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CLASSIFICATION OF MONADS AND MODULI COMPONENTS OF STABLE RANK 2 BUNDLES WITH ODD DETERMINANT AND $c_2 = 10$

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Abstract: In this paper, we provide a complete classification of the positive minimal monads whose cohomology is a stable rank 2 bundle on $\mathbb{P}^3$ with Chern classes $c_1 = -1$, $c_2 = 10$. We prove the existence of two new components of the moduli space $\mathcal{B}(-1, 10)$ of stable rank 2 bundles with the given Chern classes. We also show that Hartshorne's conditions on a sequence $\mathcal{X}$ of 10 integers are sufficient and necessary for the existence of a stable rank 2 bundle with odd determinant and spectrum $\mathcal{X}$. Furthermore, we prove that the sequence of integers $\{-2^{n-1}, -1, 0, 1^{n-1}\}$ for $n \geq 4$ is realized as the spectrum of a stable rank 2 bundle $\mathcal{E}$ of odd determinant by computing the minimal generators of its Rao module.

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1. Introduction

There are three ways to classify rank 2 vector bundles on the projective space $\mathbb{P}^3$. The first method is due to Horrocks, who proved in [**15**] that every vector bundle on $\mathbb{P}^3$ can be obtained as the cohomology of a *monad* whose terms are summands of line bundles. Another way to study stable rank 2 vector bundles on the projective space is in terms of their *spectra*, as originally introduced by Barth and Elencwajg in [**3**]. Finally, we can also classify rank 2 vector bundles on $\mathbb{P}^3$ up to flat deformation by analyzing the irreducible components of the moduli space of stable vector bundles with fixed Chern classes. It is important to note that there is no straightforward relation between these three classification schemes: a monad can be associated with two vector bundles with different spectra; a single spectrum can be associated with different monads; and an irreducible component of the moduli space may contain points corresponding to different spectra and monads.

Let $\mathcal{B}(e, m)$ denote the moduli space of stable rank 2 bundles on $\mathbb{P}^3$ with Chern classes $c_1 = e$ and $c_2 = m$. Up to normalization, it is sufficient to consider $e = -1, 0$, and when $e = -1$ the integer m must be even because the spectrum of a bundle $\mathcal{E}$ in $\mathcal{B}(-1, m)$ is symmetric, cf. C.1 of Lemma 3.

There is a complete classification of $\mathcal{B}(0, m)$ for $c_2 \leq 5$. Specifically, in [**13**, Section 5.3] it is shown that $\mathcal{B}(0, 1)$ and $\mathcal{B}(0, 2)$ are irreducible while in [**8**] and [**5**] it is proved that $\mathcal{B}(0, 3)$ and $\mathcal{B}(0, 4)$ have two irreducible components, respectively. The complete description of $\mathcal{B}(0, 5)$ was obtained in [**1**] by proving that this moduli space has exactly three irreducible components. Monads and spectra for $c_2 \leq 8$ have also been classified by Hartshorne and Rao in [**13**].

When $e = -1$, fewer results are found in the literature: Hartshorne and Sols ([**14**]) and Manolache ([**17**]) proved that $\mathcal{B}(-1, 2)$ is irreducible; Bănică and Manolache showed in [**2**] that $\mathcal{B}(-1, 4)$ has two irreducible components. More recently, the authors proved in [**10**] that both the moduli spaces $\mathcal{B}(-1,6)$ and $\mathcal{B}(-1,8)$ have at least four irreducible components.

The goal of the present paper is to list all *positive* minimal Horrocks monads (see Subsection 2.2 for precise definitions) whose cohomology is a stable rank 2 bundle with odd determinant and $c_2 = 10$, to determine which ones exist, and to study the irreducible components of $\mathcal{B}(-1,10)$. We prove that, except for one case (namely $\mathcal{X}_8^{10}$; see Subsection 2.3), all possible spectra in Table 1 are realized as the stable rank 2 bundle given by the cohomology of a positive minimal monad. The remaining case is realized as the spectrum of a stable rank 2 bundle given as the cohomology of a *negative* minimal monad, and we present it explicitly. In fact, we thank Nicolae Manolache for pointing out a mistake in our previous paper [**10**] regarding the spectrum $\mathcal{X}_6^8$; the same argument he indicated allowed us to correct our mistake and study the spectrum $\mathcal{X}_8^{10}$.

The paper is organized as follows. In Section 2 we recall three approaches to classifying rank 2 stable bundles: the Hartshorne–Serre correspondence, minimal Horrocks monads and their cohomology, and the spectrum of a rank 2 bundle. By applying the Hartshorne–Serre correspondence to certain specific cases, we obtain the families of *Hartshorne bundles* and *Ein bundles*. We also recall the definitions of positive, non-negative, and homotopy-free minimal monads, and we revisit [**10**, Lemma 3], which characterizes the minimal Horrocks monads whose cohomologies are stable rank 2 bundles on $\mathbb{P}^3$ with odd determinant. We conclude this section by listing all possible spectra for rank 2 bundles with odd determinant and $c_2 = 10$.

In Section 3, we state Theorem 4, which is the main tool for producing positive minimal monads with a stable rank 2 bundle with odd determinant as cohomology. We apply this result to list all possible positive minimal monads when $c_2 = 10$ and a fixed spectrum. We also eliminate some of the possibilities of positive minimal Horrocks monads.

Section 4 is devoted to providing a complete classification of the positive minimal Horrocks monads whose cohomology is a rank 2 stable vector bundle with Chern classes $c_1 = -1$ and $c_2 = 10$; they are summarized in Table 4. To prove the existence of these positive minimal monads, it was necessary to apply three methods: Lemma 12, an explicit construction of some positive minimal monads using Macaulay2, and the Hartshorne–Serre correspondence. Section 5 is devoted to showing that the sequence of integers $\mathcal{X}_8^{10} := \{-2^4, -1, 0, 1^4\}$ is realized as the spectrum of a negative minimal Horrocks monad, and thus all possible spectra in Table 1 are realized as the spectrum of a stable rank 2 bundle on $\mathbb{P}^3$. We also explicitly present the negative minimal monad whose cohomology is a stable bundle with spectrum $\{-2^{n-1}, -1, 0, 1^{n-1}\}$ for $n \geq 4$.

In Section 6, we list all homotopy-free, positive minimal monads and compute the dimensions of the induced families of stable rank 2 bundles. Using the lower semicontinuity of the dimension of cohomology groups, we identify two new components of $\mathcal{B}(-1,10)$ and we show that $\mathcal{B}(-1,10)$ has at least five irreducible components; see Theorem 20.

2. Constructing stable rank 2 bundles on $\mathbb{P}^3$

We work over an algebraically closed field of characteristic zero. For vector bundles of rank 2 on $\mathbb{P}^3$, and more generally for a reflexive sheaf of rank 2 on $\mathbb{P}^n$, there is a stability criterion that we will use. If $\mathcal{E}$ is a vector bundle of rank 2 on $\mathbb{P}^3$, then there is a unique integer $\eta_{\mathcal{E}}$ such that $c_1(\mathcal{E}(\eta_{\mathcal{E}})) \in \{-1, 0\}$. We denote $\mathcal{E}_{\text{norm}} := \mathcal{E}(\eta_{\mathcal{E}})$ and we say that $\mathcal{E}$ is *normalized* if $\mathcal{E} = \mathcal{E}_{\text{norm}}$ and we have that $\mathcal{E}$ is stable if and only if $\mathcal{E}_{\text{norm}}$ has no sections; see [**18**, Lemma 1.2.5].

2.1. Hartshorne–Serre correspondence. The Hartshorne–Serre correspondence provides a tool to construct stable rank 2 bundles on $\mathbb{P}^3$ by establishing a relation between locally complete intersection subschemes $X \subset \mathbb{P}^3$ of codimension 2 and rank 2 bundles on $\mathbb{P}^3$ equipped with a section whose zero set is X. More generally:

Theorem 1 ([**12**, Theorem 4.1])**.** *Fix an integer e. There is a one-to-one correspondence between:*

(1) *pairs $(\mathcal{E}, s)$, where $\mathcal{E}$ is a rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = e$ and $s \in H^0(\mathcal{E})$ is a global section whose zero set has codimension 2, and*
(2) *pairs (X, η), where X is a Cohen–Macaulay curve on $\mathbb{P}^3$, generically locally complete intersection, and $\eta \in H^0(\omega_X(4-e))$ is a global section that generates the sheaf $\omega_X(4-e)$ except at finitely many points.*

For this paper, it is sufficient to consider $\mathcal{E}$ a rank 2 bundle on $\mathbb{P}^3$ and X a local complete intersection of codimension 2. The section s in Theorem 1 induces the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{E} \longrightarrow \mathcal{I}_X(k) \longrightarrow 0,$$

where $c_1(\mathcal{E}) = k$ and $c_2(\mathcal{E}) = \deg X$.

As a naive way to apply the correspondence in Theorem 1, we recall two families of vector bundles on $\mathbb{P}^3$ with $c_1 = -1$ found in the literature: the first family of bundles is obtained by considering a curve X being a disjoint union of r conics X_i on $\mathbb{P}^3$ and η a section of $H^0(\omega_X(1))$. Then the vector bundle associated with $\mathcal{E}$ is such that $c_1 = 3$ and $c_2 = 2r$. If we normalize $\mathcal{E}$, we obtain $\mathcal{E}' := \mathcal{E}(-2)$ such that $c_1(\mathcal{E}') = -1$ and $c_2(\mathcal{E}') = 2r-2$. The family of vector bundles so obtained is called a *Hartshorne family* and a bundle from this family is called a *Hartshorne bundle*. A vector bundle from the second family is obtained by considering a curve $X = X_1 \cup X_2$, where X_i is a complete intersection curve of bidegree $(n-b_i, n+b_i+1)$ for integers b_i, $i = 1, 2$. We have $\omega_X(4) \simeq \mathcal{O}_X(2n+1)$ and from Theorem 1 we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_X(2n+1) \longrightarrow 0,$$

where $\mathcal{E}' = \mathcal{E}(-n-1)$ is such that $c_1(\mathcal{E}') = -1$ and $c_2(\mathcal{E}') = n^2 - b_1^2 - b_2^2 + n - b_1 - b_2$. We will call a vector bundle from this family an *Ein bundle*.

