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ON A MIXED BOUNDARY VALUE PROBLEM ON BOUNDED DOMAINS OF $\mathbb{R}^3$

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Abstract: We are interested in a nonlinear Laplace equation with critical nonlinearity and mixed Dirichlet–Neumann boundary conditions on bounded domains of $\mathbb{R}^3$. Building on the analysis of [**1**] and on further developments of the theory of critical points at infinity [**6**], we study the lack of compactness of the associated variational problem and prove general existence theorems.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain such that $\partial\Omega$ is continuous-Lipschitz and is given by the union of two closed parts $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$, where Γ_0 and Γ_1 are disjoint smooth $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$ having positive Hausdorff measure. Let us take a function $K\colon \overline{\Omega} \to \mathbb{R}$. We are looking for a positive map $u\colon \Omega \to \mathbb{R}$ solving the critical nonlinear elliptic problem with mixed Dirichlet–Neumann boundary conditions

$$
(1.1) \qquad \begin{cases} -\Delta u = K|u|^{\frac{4}{n-2}}u & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases}
$$

where ν is the outward unit normal to the boundary part Γ_1.

Problems of this kind appear in the modeling of the boundary control of flows in domains Ω whose boundary $\partial\Omega$ is split into several parts which differ in physical properties; see [**25**] and [**26**]. They also appear in Kelvin–Voigt fluid models in the theory of viscoelastic fluids; see [**27**], [**28**].

Problem (1.1) has a variational structure. The difficulty arising when looking for positive solutions by variational methods is the presence of the critical exponent which generates a lack of compactness of the associated variational problem and therefore the occurrence of blow-up points, also called critical points at infinity. These points constitute obstacles in the search for solutions using standard variational techniques. These points, which were introduced by Bahri–Coron [**8**], are the ends of non-precompact flow lines of the gradient flow of the associated energy functional. Besides, the mixed boundary condition creates real additional difficulties when compared to the well-studied case where Γ_1 is empty; see [**8**], [**9**], [**11**], [**12**], [**14**], [**18**], [**31**], and references therein.

For $K = 1$ on $\overline{\Omega}$, Lions–Pacella–Tricarico ([**23**]) studied the minimizing sequences of the energy functional associated to (1.1). As a consequence, some existence results have been derived under some geometrical assumptions on Ω, Γ_0, and Γ_1. In [**21**], Grossi–Pacella considered problem (1.1) when $K = 1$ on $\overline{\Omega}$. In their paper, Grossi–Pacella imposed the following hypothesis on Γ_0 and Γ_1:

(H) Either $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ or $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ intersect orthogonally.

A complete study of positive sequences failing the Palais–Smale condition was established in [**21**] under condition (H). In particular an existence theorem was proved under further topological hypothesis on Ω, Γ_0, and Γ_1. The result of [**21**] is motivated by the celebrated paper of Coron [**18**] on the homogeneous Dirichlet problem. For more results on problem (1.1) for $K = 1$ on $\overline{\Omega}$, we refer to [**2**], [**17**], [**19**], [**20**], and [**29**].

Adimurthi–Mancini ([**1**]) studied problem (1.1) for $K = 1$ on $\overline{\Omega}$. They were able to prove an existence theorem under some conditions on the mean curvature of the boundary part Γ_1. They proved using suitable test functions that the infimum of the energy functional associated to problem (1.1) is below the first level at which the Palais–Smale condition is not satisfied, and by standard proofs of concentration-compactness type ([**14**]) they obtained the existence of positive solutions. The analysis of Adimurthi–Mancini shows that there is a qualitative difference in the study of problem (1.1) according to the dimensions n. Indeed, the asymptotic estimates of the test functions and related developments in dimension 3 differ from those in dimension $n \geq 4$. For more precision, see [**1**, Section 2].

In a recent paper [**4**], the authors studied problem (1.1) in dimensions $n \geq 4$. They studied the lack of compactness of the associated variational functional and established perturbation theorems through degree-type criteria. Their results require in particular that K is smooth, uniformly close to 1, and satisfies non-degeneracy conditions on its critical points.

In the present paper we are interested in problem (1.1) in dimension 3, and K is an arbitrary positive function on $\overline{\Omega}$. The first two theorems extend the existence result of Adimurthi–Mancini ([**1**]) to any positive and locally Lipschitz function K. More precisely, we use the test functions of [**1**] to prove a strict inequality between the first noncompact energy level and the infimum of the energy functional associated to problem (1.1). A standard concentration-compactness type argument allows us to prove general existence results. Namely,

Theorem 1.1. *Let $n = 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying condition* (H). *Let us take a positive and locally Lipschitz function K on $\overline{\Omega}$. Let y_0 be a point in $\overline{\Omega}$ such that*

$$K(y_0) = \max_{x \in \overline{\Omega}} K(x).$$

If $y_0 \in \Gamma_1$ and $\mathcal{M}(y_0) > 0$, then (1.1) *admits a positive solution. Here $\mathcal{M}(y_0)$ is the mean curvature of Γ_1 at y_0.*

Theorem 1.2. *Let Ω and K be as in Theorem 1.1. Let $z_0 \in \overline{\Gamma}_1$ be such that*

$$K(z_0) = \max_{x \in \overline{\Gamma}_1} K(x).$$

If $z_0 \in \Gamma_1$ and $\mathcal{M}(z_0) > 0$, then problem (1.1) *has a positive solution provided that K is uniformly close to* 1.

