{t}_i < \tilde{t}_{i-1}$ simultaneously in each of the cases above.

Case $k^n_{i-1} = i - 2$. By Remark 2.28, we see that $u_{i-1} = u^n_{i-1}$ and $\tilde{u}_{i-1} = u^m_{i-1}$. Hence we can write:

$$\tilde{u}_i = u^n_i = \lambda_{i-1} + n\ell^n_i = \lambda_{i-2} + mb_i, \tag{2}$$

$$\tilde{u}_{i-1} = u^m_{i-1} = \lambda_{i-2} + m\ell^m_{i-1} = \lambda_k + na_{i-1}, \quad \text{with } k < i - 2. \tag{3}$$

In order to see that $\tilde{u}_i < \tilde{u}_{i-1}$, we need to show that $b_i < \ell^m_{i-1}$. To do this, we are going to exclude the possibility $b_i \geq \ell^m_{i-1}$:

- If $\ell^m_{i-1} = b_i$, we deduce that $\tilde{u}_{i-1} = \tilde{u}_i$ from equations (2) and (3). Hence, we have

$$\tilde{u}_i = \lambda_{i-1} + n\ell^n_i = \lambda_k + na_{i-1} = \tilde{u}_{i-1}, \quad k < i - 2.$$

  Then $\rho(\lambda_{i-1}) = \rho(\lambda_k) \in \mathbb{Z}/n\mathbb{Z}$, contradicting the fact that $\mathcal{B}$ is a basis.

- If $\ell^m_{i-1} < b_i$, by equation (2) and Corollary 2.17, we have
$$\begin{aligned}\tilde{u}_i - n = \lambda_{i-1} + n(\ell^n_i - 1) = \lambda_{i-2} + mb_i - n &\geq \lambda_{i-2} + m\ell^m_{i-1} + m - n \\ &= \tilde{u}_{i-1} + m - n \geq c_{\Lambda_{i-2}} + m > c_{\Lambda_{i-2}}.\end{aligned}$$
We get that $\lambda_{i-1} + n(\ell^n_i - 1) > c_{\Lambda_{i-2}}$ and thus $\lambda_{i-1} + n(\ell^n_i - 1) \in \Lambda_{i-2}$, contradicting the minimality of ℓ^n_i.

We conclude that $b_i < \ell^m_{i-1}$ and then we have that $\tilde{u}_i < \tilde{u}_{i-1}$, in the case $k^n_{i-1} = i-2$.

Let us see now that $\tilde{t}_i < \tilde{t}_{i-1}$ in this case $k^n_{i-1} = i-2$. From equations (2) and (3), using the fact that $\tilde{u}_i < \tilde{u}_{i-1}$ and the increasing semimodule property, we have
$$\lambda_{i-2} + m\ell^m_{i-1} = \tilde{u}_{i-1} > \tilde{u}_i = \lambda_{i-1} + n\ell^n_i > u_{i-1} + n\ell^n_i.$$
Consequently, $m\ell^m_{i-1} > u_{i-1} - \lambda_{i-2} + n\ell^n_i$ and
$$\tilde{t}_{i-1} = t_{i-2} + m\ell^m_{i-1} > t_{i-2} + u_{i-1} - \lambda_{i-2} + n\ell^n_i = t_{i-1} + n\ell^n_i = \tilde{t}_i.$$
This ends the proof that $\tilde{t}_i < \tilde{t}_{i-1}$ in this case.

Case $k^m_{i-1} = i-2$. Note that $\tilde{u}_{i-1} = u^n_{i-1}$ and $k^n_{i-1} = k^n_{i-2}$ in view of Remark 2.28 and Proposition 2.27. Thus, we can write
$$\tilde{u}_i = u^n_i = \lambda_{i-1} + n\ell^n_i = \lambda_k + mb_i, \qquad \text{with } k = k^n_{i-1} = k^n_{i-2} < i-2, \tag{4}$$
$$\tilde{u}_{i-1} = u^n_{i-1} = \lambda_{i-2} + n\ell^n_{i-1} = \lambda_k + mb_{i-1}, \quad \text{with } k = k^n_{i-2} < i-2. \tag{5}$$
Let us proceed in a similar way as before to show that $b_{i-1} > b_i$:

- Assume that $b_{i-1} = b_i$. Then $\rho(\lambda_{i-1}) = \rho(\lambda_{i-2})$, which is not possible.
- Assume that $b_{i-1} < b_i$. Then, we have
$$\begin{aligned}\tilde{u}_i - n = \lambda_{i-1} + n(\ell^n_i - 1) &= \lambda_k + mb_i - n \\ &> \lambda_k + (b_i - 1)m = \tilde{u}_{i-1} + (b_i - b_{i-1} - 1)m \\ &\geq \tilde{u}_{i-1} \geq n + c_{\Lambda_{i-2}}.\end{aligned}$$
Then $\lambda_{i-1} + n(\ell^n_i - 1) \in \Lambda_{i-2}$, in contradiction with the minimality of ℓ^n_i.

We conclude that $b_{i-1} > b_i$ and thus $\tilde{u}_{i-1} > \tilde{u}_i$.

Let us see now that $\tilde{t}_i < \tilde{t}_{i-1}$ in this case $k^m_{i-1} = i-2$. From equations (4) and (5), using the fact that $\tilde{u}_i < \tilde{u}_{i-1}$ and the increasing semimodule property, we have
$$\lambda_{i-2} + n\ell^n_{i-1} = \tilde{u}_{i-1} > \tilde{u}_i = \lambda_{i-1} + n\ell^n_i > u_{i-1} + n\ell^n_i.$$
Consequently, $n\ell^n_{i-1} > u_{i-1} - \lambda_{i-2} + n\ell^n_i$ and
$$\tilde{t}_{i-1} = t_{i-2} + n\ell^n_{i-1} > t_{i-2} + u_{i-1} - \lambda_{i-2} + n\ell^n_i = t_{i-1} + n\ell^n_i = \tilde{t}_i.$$
This ends the proof. □

Corollary 2.30. *Let* Λ *be a cuspidal increasing semimodule with basis*
$$\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$$
such that $\lambda_{-1} = n$ *and* $\lambda_0 = m$. *We have that* $\tilde{t}_1 = nm$ *and the following holds:*
$$t^n_{i+1} < \tilde{t}_1 = nm \quad \text{and} \quad t^m_{i+1} < \tilde{t}_1 = nm,$$
for any $1 \leq i \leq s$.

Proof: It is enough to recall that $\tilde{t}_1 = nm$ in view of Example 2.22. □

We end this section with a proposition that connects the limits and the colimits.

Proposition 2.31. *Consider* $1 \leq i \leq s$. *We have*

(a) *If* $k^n_i = i-1$, *then* $\ell^n_{i+1} + a_{i+1} = a_i$ *and* $\ell^m_{i+1} + b_{i+1} = \ell^m_i$.
(b) *If* $k^m_i = i-1$, *then* $\ell^n_{i+1} + a_{i+1} = \ell^n_i$ *and* $\ell^m_{i+1} + b_{i+1} = b_i$.

Proof: Notice that shifting the semimodule any integer number does not change the value of the limits and the colimits. Therefore, we can assume without loss of generality that Λ is normalized and thus $\lambda_{-1}=0$.

Let us prove statement (a). By hypothesis, we have that $k_i^n=i-1$. In view of Remark 2.28 and Proposition 2.27, we also have that $k_i^m=k_{i-1}^m$. Let us write:

$$u_{i+1}^n=\lambda_i+n\ell_{i+1}^n=\lambda_{i-1}+mb_{i+1}, \tag{6}$$
$$u_{i+1}^m=\lambda_i+m\ell_{i+1}^m=\lambda_{k_i^m}+na_{i+1}, \tag{7}$$
$$u_i^m=\lambda_{i-1}+m\ell_i^m=\lambda_{k_{i-1}^m}+na_i=\lambda_{k_i^m}+na_i. \tag{8}$$

From equations (6) and (7) we obtain that

$$n\ell_{i+1}^n+na_{i+1}+\lambda_{k_i^m}=mb_{i+1}+m\ell_{i+1}^m+\lambda_{i-1}. \tag{9}$$

By equation (8) we can substitute $\lambda_{k_i^m}=\lambda_{i-1}+m\ell_i^m-na_i$ in equation (9) to obtain

$$n(\ell_{i+1}^n+a_{i+1}-a_i)=m(\ell_{i+1}^m+b_{i+1}-\ell_i^m). \tag{10}$$

Since n and m have no common factor, we have that n divides $\ell_{i+1}^m+b_{i+1}-\ell_i^m$.

Let us see that $\ell_{i+1}^m+b_{i+1}-\ell_i^m=0$ and hence $\ell_{i+1}^m+b_{i+1}=\ell_i^m$ as desired. If $\ell_{i+1}^m+b_{i+1}-\ell_i^m\neq 0$, we are in one of the following three cases:

(i) $|\ell_{i+1}^m+b_{i+1}-\ell_i^m|\geq 2n$; (ii) $\ell_{i+1}^m+b_{i+1}-\ell_i^m=-n$; (iii) $\ell_{i+1}^m+b_{i+1}-\ell_i^m=n$.

Let us see that each of these cases leads to a contradiction.

Assume first that we are in case (i). Noting that $\ell_{i+1}^m, b_{i+1}, \ell_i^m\geq 1$, there is at least one of them that is strictly bigger than n. Let us consider the three possibilities:

- If $\ell_{i+1}^m>n$, we have that $m\ell_{i+1}^m>nm$ and then $\lambda_i+m\ell_{i+1}^m>nm$. This implies that $\lambda_i+m(\ell_{i+1}^m-1)>(n-1)m\geq c_\Gamma\geq c_{\Lambda_{i-1}}$. Then, we obtain $\lambda_i+m(\ell_{i+1}^m-1)\in\Lambda_{i-1}$, contradicting the minimality of ℓ_{i+1}^m.
- If $\ell_i^m>n$, we apply the same argument as before.
- If $b_{i+1}>n$, we have that $\lambda_i+n\ell_{i+1}^n=\lambda_{i-1}+mb_{i+1}>nm$ and then
$$\lambda_i+n(\ell_{i+1}^n-1)>(m-1)n\geq c_\Gamma\geq c_{\Lambda_{i-1}}.$$
Then $\lambda_i+n(\ell_{i+1}^n-1)\in\Lambda_{i-1}$ and this contradicts the minimality of ℓ_{i+1}^n.

Assume that we are in case (ii), that is, $\ell_{i+1}^m+b_{i+1}-\ell_i^m=-n$. This implies that $\ell_i^m>n$ and we use the same argument as before to obtain a contradiction.

Assume that we are in case (iii), that is, $\ell_{i+1}^m+b_{i+1}-\ell_i^m=n$. We have that $\ell_{i+1}^m+b_{i+1}>n$. By Remark 2.21 we see that the separation $S(z_{i-1},z_i)$ is given by $S(z_{i-1},z_i)=b_{i+1}$ (recall that $k_i^n=i-1$) and that the separation $S(z_i,z_{k_i^m})$ is given by $S(z_i,z_{k_i^m})=\ell_{i+1}^m$. Noting that $z_i\in\langle z_{i-1},z_{k_i^m}\rangle$ and $z_{i-1}\neq z_{k_i^m}$, we conclude that

$$n>S(z_{i-1},z_i)+S(z_i,z_{k_i^m})=b_{i+1}+\ell_{i+1}^m.$$

This contradicts $b_{i+1}+\ell_{i+1}^m>n$. The proof of the equality $\ell_{i+1}^m+b_{i+1}=\ell_i^m$ is ended. Moreover, since $\ell_{i+1}^m+b_{i+1}-\ell_i^m=0$, by equation (10), we conclude that $\ell_{i+1}^n+a_{i+1}=a_i$, as desired.

The proof of statement (b) runs in a similar way to the above arguments. □

The next corollary will be useful in our computation of Saito bases:

Corollary 2.32. *Consider* $2 \leq j+1 < q \leq s+1$. *Then*

$$\ell^m_{j+1} - (\ell^m_{j+2} + \ell^m_{j+3} + \cdots + \ell^m_q) = b_q > 0,$$

under the assumption that $\tilde{t}_{j+1} = t^m_{j+1}$ *and* $\tilde{t}_\ell = t^n_\ell$, *for* $j+2 \leq \ell \leq q-1$. *Symmetrically we have*

$$\ell^n_{j+1} - (\ell^n_{j+2} + \ell^n_{j+3} + \cdots + \ell^n_q) = a_q > 0,$$

under the assumption that $\tilde{t}_{j+1} = t^n_{j+1}$ *and* $\tilde{t}_\ell = t^m_\ell$, *for* $j+2 \leq \ell \leq q-1$.

Proof: We prove the first assertion; the second one is similar. Let us consider the difference

$$\ell^m_{j+1} - \ell^m_{j+2}.$$

Since $\tilde{t}_{j+1} = t^m_{j+1}$, by Proposition 2.27, we have that $k^n_{j+1} = j$. By Proposition 2.31, we conclude that

$$\ell^m_{j+1} - \ell^m_{j+2} = b_{j+2}.$$

Now, let us study the difference $b_{j+2} - \ell^m_{j+3}$. Since $t_{j+2} = t^m_{j+2}$, we have that $k^m_{j+2} = j+1$. By Proposition 2.31, we conclude

$$b_{j+2} - \ell^m_{j+3} = b_{j+3}.$$

Continuing in this way, we obtain

$$\ell^m_{j+1} - (\ell^m_{j+2} + \ell^m_{j+3} + \cdots + \ell^m_q) = b_q > 0,$$

as desired. □

3. Cuspidal standard bases

3.1. Semigroup and semimodule of an analytic branch. Let us consider the local ring $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$ of the analytic space $\mathbb{C}^2$ at the origin. Denote by x, y the coordinates of $\mathbb{C}^2$, that we consider as elements $x, y \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$. We recall that there is an identification $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}} = \mathbb{C}\{x,y\}$ between the local ring $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$ and the ring of convergent power series in x, y with complex coefficients. By definition, an *analytic plane branch* C at the origin of $\mathbb{C}^2$ is a principal prime ideal $C \subset \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$. Any generator f of C is called an *equation of* C. It is known that there is a morphism

$$\varphi\colon \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \longrightarrow \mathbb{C}\{t\}$$

such that $C = \ker\varphi$. Such morphisms are given in terms of convergent series by

$$\varphi(g(x,y)) = g(a(t), b(t)), \quad a(t), b(t) \in \mathbb{C}\{t\}.$$

We call them *parametrizations of* C. The subset

$$\Gamma_\varphi = \{\mathrm{ord}_t(\varphi(g));\, g \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\} \subset \mathbb{Z}_{\geq 0}$$

is a semigroup of $\mathbb{Z}_{\geq 0}$. A parametrization φ is called *primitive* if and only if the following equivalent properties hold:

- The semigroup Γ_φ has a conductor, that is, there is $c_{\Gamma_\varphi} \in \mathbb{Z}$ minimal with the property that $n \in \Gamma_\varphi$ for any $n \geq c_{\Gamma_\varphi}$.
- There is no series $\psi(t) \neq 0$ with $\mathrm{ord}_t(\psi(t)) \geq 2$ such that

  $$\varphi(g(x,y)) = g(a(t), b(t)),$$

  where $a(t) = \tilde{a}(\psi(t))$ and $b(t) = \tilde{b}(\psi(t))$. Thus, we have another parametrization $\tilde{\varphi}$ given by $\tilde{\varphi}(g(x,y)) = g(\tilde{a}(t), \tilde{b}(t))$.

There are always primitive parametrizations. If φ and φ' are primitive parametrizations of the plane branch C, we have that

$$\mathrm{ord}_t(\varphi(g)) = \mathrm{ord}_t(\varphi'(g)), \quad g \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}.$$

We denote $\nu_C(g) = \mathrm{ord}_t(\varphi(g))$. We also conclude that $\Gamma_\varphi = \Gamma_{\varphi'}$ and we call this semigroup the *semigroup* Γ_C *of the plane branch* C.

We say that C is a *plane cusp* if the semigroup Γ_C is generated by two integer numbers $2 \leq n < m$ without common factors. In this paper we mainly deal with plane cusps.

Let C be a plane cusp. We know that, after an appropriate coordinate change, we can choose coordinates and a primitive parametrization

$$x = t^n, \quad y = b(t), \quad \mathrm{ord}_t\, b(t) = m,$$

where $2 \leq n < m$ and n, m are without common factors. These coordinates are called *adapted coordinates* and the above parametrization is a *Puiseux parametrization associated to the adapted coordinates.*

Remark 3.1. Zariski's equisingularity theory concerns all the plane branches, but we are paying special attention to the cusps, which are the branches with a single Puiseux pair (n, m).

Let us denote by $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$ the $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module of germs of holomorphic differential 1-forms at the origin of $\mathbb{C}^2$. We know that $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$ is a free rank-two $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module generated by dx and dy. Similarly, we denote by $\Omega^1_{\mathbb{C},0}$ the $\mathcal{O}_{\mathbb{C},0}$-module of germs of holomorphic differential 1-forms at the origin of $\mathbb{C}$. We also denote by $\Omega^2_{\mathbb{C}^2,\mathbf{0}}$ the $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module of germs of holomorphic differential 2-forms at the origin of $\mathbb{C}^2$; in this case it is a free module of rank 1 generated by $dx \wedge dy$.

An element $\alpha \in \Omega^1_{\mathbb{C},0}$ is written as $\alpha = \psi(t)\, dt$. The *order* of α is by definition the order of $\psi(t)$. More precisely, we write

$$\mathrm{ord}_t(\alpha) = \mathrm{ord}_t(\psi(t)).$$

Given a primitive parametrization φ of an analytic plane branch C, we have a "pull-back" application

$$\begin{aligned} \varphi^*\colon \Omega^1_{\mathbb{C}^2,\mathbf{0}} &\longrightarrow \Omega^1_{\mathbb{C},\mathbf{0}} \\ \omega &\longmapsto \varphi^*\omega \end{aligned}$$

defined by the properties that $\varphi^*(\omega + \omega') = \varphi^*(\omega) + \varphi^*(\omega')$, $\varphi^*(h\omega) = \varphi(h)\varphi^*(\omega)$, and

$$\varphi^*(dx) = \frac{\partial \varphi(x)}{\partial t}\, dt, \quad \varphi^*(dy) = \frac{\partial \varphi(y)}{\partial t}\, dt.$$

The set

$$\Lambda_C = \{\mathrm{ord}_t(\varphi^*\omega) + 1;\, \omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}})\} \setminus \{\infty\} \subset \mathbb{Z}_{\geq 0}$$

is the so-called *semimodule of differential values for* C. This set is independent of the chosen primitive parametrization and also of the analytic class of C, where two analytic plane branches are analytically equivalent when they correspond to one another via an automorphism of the local ring $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$.

We take the notation $\nu_C(\omega) = \mathrm{ord}_t(\varphi^*(\omega)) + 1$ and we call it the *differential value of* ω *with respect to* C.