2.2. Minimal Horrocks monads. Another way to construct rank 2 bundles on $\mathbb{P}^3$ is due to Horrocks [**15**]. Recall that a *monad* on $\mathbb{P}^3$ is a complex

$$\mathbf{M}\colon \mathcal{C} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{A}, \tag{2.1}$$

of vector bundles on $\mathbb{P}^3$ such that the map α is injective and β is surjective. The sheaf $\mathcal{E} := \ker\beta/\operatorname{Im}\alpha$ is the *cohomology of the monad* $\mathbf{M}$. Let us assume that the morphism α in (2.1) is locally left-invertible, so the cohomology of the monad $\mathcal{E}$ of $\mathbf{M}$ is a vector bundle. Horrocks in [**15**] proved that all vector bundles on $\mathbb{P}^n$ can be obtained as cohomology of a monad of the form (2.1), where the vector bundles $\mathcal{C}$, $\mathcal{B}$, and $\mathcal{A}$ are sums of line bundles.

In this paper, we always assume that the morphism α is locally left-invertible, so that $\mathcal{E}$ is a vector bundle. A monad $\mathbf{M}$ of the form (2.1) is called *minimal* if none of the entries of the associated matrices for the maps α and β, as homogeneous forms, is a non-zero constant; in other words, if no direct summand of $\mathcal{A}$ is the image of a line subbundle of $\mathcal{B}$ and if no direct summand of $\mathcal{C}$ maps onto a direct summand of $\mathcal{B}$. In addition, $\mathbf{M}$ is said to be *homotopy-free* if

$$\operatorname{Hom}(\mathcal{B}, \mathcal{C}) = \operatorname{Hom}(\mathcal{A}, \mathcal{B}) = 0.$$

We can observe that every homotopy-free monad is a minimal monad. By considering $\mathcal{E}$ to be a stable rank 2 bundle on $\mathbb{P}^3$, there is an unique isomorphism $f\colon \mathcal{E} \to \mathcal{E}^\vee(-1)$ with a twisted symplectic structure, i.e., that is $f^\vee(-1) = -f$. In [**10**, Lemma 3], a minimal monad whose cohomology is a stable rank 2 bundle on $\mathbb{P}^3$ with $c_1 = -1$ was characterized, cf. [**16**, Theorem 2.3] and [**19**, Proposition 2.2].

Lemma 2. *Given a stable rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = -1$, we can find two sequences of integers $\boldsymbol{a} = (a_1, \dots, a_s)$ and $\boldsymbol{b} = (b_1, \dots, b_{s+1})$ such that $\mathcal{E}$ is the cohomology of a monad of the form*

$$\bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^3}(-a_i - 1) \xrightarrow{\alpha} \bigoplus_{j=1}^{s+1} (\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j - 1)) \xrightarrow{\beta} \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^3}(a_i), \tag{2.2}$$

where we order the tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $a_1 \leq \cdots \leq a_s$ and $0 \leq b_1 \leq \cdots \leq b_{s+1}$. In addition,

$$c_2(\mathcal{E}) = \sum_{i=1}^{s} a_i(a_i + 1) - \sum_{j=1}^{s+1} b_j(b_j + 1). \tag{2.3}$$

From Lemma 4 in [**10**], we also have $\beta = \alpha^\vee(-1) \circ \Omega$ with $\alpha^\vee(-1) \circ \Omega \circ \alpha = 0$, where $\Omega = \bigoplus_{j=1}^{s+1} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the standard symplectic form, with each 2×2 block regarded as a morphism

$$\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-b_j - 1) \oplus \mathcal{O}_{\mathbb{P}^3}(b_j).$$

Let us consider $\mathcal{A} = \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^3}(a_i)$ and $\mathcal{B} = \bigoplus_{j=1}^{s+1} (\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j - 1))$. Let $A = \mathbf{k}[X_0, X_1, X_2, X_3]$ be the graded module and set

$$M := H^1_*(\mathcal{E}) = \bigoplus_{l \in \mathbb{Z}} H^1(\mathcal{E}(l))$$

to be the *first cohomology module* of $\mathcal{E}$ as a module over A. If M has a minimal free presentation of the form

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

it is known that (see [**19**, Proposition 2.2]) $\mathrm{rk}(F_0) = s$ and $\mathrm{rk}(F_1) = 2s+2$. Conversely, if M has a free minimal resolution as above, then, with an argument similar to [**13**, Proposition 3.2], one can show that $\mathcal{E}$ is given as the cohomology of a minimal monad of the form (2.2) with $F_0 = H^0_*(\mathcal{A})$ and $F_1 = H^0_*(\mathcal{B})$ and thus $\mathcal{A}$ and $\mathcal{B}$ are the sheafifications of F_0 and F_1, respectively.

A minimal monad as in display (2.2) is also called a *minimal Horrocks monad*. Two special classes of minimal Horrocks monads have been studied in the literature:

- monads with $\boldsymbol{a} = (1, \dots, 1)$ and $\boldsymbol{b} = (0, \dots, 0)$ for any $s \geq 1$ are called *Hartshorne monads*;
- monads with $s = 1$ and $a_1 > b_2 \geq b_1 \geq 0$ are called *Ein monads*.

The cohomology bundles of these monads are the Hartshorne and Ein bundles previously discussed in Subsection 2.1.

Finally, recall from [**10**] that a monad as in display (2.2) is said to be *positive* if all summands of its right-hand term have a positive degree; that is, $a_i > 0$ for each $i = 1, \dots, s$. If all summands of its right-hand term have a non-negative degree, then the monad is said to be *non-negative*. If any of the summands in the right-hand term is negative, then the monad is said to be *negative*.

2.3. Spectrum of a vector bundle. The *spectrum* of a vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ is a sequence of integers that encodes partial information about the cohomology modules $H^1_*(\mathcal{E})$ and $H^2_*(\mathcal{E})$. It was first defined by Barth and Elencwajg [**3**], and later extended by Hartshorne to rank 2 reflexive sheaves; see [**12**, Theorem 7.1].

Abstractly, a *spectrum of length* l is a sequence of integers $\mathcal{X} = \{k_1, k_2, \ldots, k_l\}$ satisfying the following properties:

C.1 (Symmetry) $\{k_i\} = \{-k_i - 1\}$.

C.2 (Connectness) Any integer k between two elements of $\mathcal{X}$ also belongs to $\mathcal{X}$.

C.3 If $k = \max\{-k_i\}$ and there is an integer u with $-k \leq u \leq -2$ that occurs just once in $\mathcal{X}$, then each $k_i \in \mathcal{X}$ with $-k \leq k_i \leq u$ occurs exactly once.

Observe that condition C.1 implies that a spectrum must have even length.

The motivation behind this definition is the following result; see [**12**, Theorem 7.1].

Lemma 3. *Let $\mathcal{E}$ be a stable rank 2 bundle on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 2n$. Then there is a unique spectrum $\mathcal{X}(\mathcal{E}) = \{k_1, \ldots, k_{2n}\}$ of length $2n$ such that*

(1) $h^1(\mathbb{P}^3, \mathcal{E}(p)) = h^0(\mathbb{P}^1, \mathcal{H}(p+1))$ *for* $p \leq -1$ *and*

(2) $h^2(\mathbb{P}^3, \mathcal{E}(p)) = h^1(\mathbb{P}^1, \mathcal{H}(p+1))$ *for* $p \geq -2$,

where $\mathcal{H} := \bigoplus_{i=1}^{2n} \mathcal{O}_{\mathbb{P}^3}(k_i)$.

It is important to note that it is not known whether every spectrum as defined above is the spectrum of a stable rank 2 bundle with odd determinant. A spectrum $\mathcal{X}$ is said to be *realized* by a stable rank 2 bundle with odd determinant $\mathcal{E}$ if $\mathcal{X} = \mathcal{X}(\mathcal{E})$. Coandă recently provided in [**6**, Example 3.13] two examples of integral sequences of length 21 that cannot be the spectra of stable rank 2 bundles with even determinant; therefore, it is reasonable to assume that conditions C.1–C.3 are only necessary but not sufficient for realization by stable rank 2 bundles with odd determinant, though precise examples are not yet known.

Before we list all the possibilities of spectra of length 10, we recall the notion of order between two spectra of the same length: if $\mathcal{X}^l = \{k_1, k_2, \ldots, k_l\}$ and $\mathcal{S}^l = \{k'_1, k'_2, \ldots, k'_l\}$ are spectra of length l in non-descending order, then we say that $\mathcal{X}^l > \mathcal{S}^l$ provided the leftmost non-zero integer $k_i - k'_i$ is positive. Let us denote $r_j := \{-j-1, j\}$, $r_j r_i := \{-j-1, -i-1, i, j\}$, and so on.

We list in Table 1 all possible spectra of length 10. The first column represents the power of r_0 in the spectrum, while the spectra in each row are listed in decreasing order, from left to right.

i	Spectrum
5	$\mathcal{X}_1^{10} = \{r_0^5\}$
4	$\mathcal{X}_2^{10} = \{r_0^4 r_1\}$
3	$\mathcal{X}_3^{10} = \{r_0^3 r_1^2\}$, $\mathcal{X}_4^{10} = \{r_0^3 r_1 r_2\}$
3	$\mathcal{X}_5^{10} = \{r_0^2 r_1^3\}$, $\mathcal{X}_6^{10} = \{r_0^2 r_1^2 r_2\}$, $\mathcal{X}_7^{10} = \{r_0^2 r_1 r_2 r_3\}$
1	$\mathcal{X}_8^{10} = \{r_0 r_1^4\}$, $\mathcal{X}_9^{10} = \{r_0 r_1^3 r_2\}$, $\mathcal{X}_{10}^{10} = \{r_0 r_1^2 r_2^2\}$, $\mathcal{X}_{11}^{10} = \{r_0 r_1^2 r_2 r_3\}$, $\mathcal{X}_{12}^{10} = \{r_0 r_1 r_2 r_3 r_4\}$

TABLE 1. Possible spectra of length 10.

3. Possible positive minimal Horrocks monads

Let $\mathcal{E}$ be a stable rank 2 vector bundle on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2n$. From the properties of the spectrum, it follows that $\mathcal{X}(\mathcal{E})$ can be written as

$$\mathcal{X}(\mathcal{E}) = \{(-k-1)^{s(k)}, \ldots, 0^{s(0)}, \ldots, k^{s(k)}\}. \tag{3.1}$$

For each integer l, let M_l be the l-th graded component of the Rao module of M and $m_l = \dim M_l$. The number of minimal generators for M in degree l is defined as

$$\rho(l) := \dim[\operatorname{coker}(H^1(\mathcal{E}(l-1)) \times H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^1(\mathcal{E}(l)))].$$

Given one of the possible spectra listed in Table 1, we recall [**10**, Theorem 8], which is applied to list all *possible positive minimal Horrocks monads* having as cohomology a stable rank 2 bundle $\mathcal{E}$ with the given spectrum.