Observe that for $K = 1$ on Ω, the above two theorems are exactly the existence result of [**1**] in dimension 3. To our knowledge the results of Theorems 1.1 and 1.2 are of new type compared with previous existence results on related critical nonlinear problems involving nonconstant prescribed function K. Indeed, our results do not require any assumptions on the critical points of K. In particular, no differentiability condition is assumed on the function K.

While Theorem 1.1 is not a perturbation result, Theorem 1.2 is an interesting perturbation result since it is optimal for a certain class of domains in $\mathbb{R}^3$. Namely,

Theorem 1.3. *For any ball $\Omega \subset \mathbb{R}^3$ with a finite number of wholes such that Γ_1 describes the exterior sphere, problem* (1.1) *has a positive solution provided that K is uniformly close to* 1.

In the remainder of this paper we consider the case where the main curvature at any maximum z_0 of K on $\overline{\Gamma}_1$ is negative. For this we introduce the so-called isoperimetric constant relative to Γ_1, denoted $Q(\Gamma_1,\Omega)$ and defined by

$$Q(\Gamma_1,\Omega) = \sup_{E\in P(\Omega,\Gamma_1)} \frac{|E|^{\frac{2}{3}}}{P_\Omega(E)},$$

where $P(\Omega,\Gamma_1)$ is the set of measurable subsets E of Ω such that $\partial E \cap \Gamma_0$ does not contain any set having positive 2-dimensional Hausdorff measure, $|E|$ is the Lebesgue measure of E, and $P_\Omega(E)$ denotes the perimeter of E relative to Ω. In particular, if we denote by ω the measure of the unit ball of $\mathbb{R}^3$, then

$$Q(\Gamma_1,\Omega) \geq \frac{1}{3}\left(\frac{2}{\omega}\right)^{\frac{1}{3}}.$$

For more detail on $Q(\Gamma_1,\Omega)$ and related properties, we refer to [**30**].

Let $K|_{\Gamma_1}$ be the restriction of K on Γ_1. Denote

$$\mathrm{Crit}(K|_{\Gamma_1}) = \left\{ y \in \Gamma_1,\ \nabla(K|_{\Gamma_1})(y) = 0 \right\},$$

and let $\chi(\Gamma_1)$ be the Euler–Poincaré characteristic of Γ_1. We shall prove the following result:

Theorem 1.4. *Let Ω be a bounded domain of $\mathbb{R}^3$ with the following two properties:*

(i) $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$,

(ii) $Q(\Gamma_1,\Omega) = \frac{1}{3}\left(\frac{2}{\omega}\right)^{\frac{1}{3}}$.

Let $K\colon \overline{\Omega} \to \mathbb{R}$ be a C^2-function with the following three properties:

(iii) $K|_{\Gamma_1}$ *is a Morse function,*

(iv) $\mathcal{M}(y) \neq 0$, $\forall y \in \mathrm{Crit}(K|_{\Gamma_1})$,

(v) $$\sum_{y\in \mathrm{Crit}(K|_{\Gamma_1}),\, \mathcal{M}(y)<0} (-1)^{\mathrm{ind}(K|_{\Gamma_1},y)} \neq \chi(\Gamma_1),$$

where $\mathrm{ind}(K|_{\Gamma_1},y)$ *is the Morse index of $K|_{\Gamma_1}$ at y.*

Then problem (1.1) *has a positive solution provided that K is uniformly close to* 1.

As examples of domains satisfying condition (ii), we may consider the domains of $\mathbb{R}^3$ bounded by two concentric spheres with Γ_1 describing the interior sphere; see [**30**]. For other examples of domains, we refer to [**19**]. The proof of Theorem 1.4 relies on a careful analysis of the topological contributions of some critical points at infinity and a precise study of the change of topology of the sublevels of the associated energy functional near the first level at which a lack of compactness occurs. Prescribing the topology of Γ_1 by means of a Hopf–Poincaré counting index formula, we get the existence of positive solutions. For previous perturbation results like Theorem 1.4 on homogeneous boundary value problems, we refer the reader to [**3**], [**5**], [**15**], [**22**], and [**24**].

The rest of this paper is organized as follows. In Section 2 we recall some preliminaries related to the variational structure associated to problem (1.1). In Section 3 we expand the energy functional associated to (1.1) near critical points at infinity of p-masses, $p \geq 1$, and we determine the possible levels of the energy functional at which the lack of compactness occurs. In Section 4 we refine the expansions near critical points at infinity of only one mass and we provide precise characterization of the locations of these points as well as their energy levels. Section 5 of this paper is devoted to the proof of the existence results.

2. Variational framework

Here and in the following, we assume that $n = 3$. It is well known that problem (1.1) has a variational structure. The space of variation is

$$V(\Omega) = \{u \in H^1(\Omega), \text{ such that } u = 0 \text{ on } \Gamma_0\}.$$

For $u \in V(\Omega)$,

$$||u|| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}}$$

defines a norm on $V(\Omega)$, since the Hausdorff measure $H_2(\Gamma_0)$ is positive. Let

$$\Sigma = \{u \in V(\Omega), \, ||u|| = 1\} \quad \text{and} \quad \Sigma^+ = \{u \in \Sigma, \, u \geq 0\}.$$

We consider the functional

$$J(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\left(\displaystyle\int_\Omega K u^6 \, dx\right)^{\frac{1}{3}}},$$

whose critical points in Σ^+ are solutions of (1.1), up to a positive multiplicative constant.

It is well known that the exponent 6 is critical for the Sobolev embedding $V(\Omega) \hookrightarrow L^q(\Omega)$. For $q = 6$, the embedding is continuous and not compact. The energy functional J fails to satisfy the Palais–Smale condition on Σ^+. Based on the works [**21**] and [**23**], we describe in the following the sequences of Σ^+ which violate the Palais–Smale condition.