Remark 3.2. The set Λ_C is a semimodule over the semigroup Γ_C, that is, we have

$$\nu_C(h) + \nu_C(\omega) = \nu_C(h\omega).$$

Moreover, if $\nu_C(h) > 0$, we have that $\nu_C(dh) = \nu_C(h)$, where the notation dh stands for the differential

$$dh = (\partial h/\partial x)\, dx + (\partial h/\partial y)\, dy.$$

This means that $\Gamma_C \setminus \{0\} \subset \Lambda_C$ and $c_{\Lambda_C} \leq c_{\Gamma_C}$.

Consequently, there is a basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ for Λ_C. A set of differential forms

$$\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$$

such that $\nu_C(\omega_i) = \lambda_i$, for $i = -1, 0, 1, \ldots, s$, will be called a *minimal standard basis of differential* 1*-forms* for the analytic plane branch C.

3.2. Minimal standard bases. From now on, we consider a plane cusp C with Puiseux pair (n, m), where $2 \leq n < m$, without common factors. We denote its semigroup by $\Gamma = \Gamma_C = \langle n, m \rangle$ and its semimodule of differential values by $\Lambda = \Lambda_C$. We fix the notation

$$\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$$

for the basis of Λ. We recall that $\Gamma \setminus \{0\} \subset \Lambda$ and we have that $\lambda_{-1} = n$, $\lambda_0 = m$.

We also set adapted coordinates (x, y) and a Puiseux parametrization

$$x = t^n, \quad y = b(t), \quad \operatorname{ord}_t b(t) = m.$$

Note that $\nu_C(x) = \nu_C(dx) = n$ and $\nu_C(y) = \nu_C(dy) = m$.

3.2.1. Divisorial order. Let $\pi_C \colon M \to (\mathbb{C}^2, \mathbf{0})$ be the minimal reduction of the singularities of the cusp C. That is, the morphism π_C is the minimal finite composition of blow-ups centered at points in the successive strict transforms of C in such a way that the strict transform C' of C by π_C in M has normal crossings with the total exceptional divisor $E = \pi^{-1}(\mathbf{0})$. Let us denote by D_C the only irreducible component of E such that $C' \cap D_C \neq \emptyset$.

Lemma 3.3. *Let C, $\tilde{C}$ be two cusps with the same Puiseux pair (n, m). The following statements are equivalent:*

(a) *The cusps C and $\tilde{C}$ have the same minimal reduction of singularities.*
(b) *Given a local coordinate system (x, y) of $(\mathbb{C}^2, \mathbf{0})$, we have that (x, y) is adapted to C if and only if (x, y) is adapted to $\tilde{C}$.*

Proof: We leave this proof to the reader. □

Denote $D = D_C$ and $\pi = \pi_C$. Let us define the *divisorial order* ν_D as follows. Given an adapted coordinate system (x, y) and a point $Q \in D$, there are local coordinates (u, v) centered at Q such that $D = (u = 0)$ locally at Q. Consider $h \in \mathcal{O}_{\mathbb{C}^2, \mathbf{0}}$. Let us write the germ $(h \circ \pi)_Q$ of h at Q as

$$(h \circ \pi)_Q = (u)^\beta h',$$

where u does not divide h'. Then, we define $\nu_D(h) = \beta$. In a similar way, given a 1-form $\omega \in \Omega^1_{\mathbb{C}^2, \mathbf{0}}$, let us write the germ $(\pi^* \omega)_Q$ as

$$(\pi^* \omega)_Q = (u)^\beta \left\{ a' \frac{du}{u} + b'\, dv \right\},$$

where u does not divide the pair (a', b'). Then, we define $\nu_D(\omega) = \beta$.

The above definition of a divisorial order depends neither on the chosen adapted coordinates nor on the particular point $Q \in D$. We also call ν_D the *monomial order* in view of the next statement:

Lemma 3.4. *Let (x, y) be an adapted coordinate system for C and consider $h \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} = \mathbb{C}\{x, y\}$, which we write as*

$$h = \sum_{i,j\geq 0} h_{ij} x^i y^j.$$

Then we have that $\nu_D(h) = \min\{ni + mj;\, h_{ij} \neq 0\} = \min\{\nu_C(x^i y^j);\, h_{ij} \neq 0\}$. Similarly, take $\omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}$, which we write as

$$\omega = \sum_{i,j\geq 0} x^i y^j \left\{ \lambda_{ij} \frac{dx}{x} + \mu_{ij} \frac{dy}{y} \right\},$$

where $(\lambda_{ij}, \mu_{ij}) \in \mathbb{C}$. Then, we have that $\nu_D(h) = \min\{ni + mj;\, (\lambda_{ij}, \mu_{ij}) \neq (0, 0)\}$.

Proof: Left to the reader (see [**4**]). □

Remark 3.5. Take $\omega = ny\,dx - mx\,dy = xy(n\,dx/x - m\,dy/y)$; note that

$$\nu_D(\omega) = n + m < \nu_C(\omega).$$

Let us also note that if $\omega = a\,dx + b\,dy$, then $\nu_D(\omega) = \min\{\nu_D(xa), \nu_D(yb)\}$.

Remark 3.6. For the case of functions as well as for differential 1-forms, we have that $\nu_C(h) \geq \nu_D(h)$ and $\nu_C(\omega) \geq \nu_D(\omega)$.

3.2.2. Initial parts. Here, we fix a coordinate system (x, y) adapted to the cusp C. Given a function $h = \sum_{i,j} h_{ij} x^i y^j \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} = \mathbb{C}\{x, y\}$ with $\nu_D(h) = p$, we define the initial part $\mathrm{In}^{x,y}_{n,m}(h)$ by

$$\mathrm{In}^{x,y}_{n,m}(h) = \sum_{ni+mj=p} h_{ij} x^i y^j.$$

If there is no confusion, we write $\mathrm{In}(h) = \mathrm{In}^{x,y}_{n,m}(h)$. In a similar way, given a 1-differential form $\omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}$, with $p = \nu_D(\omega)$, which we write as

$$\omega = \sum_{ni+mj\geq p} x^i y^j \left(\lambda_{ij} \frac{dx}{x} + \mu_{ij} \frac{dy}{y} \right),$$

we define the initial part $\mathrm{In}(\omega)$ by

$$\mathrm{In}(\omega) = \sum_{ni+mj=p} x^i y^j \left(\lambda_{ij} \frac{dx}{x} + \mu_{ij} \frac{dy}{y} \right).$$

Remark 3.7. Assume that $\nu_D(\omega) = p < nm$. Then there are unique $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, with $(\alpha, \beta) \neq (0, 0)$, such that $n\alpha + m\beta = p$ and hence the initial part is a single differential monomial

$$\mathrm{In}(\omega) = x^\alpha y^\beta \left(\lambda \frac{dx}{x} + \mu \frac{dy}{y} \right).$$

We can employ a similar argument for the case of a function $h \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$.

Let $\omega = a\,dx + b\,dy$ be a differential 1-form in $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$ such that $\nu_D(\omega) < nm$. If $\nu_C(\omega) > \nu_D(\omega)$, then the initial part $\mathrm{In}(\omega)$ is given by

$$\mathrm{In}(\omega) = \mu x^\alpha y^\beta \left(m \frac{dx}{x} - n \frac{dy}{y} \right),$$

where $\mu \neq 0$, and $\alpha, \beta \geq 1$ are such that $\nu_D(\omega) = n\alpha + m\beta < nm$. Any differential 1-form as above will be called a *resonant differential* 1*-form*.

Given two differential 1-forms ω and η, we say that η is *reachable* from ω if there is a monomial $\mu x^\alpha y^\beta$ such that

$$\operatorname{In}(\eta) = \mu x^\alpha y^\beta \operatorname{In}(\omega).$$

3.2.3. Semimodule versus minimal standard bases. Let us consider a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ of the cusp C. Recall that $\nu_C(\omega_i) = \lambda_i$, for $i = -1, 0, 1, \ldots, s$.

Lemma 3.8. *We have that $\lambda_{-1} = n$ and $\lambda_0 = m$. More precisely, the initial parts of ω_{-1} and ω_0 are respectively given by $\operatorname{In}(\omega_{-1}) = \lambda\, dx$, with $\lambda \neq 0$, and $\operatorname{In}(\omega_0) = \mu\, dy$, with $\mu \neq 0$.*

Proof: Note that $\nu_C(a\, dx) = \nu_C(a) + \nu_C(dx)$, $\nu_C(b\, dy) = \nu_C(b) + \nu_C(dy)$ and

$$\nu_C(a\, dx + b\, dy) \geq \min\{\nu_C(a\, dx), \nu_C(b\, dy)\}.$$

Since $\nu_C(dx) = n$ and $\nu_C(dy) = m$, with $n < m$, we conclude that $n = \min \Lambda = \lambda_{-1}$. We also have that $\nu_C(\omega_{-1}) = n$. Let us write

$$\omega_{-1} = \lambda\, dx + \eta, \quad \eta = x\eta_1 + y\eta_2 + h\, dy.$$

We have that $\nu_D(\eta) > n$ and hence $\nu_C(\eta) \geq \nu_D(\eta) > n$. The only possibility to have that $\nu_C(\omega_{-1}) = n$ is that $\lambda \neq 0$ and, in this case, we see that $\operatorname{In}(\omega_{-1}) = \lambda\, dx$.

Let us show that $\lambda_0 = m$ and that $\operatorname{In}(\omega_0) = \mu\, dy$. Let $k \geq 1$ be the integer number defined by the property that $kn < m < (k+1)n$. Take a differential 1-form ω that we write as

$$\omega = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})\, dx + \eta, \quad \eta = x^k \eta_1 + y\eta_2 + h\, dy.$$

We have that $m \leq \nu_D(\eta) \leq \nu_C(\eta)$. If $(c_1, c_2, \ldots, c_k) \neq (0, 0, \ldots, 0)$, taking the first j such that $c_j \neq 0$ we conclude that

$$\nu_C(\omega) = jn \in \lambda_{-1} + \Gamma.$$

Thus, the next differential value λ_0 is given by a differential 1-form η written as $\eta = x^k\eta_1 + y\eta_2 + h\, dy$. Let us decompose

$$\eta = \mu\, dy + \tilde{\eta}, \quad \tilde{\eta} = x^k\eta_1 + y\eta_2 + (xh_1 + yh_2)\, dy.$$

We obtain that $m < \nu_D(\tilde{\eta}) \leq \nu_C(\tilde{\eta})$. Thus, if $\mu \neq 0$, we get $m = \nu_D(\eta) = \nu_C(\eta)$. The desired result follows. □

Assume that $s \geq 1$ and let us describe the initial part $\operatorname{In}(\omega_1)$ of the element ω_1 in the minimal standard basis $\mathcal{S}$. Take ω such that $\nu_C(\omega) \notin \Lambda_0 = \Gamma \setminus \{0\}$; the value λ_1 is the minimum of the differential values $\nu_C(\omega)$ for such 1-forms ω. The first remark is that $\nu_C(\omega) < nm$, since $c_\Gamma = (n-1)(m-1)$ and hence $\nu_D(\omega) < nm$; the second remark is that ω is resonant. In view of Remark 3.7, we can write

$$\operatorname{In}(\omega) = x^\alpha y^\beta \left\{\lambda \frac{dx}{x} + \mu \frac{dy}{y}\right\}, \quad n\alpha + m\beta = \nu_D(\omega) < nm.$$

If $\lambda n + \mu m \neq 0$, we have that $\nu_C(\omega) = \nu_D(\omega) = n\alpha + m\beta \in \Lambda_0$. Then, the 1-form ω is necessarily resonant, that is,

$$\operatorname{In}(\omega) = \mu x^\alpha y^\beta \left\{m \frac{dx}{x} - n \frac{dy}{y}\right\}.$$

Assume now that $\nu_C(\omega) = \lambda_1$ (this property is satisfied by ω_1), let us show that $\alpha = \beta = 1$ (note that we necessarily have that $\alpha \geq 1$ and $\beta \geq 1$). Let us reason by contradiction, assuming that $n\alpha + m\beta > n + m$. We start with the differential 1-form

$$\eta = my\, dx - nx\, dy.$$

We know that $\nu_D(\eta) = n+m < \nu_C(\eta)$. If $\nu_C(\eta) = a_1 n + b_1 m \in \Lambda_0$ and $\nu_C(\eta) < nm$, there is $\mu_1 \neq 0$, such that

$$\nu_C(\eta + \mu_1 d(x^{a_1} y^{b_1})) > \nu_C(\eta).$$

Put $\eta^1 = \eta + \mu_1 d(x^{a_1} y^{b_1})$ and start the procedure again with η^1. In this way, we obtain a differential 1-form $\tilde{\eta}$ with the following properties:

- Either $\nu_C(\tilde{\eta}) \notin \Lambda_0$ or $\nu_C(\tilde{\eta}) > nm$.
- $\mathrm{In}(\tilde{\eta}) = \eta = my\,dx - nx\,dy$ and hence $\nu_D(\tilde{\eta}) = n+m$.

Now, we compare $\tilde{\eta}$ with ω as follows. We know that $\lambda_1 \leq \nu_C(\tilde{\eta})$. Moreover, there is a constant $\mu \neq 0$ such that

$$\nu_D(\omega - \mu x^{\alpha-1} y^{\beta-1} \tilde{\eta}) > \nu_D(\omega) = \alpha n + \beta m.$$

Put $\omega^1 = \omega - \mu x^{\alpha-1} y^{\beta-1}\tilde{\eta}$. We have that $\nu_C(\omega^1) = \lambda_1$, since $\nu_C(x^{\alpha-1} y^{\beta-1}\tilde{\eta}) > \lambda_1$. We restart with ω^1. Repeating the procedure, we can get $\tilde{\omega}$ such that

$$\nu_D(\tilde{\omega}) > nm, \quad \nu_C(\tilde{\omega}) = \lambda_1.$$

Since $\lambda_1 = \nu_C(\tilde{\omega}) \geq \nu_D(\tilde{\omega}) > nm$, this should imply that $\lambda_1 \in \Lambda_0$, which is a contradiction.

As a consequence, we have that $\mathrm{In}(\omega_1) = \mu(my\,dx - nx\,dy)$ as desired. The above arguments are generalized in [**4**, Proposition B.1] to obtain the following statement:

Theorem 3.9. *For each $1 \leq i \leq s$, we have the following statements:*

(a) $\lambda_i = \sup\{\nu_C(\omega);\ \omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}},$ *with* $\nu_D(\omega) = t_i\}$.

(b) *If $\nu_C(\omega) = \lambda_i$, then $\nu_D(\omega) = t_i$.*

(c) *For each 1-form ω with $\nu_C(\omega) \notin \Lambda_{i-1}$, there is a unique pair $a, b \geq 0$ such that $\nu_D(\omega) = \nu_D(x^a y^b \omega_i)$. Moreover, we have that $\nu_C(\omega) \geq \lambda_i + na + mb$.*

(d) *Let $k = \lambda_i + na + mb$; then $k \notin \Lambda_{i-1}$ if and only if for all ω such that $\nu_C(\omega) = k$ we have that $\nu_D(\omega) \leq \nu_D(x^a y^b \omega_i)$.*

(e) *We have that $\lambda_i > u_i$.*

Let us note that the critical values t_i of Λ correspond exactly to the divisorial values of the elements ω_i of any minimal standard basis. Let us also note that the semimodule of differential values Λ is an increasing cuspidal semimodule.

Corollary 3.10. *For each $1 \leq i \leq s$, the 1-forms ω_i are resonant. In particular, taking an adapted coordinate system (x, y), the initial parts can be written as*

$$\mathrm{In}(\omega_i) = \mu_i x^{e_i} y^{f_i} \left(m\frac{dx}{x} - n\frac{dy}{y} \right), \quad ne_i + mf_i = \nu_D(\omega_i) = t_i < nm.$$

Proof: Applying Lemma 2.7, since Λ is increasing, we have

$$\nu_D(\omega_i) = t_i < \lambda_i = \nu_C(\omega_i) < nm.$$

The statement follows from these inequalities. □

3.3. Generalized Delorme's decomposition. In this subsection, we state and prove a decomposition result for 1-forms which generalizes Delorme's decomposition [**5**] and Theorem 8.5 in [**4**]. Throughout this subsection, we fix a cusp C with semimodule Λ and basis $\mathcal{B} = (\lambda_{-1}, \ldots, \lambda_s)$. We consider a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$ of C. We also fix an element $* \in \{n, m\}$ (that is, $*$ is equal either to n or to m). Let us recall that we denote by

$$k_i, \quad 0 \leq i \leq s,$$

the bounds corresponding to the axes u_{i+1}, as introduced in Section 2. In the same way, we denote by k_i^* the bounds corresponding to the axes u_{i+1}^*. That is, we have

$$k_i = \begin{cases} k_i^n, & \text{if } u_{i+1} = u_{i+1}^n, \\ k_i^m, & \text{if } u_{i+1} = u_{i+1}^m; \end{cases} \quad k_i^* = \begin{cases} k_i^n, & \text{if } u_{i+1}^* = u_{i+1}^n, \\ k_i^m, & \text{if } u_{i+1}^* = u_{i+1}^m. \end{cases}$$

Theorem 3.11. *Consider indices $0 \leq j \leq i \leq s$ and let ω be a 1-form such that $\nu_D(\omega) = t_{i+1}^*$ and $\nu_C(\omega) > u_{i+1}^*$. Then there is a decomposition of the 1-form ω given by*

$$\omega = \sum_{\ell=-1}^{j} f_\ell^{ij} \omega_\ell,$$

such that the following properties hold. Let v_{ij}^* be defined by $v_{ij}^* = \nu_C(f_j^{ij}\omega_j)$. Then we obtain

(a) $v_{ij}^* = \min\{\nu_C(f_\ell^{ij}\omega_\ell);\ -1 \leq \ell < j\}$.