Theorem 4. *Let $\mathcal{E}$ be a stable rank 2 bundle on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = -1$ and spectrum as in display* (3.1)*. We have*

$$\rho(-k-1) = m_{-k-1} = s(k)$$

and for $0 \le i < k$,

$$s(i) - 2\sum_{j \ge i+1} s(j) \le \rho(-i-1) \le s(i) - 1.$$

The problem facing us is whether all of the possibilities proposed by Theorem 4 exist. From Table 1 and Theorem 4, we easily determine the existence of stable rank 2 bundles with spectra $\mathcal{X}_1^{10}$ and $\mathcal{X}_{12}^{10}$. For the spectrum $\mathcal{X}_1^{10}$, we have $k = 0$ and $\rho(-1) = 5$, hence we get the Hartshorne monad with $s = 5$ and $\mathbf{b} = (0,0,0,0,0,0)$

$$\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 5} \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 5} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 5},$$

which exists according to Subsection 2.1 with $r = 6$ or [**11**, Example 3.1.3] and its cohomology is a stable rank 2 bundle $\mathcal{E}$ with spectrum $\mathcal{X}_1^{10}$.

Next, if we look at the spectrum $\mathcal{X}_{12}^{10} = \{r_0 r_1 r_2 r_3 r_4\}$, we obtain $k = 4$, $\rho(-5) = 1$, and $\rho(-i-1) = 0$ for $i = 0,1,2,3$. According to the equation of c_2 with $s = 1$ it follows that $b_1^2 + b_2^2 + b_1 + b_2 = 20$, which has the solution $b_1 = 4$ and $b_2 = 0$. In this way, we obtain the Ein monad ($s = 1$, $\mathbf{b} = (0,4)$) that exists, c.f. [**7**], and we write

$$\mathcal{O}_{\mathbb{P}^3}(-6) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(5),$$

whose cohomology is a stable rank 2 bundle on $\mathbb{P}^3$ with spectrum $\mathcal{X}_{12}^{10}$.

For each of the other spectra in Table 1, there is more than one possible minimal Horrocks monad; we will list all of these possibilities in Table 2.

Our goal in this section is to list all possible positive minimal Horrocks monads with a fixed spectrum. For this, we take each possibility of the number of minimal generators of $H^1_*(\mathcal{E})$ listed in Table 2 (i.e., for each sequence of integers $\mathbf{a}$) and substitute them into equation (2.3) to find its solutions, which are the sequences of integers $\mathbf{b}$. To simplify the notation, we denote

$$B_s := \sum_{j=1}^{s+1} b_j(b_j + 1).$$

We list all the possibilities of positive minimal Horrocks monads for each fixed spectrum, cf. Table 2.

Spectrum	k	$\rho(-k-1)$	i	$\rho(-i-1)$
$\mathcal{X}_2^{10}$	1	1	0	$\{2,3\}$
$\mathcal{X}_3^{10}$	1	2	0	$\{0,1,2\}$
$\mathcal{X}_4^{10}$	2	1	0	$\{0,1,2\}$
			1	$\{0\}$
$\mathcal{X}_5^{10}$	1	3	0	$\{0,1\}$
$\mathcal{X}_6^{10}$	2	1	0	$\{0,1\}$
			1	$\{0,1\}$
$\mathcal{X}_7^{10}$	3	1	0	$\{0,1\}$
			$\{1,2\}$	$\{0\}$
$\mathcal{X}_8^{10}$	1	4	0	$\{0\}$
$\mathcal{X}_9^{10}$	2	1	0	$\{0\}$
			1	$\{1,2\}$
$\mathcal{X}_{10}^{10}$	2	2	0	$\{0\}$
			1	$\{0,1\}$
$\mathcal{X}_{11}^{10}$	3	1	0	$\{0\}$
			1	$\{0,1\}$
			2	$\{0\}$

TABLE 2. Number of minimal generators of $H^1_*(\mathcal{E})$ for a stable rank 2 bundle with $\mathcal{X}(\mathcal{E}) = \mathcal{X}_p^{10}$ for $p = 2, \ldots, 11$.

$(\boldsymbol{\mathcal{X}_2^{10}}, \rho(-2) = 1)$

- $\rho(-1) = 2$ implies $\boldsymbol{a} = (1,1,2)$ and

$$B_3 = 0 \Longleftrightarrow \boldsymbol{b} = (0,0,0,0).$$

- $\rho(-1) = 3$ implies $\boldsymbol{a} = (1,1,1,2)$ and

$$B_4 = 2 \Longleftrightarrow \boldsymbol{b} = (0,0,0,0,1).$$

$(\boldsymbol{\mathcal{X}_3^{10}}, \rho(-2) = 2)$

- $\rho(-1) = 0$ implies $\boldsymbol{a} = (2,2)$ and

$$B_2 = 2 \Longleftrightarrow \boldsymbol{b} = (0,0,1).$$

- $\rho(-1) = 1$ implies $\boldsymbol{a} = (1,2,2)$ and

$$B_3 = 4 \Longleftrightarrow \boldsymbol{b} = (0,0,1,1).$$

- $\rho(-1) = 2$ implies $\boldsymbol{a} = (1,1,2,2)$ and

$$B_4 = 6 \Longleftrightarrow \boldsymbol{b} = (0,0,1,1,1) \text{ or } \boldsymbol{b} = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{2}).$$

$(\mathcal{X}_{\mathbf{4}}^{\mathbf{10}}, \rho(-3) = 1)$

- $\rho(-1) = 0$ implies $\boldsymbol{a} = (3)$ and
$$B_1 = 2 \Longleftrightarrow \boldsymbol{b} = (0, 1).$$
- $\rho(-1) = 1$ implies $\boldsymbol{a} = (1, 3)$ and
$$B_2 = 4 \Longleftrightarrow \boldsymbol{b} = (0, 1, 1).$$
- $\rho(-1) = 2$ implies $\boldsymbol{a} = (1, 1, 3)$ and
$$B_3 = 6 \Longleftrightarrow \boldsymbol{b} = (0, 1, 1, 1) \text{ or } \boldsymbol{b} = (0, 0, 0, 2).$$

$(\mathcal{X}_{\mathbf{5}}^{\mathbf{10}}, \rho(-2) = 3)$

- $\rho(-1) = 0$ implies $\boldsymbol{a} = (2, 2, 2)$ and
$$B_3 = 8 \Longleftrightarrow \boldsymbol{b} = (1, 1, 1, 1) \text{ or } \boldsymbol{b}= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{2}).$$
- $\rho(-1) = 1$ implies $\boldsymbol{a} = (1, 2, 2, 2)$ and
$$B_4 = 10 \Longleftrightarrow \boldsymbol{b} = (1, 1, 1, 1, 1) \text{ or } \boldsymbol{b}= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{2}).$$

$(\mathcal{X}_{\mathbf{6}}^{\mathbf{10}}, \rho(-3) = 1)$

- $\rho(-1) = \rho(-2) = 0$ implies $\boldsymbol{a} = (3)$ and
$$B_1 = 2 \Longleftrightarrow \boldsymbol{b} = (0, 1).$$
- $\rho(-1) = 1$ and $\rho(-2) = 0$ implies $\boldsymbol{a} = (1, 3)$ and
$$B_2 = 4 \Longleftrightarrow \boldsymbol{b} = (0, 1, 1).$$
- $\rho(-1) = 0$ and $\rho(-2) = 1$ implies $\boldsymbol{a} = (2, 3)$ and
$$B_2 = 8 \Longleftrightarrow \boldsymbol{b} = (0, 1, 2).$$
- $\rho(-1) = \rho(-2) = 1$ implies $\boldsymbol{a} = (1, 2, 3)$ and
$$B_3 = 10 \Longleftrightarrow \boldsymbol{b} = (0, 1, 1, 2).$$

$(\mathcal{X}_{\mathbf{7}}^{\mathbf{10}}, \rho(-4) = 1)$

- $\rho(-1) = 0$ implies $\boldsymbol{a} = (4)$ and $B_1 = 10$, which has no solution.
- $\rho(-1) = 1$ implies $\boldsymbol{a} = (1, 4)$ and
$$B_2 = 12 \Longleftrightarrow \boldsymbol{b} = (0, 2, 2) \text{ or } \boldsymbol{b} = (0, 0, 3).$$

$(\mathcal{X}_{\mathbf{8}}^{\mathbf{10}}, \rho(-2) = 4)$

- $\rho(-1) = 0$ implies $\boldsymbol{a} = (2, 2, 2, 2)$ and
$$B_4 = 14 \Longleftrightarrow \boldsymbol{b}= (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{3}) \text{ or } \boldsymbol{b}= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{2}) \text{ or } \boldsymbol{b}= (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}).$$

$(\mathcal{X}_{\mathbf{9}}^{\mathbf{10}}, \rho(-3) = 1)$

- $\rho(-2) = 1$ implies $\boldsymbol{a} = (2, 3)$ and
$$B_2 = 8 \Longleftrightarrow \boldsymbol{b} = (0, 1, 2).$$
- $\rho(-2) = 2$ implies $\boldsymbol{a} = (2, 2, 3)$ and
$$B_3 = 14 \Longleftrightarrow \boldsymbol{b} = (0, 1, 2, 2) \text{ or } \boldsymbol{b}= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{3}).$$

$(\mathcal{X}^{\mathbf{10}}_{\mathbf{10}}, \rho(-3) = 2)$

- $\rho(-2) = 0$ implies $\boldsymbol{a} = (3,3)$ and

$$B_2 = 14 \Longleftrightarrow \boldsymbol{b} = (1,2,2) \text{ or } \boldsymbol{b}= (\mathbf{0},\mathbf{1},\mathbf{3}).$$

- $\rho(-2) = 1$ implies $\boldsymbol{a} = (2,3,3)$ and

$$B_3 = 20 \Longleftrightarrow \boldsymbol{b} = (1,2,2,2) \text{ or } \boldsymbol{b}= (\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}) \text{ or } \boldsymbol{b}= (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{4}).$$

$(\mathcal{X}^{\mathbf{10}}_{\mathbf{11}}, \rho(-4) = 1)$

- $\rho(-2) = 0$ implies $\boldsymbol{a} = (4)$ and $B_1 = 10$, which has no solution.
- $\rho(-2) = 1$ implies $\boldsymbol{a} = (2,4)$ and

$$B_2 = 16 \Longleftrightarrow \boldsymbol{b} = (1,1,3).$$

Proposition 5. *If one of the 10 possibilities of positive minimal Horrocks monads in bold listed above exists, then its cohomology is not a stable rank 2 bundle.*

Proof: Suppose, for example, that a positive minimal Horrocks monad $\mathbf{M}$ with $\mathbf{a} = (1,1,2,2)$ and $\mathbf{b} = (0,0,0,0,2)$ exists and that its cohomology is a bundle $\mathcal{E}$. We can write it as

$$(\mathcal{O}_{\mathbb{P}^3}(-2)\oplus\mathcal{O}_{\mathbb{P}^3}(-3))^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(2)\oplus(\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(-1))^{\oplus 4}\oplus\mathcal{O}_{\mathbb{P}^3}(-3)$$
$$\xrightarrow{\beta} (\mathcal{O}_{\mathbb{P}^3}(2)\oplus\mathcal{O}_{\mathbb{P}^3}(1))^{\oplus 2},$$

where α and β are injective and surjective maps, respectively. However, from the minimality of $\mathbf{M}$, the first column of the map β is zero, which implies $\beta\circ\imath = 0$, where $\imath$ denotes the inclusion of $\mathcal{O}_{\mathbb{P}^3}(2)$ into the first summand of the middle term of $\mathbf{M}$. This means that $\imath \in H^0(\ker\beta(-2)) = H^0(\mathcal{E}(-2))$. Therefore, $\mathcal{E}$ cannot be stable.