Let $a \in \Omega \cup \Gamma_1$. We define a C^∞-positive cut-off function $\varphi_a(x)$, $x \in \mathbb{R}^3$, by

$$\varphi_a(x) = 1, \text{ if } x \in B(a, \rho) \quad \text{and} \quad \varphi_a(x) = 0, \text{ if } x \in B(a, 2\rho)^c, \tag{2.1}$$

where $\rho = \rho(a)$ is positive chosen so that $B(a, 2\rho) \cap \overline{\Gamma}_0 = \emptyset$. The considered test functions are $U_{(a,\lambda)}$, $a \in \Omega \cup \Gamma_1$, and $\lambda > 0$, defined by

$$U_{(a,\lambda)}(x) = \varphi_a(x) \, \delta_{(a,\lambda)}(x), \quad x \in \Omega, \tag{2.2}$$

where $\delta_{(a,\lambda)}$ are the solutions of the Yamabe equation

$$-\Delta u = u^5, \quad u > 0 \text{ in } \mathbb{R}^3.$$

Up to the multiplicative constant $c_0 = 3^{\frac{1}{4}}$, $\delta_{(a,\lambda)}$ may be written as

$$\delta_{(a,\lambda)}(x) = \frac{\lambda^{\frac{1}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{1}{2}}}, \quad x \in \mathbb{R}^3.$$

Let $p \in \mathbb{N}$ and $\varepsilon > 0$; we set

$$V(p,\varepsilon) = \Big\{ u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in \Sigma, \text{ such that } a_1,\dots,a_p \in \Omega \cup \Gamma_1,$$
$$\lambda_1,\dots,\lambda_p > \varepsilon^{-1}, |\alpha_i^4 J(u)^3 K(a_i) - 3| < \varepsilon, \forall i = 1,\dots,p, \|v\| < \varepsilon, \text{ with}$$
$$\lambda_i\, d(a_i,\partial\Omega) > \varepsilon^{-1}, \text{ if } a_i \in \Omega, \varepsilon_{ij} < \varepsilon, \forall 1 \le i \ne j \le p \text{ and } v \text{ satisfies } (V_0)\Big\},$$

where

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{-\frac{1}{2}},$$

and

$$(V_0) : \langle v, \xi\rangle := \int_\Omega \nabla\xi\nabla v\, dx = 0, \quad \forall \xi = \left\{U_{(a_i,\lambda_i)}, \frac{\partial U_{(a_i,\lambda_i)}}{\partial a_i}, \frac{\partial U_{(a_i,\lambda_i)}}{\partial \lambda_i}, i = 1,\dots,p\right\}.$$

Following [**21**] and [**23**], the failure of the Palais–Smale condition can be described as follows.

Proposition 2.1. *Assume that* (1.1) *has no positive solution and assume that* Ω *satisfies condition* (H). *Let* $(u_k)_k$ *be a sequence of* Σ^+ *such that* $J(u_k)$ *is bounded and* $\partial J(u_k)$ *tends to zero. Then there exist* $p \in \mathbb{N}$ *and a subsequence* $(u_{k_l})_l$ *of* $(u_k)_k$ *such that* $u_{k_l} \in V(p,\varepsilon_l)$, $\forall l$, *where* (ε_l) *is a positive sequence tending to zero.*

Let

$$\eta\colon \mathbb{R}^+ \times \Sigma \longrightarrow \Sigma$$
$$(s,u) \longrightarrow \eta(s,u)$$

be the gradient flow of J. Therefore, for any $u \in \Sigma$, $s \mapsto \eta(s,u)$ solves the following differential equation:

$$\begin{cases} \dot\eta(s,u) = -\partial J(\eta(s,u)), \\ \eta(0) = u. \end{cases}$$

By the same argument of [**10**, Proposition 1], we have

Proposition 2.2. *Assume that* (1.1) *has no positive solutions. For any* $u \in \Sigma^+$, *there exists a positive integer* $p = p(u)$ *and a positive function* $\varepsilon(s)$ *which tends to zero as* $s \to \infty$, *such that* $\eta(s,u) \in V(p,\varepsilon(s))$, *for* s *large enough.*

As a consequence of the above proposition, any gradient flow line $\eta(s,u)$, $u \in \Sigma^+$, can be written as:

$$\eta(s,u) = \sum_{i=1}^{p} \alpha_i(s) U_{(a_i(s),\lambda_i(s))} + v(s),$$

where

$$|\alpha_i(s)^4 J(u(s))^3 K(a_i(s)) - 3| \longrightarrow 0,\ \lambda_i(s) \longrightarrow \infty, \text{ and } \|v(s)\| \longrightarrow 0, \text{ as } s \longrightarrow \infty.$$

For $i = 1, \dots, p$, denote y_i the limit of $a_i(s)$, as $s \to \infty$, then according to [**6**],

$$(y_1, \dots, y_p)_\infty := \sum_{i=1}^{p} \frac{1}{K(y_i)^{\frac{1}{3}}} U_{(y_i,\infty)}$$

is called a critical point at infinity of J. Its level

$$C_\infty(y_1, \dots, y_p)_\infty := \lim_{s \to +\infty} J(\eta(s,u)) \tag{2.3}$$

is called critical value at infinity of J associated to $(y_1, \dots, y_p)_\infty$.

In the next section we determine all the possible critical values at infinity of J.

3. Critical values at infinity

In order to identify the possible critical values at infinity of J, we need to perform asymptotic expansions of J near all the possible neighborhoods $V(p, \varepsilon)$, $p \geq 1$, of the critical points at infinity. This leads us to apply the following deformation lemma.