(b) $v_{ij}^* = \lambda_j + t_{i+1}^* - t_j$. *In particular, if $j = i$, we have that $v_{ii}^* = \lambda_i + t_{i+1}^* - t_i = u_{i+1}^*$.*

(c) *If $j < i$, we have that $\nu_C(f_\ell^{ij}\omega_\ell) = v_{ij}^*$ for $\ell = k_j$, and $\nu_C(f_\ell^{ij}\omega_\ell) > v_{ij}^*$ for any $\ell \neq k_j$ and $-1 \leq \ell < j$.*

(d) *If $j = i$, we have that $\nu_C(f_\ell^{ii}\omega_\ell) = v_{ii}^*$ for $\ell = k_j^*$, and $\nu_C(f_\ell^{ii}\omega_\ell) > v_{ii}^*$ for any $\ell \neq k_j^*$ and $-1 \leq \ell < j$.*

Remark 3.12. Let $\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \dots, \omega_s, \omega_{s+1})$ be an extended standard basis for C, that is, $(\omega_{-1}, \omega_0, \omega_1, \dots, \omega_s)$ is a minimal standard basis of C and ω_{s+1} is a 1-form with $\nu_D(\omega_{s+1}) = t_{s+1}$ and C is invariant for ω_{s+1} (see Section 4). Notice that Theorem 3.11 can be used to write ω_{i+1} in terms of $\omega_{-1}, \omega_0, \dots, \omega_i$ for $0 \leq i \leq s$. Indeed, let us choose $* \in \{n, m\}$ such that $u_{i+1}^* = u_{i+1}$, and hence $t_{i+1}^* = t_{i+1}$. We have that $\nu_D(\omega_{i+1}) = t_{i+1}$ and

$$\nu_C(\omega_{i+1}) = \begin{cases} \lambda_{i+1} > u_{i+1}, & \text{if } i \leq s-1, \\ \infty > u_{s+1}, & \text{if } i = s. \end{cases}$$

Now, by a direct application of Theorem 3.11, if we fix j with $0 \leq j \leq i$, we have an expression

$$\omega_{i+1} = f_j^{ij}\omega_j + f_{j-1}^{ij}\omega_{j-1} + \cdots + f_0^{ij}\omega_0 + f_{-1}^{ij}\omega_{-1},$$

such that $\lambda_j + t_{i+1} - t_j = \nu_C(f_j^{ij}\omega_j) = \nu_C(f_{k_j}^{ij}\omega_{k_j}) < \nu_C(f_\ell^{ij}\omega_\ell)$ for any $\ell \neq k_j$, with $-1 \leq \ell \leq j-1$.

Before starting the proof of Theorem 3.11, we state the following lemma, whose proof follows closely the proof of [**4**, Lemma C.1].

Lemma 3.13. *Consider $0 \leq i \leq s$. Given a 1-form η with $\nu_C(\eta) > u_{i+1}^*$ and $\nu_D(\eta) > t_{i+1}^*$, we have*

(a) *If $\nu_D(\eta) < nm$, there is a 1-form α such that:*

  (i) $\nu_D(\eta - \alpha) > \nu_D(\eta)$.

  (ii) *There is a decomposition $\alpha = \sum_{\ell=-1}^{i} g_\ell\omega_\ell$, where $\nu_C(g_\ell\omega_\ell) > u_{i+1}^*$ and $\nu_D(g_\ell\omega_\ell) > t_{i+1}^*$, for any $-1 \leq \ell \leq i$.*

(b) *If $\nu_D(\eta) \geq nm$, there is a decomposition $\eta = \sum_{\ell=-1}^{i} h_\ell\omega_\ell$, where each term satisfies that $\nu_C(h_\ell\omega_\ell) > u_{i+1}^*$.*

Proof: Let us first prove statement (b). Since $\{\omega_{-1}, \omega_0\}$ is a basis of $\Omega^1_{(\mathbb{C}^2,0)}$, we can write in a unique way

$$\eta = g_{-1}\omega_{-1} + g_0\omega_0. \tag{11}$$

Moreover, since $\mathrm{In}(\omega_{-1}) = \lambda\, dx$ and $\mathrm{In}(\omega_0) = \mu\, dy$, we have

$$\nu_D(\eta) = \min\{\nu_D(g_{-1}\omega_{-1}),\, \nu_D(g_0\omega_0)\}.$$

Noting that $\nu_D(\eta) \geq nm$, we have that $\nu_D(g_{-1}\omega_{-1}) \geq nm$ and $\nu_D(g_0\omega_0) \geq nm$. By Lemma 2.29, we have that $\tilde{u}_{i+1} < \tilde{u}_1$, besides $u^*_{i+1} \leq \tilde{u}_{i+1}$, hence

$$u^*_{i+1} \leq \tilde{u}_{i+1} < \tilde{u}_1 = nm.$$

We conclude that $\nu_C(g_\ell\omega_\ell) \geq \nu_D(g_\ell\omega_\ell) \geq nm > u^*_{i+1}$, for $\ell = -1, 0$. Then the decomposition in equation (11) satisfies the required properties.

Let us now prove statement (a). By Remark 3.7, the initial part of η is equal to

$$\mathrm{In}(\eta) = x^a y^b \left(\mu_{-1}\frac{dx}{x} + \mu_0\frac{dy}{y}\right), \quad \nu_D(\eta) = na + mb.$$

There are two possibilities: η is either resonant or not. If η is not resonant, we have that $\nu_C(\eta) = \nu_D(\eta) = na + mb$. Let us consider

$$\alpha = \mathrm{In}(\eta) = g_{-1}\,\mathrm{In}(\omega_{-1}) + g_0\,\mathrm{In}(\omega_0), \quad g_{-1} = \frac{\mu_{-1}}{\lambda}x^{a-1}y^b, \quad g_0 = \frac{\mu_0}{\mu}x^a y^{b-1}. \tag{12}$$

We have that $\nu_D(\eta - \alpha) > \nu_D(\eta)$. Moreover, we have that $\nu_D(\alpha) = \nu_D(\eta) > u^*_{i+1}$. Since

$$\nu_D(\alpha) = \min\{\nu_D(g_{-1}\omega_{-1}), \nu_D(g_0\omega_0)\},$$

we conclude that $\nu_C(g_\ell\omega_\ell) \geq \nu_D(g_\ell\omega_\ell) \geq \nu_D(\alpha) > u^*_{i+1}$, for $\ell = -1, 0$. In view of Corollary 2.8, we have that $u^*_{i+1} > t^*_{i+1}$. Hence, we also get that $\nu_D(g_\ell\omega_\ell) > t^*_{i+1}$, for $\ell = -1, 0$. Thus, the expression in equation (12) satisfies the desired properties.

Now, let us assume that η is resonant. Up to multiplying η by a non-null scalar, we have

$$\mathrm{In}(\eta) = x^a y^b \left(m\frac{dx}{x} - n\frac{dy}{y}\right), \quad \nu_D(\eta) = na + mb > t^*_{i+1}.$$

Let us define the index $k := \max\{\ell \leq i : \eta \text{ is reachable by } \omega_\ell\}$. Since η is resonant, then $k \geq 1$. By definition of k, there exists a monomial $\mu x^c y^d$ such that $\nu_D(\mu x^c y^d \omega_k) = \nu_D(\eta)$ and

$$\nu_D(\eta') > \nu_D(\eta), \quad \text{where } \eta' = \eta - \mu x^c y^d \omega_k.$$

The desired decomposition will be given just by the expression $\alpha = \mu x^c y^d \omega_k$. Since $\nu_D(\alpha) = \nu_D(\eta) > t^*_{i+1}$, we only need to verify that $\nu_C(x^c y^d \omega_k) > u^*_{i+1}$.

First, let us assume that $k = i$ and hence $\alpha = \mu x^c y^d \omega_i$. Write

$$\nu_D(\alpha) = nc + md + t_i = \nu_D(\eta) > t^*_{i+1}.$$

Recalling that $t^*_{i+1} = t_i + u^*_{i+1} - \lambda_i$, we obtain that $nc + md + \lambda_i > u^*_{i+1}$. Hence, we conclude by noting that

$$\nu_C(x^c y^d \omega_i) = nc + md + \lambda_i > u^*_{i+1}.$$

Now, let us consider the case when $1 \leq k \leq i - 1$. Assume by contradiction that $\nu_C(x^c y^d \omega_k) \leq u^*_{i+1} < \nu_C(\eta)$. Taking into account that $\eta' = \eta - \mu x^c y^d \omega_k$, we see the following:

$$\nu_D(\eta') > \nu_D(x^c y^d \omega_k) = nc + md + t_k;$$
$$\nu_C(\eta') = \nu_C(x^c y^d \omega_k) = nc + md + \lambda_k.$$

By statement (d) in Theorem 3.9, we have that $nc + md + \lambda_k \in \Lambda_{k-1}$. In view of Lemma 2.4, this implies that either $c \geq \ell^n_{k+1}$ or $d \geq \ell^m_{k+1}$. There are four possibilities:

$$u_{k+1} = \lambda_k + n\ell^n_{k+1} \text{ and } c \geq \ell^n_{k+1}; \quad u_{k+1} = \lambda_k + n\ell^n_{k+1} \text{ and } d \geq \ell^m_{k+1};$$

$$u_{k+1} = \lambda_k + m\ell^m_{k+1} \text{ and } c \geq \ell^n_{k+1}; \quad u_{k+1} = \lambda_k + m\ell^m_{k+1} \text{ and } d \geq \ell^m_{k+1}.$$

The cases from the first line behave similarly to those in the second one, therefore, we will only show what happens in the first two cases.

Case $u_{k+1} = u^n_{k+1} = \lambda_k + n\ell^n_{k+1}$ *and* $c \geq \ell^n_{k+1}$. In this case we have that η is reachable from $x^{\ell^n_{k+1}}\omega_k$. If we show that $x^{\ell^n_{k+1}}\omega_k$ is reachable from ω_{k+1}, we contradict the maximality of k, as desired. In view of Corollary 3.10, noting that both ω_{k+1} and ω_k are resonant, it is enough to show that

$$\nu_D(x^{\ell^n_{k+1}}\omega_k) = \nu_D(\omega_{k+1}).$$

We have that $\nu_D(\omega_{k+1}) = t_{k+1}$ and $\nu_D(x^{\ell^n_{k+1}}\omega_k) = t_k + n\ell^n_{k+1}$. Let us see that $t_{k+1} = t_k + n\ell^n_{k+1}$ in our case. In a general way, we have that $t^n_{k+1} = t_k + n\ell^n_{k+1}$; moreover, the fact that $u_{k+1} = u^n_{k+1}$ implies also that $t_{k+1} = t^n_{k+1}$ and we finish.

Case $u_{k+1} = u^n_{k+1} = \lambda_k + n\ell^n_{k+1}$ *and* $d \geq \ell^m_{k+1}$. By Lemma 2.29, we see that

$$nc + md + \lambda_k \geq \lambda_k + m\ell^m_{k+1} = \tilde{u}_{k+1} \geq \tilde{u}_i > u^*_{i+1}.$$

This ends the proof. □

Proof of Theorem 3.11: Let us take a 1-form ω such that $\nu_D(\omega) = t^*_{i+1}$ and $\nu_C(\omega) > u^*_{i+1}$ as in the statement. We will consider three cases:

$$\text{(i) } i = 0; \quad \text{(ii) } i > 0,\ j = i; \quad \text{(iii) } i > 0,\ 0 \leq j < i.$$

Case (i): $i = 0$. Since $\{\omega_{-1}, \omega_0\}$ is a basis of $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$, the 1-form ω can be written as

$$\omega = f^{00}_{-1}\omega_{-1} + f^{00}_0\omega_0.$$

Looking at the computations in Example 2.22, we see that $t^*_1 = u^*_1 \leq nm$ and $k^*_0 = -1$. Therefore, we need to prove that $\nu_C(f^{00}_{-1}\omega_{-1}) = \nu_C(f^{00}_0\omega_0) = u^*_1$.

Recall that, up to constant, we have that $\mathrm{In}(\omega_{-1}) = dx$ and $\mathrm{In}(\omega_0) = dy$. Hence, one of the following cases occurs:

(I) $\nu_D(f^{00}_0\omega_0) = t^*_1$ and $\nu_D(f^{00}_{-1}\omega_{-1}) \geq t^*_1$.

(II) $\nu_D(f^{00}_{-1}\omega_{-1}) = t^*_1$ and $\nu_D(f^{00}_0\omega_0) \geq t^*_1$.

Assume that we are in case (I). Since $\nu_D(f^{00}_0) + \nu_D(\omega_0) \leq nm$, we have

$$\nu_D(f^{00}_0) < nm.$$

This implies that $\nu_D(f^{00}_0) = \nu_C(f^{00}_0)$ (left to the reader). Therefore, we can write

$$\nu_C(f^{00}_0\omega_0) = \nu_C(f^{00}_0) + \nu_C(\omega_0) = \nu_D(f^{00}_0) + \nu_D(\omega_0) = \nu_D(f^{00}_0\omega_0) = t^*_1 = u^*_1.$$

Moreover, since $\nu_D(f^{00}_{-1}\omega_{-1}) \geq t^*_1 = u^*_1$, we have

$$\nu_C(f^{00}_{-1}\omega_{-1}) \geq \nu_D(f^{00}_{-1}\omega_{-1}) \geq t^*_1 = u^*_1.$$

Noting that $\nu_C(f^{00}_{-1}\omega_{-1} + f^{00}_0\omega_0) > u^*_1$, that $\nu_C(f^{00}_0\omega_0) = u^*_1$, and that $\nu_C(f^{00}_0\omega_0) \geq u^*_1$, we conclude that $\nu_C(f^{00}_{-1}\omega_{-1}) = \nu_C(f^{00}_0\omega_0) = u^*_1$.

We apply a similar argument in the case that $\nu_C(f^{00}_{-1}\omega_{-1}) = t^*_1$.

Case (ii): $i > 0$ *and* $j = i$. We use the proof in the case $* = n$; the case $* = m$ runs in a similar way. Note that

$$\nu_D(\omega) = t^n_{i+1} < nm, \quad \nu_D(\omega) = t^n_{i+1} < u^n_{i+1} < \nu_C(\omega),$$

in view of Corollary 2.30 and Corollary 2.8. We deduce that the 1-form ω is resonant. Since ω_i is also resonant and we have

$$\nu_D(\omega) = \nu_D(x^{\ell^n_{i+1}}\omega_i) \quad (\text{recall that } t^n_{i+1} = t_i + n\ell^n_{i+1}),$$

we deduce that there is a non-null scalar $\mu \neq 0$ such that

$$\mathrm{In}(\omega) = \mu\,\mathrm{In}(x^{\ell^n_{i+1}}\omega_i) = \mu x^{\ell^n_{i+1}}\,\mathrm{In}(\omega_i).$$

Thus, the 1-form $\eta_1 = \omega - \mu x^{\ell^n_{i+1}}\omega_i$ satisfies the following two properties:

$$\nu_D(\eta_1) > t^n_{i+1}, \quad \nu_C(\eta_1) = \nu_C(x^{\ell^n_{i+1}}\omega_i) = u^n_{i+1} \quad (\text{recall that } u^n_{i+1} = \lambda_i + n\ell^n_{i+1}).$$

The second one comes from the fact that $\nu_C(\omega) > u^n_{i+1}$. Take the bound $k = k^n_i$ and the colimit $b = b_{i+1}$. We recall that

$$u^n_{i+1} = \lambda_i + n\ell^n_{i+1} = \lambda_k + mb.$$

Hence, the 1-form $y^b\omega_k$ satisfies that $\nu_C(y^b\omega_k) = u^n_{i+1}$. On the other hand, the divisorial value $\nu_D(y^b\omega_k)$ is given by

$$\nu_D(y^b\omega_k) = mb + t_k.$$

Let us show that $\nu_D(y^b\omega_k) > t^n_{i+1} = \nu_D(\omega)$. We have

$$\begin{aligned} t^n_{i+1} < mb + t_k &\iff t_i + n\ell^n_{i+1} < t_k + mb \\ &\iff t_i - t_k < mb - n\ell^n_{i+1} = mb - n\ell^n_{i+1} + u^n_{i+1} - u^n_{i+1} \\ &\iff t_i - t_k < (u^n_{i+1} - n\ell^n_{i+1}) - (u^n_{i+1} - mb) = \lambda_i - \lambda_k. \end{aligned}$$

We conclude, since $t_i - t_k < \lambda_i - \lambda_k$ in view of Lemma 2.7.

Take $\eta_2 = \eta_1 - \mu_2 y^b\omega_k$ such that $\nu_C(\eta_2) > u^n_{i+1}$. Note that $\nu_D(\eta_2) > t^n_{i+1}$. Applying Lemma 3.13, we get a decomposition

$$\eta_2 = \omega - \mu x^{\ell^n_{i+1}}\omega_i - \mu_2 y^b\omega_k = \sum_{\ell=-1}^{i} h_\ell\omega_\ell, \quad \nu_C(h_\ell\omega_\ell) > u^n_{i+1}, \quad \nu_D(h_\ell\omega_\ell) > t^n_{i+1},$$

having the desired properties.

Case (iii): $i > 0$, $0 \leq j < i$. Let us reason by inverse induction on j, recalling that we have the result when $j = i$. By the induction hypothesis, we can decompose ω as:

$$\omega = \sum_{\ell=-1}^{j+1} f^{ij+1}_\ell \omega_\ell, \tag{13}$$

where $v^*_{ij+1} = \nu_C(f^{ij+1}_{j+1}\omega_{j+1}) = \min\{\nu_C(f^{ij+1}_\ell\omega_\ell);\ -1 \leq \ell < j+1\}$. Notice that in the case (ii) we have proved the case where $j+1 = i$. In view of Remark 3.12, we can apply case (ii) to ω_{j+1} to obtain a decomposition:

$$\omega_{j+1} = \sum_{\ell=-1}^{j} f^{jj}_\ell \omega_\ell, \tag{14}$$

where $u_{j+1} = \nu_C(f^{jj}_j\omega_j) = \min\{\nu_C(f^{jj}_\ell\omega_\ell);\ \ell < j\}$, and the minimum is only reached at the bound $k = k_j$. If we substitute the expression of ω_{j+1} given in (14) into the expression of ω given in (13), we obtain

$$\omega = \sum_{\ell=-1}^{j} (f^{ij+1}_\ell + f^{ij+1}_{j+1} f^{jj}_\ell)\omega_\ell. \tag{15}$$

Let us show that equation (15) gives the desired decomposition. In order to do this, we only have to show that

(I) $\nu_C((f_j^{ij+1} + f_j^{jj} f_{j+1}^{ij+1})\omega_j) = \nu_C((f_k^{ij+1} + f_k^{jj} f_{j+1}^{ij+1})\omega_k) = v_{ij}^*$.

(II) $\nu_C((f_\ell^{ij+1} + f_\ell^{jj} f_{j+1}^{ij+1})\omega_\ell) > v_{ij}^*$ for $\ell \neq j, k$.