The other possibilities can be treated analogously: the main point is that $\mathbf{b}$ contains an entry (b_{s+1}) that is equal to or larger than all of the entries of $\mathbf{a}$, inducing a non-trivial section in $H^0(\mathcal{E}(-b_{s+1}))$. □

Lemma 6. *Let a, b be positive integers such that $b < a$. Consider a possible positive minimal Horrocks monad of the form:*

$$\mathcal{O}_{\mathbb{P}^3}(-b-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a-1)^{\oplus 3} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^3}(-b-1)^{\oplus 5}$$
$$\xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3}. \tag{3.2}$$

(1) *If $6b+1 \geq 4a$, then the possible monad in (3.2) does not exist.*
(2) *For $6b+1 \geq 3a$, if the monad as in display (3.2) exists, then its cohomology is an unstable vector bundle.*

Proof: From the minimally of the monad, it follows that $\mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 5}$ maps to $\mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3}$, say $\phi\colon \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3}$; set $j := \mathrm{rk}(\mathrm{Im}\,\phi) = 1,2,3$; note that $j = 3$ does not imply that ϕ is surjective.

If $j = 1, 2$, then [**13**, Lemma 3.7] yields the inequality

$$h^0((\operatorname{Im}\phi)(1)) \le j \cdot h^0(\mathcal{O}_{\mathbb{P}^3}(a)) = j\begin{pmatrix} a+4 \\ 3 \end{pmatrix},$$

which implies $h^0(\ker\phi(1)) \ge 5\binom{b+4}{3} - j\binom{a+4}{3}$. Therefore,

$$h^0(\mathcal{E}(1)) \ge h^0(\ker\phi(1)) \ge 5\begin{pmatrix} b+4 \\ 3 \end{pmatrix} - j\begin{pmatrix} a+4 \\ 3 \end{pmatrix} \ge 10,$$

thus $h^0(\mathcal{I}_Y(1)) = h^0(\mathcal{E}(1)) - 1 \ge 9$; however, for any closed subscheme $Y \subset \mathbb{P}^3$, we must have $h^0(\mathcal{I}_Y(1)) \le h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, providing a contradiction.

Now assume that $j = 3$. We can write:

$$\beta = \begin{pmatrix} 0 & \beta' \\ \phi & \beta'' \end{pmatrix},$$

where $\beta' \colon \mathcal{O}_{\mathbb{P}^3}(-b-1)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^3}(b)$ and $\beta'' \colon \mathcal{O}_{\mathbb{P}^3}(-b-1)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3}$. Since

$$\alpha = \Omega^{-1} \circ \beta^*(-1) = \begin{pmatrix} -\beta'^*(-1) & -\beta''^*(-1) \\ 0 & \phi^*(-1) \end{pmatrix},$$

it follows that

$$\beta\alpha = \begin{pmatrix} 0 & \beta'\phi^*(-1) \\ -\phi\beta'^*(-1) & \beta''\phi^*(-1) - \phi\beta''^*(-1) \end{pmatrix}.$$

In particular, $\phi\beta'^*(-1) = 0$, which implies $\operatorname{Im}\beta'^*(-1) \subset \ker\phi$ and we get the diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^3}(-b-1) & \xrightarrow{\beta'^*(-1)} & \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 5} & \longrightarrow & G := \operatorname{coker}\beta'^*(-1) \\ & \overset{0}{\searrow} & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\overline{\phi}} \\ & & \mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3} \end{array}$$

Since β' is surjective, G must be a locally free sheaf of rank 4; since $\operatorname{rk}(\phi) = 3$, it follows that $\operatorname{rk}(\ker\overline{\phi}) = 1$, thus $\ker\overline{\phi} \simeq \mathcal{O}_{\mathbb{P}^3}(k)$ for some integer k. We observe that $H^0(G(-b-1)) = 0$, and analyzing the diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^3}(k) & \hookrightarrow & G & \xrightarrow{\overline{\phi}} & \mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3} \\ & & \searrow\!\!\searrow & & \nearrow \\ & & & \operatorname{Im}\overline{\phi} & \end{array}$$

we get $H^0(G(-k)) \ne 0$, thus $k \le b$, therefore

$$\mu(\mathcal{O}(k)) \le b < \frac{6b+1}{4} = \mu(G).$$

By the seesaw property, it follows that $\mu(G) < \mu(\operatorname{Im}\overline{\phi})$; since $\mathcal{O}_{\mathbb{P}^3}(a)^{\oplus 3}$ is semi-stable with slope equal to 3, we have that $\mu(\operatorname{Im}\overline{\phi}) \le a$, so $6b+1 < 4a$.

This means that if a monad like the one in display (3.2) exists, then $6b+1 < 4a$.

Furthermore, we have $c_1(\operatorname{Im}\overline{\phi}) = 6b+1-k$, and so $k \ge 6b+1-3a \ge 0$ whenever $6b+1 \ge 3a$, which implies $H^0(\mathcal{E}(-k)) \ne 0$ and $\mathcal{E}$ is not stable. $\square$

Corollary 7. *There is no stable rank 2 bundle with spectrum $\mathcal{X}_5^{10}$, which is the cohomology of a positive minimal monad of the form*

$$\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 3} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 3}.$$

Proof: Take $b = 1$, $a = 2$ in Lemma 6. □

Proposition 8. *Let a_2, a_1, b be integers such that $a_2 \geq b > a_1 \geq 0$ and $2b - a_2 \geq 0$. The vector bundle $\mathcal{E}$ that is the cohomology of the monad*

$$\mathcal{O}_{\mathbb{P}^3}(-a_2 - 1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a_1 - 1)^{\oplus g} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 2} \oplus B \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a_2) \oplus \mathcal{O}_{\mathbb{P}^3}(a_1)^{\oplus g}$$

is not stable, where B is the sum of $(2g+2)$ line bundles.

Proof: The display of the monad implies $H^0(\ker \beta(-p)) \simeq H^0(\mathcal{E}(-p))$ whenever $p \geq 0$. By the minimality of the monad, we have:

$$\beta = \begin{pmatrix} \psi & \phi \\ 0 & \eta \end{pmatrix},$$

where $\psi\colon \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(a_2)$, $\phi\colon B \to \mathcal{O}_{\mathbb{P}^3}(a_2)$, and $\eta\colon B \to \mathcal{O}_{\mathbb{P}^3}(a_1)^{\oplus g}$. Since $\ker \psi$ is a reflexive sheaf of rank 1, it follows that $\ker \psi \simeq \mathcal{O}_{\mathbb{P}^3}(k)$ for $k = 2b - c_1(\operatorname{Im} \psi) \geq 2b - a_2 \geq 0$. In addition, we get the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^3}(k) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 2} & \xrightarrow{\quad\psi\quad} & \mathcal{O}_{\mathbb{P}^3}(a_2) \\ \big\downarrow{\scriptstyle s} & & \big\downarrow & & \big\downarrow \\ \ker \beta & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(b)^{\oplus 2} \oplus B & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(a_2) \oplus \mathcal{O}_{\mathbb{P}^3}(a_1)^{\oplus g} \end{array}$$

Thus $s \in H^0(\ker \beta(-k))$ with $k = 2b - c_1(\operatorname{Im} \psi) \geq 2b - a_2 \geq 0$. Therefore, $s \in H^0(\mathcal{E}(-k))$ for some $k \geq 0$ and so $\mathcal{E}$ cannot be stable. □

Corollary 9. *There is no stable rank 2 bundle with spectrum $\mathcal{X}_7^{10}$, which is the cohomology of a positive minimal monad of the form*

$$\mathcal{O}_{\mathbb{P}^3}(-5) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2}$$

$$\xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3}(1).$$

Proof: Take $a_2 = 4$, $a_1 = 1$, $b = 2$, and $g = 1$ in Proposition 8. □

Lemma 10. *Let E, F be torsion-free semi-stable (stable) sheaves such that $\mu(E) > \mu(F)$ (resp. $\mu(E) \geq \mu(F)$). Then,* $\operatorname{Hom}(E, F) = 0$.

Proof: Let $\varphi\colon E \to F$ be a morphism in $\operatorname{Hom}(E, F)$. If this morphism is injective, then $\mu(E) \leq \mu(F)$ (resp. $\mu(E) < \mu(F)$), which is a contradiction. If φ is not injective, we consider the commutative diagram

$$\begin{array}{ccccc} \ker \varphi & \longrightarrow & E & \xrightarrow{\quad\varphi\quad} & F \\ & & \searrow & & \nearrow \\ & & & \operatorname{Im} \varphi & \end{array}$$

and we get $\mu(E) \leq \mu(\operatorname{Im} \varphi) \leq \mu(F)$ (resp. $\mu(E) < \mu(\operatorname{Im} \varphi) < \mu(F)$), which is also a contradiction. Thus, $\operatorname{Hom}(E, F) = 0$. □

Proposition 11. *The cohomology of a monad of the form*

$$\mathcal{O}_{\mathbb{P}^3}(-4)\oplus\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2}\xrightarrow{\alpha}\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}\oplus B\xrightarrow{\beta}\mathcal{O}_{\mathbb{P}^3}(3)\oplus\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$$

with $B=\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(-1)\oplus\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3}$ *is not a stable bundle.*

Proof: Let $\mathcal{E}$ be the cohomology of this monad and $K=\ker\beta$. Suppose that the vector bundle $\mathcal{E}$ is stable. By the minimality of the monad, we can write

$$\beta=\begin{pmatrix}\psi & \eta\\ 0 & \phi\end{pmatrix},$$

where $\psi\colon\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}\to\mathcal{O}_{\mathbb{P}^3}(3)$. If we denote $P:=\ker\psi$, then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3} & \xrightarrow{\psi} & \mathcal{O}_{\mathbb{P}^3}(3) & \longrightarrow & \operatorname{coker}\psi \longrightarrow 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}\oplus B & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(3)\oplus\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & B & \xrightarrow{\phi} & \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} & \longrightarrow & 0
\end{array}$$

Note that P is a rank 2 reflexive sheaf. From the top line in the previous diagram, we get that $c_1(P)=c_1(\operatorname{coker}\psi)\geq 0$; since $\psi=(\psi_1,\psi_2,\psi_3)$ consists of three quadrics, we know that $c_1(\operatorname{coker}\psi)=\deg(h)$, where $h=\gcd(\psi_1,\psi_2,\psi_3)$, see for instance [**9**, Lemma 4]; note that $\deg(h)=0,1,2$. In addition, $h^0(P(k))=0$ when $k\leq -2$.