Deformation lemma. *Let $C_{1\infty} < C_{2\infty}$ be two subsequent critical values at infinity of J and let $a < b$ be two reals such that $[a,b] \subset (C_{1\infty}, C_{2\infty})$. Then there exists a positive constant $c = c(a,b)$, such that*

$$||\partial J(u)|| \geq c, \quad \forall u \in J^{-1}([a,b]).$$

Moreover,

$$J_b \text{ retracts by deformation on } J_a,$$

where J_α, $\alpha \in \mathbb{R}$, is the sublevel set defined by

$$J_\alpha = \{u \in \Sigma^+,\ J(u) \leq \alpha\}.$$

Let S be the best Sobolev constant. Namely,

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\left(\displaystyle\int_\Omega u^6\, dx\right)^{\frac{1}{3}}}.$$

It is well known (see for example [**13**]) that S does not depend on Ω and it is never achieved except if $\Omega = \mathbb{R}^3$ and $H_0^1(\Omega)$ is replaced with

$$\mathcal{H} = \left\{u \in L^6(\mathbb{R}^3),\ \frac{\partial u}{\partial x_i} \in L^6(\mathbb{R}^3),\ i = 1,2,3\right\}.$$

Moreover, the extremal functions of S are $u = (\text{const.})\, \delta_{(a,\lambda)}$, where $a \in \mathbb{R}^3$ and $\lambda > 0$. Using the fact that $-\Delta \delta_{(a,\lambda)} = 3\, \delta_{(a,\lambda)}^5$ in $\mathbb{R}^3$, we obtain the following two estimates for any $(a, \lambda) \in \mathbb{R}^3 \times \mathbb{R}^+$.

$$\int_{\mathbb{R}^3} \delta_{(a,\lambda)}^6\, dx = \frac{S^{\frac{3}{2}}}{3\sqrt{3}}, \tag{3.1}$$

$$\int_{\mathbb{R}^3} |\nabla \delta_{(a,\lambda)}|^2\, dx = \frac{S^{\frac{3}{2}}}{\sqrt{3}}. \tag{3.2}$$

We now prove the following expansion. We will denote by $O(g(\alpha, a, \lambda))$ any function on α, a, and λ such that $|O(g(\alpha, a, \lambda))| \leq c|g(\alpha, a, \lambda)|$, where c is a positive constant independent of α, a, and λ.

Proposition 3.1. *Let $p \geq 1$, $\varepsilon > 0$, small enough and let $u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$. We then have*

$$J(u) = \Big(\sum_{a_i \in \Gamma_1} K(a_i)^{-\frac{1}{2}} \frac{S^{\frac{3}{2}}}{2} + \sum_{a_i \in \Omega} K(a_i)^{-\frac{1}{2}} S^{\frac{3}{2}} \Big)^{\frac{2}{3}}$$

$$\times \Bigg[1 - \Big(\sum_{a_i \in \Gamma_1} \alpha_i^2 \frac{S^{\frac{3}{2}}}{2\sqrt{3}} + \sum_{a_i \in \Omega} \alpha_i^2 \frac{S^{\frac{3}{2}}}{\sqrt{3}} \Big)^{-1}$$

$$\times \Bigg(\sum_{a_i \in \Gamma_1} \alpha_i^2 \frac{\sigma_1}{2} \mathcal{M}(a_i) \frac{\log \lambda_i}{\lambda_i} + \frac{2}{9} J(u)^3 \int_\Omega K \Big(\sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} \Big)^5 v\,dx - ||v||^2$$

$$+ \frac{5}{9} J^3(u) \int_\Omega K \Big(\sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} \Big)^4 v^2\,dx + \sum_{i \neq j} O(\varepsilon_{ij}) + \sum_{i=1}^{p} O\Big(\frac{1}{\lambda_i\,\rho_i} \Big)$$

$$+ \sum_{i=1}^{p} O(|\alpha_i^4 J^3(u) K(a_i) - 3|) \Bigg) \Bigg],$$

where $\mathcal{M}(a_i)$, $a_i \in \Gamma_1$, is the mean curvature of Γ_1 at a_i, $\sigma_1 = \mathrm{mes}(S^1)$, and $\rho_i = \rho(a_i)$ is the positive constant defined in (2.1)*.*

In order to prove Proposition 3.1, we need to establish some preparatory estimates.

Lemma 3.2. *Let $u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$. For an index i such that $a_i \in \Omega$, we have*

$$\|U_{(a_i,\lambda_i)}\|^2 = \frac{S^{\frac{3}{2}}}{\sqrt{3}} + O\Big(\frac{1}{\lambda_i\,\rho_i} \Big).$$

Proof: From the expression of $U_{(a_i,\lambda_i)}$ given in (2.2), we have

$$\|U_{(a_i,\lambda_i)}\|^2 = \int_\Omega \varphi_{a_i}^2 |\nabla \delta_{(a_i,\lambda_i)}|^2\,dx + 2 \int_\Omega \varphi_{a_i} \delta_{(a_i,\lambda_i)} \nabla \delta_{(a_i,\lambda_i)} \nabla \varphi_{a_i}\,dx$$

$$+ \int_\Omega \delta_{(a_i,\lambda_i)}^2 |\nabla \varphi_{a_i}|^2\,dx := I_1 + 2I_2 + I_3.$$