Recall that $v_{ij+1}^* = \lambda_{j+1} + t_{i+1}^* - t_{j+1}$ and $v_{ij}^* = \lambda_j + t_{i+1}^* - t_j$. Hence, by Lemma 2.7, we have that $v_{ij}^* < v_{ij+1}^*$. Moreover, by the properties of the decomposition given in equation (13), we get that:

$$\nu_C(f_{j+1}^{ij+1}) = v_{ij+1}^* - \lambda_{j+1}; \tag{16}$$

$$\nu_C(f_\ell^{ij+1}\omega_\ell) \geq v_{ij+1}^* > v_{ij}^*, \qquad \text{for } \ell < j+1. \tag{17}$$

Using the expression given in (16) and the properties of the decomposition given in (14), it follows that

$$\begin{aligned}\nu_C(f_{j+1}^{ij+1} f_\ell^{jj}\omega_\ell) &= \nu_C(f_{j+1}^{ij+1}) + \nu_C(f_\ell^{jj}\omega_\ell)\\ &= v_{ij+1}^* - \lambda_{j+1} + \nu_C(f_\ell^{jj}\omega_\ell)\\ &\geq v_{ij+1}^* - \lambda_{j+1} + u_{j+1},\end{aligned}$$

where the last inequality is an equality just for $\ell = j, k$. Now, taking into account that $u_{j+1} = \lambda_j + t_{j+1} - t_j$ and that $v_{ij+1}^* = t_{i+1}^* + \lambda_{j+1} - t_{j+1}$, we obtain that

$$v_{ij+1}^* - \lambda_{j+1} + u_{j+1} = \lambda_j + t_{i+1}^* - t_j = v_{ij}^*.$$

Finally, since $\nu_C(f_\ell^{ij+1}\omega_\ell) > v_{ij}^*$ for $\ell < j+1$, by expression (17), we get that

$$\nu_C((f_\ell^{ij+1} + f_\ell^{jj} f_{j+1}^{ij+1})\omega_\ell) \geq v_{ij}^*,$$

where, again, we have an equality just for $\ell = j, k$. □

Corollary 3.14. *Consider $1 \leq i \leq s$ and let ω be a 1-form such that $\nu_D(\omega) = t_{i+1}^*$ and $\nu_C(\omega) > u_{i+1}^*$. For any decomposition*

$$\omega = \sum_{\ell=-1}^{j} f_\ell^{ij}\omega_\ell, \quad 1 \leq j \leq i,$$

satisfying the stated properties in Theorem 3.11, we have that $\mathrm{In}(\omega) = \mathrm{In}(f_j^{ij}\omega_j)$.

Proof: We only need to show that $\nu_D(f_j^{ij}\omega_j) < \nu_D(f_\ell^{ij}\omega_\ell)$ for $\ell < j$. We know that $\nu_C(f_j^{ij}\omega_j) \leq \nu_C(f_\ell^{ij}\omega_\ell)$ for $\ell < j$. Besides, $\nu_D(\omega) = t_{i+1}^* < nm$, because $i \geq 1$. Therefore, we have that $nm > \nu_D(f_j^{ij})$ and consequently the monomial order and the differential value coincide: $\nu_D(f_j^{ij}) = \nu_C(f_j^{ij})$. Furthermore,

$$\nu_C(f_\ell^{ij}\omega_\ell) = \nu_C(f_\ell^{ij}) + \lambda_\ell \geq \nu_C(f_j^{ij}\omega_j) = \nu_C(f_j^{ij}) + \lambda_j.$$

By Lemma 2.7, we have that $\lambda_j - \lambda_\ell > t_j - t_\ell$, thus

$$\nu_C(f_\ell^{ij}) + t_\ell > \nu_C(f_j^{ij}\omega_j) = \nu_C(f_j^{ij}) + t_j.$$

If $\nu_C(f_\ell^{ij}) \geq nm$, then its monomial order is at least nm. Hence, we have

$$\nu_D(f_\ell^{ij}\omega_\ell) > t_{i+1}^* = \nu_D(f_j^{ij}\omega_j).$$

Indeed, if $\nu_C(f_\ell^{ij}) < nm$, we get that $\nu_C(f_\ell^{ij}) = \nu_D(f_\ell^{ij})$. With this, we conclude that

$$\nu_D(f_\ell^{ij}\omega_\ell) = \nu_D(f_\ell^{ij}) + t_\ell > \nu_D(f_j^{ij}) + t_j = \nu_D(f_j^{ij}\omega_j). \qquad □$$

Proposition 3.15. *Let ω be a 1-form such that $\nu_D(\omega) = t_1^*$ and $\nu_C(\omega) > u_1^*$. Let us write ω (in a unique way) as*

$$\omega = f_{-1}\omega_{-1} + f_0\omega_0,$$

then

(a) *if $t_1^* = t_1$, we have that $\mathrm{In}(\omega) = \mu(my\,dx - nx\,dy)$, where $\mu \neq 0$;*
(b) *if $t_1^* = \tilde{t}_1$, we have that $\mathrm{In}(\omega) = \mu(\mathrm{In}(df))$, where $f = 0$ is a reduced equation of the cusp C and $\mu \neq 0$.*

In particular, we have that $\mathrm{In}(\omega) = \mathrm{In}(f_{-1}\omega_{-1}) + \mathrm{In}(f_0\omega_0)$.

Proof: If $t_1^* = t_1$, since $t_1 = n + m$, we see that $\mathrm{In}(\omega)$ can be written as

$$\mathrm{In}(\omega) = \mu_{-1}y\,dx - \mu_0 x\,dy.$$

Moreover, we have that $t_1 = u_1 = n + m$ and hence $\nu_C(\omega) > \nu_D(\omega)$. Hence ω is resonant and we finish.

If $t_1^* = \tilde{t}_1 = nm$, we also have that $\tilde{u}_1 = nm$. The initial part $\mathrm{In}(\omega)$ has the form

$$\mathrm{In}(\omega) = \mu_{-1}x^{m-1}\,dx + \mu_0 y^{n-1}\,dy.$$

If this initial part is not a multiple of $\mathrm{In}(df)$, we get that $\nu_C(\omega) = nm$, which contradicts the hypothesis. □

4. Standard systems and Saito bases

4.1. Standard systems. Consider a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ of C. In [**4**], we have found differential 1-forms ω_{s+1} with the following properties:

- $\nu_D(\omega_{s+1}) = t_{s+1}$.
- The cusp C is invariant for ω_{s+1}, that is, $\nu_C(\omega_{s+1}) = \infty$.

We call an *extended standard basis* of C any sequence

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1}),$$

such that $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ is a minimal standard basis and ω_{s+1} satisfies the above properties.

In the next definition we present systems of differential 1-forms, where the axes $\tilde{u}_{s+1}$, instead of u_{s+1}, are essential in their construction.

Definition 4.1. A *standard system* $(\mathcal{E}, \mathcal{F})$ for a cusp C is the data of an extended standard basis $\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1})$ and a family $\mathcal{F} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1})$ of 1-forms satisfying that

$$\nu_D(\tilde{\omega}_j) = \tilde{t}_j, \quad \nu_C(\tilde{\omega}_j) = \infty, \quad 1 \leq j \leq s + 1.$$

We say that a standard system $(\mathcal{E}, \mathcal{F})$ for C is a *special standard system* if there are expressions $\tilde{\omega}_j = h_j\omega_{s+1} + f_j\tilde{\omega}_{s+1}$, where $h_j, f_j \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$ for any $1 \leq j \leq s$.

4.2. Saito bases. Let C be a cusp. Let us denote by $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ the $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-submodule of $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$ given by the 1-forms ω such that C is invariant for ω, that is, $\nu_C(\omega) = \infty$.

It is known that $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ is isomorphic to the $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[\log C]$ of logarithmic meromorphic 1-forms having poles along C. It is also known that these modules are free $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-modules of rank 2 (see [**13**]). A basis of $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ will be called a *Saito basis for C*. The rest of this section is devoted to prove the main result of this paper (Theorem 1.1 in the introduction) which shows the existence of a Saito basis for C. Let us state it.

Theorem 4.2. *Let C be a cusp and let $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ be the basis of the semimodule Λ of differential values for C. Denote by t_{s+1} and $\tilde{t}_{s+1}$ the last critical values of Λ. Then there are two 1-forms $\omega_{s+1}, \tilde{\omega}_{s+1}$ having C as an invariant curve and such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Moreover, for any pair of 1-forms as above, the set $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis for C.*

We prove Theorem 4.2 in several steps:

(a) We prove Theorem 4.2 in the case $s = 0$.

(b) We show the existence of ω_{s+1}, $\tilde{\omega}_{s+1}$ having C as an invariant curve and such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$.

(c) We show that $\mathcal{F} \cup \{\omega_{s+1}\}$ generates the $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$, for any standard system $(\mathcal{E}, \mathcal{F})$ that includes ω_{s+1} and $\tilde{\omega}_{s+1}$.

(d) We show that any pair of 1-forms ω_{s+1}, $\tilde{\omega}_{s+1}$ having C as an invariant curve and such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$ are included in at least one special standard system $(\mathcal{E}, \mathcal{F})$.

(e) We conclude as follows. We start with $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ and we consider a special standard system $(\mathcal{E}, \mathcal{F})$ containing them. By statement (c), any 1-form ω in the Saito module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ is a combination

$$\omega = h\omega_{s+1} + \sum_{\ell=-1}^{s+1} f_\ell \tilde{\omega}_\ell.$$

Since $(\mathcal{E}, \mathcal{F})$ is a special standard system, each 1-form $\tilde{\omega}_\ell$ is a combination of $\omega_{s+1}, \tilde{\omega}_{s+1}$, for any $\ell = -1, 0, 1, \ldots, s$. In this way, we find a writing $\omega = f\omega_{s+1} + g\tilde{\omega}_{s+1}$, as desired.

In the next subsections we prove Theorem 4.2 following the above steps.

4.3. The quasi-homogeneous case. The statement of Theorem 4.2 when $s = 0$ is well known; see for instance [**13**]. Let us show it, for the sake of completeness. From Zariski's introduction of the Zariski invariant, we know that the cusp C is analytically equivalent to the curve $f = 0$, where $f = y^n - x^m$. We can take

$$\omega_1 = nx\,dy - my\,dx, \quad \tilde{\omega}_1 = df = -mx^{m-1}\,dx + ny^{n-1}\,dy.$$

Let us use the Saito criterion [**13**], which states that two 1-forms ω, $\tilde{\omega}$ in $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ give a Saito basis if and only if

$$\omega \wedge \tilde{\omega} = uf\,dx \wedge dy,$$

where u is a unit. Then ω_1 and $\tilde{\omega}_1$ provide a Saito basis. Take now ω, $\tilde{\omega}$ in $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ to be such that

$$\nu_D(\omega) = t_1 = n + m, \quad \nu_D(\tilde{\omega}) = \tilde{t}_1 = nm.$$

Write

$$\omega = A\omega_1 + B\tilde{\omega}_1, \quad \tilde{\omega} = \tilde{A}\omega_1 + \tilde{B}\tilde{\omega}_1.$$

Since $n + m < nm$, we see that A is a unit. It is also obvious that $\tilde{A}$ is not a unit. If we show that $\tilde{B}$ is a unit, the determinant $A\tilde{B} - B\tilde{A}$ is a unit and hence $\{\omega, \tilde{\omega}\}$ is a Saito basis. Note that

$$\nu_D(\tilde{A}\omega_1) \neq nm.$$

Indeed, if $\nu_D(\tilde{A}\omega_1) = \nu_D(\tilde{A}) + n + m = nm$, we conclude that

$$\nu_D(\tilde{A}) = nm - n - m = c_\Gamma - 1 \in \Gamma.$$

This is a contradiction. Then, we have

$$\nu_D(\tilde{\omega}) = nm = \nu_D(\tilde{B}\tilde{\omega}_1) = \nu_D(\tilde{B}) + nm,$$

which implies that $\tilde{B}$ is a unit. This finishes the proof.

4.4. Existence of 1-forms with the last critical values. In the next proposition we show the existence of ω_{s+1}, $\tilde{\omega}_{s+1}$ with respective divisorial values t_{s+1} and $\tilde{t}_{s+1}$ and such that C is invariant for both 1-forms. The proof follows the one in [**4**, Proposition 8.3].

Proposition 4.3. *Let C be a cusp and Λ be its semimodule of differential values with basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_s)$. Assume that $s \geq 1$. There are two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ having C as an invariant curve such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$.*

Proof: Let us select a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \dots, \omega_s)$ of the cusp C. As we have already done, let us denote by $*$ a chosen element $* \in \{n, m\}$. We have to find $\omega^*_{s+1} \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ such that $\nu_D(\omega^*_{s+1}) = t^*_{s+1}$.

We provide the detailed proof for $* = n$. The case $* = m$ runs in a similar way. Then, we have to find $\omega^n_{s+1} \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ such that $\nu_D(\omega^n_{s+1}) = t^n_{s+1}$.

Let us recall that $u^n_{s+1} = \lambda_s + n\ell = \lambda_k + mb$, where we denote $\ell = \ell^n_{s+1}$, $k = k^n_s$, and $b = b_{s+1}$. Recall that $k < s$. Consider the 1-forms

$$\eta_0 = x^\ell \omega_s, \quad \eta_1 = y^b \omega_k.$$

Note that $\nu_D(\eta_0) = t^n_{s+1} = t_s + n\ell$ and $\nu_D(\eta_1) > t^n_{s+1}$. Indeed, we have

$$\begin{aligned} \nu_D(\eta_1) = bm + t_k > \nu_D(\eta_0) = t_s + n\ell \\ \iff t_s - t_k < bm - n\ell = (u^n_{s+1} - \lambda_k) - (u^n_{s+1} - \lambda_s) = \lambda_s - \lambda_k \end{aligned}$$

and note that $t_s - t_k < \lambda_s - \lambda_k$ by Lemma 2.7. Moreover, the differential values coincide:

$$\nu_C(\eta_0) = \nu_C(\eta_1) = u^n_{s+1}.$$

Thus, there is a constant $\mu \neq 0$ such that, if we take $\theta_1 = \eta_0 - \mu\eta_1$, we get that

$$\nu_D(\theta_1) = t^n_{s+1}, \quad \nu_C(\theta_1) > \nu_C(\eta_0) = \nu_C(\eta_1) = u^n_{s+1}.$$

We consider three cases:

(i) $\nu_C(\theta_1) = \infty$. In this case we finish taking $\omega^n_{s+1} = \theta_1$.
(ii) $\nu_C(\theta_1) \geq nm$.
(iii) $\nu_C(\theta_1) < nm$.

Assume that we are in case (ii) and let φ be a primitive parametrization of C. We have that $\varphi^*(\theta_1) = \psi(t)\,dt$, with $\mathrm{ord}_t(\psi(t)) \geq nm - 1 > c_\Gamma$. In view of the classical theory of equisingularity [**15**], there is a function $h(x, y)$ such that $\varphi^*(dh) = \psi(t)\,dt$. If we take $\omega^n_{s+1} = \theta_1 - dh$, we have that $\nu_C(\omega^n_{s+1}) = \infty$. In order to finish, we have to see that $\nu_D(dh) > t^n_{s+1}$. Since $t^n_{s+1} < \tilde{t}_1 = nm$ (see Lemma 2.29), it is enough to see that $\nu_D(dh) \geq nm$. If $\nu_D(dh) < nm$, we obtain that $\nu_C(dh) = \nu_D(dh)$, in contradiction with the fact that $\nu_C(dh) \geq nm$.

Assume now that we are in case (iii). Write $\nu_C(\theta_1) = \lambda_i + \alpha n + \beta m > u^n_{s+1}$, for a certain index $-1 \leq i \leq s$. Consider the 1-form η_2 given by

$$\eta_2 = x^\alpha y^\beta \omega_i, \quad \nu_D(\eta_2) = t_i + n\alpha + m\beta.$$

Let us see that $\nu_D(\eta_2) > t^n_{s+1} = t_s + \ell n = \nu_D(\theta_1)$. Assume first that $i = s$; we know that $u^n_{s+1} = \lambda_s + n\ell < \nu_C(\eta_2) = \lambda_s + \alpha n + \beta m$, hence $n\alpha + m\beta > \ell n$ as desired. Assume now that $i < s$. We have

$$\begin{aligned} \nu_C(\eta_2) = \lambda_i + \alpha n + \beta m > u^n_{s+1} = \lambda_s + \ell n \implies \alpha n + \beta m - \ell n > \lambda_s - \lambda_i > t_s - t_i \\ \implies t_i + n\alpha + m\beta > t_s + \ell n. \end{aligned}$$

On the other hand, we have that $\nu_C(\eta_2) = \nu_C(\theta_1)$. Hence, there is a constant $\mu \neq 0$ such that if we take $\theta_2 = \theta_1 - \mu\eta_2$, we obtain that

$$\nu_D(\theta_2) = \nu_D(\theta_1) = t^n_{s+1}, \quad \nu_C(\theta_2) > \nu_C(\theta_1).$$

We start the procedure again with θ_2; since the differential value is strictly increasing, in a finite number of steps we arrive at case (ii) or case (i) and we finish. □

4.5. Generators of the Saito module. Let us consider a standard system $(\mathcal{E}, \mathcal{F})$ of C, given by

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1}), \quad \mathcal{F} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}).$$

In Proposition 4.8 we describe a generator system of the Saito module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$.

Our arguments run by first considering the initial forms and finally by applying Artin's approximation theorem. Moreover, we work in an ordered way in terms of the divisorial values of the forms. In order to do this, we just need the concept of a partial standard system.

Consider an index $0 \leq j \leq s$. A *j-partial standard system associated to the extended standard basis $\mathcal{E}$* is a pair $(\mathcal{E}, \mathcal{F}^j)$, where $\mathcal{F}^j$ is a list

$$\mathcal{F}^j = (\tilde{\omega}_{j+1}, \tilde{\omega}_{j+2}, \ldots, \tilde{\omega}_{s+1}),$$

such that $\nu_D(\tilde{\omega}_\ell) = \tilde{t}_\ell$ and $\omega_\ell \in \Omega^1_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}[C]$, for $j+1 \leq \ell \leq s+1$.

We start with a lemma concerning the structure of critical values:

Lemma 4.4. *Let Λ be an increasing cuspidal semimodule of length $s \geq 1$. Assume that the basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ satisfies that $\lambda_{-1} = n$ and $\lambda_0 = m$. Consider the set*

$$T = \{t_{s+1}, \tilde{t}_2, \tilde{t}_3, \ldots, \tilde{t}_{s+1}\},$$

where t_j, $\tilde{t}_j$ are the critical values of Λ corresponding to the index j. Then, there are two nonnegative integer numbers $p, q \in \mathbb{Z}_{\geq}0$ such that

$$\{pn + n + m, qm + n + m\} \subset T.$$

Moreover, we have that $p < m - 2$ and $q < n - 2$.

Proof: We know that one of the following mutually exclusive properties holds:

(i) $\tilde{t}_2 = t_1 + n\ell^n_2 = n + m + n\ell^n_2$.
(ii) $\tilde{t}_2 = t_1 + m\ell^m_2 = n + m + m\ell^m_2$.

Let us show the proof in case (i); case (ii) has a similar proof. We can write $\tilde{t}_2 = n + m + pn \in T$, where $p = \ell^n_2$. Thus, it is enough to find an element of T of the form $n + m + qm$.

Assume first that $s = 1$. Then $t_{s+1} = t_2 = t_1 + m\ell^m_2 = n+m+m\ell^m_2$. Taking $q = \ell^m_2$, we have that $t_{s+1} = n + m + qm \in T$.

Assume now that $s > 1$. There are two cases:

(I) For any $2 \leq i \leq s$, we have that $t_{i+1} - t_i = m\ell^m_{i+1}$.
(II) There is an index (that we take to be the minimum one) with $2 \leq i \leq s$ such that $t_{i+1} - t_i = n\ell^n_{i+1}$.