Next, consider the composed map $\tau\colon P\to K\to\mathcal{E}$; since $c_1(P)\geq 0>c_1(\mathcal{E})=-1$, this map cannot be injective. If $\tau\neq 0$, then $\ker(\tau)\simeq\mathcal{O}_{\mathbb{P}^3}(t)$ for some $t\leq 1$ (since $h^0(P(-2))=0$); in addition, $\operatorname{Im}(\tau)$, being a subsheaf of $\mathcal{E}$, is a torsion-free sheaf of rank 1, so $\operatorname{Im}(\tau)=\mathcal{I}_Y(c_1(P)-t)$ for some 1-dimensional subscheme $Y\subset\mathbb{P}^3$. Since $\mathcal{E}$ is stable, it follows that $c_1(P)-t\leq -1$ so $t\geq c_1(P)+1\geq 1$, thus $t=1$. Therefore, $\ker(\tau)=\mathcal{O}_{\mathbb{P}^3}(1)$ and, since $\mu(\operatorname{Im}(\tau))=c_1(P)-1<-1/2=\mu(\mathcal{E})$, we get $c_1(P)\leq 0$ so $c_1(P)=0$.

We then obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & P & \longrightarrow & \mathcal{I}_Y(-1) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & \overset{\tau}{\searrow} & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-4)\oplus\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} & \longrightarrow & K & \longrightarrow & \mathcal{E} & \longrightarrow & 0
\end{array}$$

But the morphism in the leftmost vertical arrow must vanish, thus contradicting the assumption that $\tau\neq 0$.

If $\tau=0$, we obtain the diagram

$$\begin{array}{ccccc}
 & & P & & \\
 & \overset{\sigma}{\swarrow} & \downarrow & \overset{0}{\searrow} & \\
\mathcal{O}_{\mathbb{P}^3}(-4)\oplus\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} & \longrightarrow & K & \longrightarrow & \mathcal{E}
\end{array}$$

for some $\sigma\in\operatorname{Hom}(P,\mathcal{O}_{\mathbb{P}^3}(-4)\oplus\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2})$. However,

$$\operatorname{Hom}(P,\mathcal{O}_{\mathbb{P}^3}(-4)\oplus\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2})\simeq H^0(P(-4))\oplus H^0(P(-2))^{\oplus 2}=0,$$

with the isomorphism coming from the fact that P is a rank 2 reflexive sheaf with $c_1(P)=0$. We conclude that the map $P\to K$ must vanish, contradicting the fact that it is injective. □

4. Existence of positive minimal monads

This section aims to establish the existence of positive minimal Horrocks monads whose cohomology is a stable rank 2 bundle on $\mathbb{P}^3$ with $c_2 = 10$, $c_1 = -1$. To this end, we recall the following result (see [**10**, Lemma 6]), which allows us to construct minimal monads for higher Chern classes.

Lemma 12. *Let $(\mathcal{E}, \sigma)$ be a pair consisting of a stable rank 2 vector bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2n$ and a section $\sigma \in H^0(\mathcal{E}(r))$ with $r > 0$ such that $X := (\sigma)_0$ is a curve. If C is a complete intersection curve of type (u, v) disjoint from X satisfying $u + v = 2r - 1$, then there is a pair $(\mathcal{E}', \sigma')$ consisting of a stable rank 2 vector bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2n + uv$, and a section $\sigma' \in H^0(\mathcal{E}'(r))$ such that $(\sigma')_0 = X \sqcup C$. Moreover, if $\mathcal{E}$ is the cohomology of a minimal monad of the form*

$$\mathbf{M}\colon \mathcal{C} \longrightarrow \mathcal{B} \longrightarrow \mathcal{A},$$

then $\mathcal{E}'$ is the cohomology of a minimal monad of the form

$$\mathbf{M}'\colon \mathcal{O}_{\mathbb{P}^3}(-r) \oplus \mathcal{C} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(r-1-u) \oplus \mathcal{O}_{\mathbb{P}^3}(r-1-v) \oplus \mathcal{B} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(r-1) \oplus \mathcal{A}.$$

In order to apply Lemma 12 we tabulate in Table 3 all the positive minimal Horrocks monads whose cohomology is a stable rank 2 bundle on $\mathbb{P}^3$ with $c_2 \leq 8$ as listed in [**10**].

Spectrum	$\boldsymbol{b}$	$\boldsymbol{a}$	Label
$\mathcal{X}_1^2 = \{r_0\}$	$(0,0)$	(1)	$\mathbf{M}_1$
$\mathcal{X}_1^4 = \{r_0^2\}$	$(0,0,0)$	$(1,1)$	$\mathbf{M}_2$
$\mathcal{X}_2^4 = \{r_0 r_1\}$	$(0,1)$	(2)	$\mathbf{M}_3$
$\mathcal{X}_1^6 = \{r_0^3\}$	$(0,0,0,0)$	$(1,1,1)$	$\mathbf{M}_4$
$\mathcal{X}_2^6 = \{r_0^2 r_1\}$	$(0,0)$ $(0,0,1)$	(2) $(1,2)$	$\mathbf{M}_5$ $\mathbf{M}_6$
$\mathcal{X}_3^6 = \{r_0 r_1^2\}$	$(1,1,1)$	$(2,2)$	$\mathbf{M}_7$
$\mathcal{X}_4^6 = \{r_0 r_1 r_2\}$	$(0,2)$	(3)	$\mathbf{M}_8$
$\mathcal{X}_1^8 = \{r_0^4\}$	$(0,0,0,0,0)$	$(1,1,1,1)$	$\mathbf{M}_9$
$\mathcal{X}_2^8 = \{r_0^3 r_1\}$	$(0,0,0)$ $(0,0,0,1)$	$(1,2)$ $(1,1,2)$	$\mathbf{M}_{10}$ $\mathbf{M}_{11}$
$\mathcal{X}_3^8 = \{r_0^2 r_1^2\}$	$(0,1,1)$ $(0,1,1,1)$	$(2,2)$ $(1,2,2)$	$\mathbf{M}_{12}$ $\mathbf{M}_{13}$
$\mathcal{X}_4^8 = \{r_0^2 r_1 r_2\}$	$(0,0,2)$	$(1,3)$	$\mathbf{M}_{14}$
$\mathcal{X}_5^8 = \{r_0 r_1^2 r_2\}$	$(1,1)$ $(1,1,2)$	(3) $(2,3)$	$\mathbf{M}_{15}$ $\mathbf{M}_{16}$
$\mathcal{X}_7^8 = \{r_0 r_1 r_2 r_3\}$	$(0,3)$	(4)	$\mathbf{M}_{17}$

TABLE 3. Positive minimal Horrocks monads for $c_2 \leq 8$; see [**10**].

In Table 4 we tabulate all positive minimal Horrocks monads whose cohomology is a stable bundle on $\mathbb{P}^3$ with $c_1 = -1$ and $c_2 = 10$. Most of them are constructed via Lemma 12, and we establish the following notation in the table:

- The fourth column indicates the degree to which the global section exists for constructing the vector bundle $\mathcal{E}$, giving a curve Y as a zero scheme.
- The last column describes how we construct the curve X by applying Lemma 12; here, P_n denotes a plane curve of degree n while $C_{a,b}$ denotes a curve of bidegree (a,b) on a smooth quadric.
- Two monads are obtained via an explicit computation in Proposition 13.
- Two monads are obtained by applying Theorem 15, using the curve given; $X \cup_p Y$ and $X \cup_{2p} Y$ mean the union of two curves intersecting at one or two points, respectively.

Spectrum	$\boldsymbol{b}$	$\boldsymbol{a}$	r	Construction
$\mathcal{X}_1^{10}$	$(0,0,0,0,0,0)$	$(1,1,1,1,1)$	1	disjoint union of 6 conics
$\mathcal{X}_2^{10}$	$(0,0,0,0)$	$(1,1,2)$	3	$\mathbf{M}_2,(2,3)$
	$(0,0,0,0,1)$	$(1,1,1,2)$	3	$\mathbf{M}_4,(1,4)$
$\mathcal{X}_3^{10}$	$(0,0,1)$	$(2,2)$	3	$\mathbf{M}_3,(2,3)$
	$(0,0,1,1)$	$(1,2,2)$	3	$\mathbf{M}_6,(1,4)$
	$(0,0,1,1,1)$	$(1,1,2,2)$	2	$\mathbf{M}_{13},(1,2)$
$\mathcal{X}_4^{10}$	$(0,1)$	(3)	2	Ein
	$(0,1,1)$	$(1,3)$	2	$\mathbf{M}_{15},(1,2)$
	$\boxed{(\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{1})}$	$\boxed{(\mathbf{1},\mathbf{1},\mathbf{3})}$	-	does not exist
	$(0,0,0,2)$	$(1,1,3)$	4	$\mathbf{M}_2,(1,6)$
$\mathcal{X}_5^{10}$	$(1,1,1,1)$	$(2,2,2)$	3	$\mathbf{M}_7,(1,4)$
$\mathcal{X}_6^{10}$	$(0,1)$	(3)	2	$C_{3,3}$
	$(0,1,1)$	$(1,3)$	2	$\mathbf{M}_{15},(1,2)$
	$(0,1,2)$	$(2,3)$	3	$\mathbf{M}_8,(1,4)$
	$(0,1,1,2)$	$(1,2,3)$	2	$\mathbf{M}_{16},(1,2)$
$\mathcal{X}_7^{10}$	$\boxed{(\mathbf{0},\mathbf{2},\mathbf{2})}$	$\boxed{(\mathbf{1},\mathbf{4})}$	-	does not exist
	$(0,0,3)$	$(1,4)$	2	$\mathbf{M}_{17},(1,2)$
$\mathcal{X}_9^{10}$	$(0,1,2)$	$(2,3)$	3	$\mathbf{M}_8,(1,4)$
	$(0,1,2,2)$	$(2,2,3)$	1	Theorem 15, $P_2 \cup_p P_3$
$\mathcal{X}_{10}^{10}$	$(1,2,2)$	$(3,3)$	1	Proposition 13
	$(1,2,2,2)$	$(2,3,3)$	1	Theorem 15, $P_2 \cup_{2p} P_3$
$\mathcal{X}_{11}^{10}$	$(1,1,3)$	$(2,4)$	1	Proposition 13
$\mathcal{X}_{10}^{12}$	$(0,4)$	(5)	1	P_5

Table 4. Positive minimal Horrocks monads for $c_2 = 10$. It was shown in Corollary 9 and Proposition 11 that if the boxed possible monads in bold exist, then their cohomologies are not stable bundles.