Using the fact that $a_i \in \Omega$,

$$I_1 = \int_{B(a_i,\rho_i)} |\nabla \delta_{(a_i,\lambda_i)}|^2\,dx + O\Big(\int_{|x-a_i| \geq \rho_i} |\nabla \delta_{(a_i,\lambda_i)}|^2\,dx \Big)$$

$$= \int_{\mathbb{R}^3} |\nabla \delta_{(a_i,\lambda_i)}|^2\,dx + O\Big(\int_{|x-a_i| \geq \rho_i} |\nabla \delta_{(a_i,\lambda_i)}|^2\,dx \Big).$$

Using (3.2), we get

$$I_1 = \frac{S^{\frac{3}{2}}}{\sqrt{3}} + O\Big(\frac{1}{\lambda_i\,\rho_i} \Big).$$

Using the fact that $\nabla \varphi_{a_i} = 0$ in $B(a_i,\rho_i)$, we obtain that

$$I_3 = O\Big(\frac{1}{\lambda_i} \Big) \quad \text{and} \quad I_2 = O\Big(\frac{1}{\lambda_i \rho_i} \Big),$$

and the estimate follows. □

Lemma 3.3. *Let $u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$. For an index i such that $a_i \in \Gamma_1$, we have*

$$\|U_{(a_i,\lambda_i)}\|^2 = \frac{S^{\frac{3}{2}}}{2\sqrt{3}} - \frac{\sigma_1}{2}\mathcal{M}(a_i)\frac{\log\lambda_i}{\lambda_i} + O\left(\frac{1}{\lambda_i\,\rho_i}\right).$$

Proof: We follow the proof of Lemma 3.2. The only modification is in the computation of the integral I_1. Namely, for $a_i \in \Gamma_1$, we have

$$\begin{aligned} I_1 &= \int_{B(a_i,\rho_i)\cap\Omega} |\nabla\delta_{(a_i,\lambda_i)}|^2\,dx + O1\left(\int_{|x-a_i|\geq\rho_i} |\nabla\delta_{(a_i,\lambda_i)}|^2\,dx\right) \\ &= \int_{\Omega} |\nabla\delta_{(a_i,\lambda_i)}|^2\,dx + O\left(\int_{|x-a_i|\geq\rho_i} |\nabla\delta_{(a_i,\lambda_i)}|^2\,dx\right). \end{aligned}$$

Using [**1**, estimates (2.17) and (2.19)], we have

$$\int_{\Omega} |\nabla\delta_{(a_i,\lambda_i)}|^2\,dx = \frac{S^{\frac{3}{2}}}{2\sqrt{3}} - \frac{\sigma_1}{2}\mathcal{M}(a_i)\frac{\log\lambda_i}{\lambda_i} + O\left(\frac{1}{\lambda_i}\right).$$

It follows that

$$I_1 = \frac{S^{\frac{3}{2}}}{2\sqrt{3}} - \frac{\sigma_1}{2}\mathcal{M}(a_i)\frac{\log\lambda_i}{\lambda_i} + O\left(\frac{1}{\lambda_i\,\rho_i}\right),$$

and the estimate is valid. □

Lemma 3.4. *Let $u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$. For an index i such that $a_i \in \Omega$, we have*

$$\int_{\Omega} K(x)U^6_{(a_i,\lambda_i)}\,dx = K(a_i)\frac{S^{\frac{3}{2}}}{3\sqrt{3}} + O\left(\frac{1}{\lambda_i\,\rho_i}\right).$$

Proof: We have

$$\int_{\Omega} K(x)U^6_{(a_i,\lambda_i)}\,dx = \int_{B(a_i,\rho_i)} K(x)\delta^6_{(a_i,\lambda_i)}\,dx + O\left(\frac{1}{(\lambda_i\,\rho_i)^3}\right),$$

since $a_i \in \Omega$. Using the fact that K is locally Lipschitz on $\overline{\Omega}$, we write

$$K(x) = K(a_i) + O(|x-a_i|).$$

Therefore,

$$\begin{aligned} \int_{B(a_i,\rho_i)} K(x)\delta^6_{(a_i,\lambda_i)}\,dx &= K(a_i)\int_{\mathbb{R}^3} \delta^6_{(a_i,\lambda_i)}\,dx \\ &\quad + O\left(\frac{1}{(\lambda_i\,\rho_i)^3}\right) + O\left(\int_{\mathbb{R}^3} |x-a_i|\delta^6_{(a_i,\lambda_i)}\,dx\right). \end{aligned}$$

Using estimate (3.1), we get

$$\int_{B(a_i,\rho_i)} K(x)\delta^6_{(a_i,\lambda_i)}\,dx = K(a_i)\frac{S^{\frac{3}{2}}}{3\sqrt{3}} + O\left(\frac{1}{\lambda_i\,\rho_i}\right). \qquad □$$

Lemma 3.5. *Let* $u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$*. For an index* i *such that* $a_i \in \Gamma_1$*, we have*

$$\int_\Omega K(x) U^6_{(a_i,\lambda_i)}\,dx = K(a_i)\frac{S^{\frac{3}{2}}}{6\sqrt{3}} + O\left(\frac{1}{\lambda_i\,\rho_i}\right).$$

Proof: We have

$$\begin{aligned}\int_\Omega K(x) U^6_{(a_i,\lambda_i)}\,dx &= \int_{B(a_i,\lambda_i)\cap\Omega} K(x)\delta^6_{(a_i,\lambda_i)}\,dx + O\left(\frac{1}{(\lambda_i\,\rho_i)^3}\right)\\ &= K(a_i)\int_{B(a_i,\lambda_i)\cap\Omega} \delta^6_{(a_i,\lambda_i)}\,dx\\ &\quad + O\left(\int_{\mathbb{R}^3} |x-a_i|\delta^6_{(a_i,\lambda_i)}\,dx\right) + O\left(\frac{1}{(\lambda_i\,\rho_i)^3}\right)\\ &= K(a_i)\int_\Omega \delta^6_{(a_i,\lambda_i)}\,dx + O\left(\frac{1}{(\lambda_i\,\rho_i)^3}\right) + O\left(\frac{1}{\lambda_i}\right).\end{aligned}$$