Assume we are in case (I). Recall that $t_2 = t_1 + m\ell^m_2$, since $\tilde{t}_2 = t_1 + n\ell^n_2$. By a telescopic computation, we see that $t_{s+1} \in T$ may be written as

$$t_{s+1} = t_1 + \left(\sum_{\ell=2}^{s+1} \ell^m_\ell\right) m = n + m + qm.$$

Assume we are in case (II). For any $2 \leq j \leq i$, we have that $t_j = t_{j-1} + m\ell_j^m$. By a telescopic computation, we obtain that $t_i = t_1 + q_i m$. The element $\tilde{t}_{i+1} \in T$ is given by $\tilde{t}_{i+1} = t_i + m\ell_{i+1}^m$ and hence we have

$$\tilde{t}_{i+1} = t_1 + (q_i + \ell_{i+1}^m)m = n + m + qm,$$

as desired. This ends the proof. □

Remark 4.5. As a consequence of Lemma 4.4, we have the following property. Assume that Λ is the semimodule of differential values of a cusp C and $(\mathcal{E}, \mathcal{F})$ is a standard system, where

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1}), \quad \mathcal{F} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}).$$

Consider the set $\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}\}$. Assuming that (x, y) is a system of adapted coordinates with respect to the cusp C, there are two 1-forms $\eta_1, \eta_2 \in \mathcal{T}$ such that

$$\operatorname{In}(\eta_1) = \mu_1 x^p (my\, dx - nx\, dy), \quad \operatorname{In}(\eta_2) = \mu_2 y^q (my\, dx - nx\, dy),$$

where $\mu_1 \neq 0 \neq \mu_2$ and $p, q \in \mathbb{Z}_{\geq 0}$.

The next lemma is the key argument for finding our generator system of the Saito module. It will also be important in order to find the Saito bases we are looking for.

Lemma 4.6. *Let us consider a standard system $(\mathcal{E}, \mathcal{F})$ and a 1-form $\omega \in \Omega^1_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}[C]$. Assume that (x, y) is a system of adapted coordinates with respect to C. Then the initial form $\operatorname{In}(\omega)$ is a combination, with quasi-homogeneous coefficients, of the initial forms*

$$\operatorname{In}(\tilde{\omega}_1), \ldots, \operatorname{In}(\tilde{\omega}_{s+1}), \operatorname{In}(\omega_{s+1}).$$

Proof: The initial form $W = \operatorname{In}(\omega)$ has the invariant curve C_1 given by $y^n - \varepsilon x^m = 0$, for a certain $\varepsilon \neq 0$ (we leave this property to the reader). Let us invoke the result of Theorem 4.2 for the case of length zero established in Subsection 4.3. In this case we consider the two 1-forms

$$W_1 = nx\, dy - my\, dx, \quad \tilde{W}_1 = ny^{n-1}\, dy - \varepsilon m x^{m-1}\, dx,$$

that give a Saito basis $\{W_1, \tilde{W}_1\}$ of C_1. This gives a decomposition

$$W = HW_1 + \tilde{G}_1 \tilde{W}_1,$$

where we can take H, $\tilde{G}_1$ to be quasi-homogeneous with respect to the weights n, m. By statement (b) of Proposition 3.15 and up to multiplying $\tilde{\omega}_1$ by a constant, we have

$$\operatorname{In}(\tilde{\omega}_1) = \tilde{W}_1.$$

Now, we are going to show the existence of a decomposition

$$HW_1 = G_{s+1} W_{s+1} + \sum_{\ell=2}^{s+1} \tilde{G}_\ell \tilde{W}_\ell, \quad \text{where } \tilde{W}_\ell = \operatorname{In}(\tilde{\omega}_\ell),\ W_{s+1} = \operatorname{In}(\omega_{s+1}),$$

with all the coefficients G_{s+1} and $\tilde{G}_\ell$ being quasi-homogeneous.

Let $\delta = \nu_D(HW_1)$. Since H is a quasi-homogeneous polynomial, we can write

$$HW_1 = \sum_{\alpha n + \beta m = \delta} W_{\alpha\beta}, \quad W_{\alpha\beta} = \mu x^\alpha y^\beta \left(n\frac{dy}{y} - m\frac{dx}{x} \right), \quad \alpha, \beta \in \mathbb{Z}_{\geq 1}.$$

Now, it is enough to show that each of the 1-forms $W_{\alpha\beta}$ is reachable by one of the 1-forms in the set

$$\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_2, \ldots, \tilde{\omega}_{s+1}\}.$$

We consider two cases:

(i) There is a differential 1-form $W_{\alpha\beta} \neq 0$ such that $\alpha \geq m$ or $\beta \geq n$.

(ii) For any $W_{\alpha\beta} \neq 0$ we have that $\alpha < m$ and $\beta < n$.

Assume we are in case (i). By a straightforward verification, we see that all the 1-forms $W_{\alpha\beta} \neq 0$ satisfy the condition that either $\alpha \geq m$ or $\beta \geq n$. In view of Lemma 4.4 and Remark 4.5, we see that each $W_{\alpha\beta} \neq 0$ is reachable by an element of $\mathcal{T}$.

Assume now that we are in case (ii). Then, there is only one term $W_{\alpha\beta} \neq 0$ and hence we have

$$HW_1 = \mu x^{\alpha-1} y^{\beta-1}(my\,dx - nx\,dy), \quad 1 \leq \alpha < m,\, 1 \leq \beta < n.$$

Moreover, we have that $\tilde{G}_1\tilde{W}_1 = 0$. Indeed, we know that

$$\tilde{G}_1\tilde{W}_1 = \tilde{G}_1(ny^{n-1}\,dy - \varepsilon m x^{m-1}\,dx)$$

and, if this expression is nonzero, it contributes to terms corresponding to case (i), which is not possible. We conclude that

$$\mathrm{In}(\omega) = W = HW_1 = \mu x^{\alpha-1}y^{\beta-1}(my\,dx - nx\,dy) = \mu' x^{\alpha-1}y^{\beta-1}\omega_1.$$

Note that ω is then reachable by ω_1. Let q be the maximum index $1 \leq q \leq s+1$ such that ω is reachable by ω_q. If $q = s+1$, we obtain the result. Assume that $1 \leq q \leq s$. Write

$$\eta = \omega - \mu'' x^a y^b \omega_q, \quad \nu_D(\eta) > \nu_D(\omega).$$

We have that $\nu_C(\eta) = \nu_C(x^a y^b \omega_q)$. We can invoke property (d) in Theorem 3.9 to obtain that $\nu_C(\eta) \in \Lambda_{q-1}$, that is,

$$\lambda_q + na + mb \in \Lambda_{q-1}.$$

By Lemma 2.4, we have that either $a \geq \ell^n_{q+1}$ or $b \geq \ell^m_{q+1}$. Assume that $a \geq \ell^n_{q+1}$. If $u_{q+1} = u^n_{q+1}$, then ω is reachable by ω_{q+1}, which contradicts the maximality of q. If $u_{q+1} = u^m_{q+1}$, we obtain that ω is reachable by $\tilde{\omega}_{q+1}$ and we get the result. We can use similar arguments for the case that $b \geq \ell^m_{q+1}$. This ends the proof. □

Remark 4.7. Let $(\mathcal{E}, \mathcal{F}^j)$, with $\mathcal{F}^j = (\tilde{\omega}_{j+1}, \tilde{\omega}_{j+2}, \ldots, \tilde{\omega}_{s+1})$, be a j-partial standard system, with $j \geq 1$, and take a 1-form $\omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ such that $\nu_D(\omega) < \tilde{t}_j$. By the same arguments as in the preceding lemma, noting that $\tilde{t}_j < \tilde{t}_{j-1} < \cdots < \tilde{t}_1$, we see that there is a combination

$$\mathrm{In}(\omega) = G_{s+1}W_{s+1} + \sum_{\ell=j+1}^{s+1} \tilde{G}_\ell \tilde{W}_\ell, \quad \text{where } \tilde{W}_\ell = \mathrm{In}(\tilde{\omega}_\ell),\, W_{s+1} = \mathrm{In}(\omega_{s+1}),$$

all the coefficients being quasi-homogeneous of the corresponding degree.

Proposition 4.8. *The set $\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_{s+1}\}$ is a generator system of the Saito $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$.*

Proof: Take $\omega \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$. We know the existence of a decomposition

$$\mathrm{In}(\omega) = G_{s+1}W_{s+1} + \sum_{\ell=1}^{s+1} \tilde{G}_\ell \tilde{W}_\ell, \quad \text{where } \tilde{W}_\ell = \mathrm{In}(\tilde{\omega}_\ell),\, W_{s+1} = \mathrm{In}(\omega_{s+1}),$$

with all the coefficients G_{s+1} and $\tilde{G}_\ell$ being quasi-homogeneous. We restart the procedure of Lemma 4.6 with

$$\omega' = \omega - \left(G_{s+1}\omega_{s+1} + \sum_{\ell=1}^{s+1} \tilde{G}_\ell \tilde{\omega}_\ell \right).$$

In this way, we obtain a formal expression $\omega = \hat{g}_{s+1}\omega_{s+1} + \sum_{\ell=1}^{s+1} \widehat{\tilde{g}}_{\ell}\tilde{\omega}_{\ell}$. By a direct application of Artin's approximation theorem [**2**], we obtain the desired convergent expression

$$\omega = g_{s+1}\omega_{s+1} + \sum_{\ell=1}^{s+1} \tilde{g}_{\ell}\tilde{\omega}_{\ell}. \qquad \square$$

4.6. Existence of special standard systems. This subsection is devoted to providing a proof of the following result.

Proposition 4.9. *Assume that the length s of the semimodule Λ of differential values of the cusp C is $s \geq 1$. Take two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ in $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Then there is a special standard system $(\mathcal{E}, \mathcal{F})$ for C containing $\omega_{s+1}, \tilde{\omega}_{s+1}$ in the sense that*

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1}), \quad \mathcal{F} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}).$$

The proof of the above proposition follows directly from the next result.

Proposition 4.10. *Assume that the length s of the semimodule Λ of differential values of the cusp C is $s \geq 1$. Take two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ in $\Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. For any index $1 \leq j \leq s$ there are functions f_j, $\tilde{f}_j$ such that*

$$\nu_D(f_j\omega_{s+1} + \tilde{f}_j\tilde{\omega}_{s+1}) = \tilde{t}_j.$$

Note that the 1-forms in $\mathcal{F}$ will be the 1-forms $\tilde{\omega}_j = f_j\omega_{s+1} + \tilde{f}_j\tilde{\omega}_{s+1}$ for $1 \leq j \leq s$. Throughout the whole proof, we consider an extended standard basis

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1})$$

ending at ω_{s+1}. The proof of Proposition 4.10 is quite long. In order to make the arguments clear, we provide it in two steps:

Step 1: case $j = s$. That is, we find $\tilde{\omega}_s \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ such that $\nu_D(\tilde{\omega}_s) = \tilde{t}_s$.

Step 2: The general case.

4.6.1. Existence of special standard systems. First case. This subsection is devoted to the proof of Proposition 4.10 when $j = s$. Hence, we are going to prove that there is a combination

$$\tilde{\omega}_s = \tilde{f}_s\tilde{\omega}_{s+1} + f_s\omega_{s+1}$$

such that $\nu_D(\tilde{\omega}_s) = \tilde{t}_s$.

There are two possible cases: $t_{s+1} = t_s + n\ell^n_{s+1}$ and $t_{s+1} = t_s + m\ell^m_{s+1}$. Both cases work in a similar way. We assume from now on that $t_{s+1} = t_s + n\ell^n_{s+1}$ and hence we have $\tilde{t}_{s+1} = t_s + m\ell^m_{s+1}$. Let us write Delorme's decompositions of $\tilde{\omega}_{s+1}$ and ω_{s+1} as follows:

$$\tilde{\omega}_{s+1} = \tilde{\mu}_1 y^{\ell^m_{s+1}}\omega_s + \tilde{\mu}_2 x^{a_{s+1}}\omega_{k^m_s} + \tilde{\eta}, \quad \tilde{\eta} = \sum_{\ell=-1}^{s} \tilde{h}_\ell\omega_\ell,$$

$$\omega_{s+1} = \mu_1 x^{\ell^n_{s+1}}\omega_s + \mu_2 y^{b_{s+1}}\omega_{k^n_s} + \eta, \quad \eta = \sum_{\ell=-1}^{s} h_\ell\omega_\ell, \tag{18}$$

where we have the following properties:

- $\mathrm{In}(\omega_{s+1}) = \mu_1 \mathrm{In}(x^{\ell^n_{s+1}}\omega_s)$. Recall that $t_{s+1} = t_s + n\ell^n_{s+1}$.
- $\mathrm{In}(\tilde{\omega}_{s+1}) = \tilde{\mu}_1 \mathrm{In}(y^{\ell^m_{s+1}}\omega_s)$. Recall that $\tilde{t}_{s+1} = t_s + m\ell^m_{s+1}$.

- $\nu_C(\mu_1 x^{\ell^n_{s+1}}\omega_s+\mu_2 y^{b_{s+1}}\omega_{k^n_s}) > \nu_C(\mu_1 x^{\ell^n_{s+1}}\omega_s) = \nu_C(\mu_2 y^{b_{s+1}}\omega_{k^n_s}) = u^n_{s+1} = u_{s+1}$. Recall that $u^n_{s+1} = \lambda_s + n\ell^n_{s+1} = \lambda_{k^n_s} + mb_{s+1}$.
- $\nu_C(\tilde\mu_1 y^{\ell^m_{s+1}}\omega_s+\tilde\mu_2 x^{a_{s+1}}\omega_{k^m_s}) > \nu_C(\tilde\mu_1 y^{\ell^m_{s+1}}\omega_s) = \nu_C(\tilde\mu_2 x^{a_{s+1}}\omega_{k^m_s}) = u^m_{s+1} = \tilde u_{s+1}$. Recall that $u^m_{s+1} = \lambda_s + m\ell^m_{s+1} = \lambda_{k^m_s} + na_{s+1}$.
- For any $-1 \le \ell \le s$, we have that $\nu_C(h_\ell\omega_\ell) > u^n_{s+1}$ and $\nu_C(\tilde h_\ell\omega_\ell) > u^m_{s+1}$.

Let us consider the 1-form $\theta_{\mathbf{0}} \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C]$ defined by

$$\theta_{\mathbf{0}} = \mu_1 x^{\ell^n_{s+1}}\tilde\omega_{s+1} - \tilde\mu_1 y^{\ell^m_{s+1}}\omega_{s+1} = \xi + \zeta_{\mathbf{0}},$$

where $\xi = \tilde\mu_3 x^{\ell^n_{s+1}+a_{s+1}}\omega_{k^m_s} - \mu_3 y^{\ell^m_{s+1}+b_{s+1}}\omega_{k^n_s}$, with $\tilde\mu_3 = \mu_1\tilde\mu_2$, $\mu_3 = \tilde\mu_1\mu_2$ and such that $\zeta_{\mathbf{0}} = \sum_{\ell=-1}^{s} g^{\mathbf{0}}_\ell\omega_\ell$. In a more general way, given a pair of functions $\tilde f, f \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$, we write

$$\theta_{\tilde f,f} = \theta_{\mathbf{0}} + \tilde f\tilde\omega_{s+1} + f\omega_{s+1} = \xi + \zeta_{\tilde f,f} \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C],$$

where $\zeta_{\tilde f,f} = \zeta_{\mathbf{0}} + \tilde f\tilde\omega_{s+1} + f\omega_{s+1}$. We also write $\zeta_{\tilde f,f} = \sum_{\ell=-1}^{s} g^{\tilde f,f}_\ell\omega_\ell$. Let us note that $\theta_{\mathbf{0}} = \theta_{0,0}$, $\zeta_{\mathbf{0}} = \zeta_{0,0}$, and $g^{\mathbf{0}}_\ell = g^{0,0}_\ell$, for $-1 \le \ell \le s$.

In order to prove the desired result, we are going to show the existence of a pair $\tilde f$, f such that $\nu_D(\theta_{\tilde f,f}) = \tilde t_s$.

We have two options: $u_s = u^n_s$ and $u_s = u^m_s$. Both cases work in a similar way. We fix the case that $u_s = u^n_s$. Hence, we have $t_s = t^n_s$, $\tilde u_s = u^m_s$, and $\tilde t_s = t^m_s$. By Proposition 2.27, we know that $k^n_s = s-1$ and $k^m_s = k^m_{s-1}$.

Lemma 4.11. $\nu_D(\xi) = \tilde t_s$.

Proof: By Proposition 2.31, the colimits a_{s+1} and b_{s+1} satisfy that $b_{s+1} + \ell^m_{s+1} = \ell^m_s$ and $a_{s+1} + \ell^n_{s+1} = a_s$. Hence, we have

$$\xi = -\mu_3 y^{\ell^m_s}\omega_{k^n_s} + \tilde\mu_3 x^{a_s}\omega_{k^m_s} = -\mu_3 y^{\ell^m_s}\omega_{s-1} + \tilde\mu_3 x^{a_s}\omega_{k^m_{s-1}}.$$

Let us show that $\nu_D(\xi) = \tilde t_s$. Note that $\nu_D(y^{\ell^m_s}\omega_{s-1}) = m\ell^m_s + t_{s-1} = t^m_s = \tilde t_s$. Thus, it is enough to show that $\nu_D(x^{a_s}\omega_{k^m_{s-1}}) > \tilde t_s = t^m_s$. We have $\nu_D(x^{a_s}\omega_{k^m_{s-1}}) = na_s + t_{k^m_{s-1}}$. Since $u^m_s = m\ell^m_s + \lambda_{s-1} = na_s + \lambda_{k^m_{s-1}}$, then

$$\begin{aligned} na_s - m\ell^m_s = \lambda_{s-1} - \lambda_{k^m_{s-1}} > t_{s-1} - t_{k^m_{s-1}} \\ \Longrightarrow na_s + t_{k^m_{s-1}} > \tilde t_s = t_{s-1} + m\ell^m_s, \end{aligned}$$

by Lemma 2.7. We conclude that $\nu_D(\xi) = \tilde t_s$. □

The problem is reduced to finding $\tilde f$, f such that $\nu_D(\zeta_{f,\tilde f}) > \tilde t_s$.

We say that a pair of functions $\tilde f$, f is a *good pair* if and only if we have that $\nu_C(g^{\tilde f,f}_\ell\omega_\ell) > \tilde u_s$, for any $\ell = -1, 0, \dots, s$.