Proposition 13. *The minimal monads with* $\boldsymbol{b} = (1,1,3)$, $\boldsymbol{a} = (2,4)$ *and* $\boldsymbol{b} = (1,2,2)$, $\boldsymbol{a} = (3,3)$ *exist.*

Proof: To prove the existence of the monad with $\boldsymbol{b} = (1,1,3)$ and $\boldsymbol{a} = (2,4)$ it is sufficient to consider the maps:

$$\beta = \begin{pmatrix} y & 0 & w^3 & z^6 & 0 & x^8 \\ 0 & w & y & w^4 & x^4 & z^6 \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} -x^8 & -z^6 \\ -w^6 & -x^4 - yw^3 \\ -z^6 & 0 \\ w^3 & y \\ 0 & w \\ y & 0 \end{pmatrix},$$

where α is injective and β is surjective, satisfying $\beta \circ \alpha = 0$. Now for the other possible monad $\boldsymbol{b} = (1,2,2)$, $\boldsymbol{a} = (3,3)$, we take

$$\beta = \begin{pmatrix} w & z & x^2 & 0 & x^6 & y^6 \\ z & x & w^2 & w^5 & y^6 & 0 \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} y^6 - xzw^4 & -x^2w^4 \\ xw^5 - x^3zw^2 & y^6 - x^4w^2 \\ x^4z + xz^2w^2 & x^5 + x^2zw^2 + w^5 \\ -x^2 & -w^2 \\ -z & -x \\ -w & -z \end{pmatrix}.$$

It is easy to see that α is injective and β is surjective such that $\beta \circ \alpha = 0$. Therefore, the positive minimal Horrocks monad

$$\mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2} \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(3)^{\oplus 2}$$

indeed exists. □

Next, let X be a locally complete intersection curve in $\mathbb{P}^3$ of degree d which is a union of irreducible non-singular curves meeting quasi-transversely. From [**13**, Proposition 2.8], the sheaf $\mathcal{N}_X \otimes \omega_X(3)$ admits a general section that is nowhere vanishing, where $\mathcal{N}_X$ denotes the normal sheaf of X, and by considering Y to be the multiplicity 2 structure on X defined by the associated exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{I}_X \longrightarrow \omega_X(3) \longrightarrow 0, \tag{4.1}$$

it follows from Ferrand's theorem [**11**, Theorem 1.5] that $\omega_Y = \mathcal{O}_Y(-3)$. By Theorem 1 there is a stable rank 2 bundle $\mathcal{E}(1)$ on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 2d$ such that the curve $(Y, \mathcal{O}_Y)$ is given as a zero set of a section s of $H^0(Y, \mathcal{E}(1))$, and this section s induces the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \longrightarrow 0. \tag{4.2}$$

Since $c_1(\mathcal{E}(1)) = 1$, if $X \neq \emptyset$, then the curve Y of even degree does not lie on a hyperplane, so the vector bundle $\mathcal{E}$ is stable. In this manner, we prove the following.

Proposition 14. *Let X be a locally complete intersection curve in $\mathbb{P}^3$ of degree d, which is a union of irreducible, non-singular curves meeting quasi-transversely, where each connected component C has $H^0(C,\mathcal{O}_C) \simeq k$. Then there is a stable rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^3$ such that $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2d$ equipped with a section $\sigma \in H^0(\mathcal{E}(-1))$ whose zero scheme is a multiplicity 2 structure on X.*

We close this section with the construction of the last two cases left in Table 4.

Theorem 15. *The possible positive minimal Horrocks monads $\mathbf{M}$ and $\mathbf{M}'$ with $\boldsymbol{b} = (0,1,2,2)$, $\boldsymbol{a} = (2,2,3)$ and $\boldsymbol{b}' = (1,2,2,2)$, $\boldsymbol{a}' = (2,3,3)$, respectively, exist.*

Proof: Let X be a locally complete intersection curve on $\mathbb{P}^3$ that is a union of irreducible non-singular curves meeting quasi-transversely. From exact sequences (4.1) and (4.2) we get the following long exact sequence in cohomology

$$0 \longrightarrow H^0(\omega_X(l+3)) \longrightarrow H^1(\mathcal{E}(l)) \longrightarrow H^1(\mathcal{I}_X(l)) \longrightarrow H^1(\omega_X(l+3)) \longrightarrow \cdots \tag{4.3}$$

for each integer $l \le -1$, which implies

$$H^1(\mathcal{E}(l)) \simeq H^0(\omega_X(l+3)), \quad l \le -1.$$

If $X = X_1 \cup X_2$ with $S = X_1 \cap X_2$ containing r collinear points, then we have (see [**13**, Lemma 2.7]) the exact sequence

$$0 \longrightarrow \omega_{X_1} \oplus \omega_{X_2} \longrightarrow \omega_X \longrightarrow \omega_S \longrightarrow 0$$

that implies

$$h^0(\omega_X(m)) = h^0(\omega_{X_1}(m)) + h^0(\omega_{X_2}(m)) + h^0(\omega_S(m)) - \operatorname{rk}(\delta), \tag{4.4}$$

where δ is the connecting homomorphism

$$\delta\colon H^0(\omega_S(m)) \longrightarrow H^1(\omega_{X_1}(m)) \oplus H^1(\omega_{X_2}(m)),$$

such that

$$\operatorname{rk}(\delta) = \begin{cases} 0 & \text{if } m > 0, \\ 1-m & \text{if } 2-r \le m \le 0, \\ r & \text{if } m \le 1-r. \end{cases}$$

If we consider the curve $X = P_2 \cup P_3$, a union of two smooth plane curves P_2 and P_3 of degree 2 and 3, respectively, joined by one point x_0, then $S = \{x_0\}$ and we have

$$\omega_{P_2} = \mathcal{O}_{P_2}(-1), \omega_{P_3} = \mathcal{O}_{P_3}, \text{ and } h^0(\omega_S(m)) = 1, \quad m \in \mathbb{Z}.$$

By Proposition 14 we get a stable rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^3$ such that $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 10$. From exact sequence (4.3) and equation (4.4)

$$\begin{aligned} &h^1(\mathcal{E}(-1)) = h^0(\omega_X(2)) = h^0(\mathcal{O}_{P_2}(1)) + h^0(\mathcal{O}_{P_3}(2)) + h^0(\omega_S(2)) - 0 = 10, \\ &h^1(\mathcal{E}(-2)) = h^0(\omega_X(1)) = h^0(\mathcal{O}_{P_2}) + h^0(\mathcal{O}_{P_3}(1)) + h^0(\omega_S(1)) - 0 = 5, \\ &h^1(\mathcal{E}(-3)) = h^0(\omega_X) = h^0(\mathcal{O}_{P_3}) + h^0(\omega_S) - 1 = 1, \\ &h^1(\mathcal{E}(-k)) = 0, \quad \forall k \ge 4. \end{aligned}$$

Therefore, the stable rank 2 bundle $\mathcal{E}$ has spectrum $\mathcal{X} = \{r_0 r_1^3 r_2\} = \mathcal{X}_9^{10}$. Now, we computer the number of minimal generators:

$$\begin{aligned} &\rho(-3) = h^0(\mathcal{E}(-3)) = 1, \\ &\rho(-k) = 0, \quad k \ge 4. \end{aligned}$$

To find $\rho(-1)$ we can observe that

$$H^1(\mathcal{E}(-2)) \simeq H^0(\omega_X(1)) = H^0(\mathcal{O}_{P_2}) \oplus H^0(\mathcal{O}_{P_3}(1)) \oplus H^0(\omega_S(1)),$$
$$H^1(\mathcal{E}(-1)) \simeq H^0(\omega_X(2)) = H^0(\mathcal{O}_{P_2}(1)) \oplus H^0(\mathcal{O}_{P_3}(2)) \oplus H^0(\omega_S(2)).$$

With these descriptions of $H^1(\mathcal{E}(-2))$ and $H^1(\mathcal{E}(-1))$ it follows that the morphism $H^1(\mathcal{E}(-2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^1(\mathcal{E}(-2))$ is surjective, hence $\rho(-1) = 0$. To compute $\rho(-2)$ we look at the morphism δ with $m = 0$, which has rank 1. This means δ injective and thus

$$H^1(\mathcal{E}(-3)) \simeq H^0(\omega_X) \simeq H^0(\mathcal{O}_{P_3}).$$

On the other hand,

$$H^1(\mathcal{E}(-2)) \simeq H^0(\mathcal{O}_{P_2}) \oplus H^0(\mathcal{O}_{P_3}(1)) \oplus H^0(\omega_S(1)).$$

Therefore,

$$\rho(-2) = \dim[\operatorname{coker}(H^1(\mathcal{E}(-3)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^1(\mathcal{E}(-2)))] = 2.$$

This is sufficient to conclude the existence of the minimal positive Horrocks monad $\mathbf{M}$ whose cohomology is the vector bundle $\mathcal{E}$.

To show the existence of the second possible minimal monad, we take $X = P_2 \cup P_3$, a union of two smooth plane curves P_2 and P_3 of degree 2 and 3, respectively, joined by two points x_0, x_1, then $S = \{x_0, x_1\}$ and consequently $h^0(\omega_S(m)) = 2$, $m \in \mathbb{Z}$. From the Hartshorne–Serre correspondence, c.f. Proposition 14, there is a rank 2 bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = -1$, $c_2(\mathcal{F}) = 10$. Furthermore,

$$\begin{aligned} H^1(\mathcal{F}(-3)) &\simeq H^0(\omega_X) \simeq H^0(\mathcal{O}_{P_3}) \oplus H^0(k(x_0)), \\ H^1(\mathcal{F}(-2)) &\simeq H^0(\mathcal{O}_{P_2}) \oplus H^0(\mathcal{O}_{P_3}(1)) \oplus H^0(\omega_S(1)), \end{aligned} \tag{4.5}$$

where $k(x_0)$ is the skyscraper sheaf supported at x_0 (it could also be the skyscraper sheaf supported at x_1). By repeating the above arguments

$$h^1(\mathcal{F}(-1)) = 11,\ h^1(\mathcal{F}(-2)) = 6,\ h^1(\mathcal{F}(-3)) = 2,\ h^1(\mathcal{F}(-k)) = 0, \quad \forall k \geq 4,$$

which implies $\mathcal{X}(\mathcal{F}) = \{r_0 r_1^2 r_2^2\} = \mathcal{X}_{10}^{10}$. It is easy to see that

$$\rho(-3) = 2,\ \rho(-1) = 0,\ \text{and}\ \rho(-k) = 0, \quad k \geq 4.$$

Revisiting the isomorphisms in (4.5), we compute the generators of minimal degree 2, obtaining $\rho(-2) = 1$. Then, the positive minimal Horrocks monad $\mathbf{M}'$ exists and its cohomology is the vector bundle $\mathcal{F}$. □

5. Negative minimal Horrocks monads

The main goal of this section is to prove that, for each $n \geq 4$, the integral sequence $\{-2^{n-1}, -1, 0, 1^{n-1}\}$ can be realized as the spectrum of a stable rank 2 bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 2n$ which necessarily is given as the cohomology of a negative minimal Horrocks monad, which we explicitly display. In addition to the spectrum $\mathcal{X}_8^{10} = \{r_0 r_1^4\}$ obtained by considering $n = 5$, we also look for spectra of length 10 that can be realized as the spectrum of a stable rank 2 bundle given as cohomology of a negative minimal monad, namely: $\mathcal{X}_5^{10} = \{r_0^2 r_1^3\}$ and $\mathcal{X}_9^{10} = \{r_0 r_1^3 r_2\}$.