From [**1**, (2.18) and (2.20)] we have

$$\int_\Omega \delta^6_{(a_i,\lambda_i)}\,dx = \frac{S^{\frac{3}{2}}}{6\sqrt{3}} + O\left(\frac{1}{\lambda_i}\right).$$

This finishes the proof. □

We now state the following estimates, where both $U_{(a_i,\lambda_i)}$ and $U_{(a_j,\lambda_j)}$, $1 \le i \ne j \le p$, occur.

Lemma 3.6. *Let* $1 \le i \ne j \le p$*. We then have*

$$\langle U_{(a_i,\lambda_i)}, U_{(a_j,\lambda_j)}\rangle = O(\varepsilon_{ij}),$$
$$\int_\Omega K(x) U^5_{(a_i,\lambda_i)} U_{(a_j,\lambda_j)}\,dx = O(\varepsilon_{ij}).$$

Proof: The proof follows from the computations of [**6**, Part 1.1]. □

We now prove Proposition 3.1.

Proof of Proposition 3.1: Let $u = \sum_{i=1}^{p} \alpha_i U_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$.

$$J(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\left(\displaystyle\int_\Omega K u^6\,dx\right)^{\frac{1}{3}}} := \frac{N}{D^{\frac{1}{3}}}.$$

Using the fact that v satisfies (V_0), the numerator of $J(u)$ reduces to

$$N = \sum_{i=1}^{p} \alpha_i^2 \|U_{(a_i,\lambda_i)}\|^2 + \|v\|^2 + \sum_{i\ne j} \alpha_i\alpha_j \langle U_{(a_i,\lambda_i)}, U_{(a_j,\lambda_j)}\rangle.$$

Using Lemmas 3.2, 3.3, and 3.6, we have

$$N = \sum_{a_i\in\Gamma_1} \alpha_i^2\Big(\frac{S^{\frac32}}{2\sqrt3} - \frac{\sigma_1}{2}\mathcal{M}(a_i)\frac{\log\lambda_i}{\lambda_i}\Big) + \sum_{a_i\in\Omega}\alpha_i^2\frac{S^{\frac32}}{\sqrt3} + \sum_{i\neq j}O(\varepsilon_{ij}) + \|v\|^2 + O\Big(\frac{1}{\lambda_i\,\rho_i}\Big)$$

$$(3.3)\qquad = \Big(\sum_{a_i\in\Gamma_1}\alpha_i^2\frac{S^{\frac32}}{2\sqrt3} + \sum_{a_i\in\Omega}\alpha_i^2\frac{S^{\frac32}}{\sqrt3}\Big) \times\Bigg(1+\Big(\sum_{a_i\in\Gamma_1}\alpha_i^2\frac{S^{\frac32}}{2\sqrt3} + \sum_{a_i\in\Omega}\alpha_i^2\frac{S^{\frac32}}{\sqrt3}\Big)^{-1}\Big(-\sum_{a_i\in\Gamma_1}\alpha_i^2\frac{\sigma_1}{2}\mathcal{M}(a_i)\frac{\log\lambda_i}{\lambda_i} + \|v\|^2 + \sum_{i\neq j}O(\varepsilon_{ij}) + \sum_{i=1}^{p}O\Big(\frac{1}{\lambda_i\,\rho_i}\Big)\Big)\Bigg).$$

Concerning the denominator of $J(u)$, we first expand

$$\begin{aligned} D &= \int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)} + v\Big)^6 dx \\ &= \int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^6 dx + 6\int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^5 v\,dx \\ &\quad + 15\int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^4 v^2\,dx + o(\|v\|^2) \\ &= \sum_{i=1}^p\alpha_i^6\int_\Omega K\,U^6_{(a_i,\lambda_i)}\,dx + 6\sum_{i\neq j}\alpha_i^5\alpha_j\int_\Omega K\,U^5_{(a_i,\lambda_i)}U_{(a_j,\lambda_j)}\,dx \\ &\quad + 6\int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^5 v\,dx + 15\int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^4 v^2\,dx \\ &\quad + o(\|v\|^2) + \sum_{i\neq j}O(\varepsilon_{ij}). \end{aligned}$$

Using the estimates of Lemmas 3.4–3.6, we get

$$D = \Big(\sum_{a_i\in\Gamma_1}\alpha_i^6K(a_i)\frac{S^{\frac32}}{6\sqrt3} + \sum_{a_i\in\Omega}\alpha_i^6K(a_i)\frac{S^{\frac32}}{3\sqrt3}\Big) \times\Bigg(1+\Big(\sum_{a_i\in\Gamma_1}\alpha_i^6K(a_i)\frac{S^{\frac32}}{6\sqrt3} + \sum_{a_i\in\Omega}\alpha_i^6K(a_i)\frac{S^{\frac32}}{3\sqrt3}\Big)^{-1} \times\Big(6\int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^5 v\,dx + 15\int_\Omega K\Big(\sum_{i=1}^p\alpha_i U_{(a_i,\lambda_i)}\Big)^4 v^2\,dx + \sum_{i\neq j}O(\varepsilon_{ij}) + \sum_{i=1}^{p}O\Big(\frac{1}{\lambda_i\,\rho_i}\Big) + o(\|v\|^2)\Big)\Bigg).$$