We end the proof as a direct consequence of the following lemmas:

Lemma 4.12. *The pair $\tilde f = 0$, $f = 0$ is a good pair.*

Lemma 4.13. *If $\tilde f$, f is a good pair, then $\nu_D(g^{\tilde f,f}_\ell\omega_\ell) > \tilde t_s$ for $-1 \le \ell \le s-1$ and $\nu_D(g^{\tilde f,f}_s\omega_s) \ne \tilde t_s$.*

Corollary 4.14. *Assume that $\tilde f$, f is a good pair. Then we have that either $\nu_D(\theta_{\tilde f,f}) = \tilde t_s$ or $\nu_D(\theta_{\tilde f,f}) = \nu_D(g^{\tilde f,f}_s\omega_s) < \tilde t_s$.*

Lemma 4.15. *If $\tilde f$, f is a good pair and $\nu_D(\theta_{\tilde f,f}) < \tilde t_s$, then there is another good pair $\tilde f_1$, f_1 such that $\nu_D(g^{\tilde f_1,f_1}_s\omega_s) > \nu_D(g^{\tilde f,f}_s\omega_s)$.*

Indeed, by Lemma 4.12, there is at least one good pair. By Lemmas 4.13 and 4.11 we obtain Corollary 4.14. Now, we apply Lemma 4.15 repeatedly to get that $\nu_D(g_s^{\tilde{f},f}\omega_s) \geq \tilde{t}_s$. Hence, in view of Lemmas 4.11 and 4.12, we get that $\nu_D(g_s^{\tilde{f},f}\omega_s) > \tilde{t}_s$ and $\nu_D(\theta_{\tilde{f},f}) = \tilde{t}_s$ as desired.

The rest of this subsection is devoted to proving the above three Lemmas 4.12, 4.13, and 4.15.

Proof of Lemma 4.12: We have to prove that

$$\nu_C(g_\ell^{\mathbf{0}}\omega_\ell) > \tilde{u}_s, \quad \text{for any } \ell = -1, 0, \ldots, s.$$

Note that $\zeta_{\mathbf{0}} = \mu_1 x^{\ell_{s+1}^n}\tilde{\eta} - \tilde{\mu}_1 y^{\ell_{s+1}^m}\eta$. Then, we have that $g_\ell^{\mathbf{0}} = \mu_1 x^{\ell_{s+1}^n}\tilde{h}_\ell - \tilde{\mu}_1 y^{\ell_{s+1}^m} h_\ell$, for any $\ell = -1, 0, \ldots, s$. Now, it is enough to show that

$$\nu_C(x^{\ell_{s+1}^n}\tilde{h}_\ell\omega_\ell) > \tilde{u}_s \quad \text{and} \quad \nu_C(y^{\ell_{s+1}^m}h_\ell\omega_\ell) > \tilde{u}_s.$$

We have that

$$\begin{aligned}
\nu_C(x^{\ell_{s+1}^n}\tilde{h}_\ell\omega_\ell) &= n\ell_{s+1}^n + \nu_C(\tilde{h}_\ell\omega_\ell) > n\ell_{s+1}^n + u_{s+1}^m = n\ell_{s+1}^n + \tilde{u}_{s+1}\\
&= n\ell_{s+1}^n + na_{s+1} + \lambda_{k_s^m} = n(\ell_{s+1}^n + a_{s+1}) + \lambda_{k_{s-1}^m}\\
&= na_s + \lambda_{k_{s-1}^m} = u_s^m = \tilde{u}_s,\\
\nu_C(y^{\ell_{s+1}^m}h_\ell\omega_\ell) &= m\ell_{s+1}^m + \nu_C(h_\ell\omega_\ell) > m\ell_{s+1}^m + u_{s+1}^n = m\ell_{s+1}^m + u_{s+1}\\
&= m\ell_{s+1}^m + mb_{s+1} + \lambda_{k_s^n} = m(\ell_{s+1}^m + b_{s+1}) + \lambda_{s-1}\\
&= m\ell_s^m + \lambda_{s-1} = u_s^m = \tilde{u}_s.
\end{aligned}$$

This ends the proof of Lemma 4.12. □

Proof of Lemma 4.13: Throughout the proof of this lemma, we just write $g_\ell^{\tilde{f},f} = g_\ell$, in order to simplify the notation.

Let us first show that $\nu_D(g_\ell\omega_\ell) > \tilde{t}_s$, for any $-1 \leq \ell \leq s-1$. Recall that $\nu_C(g_\ell\omega_\ell) > \tilde{u}_s$ and write

$$\nu_C(g_\ell\omega_\ell) = \nu_C(g_\ell) + \lambda_\ell > \tilde{u}_s = u_s^m = \lambda_{s-1} + m\ell_s^m.$$

Noting that $\lambda_{s-1} - \lambda_\ell \geq t_{s-1} - t_\ell$, in view of Lemma 2.7, we have

$$\nu_C(g_\ell) + \lambda_{s-1} > \lambda_{s-1} + t_{s-1} - t_\ell + m\ell_s^m$$

and thus we have $\nu_C(g_\ell) + t_\ell > t_{s-1} + m\ell_s^m = t_s^m = \tilde{t}_s$.

There are two cases: when $\nu_D(g_\ell) < nm$, then $\nu_C(g_\ell) = \nu_D(g_\ell)$. The other possibility is $\nu_D(g_\ell) \geq nm$, but we have that $\tilde{t}_s \leq nm$. In both cases, we conclude

$$\nu_D(g_\ell\omega_\ell) = \nu_D(g_\ell) + t_\ell > \tilde{t}_s,$$

as desired.

Let us show that $\nu_D(g_s\omega_s) \neq \tilde{t}_s$. Assume by contradiction that $\nu_D(g_s\omega_s) = \tilde{t}_s$. Recalling that $t_s = t_s^n$, $\tilde{t}_s = t_s^m$, $t_s^n = t_{s-1} + n\ell_s^n$, and $t_s^m = t_{s-1} + m\ell_s^m$, we have

$$\begin{aligned}
\nu_D(g_s\omega_s) = \tilde{t}_s &\Longrightarrow \nu_D(g_s) + t_s = \tilde{t}_s \Longrightarrow \nu_D(g_s) + t_s^n = t_s^m\\
&\Longrightarrow \nu_D(g_s) + t_{s-1} + n\ell_s^n = t_{s-1} + m\ell_s^m\\
&\Longrightarrow m\ell_s^m = \nu_D(g_s) + n\ell_s^n.
\end{aligned}$$

This implies that $m\ell_s^m \in \Gamma$ is written in two different ways as a combination of n, m with nonnegative integer coefficients. This is not possible, since $m\ell_s^m < nm$, in view of Remark 2.2. The proof of Lemma 4.13 is ended. □

Proof of Lemma 4.15: Assume that $\tilde f$, f is a good pair with $\nu_D(\theta_{\tilde f,f}) < \tilde t_s$. Let us find another good pair $\tilde f_1$, f_1 such that $\nu_D(g_s^{\tilde f_1,f_1}\omega_s) > \nu_D(g_s^{\tilde f,f}\omega_s)$.

Since $\nu_D(\xi) = \tilde t_s$, $\theta_{\tilde f,f} = \xi + \zeta_{\tilde f,f}$, and $\nu_D(\theta_{\tilde f,f}) < \tilde t_s$, we know that $\operatorname{In}(\theta_{\tilde f,f}) = \operatorname{In}(\zeta_{\tilde f,f})$. In particular $\nu_D(\zeta_{\tilde f,f}) = \nu_D(\theta_{\tilde f,f})$. Applying Lemma 4.13, we get that

$$\operatorname{In}(\theta_{\tilde f,f}) = \operatorname{In}(\zeta_{\tilde f,f}) = \operatorname{In}(g_s^{\tilde f,f}\omega_s) = \operatorname{In}(g_s^{\tilde f,f})\operatorname{In}(\omega_s).$$

Noting that $\nu_D(\theta_{\tilde f,f}) < \tilde t_s \le nm$ and $\nu_C(\theta_{\tilde f,f}) = \infty$, we have that $\theta_{\tilde f,f}$ is a resonant 1-form and, by the results in Subsection 3.2.2, there is a monomial $\mu x^a y^b$ such that

$$\operatorname{In}(\theta_{\tilde f,f}) = \mu x^a y^b\left(m\frac{dx}{x} - n\frac{dy}{y}\right), \quad a,b \ge 1,\ na+mb = \nu_D(\theta_{\tilde f,f}).$$

We conclude that there are $0 \le a' < a$, $0 \le b' < b$ and $\mu'\mu'' = \mu$ such that

$$\operatorname{In}(g_s^{\tilde f,f}) = \mu' x^{a'} y^{b'}, \quad \operatorname{In}(\omega_s) = \mu'' x^{a-a'} y^{b-b'}\left(m\frac{dx}{x} - n\frac{dy}{y}\right).$$

Let us consider the decomposition

$$\theta_{\tilde f,f} = \mu' x^{a'} y^{b'}\omega_s + \eta', \quad \nu_D(\eta') > \nu_D(x^{a'}y^{b'}\omega_s).$$

Noting that $\nu_C(\theta_{\tilde f,f}) = \infty$, we have that $\nu_C(\eta') = \nu_C(\mu' x^{a'} y^{b'}\omega_s) = na' + mb' + \lambda_s$. Let us apply Theorem 3.9, statement (d), to the integer number $k = \lambda_s + na' + mb'$. Since there is η' such that $\nu_C(\eta') = k$ and $\nu_D(\eta') > \nu_D(x^{a'}y^{b'}\omega_s)$, we conclude that $k \in \Lambda_{s-1}$. By Lemma 2.4, we know that one of the following properties holds:

$$a' \ge \ell^n_{s+1} \quad \text{or} \quad b' \ge \ell^m_{s+1}.$$

Let us show that $\theta_{\tilde f,f}$ is reachable from ω_{s+1} or from $\tilde\omega_{s+1}$. Assume that $a' \ge \ell^n_{s+1}$, then we have

$$\begin{aligned}\nu_D(\theta_{\tilde f,f}) &= an + bm = (a-a')n + (b-b')m + a'n + b'm\\ &= t_s + n\ell^n_{s+1} + (a' - \ell^n_{s+1})n + b'm = t^n_{s+1} + (a' - \ell^n_{s+1})n + b'm.\end{aligned}$$

Noting that $t_{s+1} = t^n_{s+1}$, we have that $\theta_{\tilde f,f}$ and $x^{a'-\ell^n_{s+1}}y^{b'}\omega_{s+1}$ have the same initial parts (up to a constant) and thus $\theta_{\tilde f,f}$ is reachable from ω_{s+1}. In the same way, if we assume that $b' \ge \ell^m_{s+1}$, we have

$$\begin{aligned}\nu_D(\theta_{\tilde f,f}) &= an + bm = (a-a')n + (b-b')m + a'n + b'm\\ &= t_s + m\ell^m_{s+1} + a'n + (b' - \ell^m_{s+1})m = t^m_{s+1} + a'n + (b' - \ell^m_{s+1})m\\ &= \tilde t_{s+1} + a'n + (b' - \ell^m_{s+1})m.\end{aligned}$$

We conclude as above that $\theta_{\tilde f,f}$ is reachable from $\tilde\omega_{s+1}$.

Assume now that $a' \ge \ell^n_{s+1}$ and hence $\theta_{\tilde f,f}$ is reachable from ω_{s+1}. Thus, there is a constant $\mu_3 \ne 0$ such that

$$\nu_D(\theta_{\tilde f,f} - \mu_3 x^{a'-\ell^n_{s+1}}y^{b'}\omega_{s+1}) > \nu_D(\theta_{\tilde f,f}).$$

Let us put $\tilde f_1 = \tilde f$ and $f_1 = f - \mu_3 x^{a'-\ell^n_{s+1}}y^{b'}$. Note that

$$\theta_{\tilde f_1,f_1} = \theta_{\tilde f,f} - \mu_3 x^{a'-\ell^n_{s+1}}y^{b'}\omega_{s+1}$$

and hence $\nu_D(\theta_{\tilde f_1,f_1}) > \nu_D(\theta_{\tilde f,f})$.

Let us verify that $\tilde{f}_1$, f_1 is a good pair. Let us write

$$x^{a'-\ell^n_{s+1}}y^{b'}\omega_{s+1}=\sum_{\ell=-1}^{s}g'_\ell\omega_\ell,$$

coming from the decomposition of ω_{s+1} in equation (18). Noting that

$$\zeta_{\tilde{f}_1,f}=\zeta_{\tilde{f},f}-\mu_3x^{a'-\ell^n_{s+1}}y^{b'}\omega_{s+1},$$

we see that $\tilde{f}_1$, f_1 is a good pair if $\nu_C(g'_\ell\omega_\ell)>\tilde{u}_s$, for $\ell=-1,0,\dots,s$. Let us show that this is true. Since the terms $g'_\ell\omega_\ell$, for $-1\leq\ell\leq s$, come from the decomposition of ω_{s+1} times a monomial, we can apply Remark 3.12 to see that

$$\nu_C(g'_s\omega_s)\leq\nu_C(g'_\ell\omega_\ell),\quad\text{for }-1\leq\ell\leq s.$$

Hence, it is enough to show that $\nu_C(g'_s\omega_s)>\tilde{u}_s$. Notice that

$$\operatorname{In}(\zeta_{\tilde{f},f})=\operatorname{In}(g_s^{\tilde{f},f}\omega_s)=\mu_3\operatorname{In}(x^{a'-\ell^n_{s+1}}y^{b'}\omega_{s+1})=\mu_3\operatorname{In}(g'_s\omega_s),$$

where the last equality comes from Corollary 3.14. Thus, we have

$$\nu_D(g_s^{\tilde{f},f}\omega_s)=\nu_D(g'_s\omega_s)<\tilde{t}_s\leq nm.$$

Therefore, $\nu_D(g_s^{\tilde{f},f})=\nu_D(g'_s)<nm$. This implies that

$$\nu_D(g_s^{\tilde{f},f})=\nu_C(g_s^{\tilde{f},f})=\nu_C(g'_s)=\nu_D(g'_s).$$

Since $\tilde{f}$, f is a good pair, we conclude that $\nu_C(g'_s\omega_s)=\nu_C(g_s^{\tilde{f},f}\omega_s)>\tilde{u}_s$. If $b'\geq\ell^m_{s+1}$, then $\theta_{\tilde{f},f}$ is reachable by $\tilde{\omega}_{s+1}$ and we proceed in a similar way. This ends the proof of Lemma 4.15. □

4.6.2. Existence of special standard systems. Induction step. This subsection is devoted to the proof of Proposition 4.10 when $1\leq j<s$, assuming that the result is true for $j+1,j+2,\dots,s$. Consequently, we are going to prove that there is a combination

$$\tilde{\omega}_j=\tilde{f}_j\tilde{\omega}_{s+1}+f_j\omega_{s+1}$$

such that $\nu_D(\tilde{\omega}_j)=\tilde{t}_j$, under the assumption that for any $j+1\leq\ell\leq s$ there is a combination $\tilde{\omega}_\ell=\tilde{f}_\ell\tilde{\omega}_{s+1}+f_\ell\omega_{s+1}$ such that $\nu_D(\tilde{\omega}_\ell)=\tilde{t}_\ell$.

The proof is very similar to the case $j=s$. Recall that $\nu_D(\tilde{\omega}_{j+1})=\tilde{t}_{j+1}$. There are two options: either $\tilde{t}_{j+1}=t^n_{j+1}$ or $\tilde{t}_{j+1}=t^m_{j+1}$. In both cases, the proof works in a similar way. We fix from now on the option $\tilde{t}_{j+1}=t^m_{j+1}$.

Let us define the number $q\in\{j+2,\dots,s+1\}$ as follows:

$$q=\begin{cases}s+1, & \text{if } \tilde{t}_\ell=t^n_\ell,\ \text{for } \ell=j+2,j+3,\dots,s+1,\\ \min\{\ell:\tilde{t}_\ell=t^m_\ell,\ j+2\leq\ell\leq s+1\}, & \text{otherwise,}\end{cases}$$

and define the 1-form $\widehat{\omega}_q$ as follows:

$$\widehat{\omega}_q=\begin{cases}\omega_{s+1}, & \text{if } \tilde{t}_\ell=t^n_\ell,\ \text{for } \ell=j+2,j+3,\dots,s+1,\\ \tilde{\omega}_q, & \text{otherwise.}\end{cases}$$

Let us note that $\nu_D(\widehat{\omega}_q)=t^m_q$ in both cases.

Now, we proceed as follows:

- First, we find a combination $\theta_{\mathbf{0}}$ of $\tilde{\omega}_{j+1}$ and $\widehat{\omega}_q$ such that $\nu_D(\theta_{\mathbf{0}})\leq\tilde{t}_j$. Note that $\theta_{\mathbf{0}}$ should be a combination of $\tilde{\omega}_{s+1}$ and ω_{s+1}, in view of the induction hypothesis.

- Next, we find a 1-form $\tilde{\omega}_j - \boldsymbol{\theta_0}$ which is a combination of
$$\tilde{\omega}_{j+1}, \tilde{\omega}_{j+2}, \dots, \tilde{\omega}_{s+1}, \omega_{s+1},$$
in such a way that $\nu_D(\tilde{\omega}_j) = \tilde{t}_j$.

Consider Delorme's decompositions of $\tilde{\omega}_{j+1}$ and $\widehat{\omega}_q$ as introduced in Theorem 3.11, which we write as follows:
$$\tilde{\omega}_{j+1} = \tilde{\mu}_1 y^{\ell^m_{j+1}} \omega_j + \tilde{\mu}_2 x^{a_{j+1}} \omega_{k^m_j} + \tilde{\eta}, \quad \tilde{\eta} = \sum_{\ell=-1}^{j} \tilde{h}_\ell \omega_\ell,$$
$$\widehat{\omega}_q = M\omega_j + N\omega_{k^n_j} + \eta, \qquad \eta = \sum_{\ell=-1}^{j} h_\ell \omega_\ell,$$
where M, N are monomials in such a way that we have the following properties:

- $\mathrm{In}(\tilde{\omega}_{j+1}) = \tilde{\mu}_1 \mathrm{In}(y^{\ell^m_{j+1}} \omega_j) = \tilde{\mu}_1 y^{\ell^m_{j+1}} \mathrm{In}(\omega_j)$. Recall that $\tilde{t}_{j+1} = t_j + m\ell^m_{j+1}$.
- $\nu_C(\tilde{\mu}_1 y^{\ell^m_{j+1}} \omega_j + \tilde{\mu}_2 x^{a_{j+1}} \omega_{k^m_j}) > \nu_C(y^{\ell^m_{j+1}} \omega_j) = \nu_C(x^{a_{j+1}} \omega_{k^m_j}) = u^m_{j+1} = \tilde{u}_{j+1}$. Recall that $u^m_{j+1} = \lambda_j + m\ell^m_{j+1} = \lambda_{k^m_j} + na_{j+1}$.
- $\nu_C(\tilde{h}_\ell \omega_\ell) > \tilde{u}_{j+1} = u^m_{j+1}$, for $\ell = -1, 0, 1, \dots, j$.
- $\mathrm{In}(\widehat{\omega}_q) = \mathrm{In}(M\omega_j) = M \,\mathrm{In}(\omega_j)$.
- $\nu_C(M\omega_j + N\omega_{k^n_j}) > \nu_C(M\omega_j) = \nu_C(N\omega_{k^n_j}) = \lambda_j + t^m_q - t_j = v^m_{q-1,j}$.
- $\nu_C(h_\ell \omega_\ell) > \lambda_j + t^m_q - t_j = v^m_{q-1,j}$, for $\ell = -1, 0, 1, \dots, j$.