Before this, we thank Nicolae Manolache for alerting us, in private communication, to a mistake we made in our previous article, namely [**10**, Proposition 17]. In fact, the sequence $\mathcal{X}_6^8 = \{-2^3, -1, 0, 1^3\}$ *is* realized as the spectrum of a stable rank 2 bundle on $\mathbb{P}^3$ and the question proposed by Hartshorne ([**13**, (Q2), p. 806]) continues without fail cases for stable rank 2 bundles on $\mathbb{P}^3$ with odd determinant; on the other hand, Coandă recently showed in [**6**, Proposition 3.12 and Example 3.13] that the question in [**13**, (Q2), p. 806] has a negative answer.

Our starting point is the following proposition, whose proof is an adaptation of the arguments of [**13**, Lemma 2.12].

Proposition 16. *For each $n \geq 4$, the sequence $\mathcal{X}_n = \{-2^{n-1}, -1, 0, 1^{n-1}\}$ is realized as the spectrum of a stable rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2n$.*

Proof: Let X be a divisor of type $(1, n-1)$ on a smooth quadric Q in $\mathbb{P}^3$; note that X is a rational curve of degree n. Let Y be a double structure on X given by an exact sequence of the form

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{I}_X \longrightarrow \omega_X(3) \longrightarrow 0,$$

and take $\mathcal{E}$ to be given by an extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Since $h^0(\mathcal{I}_Y) = 0$, we conclude that $\mathcal{E}$ is stable. Note that $h^0(\mathcal{I}_X(-l)) = h^1(\mathcal{I}_X(-l)) = 0$ for $l \geq 1$; from the exact sequences above, we get

$$H^1(\mathcal{E}(-l)) \simeq H^1(\mathcal{I}_Y(-l)) \simeq H^0(\omega_X(-l+3))$$

on the same range. On the other hand, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_Q(-2,-2) \longrightarrow \mathcal{O}_Q(-1, n-3) \longrightarrow \omega_X \longrightarrow 0.$$

In this manner,

$$h^1(\mathcal{E}(-l)) = h^0(\omega_X(-l+3)) = \begin{cases} 0, & l > 2, \\ n-1, & l = 2. \end{cases}$$

In particular,

$$n - 1 = h^1(\mathcal{E}(-2)) - h^1(\mathcal{E}(-3)) = \#\{k_j \in \mathcal{X} \mid k_j \geq 1\}.$$

Since $0, -1 \in \mathcal{X}(\mathcal{E})$ and $c_2(\mathcal{E}) = \deg(Y) = 2n$ it follows that $\mathcal{X}(\mathcal{E}) = \mathcal{X}_n$. □

Our next step is to explicitly figure out a minimal monad whose cohomology is a rank 2 bundle as constructed in Proposition 16. First, we need two technical lemmas.

Lemma 17. *If C is a rational curve of degree $n \geq 2$ and $h^1(\mathcal{I}_C(1)) = 0$, then the evaluation morphism*

$$\rho\colon H^0(\omega_C(r)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(\omega_C(r+1))$$

is surjective, for $r \geq 1$.

Proof: Since C is a rational curve of degree n, we have isomorphisms

$$H^0(\omega_C(p)) \simeq H^0(\mathcal{O}_C((np-2)\mathrm{pt})) \text{ for each } p \in \mathbb{Z}.$$

On the other hand, since the curve C satisfies $h^1(\mathcal{I}_C(1)) = 0$, the map $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_C(1))$ is surjective. Therefore, we get the commutative diagram

$$\begin{array}{ccc} H^0(\omega_C(r)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) & \xrightarrow{\quad\rho\quad} & H^0(\omega_C(r+1)) \\ \downarrow\!\!\downarrow & & \downarrow\wr \\ H^0(\mathcal{O}_C((nr-2)\mathrm{pt})) \otimes H^0(\mathcal{O}_C(1)) & \xrightarrow{\ \mu\ }\!\!\!\!\rightarrow & H^0(\mathcal{O}_C((n(r+1)-2)\mathrm{pt})) \end{array}$$

Note that the map μ is surjective because C is a rational curve; it then follows from the diagram that ρ is also surjective. □

Next, let C be a complete intersection of a non-singular quadric hypersurface $Q \subset \mathbb{P}^3$ with a hypersurface of degree a; given another integer $b > a$, let $\{L_1, \dots, L_{b-a}\}$ be mutually disjoint lines in the same family of lines in Q. Setting $X := C \cup L_1 \cup \cdots \cup L_{b-a}$, we note that this is a curve of type (a, b) in Q.

Lemma 18. *Let X be a curve as constructed above with $a = 1$ and $b = n > 1$. Then the map*

$$\eta\colon H^0(\omega_X(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(\omega_X(k+1))$$

is surjective for $k \geq 1$.

Proof: Consider the exact sequence

$$0 \longrightarrow \omega_C \longrightarrow \omega_X \longrightarrow \bigoplus_{i=1}^{n-1} \omega_{L_i}(1) \longrightarrow 0.$$

One can check that $H^1(\omega_C(k)) = 0$ for $k \geq 1$, and this implies the surjectivity of the morphism $H^0(\omega_X(k)) \to \bigoplus_{i=1}^{n-1} H^0(\omega_{L_i}(k+1))$. We therefore have the commutative diagram

$$\begin{array}{ccccc} H^0(\omega_C(k)) \otimes V & \longrightarrow & H^0(\omega_X(k)) \otimes V & \twoheadrightarrow & \left(\bigoplus_{i=1}^{n-1} H^0(\omega_{L_i}(k+1))\right) \otimes V \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \psi} \\ H^0(\omega_C(k+1)) & \longrightarrow & H^0(\omega_X(k+1)) & \twoheadrightarrow & \bigoplus_{i=1}^{n-1} H^0(\omega_{L_i}(k+2)) \end{array}$$

where $V = H^0(\mathcal{O}_{\mathbb{P}^3}(1))$; since both C and each L_i are ACM rational curves, then Lemma 17 implies that φ and ψ are surjective when $k \geq 0$. It follows that η is also surjective. $\square$

Theorem 19. *Let $\mathcal{E}$ be a rank 2 bundle as constructed in Proposition 16 from a curve X as in Lemma 18. Then $\mathcal{E}$ is the cohomology of a negative minimal monad of the form*

$$\mathcal{A}^\vee(-1) \longrightarrow \mathcal{B} \xrightarrow{\beta} \mathcal{A},$$

where $\mathcal{A} = \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus(n-3)} \oplus \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus(n-1)}$ and $\mathcal{B} = \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus(2n-3)} \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus(2n-3)}$.

Proof: Let us compute the minimal generators of the Rao module of $\mathcal{E}$. From Theorem 4 it follows that $\rho(-2) = n-1$ and $\rho(-j) = 0$, for $j = 1$ and $j \geq 3$. Now, let us show that the natural map

$$\varphi\colon H^1(\mathcal{E}(-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^1(\mathcal{E})$$

is surjective and thus $\rho(0) = 0$. From the exact sequences (4.1) and (4.2), the surjectivity of φ is equivalent to the surjectivity of the map $H^1(\mathcal{I}_Y(-1)) \otimes H^0(O(1)) \to H^1(\mathcal{I}_Y)$ which is surjective if and only if the map

$$\rho\colon H^0(\omega_X(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(\omega_X(3))$$

is surjective, where X is a rational curve of type $(1, n-1)$. If we take $k = 2$ in Lemma 18, then it follows that the map ρ is surjective. Therefore, $\rho(0) = 0$ as desired.

On the other hand, with a slight adaptation of the results of [**6**, Section 2] for stable rank 2 bundles with odd determinants, we get

$$\rho(i) \leq \max(s(-i-1)-2, 0), \quad i \geq 1. \tag{5.1}$$

The above inequality is not true for $i = 0$. By applying the inequality (5.1) to the spectrum $\mathcal{X}_n$ we obtain

$$\rho(i) = 0,\ i \geq 2, \quad \text{and} \quad \rho(1) \leq n-3.$$

If $\rho(1) = t$, then equation (2.3) implies

$$2n = c_2 = (n-1)\cdot 2\cdot(2+1) - \sum_{i=1}^{n+t} b_i(b_i+1) \Longrightarrow \sum_{i=1}^{n+t} b_i(b_i+1) = 4n-6.$$

If $t < n-3$, then at least one of the integers b_i is greater than or equal to 2, which contradicts the minimality of the monad. Therefore, $t = n-3$ and $b_i = 1$, $i = 1, \ldots, 2n-3$, is the only solution of

$$\sum_{i=1}^{n+t} b_i(b_i+1) = 4n-6. \qquad \square$$

If we apply the formula (5.1) to each spectrum listed in Table 1, then the spectra that can be realized as the cohomology of a negative minimal monad are: $\mathcal{X}_5^{10} = \{r_0^2 r_1^3\}$, $\mathcal{X}_8^{10} = \{r_0 r_1^4\}$, and $\mathcal{X}_9^{10} = \{r_0 r_1^3 r_2\}$. The spectrum $\mathcal{X}_8^{10}$ was treated in Theorem 19 by taking $n = 5$, while the formula (5.1) applied to the spectrum $\mathcal{X}_5^{10}$ provides $\rho(1) \leq 1$ and $\rho(0) \leq 1$. In Table 5, we list all remaining possible negative minimal monads whose cohomology yields a stable rank 2 bundle with spectrum $\mathcal{X}_5^{10}$ or $\mathcal{X}_9^{10}$.