Therefore,

$$D^{\frac{1}{3}} = \Big(\sum_{a_i\in\Gamma_1} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{6\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{3\sqrt{3}} \Big)^{\frac{1}{3}}$$
$$\times \Bigg(1 + \Big(\sum_{a_i\in\Gamma_1} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{6\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{3\sqrt{3}} \Big)^{-1}$$
$$\times \Bigg(2\int_\Omega K\Big(\sum_{i=1}^p \alpha_i U_{(a_i,\lambda_i)}\Big)^5 v\,dx + 5\int_\Omega K\Big(\sum_{i=1}^p \alpha_i U_{(a_i,\lambda_i)}\Big)^4 v^2\,dx$$
$$+ o(\|v\|^2) + \sum_{i\neq j} O(\varepsilon_{ij}) + \sum_{i=1}^p O\Big(\frac{1}{\lambda_i\,\rho_i}\Big)\Bigg)\Bigg).$$

Using relations $\alpha_i^4 K(a_i) = 3J(u)^{-3} + O(\varepsilon)$, for any $i = 1, \dots, p$, we find that

$$D^{\frac{1}{3}} = \Big(\sum_{a_i\in\Gamma_1} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{6\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{3\sqrt{3}} \Big)^{\frac{1}{3}}$$
$$\times \Bigg(1 + \frac{J(u)^3}{3} \Big(\sum_{a_i\in\Gamma_1} \alpha_i^2 \frac{S^{\frac{3}{2}}}{6\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^2 \frac{S^{\frac{3}{2}}}{3\sqrt{3}} \Big)^{-1}$$
$$\times \Bigg(2\int_\Omega K\Big(\sum_{i=1}^p \alpha_i U_{(a_i,\lambda_i)}\Big)^5 v\,dx + 5\int_\Omega K\Big(\sum_{i=1}^p \alpha_i U_{(a_i,\lambda_i)}\Big)^4 v^2\,dx$$
$$+ o(\|v\|^2) + \sum_{i\neq j} O(\varepsilon_{ij}) + \sum_{i=1}^p O\Big(\frac{1}{\lambda_i\,\rho_i}\Big) + O(\varepsilon)\Bigg)\Bigg). \tag{3.4}$$

Here $O(\varepsilon) = \sum_{i=1}^p O(|\alpha_i^4 J(u)^3 K(a_i) - 3|)$. From (3.3) and(3.4), we get

$$J(u) = \psi(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p)$$
$$\times \Bigg(1 + \Big(\sum_{a_i\in\Gamma_1} \alpha_i^2 \frac{S^{\frac{3}{2}}}{2\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^2 \frac{S^{\frac{3}{2}}}{\sqrt{3}} \Big)^{-1}$$
$$\times \Bigg[- \sum_{a_i\in\Gamma_1} \alpha_i^2 \frac{\sigma_1}{2} \mathcal{M}(a_i) \frac{\log\lambda_i}{\lambda_i} + \|v\|^2 - \frac{2}{9} J(u)^3 \int_\Omega \Big(\sum_{i=1}^p \alpha_i U_{(a_i,\lambda_i)}\Big)^5 v\,dx$$
$$- \frac{5}{9} J(u)^3 \int_\Omega \Big(\sum_{i=1}^p \alpha_i U_{(a_i,\lambda_i)}\Big)^4 v^2\,dx + \sum_{i\neq j} O(\varepsilon_{ij}) + \sum_{i=1}^p O\Big(\frac{1}{\lambda_i\,\rho_i}\Big)$$
$$+ O(\varepsilon) + o(\|v\|^2)\Bigg]\Bigg), \tag{3.5}$$

where

$$\psi(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) = \frac{\sum_{a_i\in\Gamma_1} \alpha_i^2 \frac{S^{\frac{3}{2}}}{2\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^2 \frac{S^{\frac{3}{2}}}{\sqrt{3}}}{\Big(\sum_{a_i\in\Gamma_1} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{6\sqrt{3}} + \sum_{a_i\in\Omega} \alpha_i^6 K(a_i) \frac{S^{\frac{3}{2}}}{3\sqrt{3}} \Big)^{\frac{1}{3}}} := \frac{N_1}{D_1}.$$

Using again relations $\alpha_i^4 = 3J(u)^{-3}K(a_i)^{-1} + O(\varepsilon)$, $i = 1, \ldots, p$, we find that

$$N_1 = J(u)^{-\frac{3}{2}} \bigg(\sum_{a_i \in \Gamma_1} K(a_i)^{-\frac{1}{2}} \frac{S^{\frac{3}{2}}}{2} + \sum_{a_i \in \Omega} K(a_i)^{-\frac{1}{2}} S^{\frac{3}{2}} \bigg) (1 + O(\varepsilon)),$$

and

$$D_1 = J(u)^{-\frac{3}{2}} \bigg(\sum_{a_i \in \Gamma_1} K(a_i)^{-\frac{1}{2}} \frac{S^{\frac{3}{2}}}{2} + \sum_{a_i \in \Omega} K(a_i)^{-\frac{1}{2}} S^{\frac{3}{2}} \bigg)^{\frac{1}{3}} (1 + O(\varepsilon)).$$

Therefore,

$$\psi(\alpha_1, \ldots, \alpha_p, a_1, \ldots, a_p) = \bigg(\sum_{a_i \in \Gamma_1} K(a_i)^{-\frac{1}{2}} \frac{S^{\frac{3}{2}}}{2} + \sum_{a_i \in \Omega} K(a_i)^{-\frac{1}{2}} S^{\frac{3}{2}} \bigg)^{\frac{2}{3}} (1 + O(\varepsilon)). \tag{3.6}$$