Let us compute the monomials M and N. We have
$$t^m_q = \nu_D(\widehat{\omega}_q) = \nu_D(M) + \nu_D(\omega_j) \Longrightarrow \nu_D(M) = t^m_q - t_j.$$
By a telescopic argument, we obtain
$$\begin{aligned}
t^m_q - t_j &= t^m_q - t_{j+1} + (t_{j+1} - t_j)\\
&= t^m_q - t_{j+1} + n\ell^n_{j+1}\\
&= t^m_q - t_{j+2} + (t_{j+2} - t_{j+1}) + n\ell^n_{j+1}\\
&= t^m_q - t_{j+2} + m\ell^m_{j+2} + n\ell^n_{j+1}\\
&= t^m_q - t_{j+3} + (t_{j+3} - t_{j+2}) + m\ell^m_{j+2} + n\ell^n_{j+1}\\
&= t^m_q - t_{j+3} + m(\ell^m_{j+3} + \ell^m_{j+2}) + n\ell^n_{j+1}\\
&\dots\\
&= t^m_q - t_{q-1} + m(\ell^m_{q-1} + \dots + \ell^m_{j+3} + \ell^m_{j+2}) + n\ell^n_{j+1}\\
&= m(\ell^m_q + \ell^m_{q-1} + \dots + \ell^m_{j+3} + \ell^m_{j+2}) + n\ell^n_{j+1}.
\end{aligned}$$
This implies that $M = \mu_1 x^a y^b$, where
$$a = \ell^n_{j+1}, \quad b = \ell^m_q + \ell^m_{q-1} + \dots + \ell^m_{j+3} + \ell^m_{j+2}.$$
Let us now compute the monomial N. We know that
$$\nu_C(N\omega_{k^n_j}) = \nu_D(N) + \lambda_{k^n_j} = \nu_C(M\omega_j) = \lambda_j + na + mb.$$
Then, we get
$$\nu_D(N) = \lambda_j - \lambda_{k^n_j} + na + mb.$$
Recalling that $u^n_{j+1} = \lambda_j + n\ell^n_{j+1} = \lambda_{k^n_j} + mb_{j+1}$, we obtain
$$\begin{aligned}
\nu_D(N) &= \lambda_j - \lambda_{k^n_j} + na + mb\\
&= mb_{j+1} - n\ell^n_{j+1} + na + mb = m(b_{j+1} + b).
\end{aligned}$$
This implies that $N = \mu_2 y^{b_{j+1}+b}$.

Let us note that $b < \ell^m_{j+1}$, in view of Corollary 2.32. More precisely, we have that $\ell^m_{j+1} - b = b_q$. Now, we consider the 1-form $\theta_{\mathbf{0}}$ given by

$$\theta_{\mathbf{0}} = \mu_1 x^a \tilde{\omega}_{j+1} - \tilde{\mu}_1 y^{b_q} \widehat{\omega}_q = \mu_1 x^{\ell^n_{j+1}} \tilde{\omega}_{j+1} - \tilde{\mu}_1 y^{b_q} \widehat{\omega}_q.$$

We write $\theta_{\mathbf{0}} = \xi + \zeta_{\mathbf{0}}$, where

$$\begin{aligned}\xi &= \mu_1 \tilde{\mu}_2 x^{a+a_{j+1}} \omega_{k^m_j} - \tilde{\mu}_1 \mu_2 y^{b_q+b+b_{j+1}} \omega_{k^n_j} \\ &= \mu_1 \tilde{\mu}_2 x^{\ell^n_{j+1}+a_{j+1}} \omega_{k^m_j} - \tilde{\mu}_1 \mu_2 y^{\ell^m_{j+1}+b_{j+1}} \omega_{k^n_j}\end{aligned}$$

and $\zeta_{\mathbf{0}} = \sum_{\ell=-1}^{j} g^{\mathbf{0}}_\ell \omega_\ell = \sum_{\ell=-1}^{j} (\mu_1 x^{\ell^n_{j+1}} \tilde{h}_\ell - \tilde{\mu}_1 y^{b_q} h_\ell)\omega_\ell$.

In a more general way, given a list of functions $\tilde{\mathbf{f}}$, f in $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$, where

$$\tilde{\mathbf{f}} = (\tilde{f}_{j+1}, \tilde{f}_{j+2}, \ldots, \tilde{f}_{s+1}),$$

we write

$$\theta_{\tilde{\mathbf{f}},f} = \theta_{\mathbf{0}} + \sum_{\ell=j+1}^{s+1} \tilde{f}_\ell \tilde{\omega}_\ell + f\omega_{s+1} = \xi + \zeta_{\tilde{\mathbf{f}},f} \in \Omega^1_{\mathbb{C}^2,\mathbf{0}}[C],$$

where $\zeta_{\tilde{\mathbf{f}},f} = \zeta_{\mathbf{0}} + \sum_{\ell=j+1}^{s+1} \tilde{f}_\ell \tilde{\omega}_\ell + f\omega_{s+1}$. We also write $\zeta_{\tilde{\mathbf{f}},f} = \sum_{\ell=-1}^{s} g^{\tilde{\mathbf{f}},f}_\ell \omega_\ell$. Let us note that $\theta_{\mathbf{0}} = \theta_{\mathbf{0},0}$, $\zeta_{\mathbf{0}} = \zeta_{\mathbf{0},0}$, and $g^{\mathbf{0}}_\ell = g^{\mathbf{0},0}_\ell$, for $-1 \le \ell \le s$.

In order to prove the desired result, we are going to show the existence of a list $\tilde{\mathbf{f}}$, f such that $\nu_D(\theta_{\tilde{\mathbf{f}},f}) = \tilde{t}_j$.

We have two options: $u_j = u^n_j$ and $u_j = u^m_j$. Both cases work in a similar way. We consider the case that $u_j = u^n_j$. Hence, we have $t_j = t^n_j$, $\tilde{u}_j = u^m_j$, and $\tilde{t}_j = t^m_j$. By Proposition 2.27, we know that $k^n_j = j-1$ and $k^m_j = k^m_{j-1}$. We will show that

Lemma 4.16. $\nu_D(\xi) = \tilde{t}_j$.

Then, the problem is reduced to finding a list $(\tilde{\mathbf{f}}, f)$ such that $\nu_D(\zeta_{\tilde{\mathbf{f}},f}) > \tilde{t}_j$.

We say that a list of functions $(\tilde{\mathbf{f}}, f)$ is a *good list* if and only if we have that $\nu_C(g^{\tilde{\mathbf{f}},f}_\ell \omega_\ell) > \tilde{u}_j$, for any $\ell = -1, 0, \ldots, j$.

We end the proof as a direct consequence of the following lemmas:

Lemma 4.17. *The list* $(\tilde{\mathbf{f}}, f) = (\mathbf{0}, 0)$ *is a good list.*

Lemma 4.18. *If* $(\tilde{\mathbf{f}}, f)$ *is a good list, then* $\nu_D(g^{\tilde{\mathbf{f}},f}_\ell \omega_\ell) > \tilde{t}_j$ *for* $-1 \le \ell \le j-1$ *and* $\nu_D(g^{\tilde{\mathbf{f}},f}_j \omega_j) \ne \tilde{t}_j$.

Corollary 4.19. *Assume that* $(\tilde{\mathbf{f}}, f)$ *is a good list. Then we have that either* $\nu_D(\theta_{\tilde{\mathbf{f}},f}) = \tilde{t}_j$ *or* $\nu_D(\theta_{\tilde{\mathbf{f}},f}) = \nu_D(g^{\tilde{\mathbf{f}},f}_j \omega_j) < \tilde{t}_j$.

Lemma 4.20. *If* $(\tilde{\mathbf{f}}, f)$ *is a good list and* $\nu_D(\theta_{\tilde{\mathbf{f}},f}) < \tilde{t}_j$*, then there is another good list* $(\tilde{\mathbf{f}}^1, f^1)$ *such that* $\nu_D(g^{\tilde{\mathbf{f}}^1,f^1}_j \omega_j) > \nu_D(g^{\tilde{\mathbf{f}},f}_j \omega_j)$.

Indeed, by Lemma 4.17, there is at least one good list. By Lemma 4.18 and Lemma 4.16 we obtain Corollary 4.19. Now, we apply Lemma 4.20 repeatedly to get that $\nu_D(g^{\tilde{\mathbf{f}},f}_j \omega_j) \ge \tilde{t}_j$. Hence, in view of Lemmas 4.16 and 4.17, we get that $\nu_D(g^{\tilde{\mathbf{f}},f}_j \omega_j) > \tilde{t}_j$ and $\nu_D(\theta_{\tilde{\mathbf{f}},f}) = \tilde{t}_j$ as desired.

The rest of this subsection is devoted to proving the above four Lemmas 4.16, 4.17, 4.18, and 4.20.

Proof of Lemma 4.16: By Proposition 2.31, the colimits a_{j+1} and b_{j+1} satisfy that $b_{j+1}+\ell^m_{j+1}=\ell^m_j$ and $a_{j+1}+\ell^n_{j+1}=a_j$. Then, we have

$$\begin{aligned}\xi &= \mu_1\tilde\mu_2 x^{\ell^n_{j+1}+a_{j+1}}\omega_{k^m_j}-\tilde\mu_1\mu_2 y^{\ell^m_{j+1}+b_{j+1}}\omega_{k^n_j}\\ &= \mu_1\tilde\mu_2 x^{a_j}\omega_{k^m_{j-1}}-\tilde\mu_1\mu_2 y^{\ell^m_j}\omega_{j-1}.\end{aligned}$$

Note that $\nu_D(y^{\ell^m_j}\omega_{j-1})=m\ell^m_j+t_{j-1}=t^m_j=\tilde t_j$. Thus, it is enough to show that $\nu_D(x^{a_j}\omega_{k^m_{j-1}})>\tilde t_j=t^m_j$. We have $\nu_D(x^{a_j}\omega_{k^m_{j-1}})=na_j+t_{k^m_{j-1}}$. Since $u^m_j=m\ell^m_j+\lambda_{j-1}=na_j+\lambda_{k^m_{j-1}}$, in view of Lemma 2.7, we have

$$na_j-m\ell^m_j=\lambda_{j-1}-\lambda_{k^m_{j-1}}>t_{j-1}-t_{k^m_{j-1}},$$

which implies that

$$\nu_D(x^{a_j}\omega_{k^m_{j-1}})=na_j+t_{k^m_{j-1}}>\tilde t_j=t_{j-1}+m\ell^m_j.$$

We conclude that $\nu_D(\xi)=\tilde t_j$. □

Proof of Lemma 4.17: We have to prove that

$$\nu_C(g^{\mathbf{0}}_\ell\omega_\ell)>\tilde u_j,\quad \text{for any } \ell=-1,0,\dots,j.$$

Note that $g^{\mathbf{0}}_\ell\omega_\ell=(\mu_1x^{\ell^n_{j+1}}\tilde h_\ell-\tilde\mu_1y^{b_q}h_\ell)\omega_\ell$. Then, it is enough to show that

$$\nu_C(x^{\ell^n_{j+1}}\tilde h_\ell\omega_\ell)>\tilde u_j\quad\text{and}\quad \nu_C(y^{b_q}h_\ell\omega_\ell)>\tilde u_j.$$

We have

$$\begin{aligned}\nu_C(x^{\ell^n_{j+1}}\tilde h_\ell\omega_\ell)&=n\ell^n_{j+1}+\nu_C(\tilde h_\ell\omega_\ell)>n\ell^n_{j+1}+u^m_{j+1}=n\ell^n_{j+1}+\tilde u_{j+1}\\ &=n\ell^n_{j+1}+na_{j+1}+\lambda_{k^m_j}=n(\ell^n_{j+1}+a_{j+1})+\lambda_{k^m_{j-1}}\\ &=na_j+\lambda_{k^m_{j-1}}=u^m_j=\tilde u_j.\end{aligned}$$

Let us now consider $\nu_C(y^{b_q}h_\ell\omega_\ell)$. We have

$$\nu_C(y^{b_q}h_\ell\omega_\ell)>mb_q+\lambda_j+t^m_q-t_j.$$

Let us show that $mb_q+\lambda_j+t^m_q-t_j=\tilde u_j$. Recall that $\tilde u_j=u^m_j=\lambda_{j-1}+m\ell^m_j$. Thus, we have to prove that

$$mb_q+\lambda_j+t^m_q-t_j-\lambda_{j-1}-m\ell^m_j=0.$$

Note that $k^n_j=j-1$ and then $\lambda_j-\lambda_{j-1}=-n\ell^n_{j+1}+mb_{j+1}$. Then we have to verify that

$$mb_q-n\ell^n_{j+1}+mb_{j+1}+t^m_q-t_j-m\ell^m_j=0.$$

Recalling that $t^m_q-t_j=na+mb=n\ell^n_{j+1}+mb$ and that $b_q=\ell^m_{j+1}-b$, we have to verify that

$$m(\ell^m_{j+1}-b)-n\ell^n_{j+1}+n\ell^n_{j+1}+mb+mb_{j+1}-m\ell^m_j=0.$$

We have to see that $b_{j+1}+\ell^m_{j+1}=\ell^m_j$, and this follows from Proposition 2.31. □

Proof of Lemma 4.18: Throughout the proof of this lemma, we just write $g^{\tilde{\mathbf{f}},f}_\ell=g_\ell$, in order to simplify the notation.

Let us first show that $\nu_D(g_\ell\omega_\ell)>\tilde t_j$, for any $-1\le\ell\le j-1$. Recall that $\nu_C(g_\ell\omega_\ell)>\tilde u_j$ and write

$$\nu_C(g_\ell\omega_\ell)=\nu_C(g_\ell)+\lambda_\ell>\tilde u_j=u^m_j=\lambda_{j-1}+m\ell^m_j.$$

Noting that $\lambda_{j-1}-\lambda_\ell\ge t_{j-1}-t_\ell$, in view of Lemma 2.7, we have

$$\nu_C(g_\ell)+\lambda_{j-1}>\lambda_{j-1}+t_{j-1}-t_\ell+m\ell^m_j$$

and thus we have $\nu_C(g_\ell)+t_\ell>t_{j-1}+m\ell^m_j=t^m_j=\tilde t_j$.

Recall that $\nu_C(g_\ell) = \nu_D(g_\ell)$ when $\nu_D(g_\ell) < nm$. Noting that $\tilde{t}_j \leq nm$, we conclude that

$$\nu_D(g_\ell \omega_\ell) = \nu_D(g_\ell) + t_\ell > \tilde{t}_j,$$

as desired.

Let us show that $\nu_D(g_j\omega_j) \neq \tilde{t}_j$. Assume by contradiction that $\nu_D(g_j\omega_j) = \tilde{t}_j$. Recalling that $t_j = t_j^n$, $\tilde{t}_j = t_j^m$, $t_j^n = t_{j-1} + n\ell_j^n$, and $t_j^m = t_{j-1} + m\ell_j^m$, we have

$$\begin{aligned}\nu_D(g_j\omega_j) = \tilde{t}_j &\Longrightarrow \nu_D(g_j) + t_j = \tilde{t}_j \Longrightarrow \nu_D(g_j) + t_j^n = t_j^m \\ &\Longrightarrow \nu_D(g_j) + t_{j-1} + n\ell_j^n = t_{j-1} + m\ell_j^m \\ &\Longrightarrow m\ell_j^m = \nu_D(g_j) + n\ell_j^n.\end{aligned}$$

This implies that $m\ell_j^m \in \Gamma$ is written in two different ways as a combination of n, m with nonnegative integer coefficients. This is not possible, since $m\ell_j^m < nm$, in view of Remark 2.2. □

Proof of Lemma 4.20: Assume that $\tilde{\mathbf{f}}$, f is a good list with $\nu_D(\theta_{\tilde{\mathbf{f}},f}) < \tilde{t}_j$. Let us find another good list $\tilde{\mathbf{f}^1}$, f^1 such that

$$\nu_D(g_j^{\tilde{\mathbf{f}}^1,f^1}\omega_j) > \nu_D(g_j^{\tilde{\mathbf{f}},f}\omega_j).$$

Let us note that $\nu_D(\theta_{\tilde{\mathbf{f}},f}) = \nu_D(g_j^{\tilde{\mathbf{f}},f}\omega_j) < \tilde{t}_j$ and, more precisely, we have

$$W = \mathrm{In}(g_j^{\tilde{\mathbf{f}},f}\omega_j) = \mathrm{In}(\theta_{\tilde{\mathbf{f}},f}).$$

In view of Remark 4.7, there is a decomposition

$$W = G_{s+1}W_{s+1} + \sum_{\ell=j+1}^{s+1} \tilde{G}_\ell \tilde{W}_\ell,$$

where the coefficients are quasi-homogeneous. Moreover, all the 1-forms W, W_{s+1}, $\tilde{W}_\ell$, for $j+1 \leq \ell \leq s+1$ are resonant with divisorial value $< nm$. We conclude that all those 1-forms are given by the product of a monomial and the 1-form

$$m\frac{dx}{x} - n\frac{dy}{y}.$$

Up to multiplying some of the terms by an appropriate scalar number, we can assume without loss of generality that all the coefficients $G_{s+1}, \tilde{G}_{j+1}, \tilde{G}_{j+2}, \ldots, \tilde{G}_{s+1}$ are zero except exactly one of them. Hence, we have

$$W = G_{s+1}W_{s+1} \text{ or there is } \ell_0 \text{ such that } W = \tilde{G}_{\ell_0}\tilde{W}_{\ell_0}.$$

Let us write $S = W_{s+1}$ in the first case and $S = \tilde{W}_{\ell_0}$ in the second one. Then we have that $W = GS$, where $G = G_{s+1}$ in the first case and $G = \tilde{G}_{\ell_0}$ in the second one.

Now we define the list $(\tilde{\mathbf{f}}^1, f^1)$ by

$$(\tilde{f}^1_{j+1}, \tilde{f}^1_{j+1}, \ldots, \tilde{f}^1_{s+1}, f^1) = (\tilde{\mathbf{f}}, f) - (\tilde{G}_{j+1}, \tilde{G}_{j+2}, \ldots, \tilde{G}_{s+1}, G_{s+1}).$$

It is a straightforward verification that $\nu_D(g_j^{\tilde{\mathbf{f}}^1,f^1}\omega_j) > \nu_D(g_j^{\tilde{\mathbf{f}},f}\omega_j)$.