Spectrum	$\boldsymbol{b}$	$\boldsymbol{a}$
$\mathcal{X}_5^{10}$	$(0,1,1,1,1)$	$(-1,2,2,2)$
	$(0,1,1,1,1)$	$(0,2,2,2)$
	$(0,0,1,1,1,1)$	$(-1,0,2,2,2)$
	$(0,1,1,1,1,1)$	$(-1,1,2,2,2)$
	$(0,1,1,1,1,1)$	$(0,1,2,2,2)$
	$(0,0,1,1,1,1,1)$	$(-1,0,1,2,2,2)$
$\mathcal{X}_9^{10}$	$(0,0,1,2)$	$(-1,2,3)$
	$(1,1,1,1)$	$(-1,2,3)$
	$(0,0,1,2,2)$	$(-1,2,2,3)$
	$(1,1,1,1,2)$	$(-1,2,2,3)$

TABLE 5. Remaining possible negative minimal monads.

Since the main goal of this paper is to classify all positive minimal monads whose cohomology is a stable rank 2 bundle on $\mathbb{P}^3$ with $c_1 = -1$ and $c_2 = 10$, we will leave the study of these possible negative minimal monads here for future work.

6. Dimension of the moduli space

Three irreducible components of the moduli space $\mathcal{B}(-1,10)$ are known: the Hartshorne component and two Ein components, which we comment on in Theorem 20. In this section, we apply the semi-continuity of the dimension of the cohomology groups of coherent sheaves and the formula (6.1), whose proof can be found in [**10**, p. 23], to prove that there is at least one more new component in $\mathcal{B}(-1,10)$. Let us recall the notation and for more details see [**10**]. Let us recall the formula for computing the dimension of the set of isomorphism classes of stable rank 2 bundles on $\mathbb{P}^3$ with odd determinant, which is given as the cohomology of a homotopy-free monad.

Define $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ to be the family of minimal monads as in display (2.2), where $\boldsymbol{a} = (a_1,\dots,a_s) := (a_1^{r_1},\dots,a_k^{r_k})$ and $\boldsymbol{b} = (b_1,\dots,b_{s+1}) := (b_1^{t_1},\dots,b_w^{t_w})$; assume that $b_{s+1} < a_1$, so that every monad in $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ is homotopy-free. Now, we introduce the notation and recall the formula for the dimension of homotopy-free families of monads. Let $\mathcal{B} = \bigoplus_{j=1}^{s+1}(\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j-1))$, $\mathcal{A} = \bigoplus_{i=1}^{s}\mathcal{O}_{\mathbb{P}^3}(a_i)$, and

$$H := \{\alpha : \mathcal{A}^\vee(-1) \longrightarrow \mathcal{B} \mid \alpha \text{ is locally left-invertible}\}.$$

From Lemma 2,

$$\mathcal{P}(\boldsymbol{a};\boldsymbol{b}) = \{\alpha \in H \mid \alpha^\vee(-1)\circ\Omega\circ\alpha = 0\},$$

and so $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ is represented as a locally closed subset of $\mathrm{Hom}(\mathcal{A}^\vee(-1),\mathcal{B})$, which is an affine space. Since $\alpha^\vee(-1)\circ\Omega\circ\alpha$ is skew-symmetric as a morphism from $\mathcal{A}^\vee(-1)$ to $\mathcal{A}$, we conclude that

$$\dim\mathcal{P}(\boldsymbol{a};\boldsymbol{b}) = \dim H - \dim W,$$

where W denotes the subspace of skew-symmetric bilinear forms on $\mathcal{A}(-1)$. Now, we consider the following two groups:

$$G := \{\varphi \in \mathrm{End}(\mathcal{B}) \mid \varphi^\vee(-1)\circ\Omega\circ\varphi = \Omega\}, \text{ and}$$

$$\mathrm{GL}(\mathcal{A}) := \{u : \mathcal{A} \longrightarrow \mathcal{A} \mid u \text{ is an isomorphism}\}.$$

They act on $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ as follows:

$$(u,\varphi)\cdot\alpha = \varphi^{-1}\circ\alpha\circ u^\vee(-1), \text{ for } u \in \mathrm{GL}(\mathcal{A}), \text{ and } \varphi \in G.$$

The subgroup $\pm(\mathrm{id},\mathrm{id}) \subset (\mathrm{GL}(\mathcal{A})\times G)$ acts trivially on $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ and we set

$$G_0 := (\mathrm{GL}(\mathcal{A})\times G)/\pm(\mathrm{id},\mathrm{id}),$$

which acts freely on $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ (see Section 8 of [**10**]). It is not difficult to see that two monads in $\mathcal{P}(\boldsymbol{a};\boldsymbol{b})$ are isomorphic if and only if they belong to the same orbit of this action. Consequently, the cohomology bundles from monads in the same orbit are isomorphic, providing a map

$$\mathcal{P}(\boldsymbol{a};\boldsymbol{b})/G_0 \longrightarrow \mathcal{V}(\boldsymbol{a};\boldsymbol{b}).$$

We now argue that this is a bijection. Indeed, surjectivity comes from the very definition of the set $\mathcal{V}(\boldsymbol{a};\boldsymbol{b})$, while injectivity follows from the fact that monads in different orbits have non-isomorphic cohomology bundles. It follows that $\mathcal{V}(\boldsymbol{a};\boldsymbol{b})$ can be regarded as a quasi-projective variety parametrizing a family of (isomorphism classes of) stable rank 2 bundles. Therefore, we obtain an injective modular morphism

$$\Psi\colon \mathcal{V}(\boldsymbol{a};\boldsymbol{b}) \longrightarrow \mathcal{B}(-1,c_2),$$

where c_2 is given in terms of $(\boldsymbol{a};\boldsymbol{b})$ according to the formula in display (2.3).

$$\dim\mathcal{V}(\boldsymbol{a};\boldsymbol{b}) = \dim H - \dim W - \dim\mathrm{GL}(\mathcal{A}) - \dim G. \tag{6.1}$$

Below, we collect in Table 6 each family of homotopy-free monads and its respective dimension.

Spectrum	$\boldsymbol{b}$	$\boldsymbol{a}$	w	g	s	h	$\dim \mathcal{V}(\boldsymbol{a};\boldsymbol{b})$
$\mathcal{X}_1^{10}$	$(0,0,0,0,0,0)$	$(1,1,1,1,1)$	200	25	120	420	**75**
$\mathcal{X}_2^{10}$	$(0,0,0,0)$	$(1,1,2)$	90	13	56	232	73
$\mathcal{X}_3^{10}$	$(0,0,1)$	$(2,2)$	56	4	65	198	73
$\mathcal{X}_6^{10}$	$(0,1)$	(3)	0	1	40	121	80
$\mathcal{X}_5^{10}$	$(1,1,1,1)$	$(2,2,2)$	168	9	216	468	**75**
$\mathcal{X}_{10}^{10}$	$(1,2,2)$	$(3,3)$	120	4	271	484	89
$\mathcal{X}_{12}^{10}$	$(0,4)$	(5)	0	1	317	430	112

Table 6. Dimension of the families of homotopy-free Horrocks monads whose cohomology is a stable bundle with $c_1 = -1$ and $c_2 = 10$. We set $h := \dim H$, $w := \dim W$, $g := \dim \mathrm{GL}(\mathcal{A})$, and $s := \dim G = \dim S$. The families with dimensions equal to the expected one are marked in bold.

Hartshorne showed in [**11**, Section 4] that the cohomology bundles of monads in the family $\mathcal{V}(1^5;0^6)$ are generic points in an irreducible component of $\mathcal{B}(-1,10)$ with expected dimension 75. We denote this component by M_1.

In addition, Ein showed in [**7**] that the cohomology bundles of monads in the family $\mathcal{V}(3;0,1)$ and $\mathcal{V}(5;0,4)$ are generic points in two distinct irreducible components of $\mathcal{B}(-1,10)$ of dimensions 80 and 112, respectively. We denote these components by M_2 and M_3, respectively.

Our final result presents two new components of $\mathcal{B}(-1,10)$.

Theorem 20. *The moduli scheme $\mathcal{B}(-1,10)$ has at least five components:*

(1) *The* Hartshorne component M_1 *containing the family $\mathcal{V}(1^5;0^6)$ of expected dimension* 75*.*

(2) *Two* Ein components M_2 *and* M_3 *whose generic points correspond to elements of the family $\mathcal{V}(3;0,1)$ and $\mathcal{V}(5;0,4)$, respectively. These components have dimensions* 80 *and* 112*, respectively.*

(3) *Two new irreducible components M_4 and M_5 of dimension at least* 89 *and* 75*, respectively. The former contains the family $\mathcal{V}(3^2;2^2,1)$, while the latter contains the family $\mathcal{V}(2^3;1^4)$.*

Proof: As computed in Table 6, the family $\mathcal{V}(3^2;2^2,1)$ has dimension 89 and thus cannot be contained in the components M_1, M_2. On the other hand, if $\mathcal{F} \in \mathcal{V}(3^2;2^2,1)$ lies in the component M_3, then we must have $h^1(\mathcal{F}(-5)) \geq 1$ and by lower semicontinuity, since every Ein bundle $\mathcal{E}$ in the family $\mathcal{V}(5;0,4)$ has $h^1(\mathcal{E}(-5)) = 1$. Therefore, the family $\mathcal{V}(3^2;2^2,1)$ must lie in a different irreducible component of dimension larger than or equal to 89, which we denote by M_4.

Since $\dim \mathcal{V}(2^3;1^4) = 75$, the family $\mathcal{V}(2^3;1^4)$ cannot be contained in the Hartshorne component M_1. We will argue that this family cannot be contained in the other components described here.

Indeed, take $\mathcal{G} \in \mathcal{V}(2^3;1^4)$; it has spectrum $\mathcal{X}_5^{10}$, thus $h^1(\mathcal{G}(-3)) = 0$. On the other hand, a generic point $\mathcal{E}$ in the Ein component M_2 must have $h^1(\mathcal{E}(-3)) \geq 1$, while a generic point $\mathcal{E}$ in M_3 must have $h^1(\mathcal{E}(-3)) \geq 6$. Finally, a generic point $\mathcal{F}$ in the component M_4 must have $h^1(\mathcal{F}(-3)) \geq 2$, since it contains a family of bundles with spectrum $\mathcal{X}_{10}^{10}$ of dimension larger than $\dim \mathcal{V}(2^3; 1^4) = 75$. Therefore, the family $\mathcal{V}(2^3; 1^4)$ must lie in yet another irreducible component of $\mathcal{B}(-1, 10)$ of dimension at least 75. □

Remark 21. In Table 4, we see that there are two families of Ein bundles with different spectra that are cohomology of the Ein monads with $b = (0, 1)$ and $a = (3)$. By lower semi-continuity, a general point of the Ein component M_2 is a stable vector bundle with spectrum $\mathcal{X}_6^{10}$.

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