We just have to verify that $(\tilde{\mathbf{f}}^1, f^1)$ is a good list. We do so in the case that $S = W_{s+1}$; the other cases work in a similar way. Note that

$$\zeta_{\tilde{\mathbf{f}}^1,f^1} = \zeta_{\tilde{\mathbf{f}},f} - G_{s+1}\omega_{s+1} = \sum_{\ell=-1}^{j} g_\ell^{\tilde{\mathbf{f}}^1,f^1}\omega_\ell.$$

Let us consider a Delorme's decomposition of ω_{s+1} given by $\omega_{s+1} = \sum_{\ell=-1}^{j} c_\ell \omega_\ell$, where we know that

- $\mathrm{In}(\omega_{s+1}) = \mathrm{In}(c_j\omega_j)$.
- $\nu_C(c_j\omega_j) \le \nu_C(c_\ell\omega_\ell)$, for $\ell = -1, 0, 1, \ldots, j$.

Note that $g_\ell^{\tilde{\mathbf{f}}^1, f^1} = g_\ell^{\tilde{\mathbf{f}}, f} - G_{s+1}c_\ell$, for $\ell = -1, 0, 1, \ldots, j$. Then, in order to show that we have a good list, it is enough to show that $\nu_C(G_{s+1}c_j\omega_j) > \tilde{u}_j$.

We know that $\nu_D(g_j^{\tilde{\mathbf{f}}, f}\omega_j) = \nu_D(G_{s+1}c_j\omega_j) < nm$, since they share the initial part. Noting that the divisorial values are under nm, we have

$$\nu_D(g_j^{\tilde{\mathbf{f}}, f}) = \nu_C(g_j^{\tilde{\mathbf{f}}, f}), \quad \nu_D(G_{s+1}c_j) = \nu_C(G_{s+1}c_j).$$

We conclude that $\nu_C(G_{s+1}c_j\omega_j) = \nu_C(g_j^{\tilde{\mathbf{f}}, f}\omega_j) > \tilde{u}_j$, as desired. □

5. New discrete analytic invariants

Let $\pi\colon M \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of a cusp with Puiseux pair (n, m). We know that π is the composition

$$\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N$$

of blow-ups $\pi_j\colon M_j \to M_{j-1}$ centered at points $P_{j-1} \in M_{j-1}$, for $j = 1, 2, \ldots, N$, where $P_0 = 0 \in \mathbb{C}^2$. Hence $M_0 = (\mathbb{C}^2, \mathbf{0})$ and $M_N = M$. We also know that each infinitely near point P_j belongs to the divisor

$$D_j = \pi_j^{-1}(P_{j-1}), \quad j = 1, 2, \ldots, N-1.$$

We also put $D = D_N$ the last divisor of π. Let us denote by $\mathcal{C}_\pi$ the set of all cusps C such that π is the minimal reduction of singularities of C.

Remark 5.1. For any Saito basis ω, ω' of a cusp $C \in \mathcal{C}_\pi$, we have

$$\nu_{D_j}(\omega) + \nu_{D_j}(\omega') \le \nu_{D_j}(xyf), \quad j = 1, 2, \ldots, N,$$

where $f = 0$ is a reduced equation of the cusp C. Indeed, since ω, ω' is a Saito basis, we have

$$\omega \wedge \omega' = uf\, dx \wedge dy = uxyf\left(\frac{dx}{x} \wedge \frac{dy}{y}\right),$$

where u is a unit. The property follows from the fact that

$$\nu_{D_j}(\omega) + \nu_{D_j}(\omega') \le \nu_{D_j}(\omega \wedge \omega').$$

Given a divisor D_j, for $j = 1, 2, \ldots, N$, and a cusp $C \in \mathcal{C}_\pi$, we define *the pair* $(\mathfrak{s}_{D_j}(C), \tilde{\mathfrak{s}}_{D_j}(C))$ *of Saito multiplicities at* D_j by

$$\mathfrak{s}_{D_j}(C) = \min\{\nu_{D_j}(\omega);\ \omega \text{ belongs to a Saito basis of } C\},$$
$$\tilde{\mathfrak{s}}_{D_j}(C) = \max\{\nu_{D_j}(\omega);\ \omega \text{ belongs to a Saito basis of } C\}.$$

Note that $\mathfrak{s}_{D_j}(C)$ is equal to the minimal divisorial order of the elements of any Saito basis, whereas $\tilde{\mathfrak{s}}_{D_j}(C)$ does not follow directly from a given Saito basis.

The pair of Saito multiplicities is an analytic invariant of the cusp C. In [**6**], the author introduces an invariant directly related to the first pair $(\mathfrak{s}_{D_1}(C), \tilde{\mathfrak{s}}_{D_1}(C))$.

A natural question is to know if the pairs of Saito multiplicities may be deduced from the knowledge of the semimodule of differential values. The answer is positive for the last pair $(\mathfrak{s}_D(C), \tilde{\mathfrak{s}}_D(C))$. On the other hand, we present here an example of two cusps in $\mathcal{C}_\pi$ having the same semimodule of differential values such that the first pairs of Saito multiplicities do not coincide.

Theorem 5.2. *Take $C \in \mathcal{C}_\pi$, then $(\mathfrak{s}_D(C), \tilde{\mathfrak{s}}_D(C)) = (t_{s+1}, \tilde{t}_{s+1})$, where t_{s+1} and $\tilde{t}_{s+1}$ are the last critical values of the semimodule of differential values of C.*

Proof: We know that ω_{s+1} and $\tilde{\omega}_{s+1}$ are a Saito basis of C and

$$\nu_D(\omega_{s+1}) = t_{s+1} < \tilde{t}_{s+1} = \nu_D(\tilde{\omega}_{s+1}).$$

This proves that $\mathfrak{s}_D(C) = t_{s+1}$ and $\tilde{t}_{s+1} \leq \tilde{\mathfrak{s}}_D(C)$. Now, let ω, ω' be another Saito basis, with $\nu_D(\omega) = t_{s+1}$ and $\nu_D(\omega') \geq \nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Let us write

$$\omega = h\omega_{s+1} + \tilde{h}\tilde{\omega}_{s+1}, \quad \omega' = g\omega_{s+1} + \tilde{g}\tilde{\omega}_{s+1},$$

where $\delta = h\tilde{g} - g\tilde{h}$ is a unit in $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$. By taking into consideration the divisorial order ν_D, we have that $\nu_D(h) = 0$ and $\nu_D(g) > 0$; hence h is a unit and g is not a unit. Since δ is a unit, we have that $\tilde{g}$ is a unit. If $\nu_D(\omega') > \tilde{t}_{s+1} = \nu_D(\tilde{\omega}_{s+1})$, we necessarily have that

$$\nu_D(g\omega_{s+1}) = \nu_D(\tilde{g}\tilde{\omega}_{s+1}) = \nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}.$$

Let us see that this is not possible. Assume that $t_{s+1} = t_s + n\ell_{s+1}^n$ and hence $\tilde{t}_{s+1} = t_s + m\ell_{s+1}^m$ (the case $t_{s+1} = t_s + m\ell_{s+1}^m$ is similar). We have

$$\nu_D(g) + t_{s+1} = \tilde{t}_{s+1} \Rightarrow \nu_D(g) + n\ell_{s+1}^n = m\ell_{s+1}^m.$$

Noting that $\nu_D(g) \in \Gamma$, we obtain two different ways of writing $m\ell_{s+1}^m < nm$ as a linear combination of n, m with nonnegative integer coefficients. This is a contradiction. □

We are now going to present the example of two cusps C_1 and C_2 corresponding to the Puiseux pair $(7, 36)$, such that the (common) semimodule of differential values has a basis $\mathcal{B} = (7, 36, 123)$ and such that the Saito pairs of multiplicities with respect to the first divisor D_1 are different for C_1 and C_2.

Remark 5.3. Let us note that for any 1-form ω, we have

$$\nu_{\mathbf{0}}(\omega) = \nu_{D_1}(\omega) - 1,$$

where $\nu_{\mathbf{0}}$ means the minimum of the multiplicity of the coefficients of ω.

First example. Consider the cusp C_1 invariant for the 1-form

$$\omega = 36x^3(7x\,dy - 36y\,dx) - 560y^3\,dy,$$

with a parametrization $\phi_1(t) = (t^7, t^{36} + t^{116} + \frac{28}{9}t^{196} + \text{h.o.t.})$. The basis of the semimodule of differential values of C_1 is $(7, 36, 123)$, with a minimal standard basis given by

$$\mathcal{S} = (\omega_{-1} = dx, \omega_0 = dy, \omega_1 = 7x\,dy - 36y\,dx).$$

We have $u_2^n = \lambda_1 + n\ell_2^n = \lambda_0 + mb_2$, that is, $123 + 7\ell_2^n = 36 + 36b_2$; we obtain that

$$\ell_2^n = b_2 = 3, \quad u_2^n = 144.$$

Similarly, we find that

$$u_2^m = 231 = 123 + 36\ell_2^m = 7 + 7a_2, \quad \ell_2^m = 3, \quad a_2 = 32.$$

Hence $u_2 = u_2^n$ and $\tilde{u}_2 = u_2^m$. Moreover, we have

$$t_2 = t_2^n = t_1 + n\ell_2^n = 43 + 7 \cdot 3 = 64, \quad \tilde{t}_2 = t_2^m = t_1 + m\ell_2^m = 43 + 36 \cdot 3 = 151.$$

We see that $\nu_D(\omega) = t_2 = 64$. Hence we can take $\omega_2 = \omega$ to obtain an extended standard basis and as being one of the generators of a Saito basis of C_1. Notice that $\nu_{D_1}(\omega) = 4$, since $\nu_{\mathbf{0}}(\omega) = 3$. We can take $\tilde{\omega}_2$ to be a 1-form with divisorial order $\nu_D(\tilde{\omega}_2) = \tilde{t}_2 = 151$ and C_1 to be invariant by $\tilde{\omega}_2$. By Delorme's decomposition in Theorem 3.11, we can write $\tilde{\omega}_2$ as

$$\tilde{\omega}_2 = y^3\omega_1 + \mu x^{32}\,dx + \eta_2; \quad \eta_2 = f_{-1}\,dx + f_0\,dy + f_1(7x\,dy - 36y\,dx),$$

for an appropriate constant μ and such that $\nu_{C_1}(f_\ell\omega_\ell) > \tilde{u}_2 = 231$, for $\ell = -1, 0, 1$.

Let us compute $\nu_{D_1}(\tilde{\omega}_2)$. Assume that we have $\nu_{D_1}(f_\ell\omega_\ell) > 5$, for $\ell = -1, 0, 1$, then we obtain that $\nu_{D_1}(\tilde{\omega}_2) = 5$. In view of Remark 5.1, we know that

$$\mathfrak{s}_{D_1}(C_1) + \tilde{\mathfrak{s}}_{D_1}(C_1) \leq \nu_{D_1}(xyf) = 7 + 2 = 9.$$

Thus, we have $(\mathfrak{s}_{D_1}(C_1), \tilde{\mathfrak{s}}_{D_1}(C_1)) = (4, 5)$ since the Saito basis ω, $\tilde{\omega}_2$ gives the maximal pair $(4, 5)$.

It remains to show that $\nu_{D_1}(f_\ell\omega_\ell) > 5$, for $\ell = -1, 0, 1$. We consider two situations: $\nu_D(f_\ell) \geq nm$ and $\nu_D(f_\ell) < nm$. In the first situation, we have

$$\nu_{\mathbf{0}}(f_\ell) \geq n = 7.$$

In the case that $\nu_D(f_\ell) < nm$, we have

$$\nu_D(f_\ell) = \nu_{C_1}(f_\ell) > 231 - \lambda_\ell.$$

Moreover, looking at the monomials in the expression of f_ℓ, we obtain

$$\nu_D(f_\ell) \leq \nu_{\mathbf{0}}(f_\ell)m = 36\nu_{\mathbf{0}}(f_\ell).$$

Thus, we have

$$\nu_{D_1}(f_\ell\omega_\ell) = \begin{cases} \nu_0(f_{-1}) + 1 \geq \dfrac{\nu_D(f_{-1})}{36} + 1 > \dfrac{231 - \lambda_{-1}}{36} + 1 = \dfrac{260}{36} \geq 5, & \ell = -1, \\[2ex] \nu_0(f_0) + 1 \geq \dfrac{\nu_D(f_0)}{36} + 1 > \dfrac{231 - \lambda_0}{36} + 1 = \dfrac{231}{36} \geq 5, & \ell = 0, \\[2ex] \nu_0(f_1) + 2 \geq \dfrac{\nu_D(f_1)}{36} + 2 > \dfrac{231 - \lambda_1}{36} + 2 = \dfrac{180}{36} = 5, & \ell = 1. \end{cases}$$

Second example. Take the cusp C_2 with Puiseux pair $(7, 36)$ invariant by the 1-form

$$\omega' = 36x^3(7x\,dy - 36y\,dx) - 560y^3\,dy + y(7x\,dy - 36y\,dx)$$

and defined by a parametrization as follows:

$$\phi_2(t) = \left(t^7, t^{36} + t^{116} - \frac{4}{171}t^{131} + \frac{1}{1782}t^{146} - \frac{1}{72900}t^{161} + \text{h.o.t.}\right).$$

The basis of the semimodule of differential values is $(7, 36, 123)$. We can take

$$\mathcal{S} = (\omega_{-1} = dx, \omega_0 = dy, \omega_1 = 7x\,dy - 36y\,dx)$$

as a minimal standard basis for C_2 (thus, it is the same one as for C_1). We repeat the arguments used for the curve C_1. We can take $\omega'_2 = \omega'$ as one of the generators of a Saito basis of C_2, with $\nu_D(\omega') = t_2$. Again, we obtain a partial standard system $(\omega_{-1}, \omega_0, \omega_1, \omega'_2 = \omega', \tilde{\omega}'_2)$, where $\tilde{\omega}'_2$ can be written as

$$\tilde{\omega}'_2 = y^3\omega_1 + \mu' x^{32}\,dx + \eta'_2; \quad \eta'_2 = \sum_{\ell=-1}^{1} f'_\ell\omega_\ell,$$

with μ' being an appropriate constant and $\nu_{C_2}(f'_\ell\omega_\ell) > 231$. Thus, again we find that $\nu_{D_1}(f'_\ell\omega_\ell) > 5$. We have that $\nu_{D_1}(\tilde{\omega}'_2) = 5$.

Now, we have

$$(\nu_{D_1}(\omega'), \nu_{D_1}(\tilde{\omega}'_2)) = (3, 5).$$

This implies that $\mathfrak{s}_{D_1}(C_2) = 3 < 4 = \mathfrak{s}_{D_1}(C_1)$. Hence the Saito pairs of multiplicities for C_1 and C_2 are different.

Moreover, the pair $(3,5)$ is not maximal yet: the 1-form $\eta = \tilde{\omega}_2' - y^2\omega_2'$ satisfies that $\{\eta, \omega_2'\}$ is a Saito basis and $\nu_{D_1}(\eta) = 6$. Hence the Saito pair of multiplicities for the first divisor and the cusp C_2 is equal to $(\mathfrak{s}_{D_1}(C_2), \tilde{\mathfrak{s}}_{D_1}(C_2)) = (3,6)$.

Remark 5.4. Given a plane curve $S \subset (\mathbb{C}^2, \mathbf{0})$ and a finite sequence of blow-ups

$$\sigma \colon M \longrightarrow (\mathbb{C}^2, \mathbf{0})$$

with last exceptional divisor D, we can define in a similar way the Saito pair of multiplicities $(\mathfrak{s}_D(S), \tilde{\mathfrak{s}}_D(S))$. In this way we have infinitely many analytic invariants of S. An interesting question would be to describe the set of these invariants as a subset of the moduli of plane curves given in [**10**].

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References

[1] M. Alberich-Carramiñana, P. Almirón, and J.-J. Moyano-Fernández, Curve singularities with one Puiseux pair and value sets of modules over their local rings, *J. Algebraic Combin.* **61(2)** (2025), Paper no. 20, 20 pp. DOI: 10.1007/s10801-025-01382-x.

[2] M. Artin, On the solutions of analytic equations, *Invent. Math.* **5** (1968), 277–291. DOI: 10.1007/BF01389777.

[3] R. Berger, Differentialmoduln eindimensionaler lokaler Ringe, *Math. Z.* **81** (1963), 326–354. DOI: 10.1007/BF01111579.

[4] F. Cano, N. Corral, and D. Senovilla-Sanz, Analytic semiroots for plane branches and singular foliations, *Bull. Braz. Math. Soc. (N.S.)* **54(2)** (2023), Paper no. 27, 49 pp. DOI: 10.1007/s00574-023-00344-w.

[5] C. Delorme, Sur les modules des singularités des courbes planes, *Bull. Soc. Math. France* **106(4)** (1978), 417–446.

[6] Y. Genzmer, Dimension of the moduli space of a germ of curve in $\mathbb{C}^2$, *Int. Math. Res. Not. IMRN* **2022(5)** (2022), 3805–3859. DOI: 10.1093/imrn/rnaa237.

[7] Y. Genzmer, Number of moduli for a union of smooth curves in $(\mathbb{C}^2, 0)$, *J. Symbolic Comput.* **113** (2022), 148–170. DOI: 10.1016/j.jsc.2022.03.002.

[8] Y. Genzmer, The Saito module and the moduli of a germ of curve in $(\mathbb{C}^2, 0)$, *Ann. Inst. Fourier (Grenoble)* **75(3)** (2025), 1053–1107. DOI: 10.5802/aif.3655.

[9] Y. Genzmer and M. E. Hernandes, On the Saito basis and the Tjurina number for plane branches, *Trans. Amer. Math. Soc.* **373(5)** (2020), 3693–3707. DOI: 10.1090/tran/8019.

[10] A. Hefez and M. E. Hernandes, The analytic classification of plane branches, *Bull. Lond. Math. Soc.* **43(2)** (2011), 289–298. DOI: 10.1112/blms/bdq113.

[11] A. Hefez and M. E. Hernandes, The analytic classification of irreducible plane curve singularities, in: *Handbook of Geometry and Topology of Singularities II*, Springer, Cham, 2021, pp. 1–65. DOI: 10.1007/978-3-030-78024-1_1.

[12] M. E. Hernandes and M. E. R. Hernandes, The analytic classification of plane curves, *Compos. Math.* **160(4)** (2024), 915–944. DOI: 10.1112/S0010437X24007061.

[13] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27(2)** (1980), 265–291.

[14] O. Zariski, Characterization of plane algebroid curves whose module of differentials has maximum torsion, *Proc. Nat. Acad. Sci. U.S.A.* **56(3)** (1966), 781–786. DOI: 10.1073/pnas.56.3.781.

[15] O. Zariski, *The Moduli Problem for Plane Branches*, With an appendix by Bernard Teissier, Translated from the 1973 French original by Ben Lichtin Univ. Lecture Ser. **39**, American Mathematical Society, Providence, RI, 2006. DOI: 10.1090/ulect/039.

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