The expansion of $J(u)$ follows from (3.5) and (3.6). □

As a consequence of the expansion above, we have

Corollary 3.7. *Assume condition* (H) *and assume that J has no critical point in Σ^+. According to definition* (2.3) *and the expansion of Proposition 3.1, the critical values at infinity of J are of the form:*

$$C_\infty = \bigg(\sum_{a_i \in E_{\overline{\Gamma}_1}} K(a_i)^{-\frac{1}{2}} \frac{S^{\frac{3}{2}}}{2} + \sum_{a_i \in E_{\overline{\Omega}}} K(a_i)^{-\frac{1}{2}} S^{\frac{3}{2}} \bigg)^{\frac{2}{3}},$$

where $E_{\overline{\Gamma}_1}$ is a finite set in Γ_1 and $E_{\overline{\Omega}}$ is a finite set in $\overline{\Omega}$.

Of course the above corollary does not provide precise identification of the critical values at infinity of J nor precise characterization of the critical points at infinity. It only provides necessary forms of the critical values at infinity and the associated critical points at infinity.

Our aim in the next section is to identify the critical points at infinity of a single bubble concentrating at a point in the boundary part Γ_1. Denote

$$V_{\Gamma_1}(1, \varepsilon) = \{u = \alpha U_{(a,\lambda)} + v \in V(1, \varepsilon), \text{ such that } a \in \Gamma_1\}.$$

Our topological method avoids all the neighborhoods of critical points at infinity $V(p, \varepsilon)$, $p \geq 1$, such that $V(p, \varepsilon) \neq V_{\Gamma_1}(1, \varepsilon)$.

4. Variational analysis in $V_{\Gamma_1}(1, \varepsilon)$

We first refine the expansion of $J(u)$, $u \in V_{\Gamma_1}(1, \varepsilon)$, in order to show that the v-part of u is negligible to the concentration phenomenon and hence provide a more suitable parametrization of the set $V_{\Gamma_1}(1, \varepsilon)$. After that we study the variations of $J(u)$ with respect to the concentration speed λ and the concentration point a. This is the purpose of the first subsection.

4.1. Asymptotic expansions $V_{\Gamma_1}(1, \varepsilon)$. For $u = \alpha U_{(a,\lambda)} \in V(1, \varepsilon)$, we set

$$E(u) = \{v \in V(\Omega), \text{ such that } u + v \in V_{\Gamma_1}(1, \varepsilon)\}.$$

Consider the minimization problem

$$\min_{v \in E(u)} J(u + v). \tag{4.1}$$

We then have

Proposition 4.1. *For $\varepsilon > 0$ small enough, problem* (4.1) *admits a unique minimizer $\overline{v} = \overline{v}(u)$, satisfying*

$$\|\overline{v}\| = O\left(\frac{1}{\sqrt{\lambda\rho}}\right).$$

Moreover, there exists a change of variables $V = v - \overline{v}$ such that

$$J(u+v) = J(u+\overline{v}) + \|V\|^2.$$

Proof: We follow the argument of [**6**, Proposition 5.4]. For $\varepsilon > 0$ small enough, the expansion of Proposition 3.1 shows that the mapping $v \mapsto J(u+v)$ behaves as $g(v) := f(v) + Q(v,v)$, where

$$f(v) = -\int_\Omega K(\alpha U_{(a,\lambda)})^5 v\,dx \quad \text{and} \quad Q(v,v) = \|v\|^2 - \frac{5}{9}J^3(u)\int_\Omega K(\alpha U_{(a,\lambda)})^4 v^2\,dx.$$

The coercivity of $Q(v,v)$, $v \in E(u)$ (see [**6**]), implies the existence of a unique minimizer $\overline{v}$ of $J(u+v)$, $v \in E(u)$. Therefore, $\overline{v}$ satisfies

$$Q(\overline{v},h) = -\frac{1}{2}f(h), \quad \forall h \in V(\Omega).$$

It follows that

$$\|\overline{v}\| \leq c|f|, \tag{4.2}$$

where $|f|$ is the norm of f in the set of linear mapping on $\widetilde{E}(u) := \{v \in V(\Omega),$ such that v satisfies $(V_0)\}$. Let $v \in \widetilde{E}(u)$. We have

$$\begin{aligned} -f(v) &= \int_\Omega K(\alpha U_{(a,\lambda)})^5 v\,dx = \int_{B(a,\rho)\cap\Omega} K(\alpha\delta_{(a,\lambda)})^5 v\,dx + O\left(\frac{\|v\|}{(\lambda\rho)^{\frac{5}{2}}}\right) \\ &= \alpha^5 K(a)\int_{B(a,\rho)\cap\Omega} \delta^5_{(a,\lambda)} v\,dx + O\left(\frac{\|v\|}{\lambda\rho}\right). \end{aligned} \tag{4.3}$$

We compute the integral of (4.3) as follows.

$$\begin{aligned} \int_{B(a,\rho)\cap\Omega} \delta^5_{(a,\lambda)} v\,dx &= \int_\Omega \delta^5_{(a,\lambda)} v\,dx + O\left(\frac{\|v\|}{(\lambda\rho)^{\frac{5}{2}}}\right) \\ &= \frac{1}{3}\int_\Omega -\triangle\delta_{(a,\lambda)} v\,dx + O\left(\frac{\|v\|}{(\lambda\rho)^{\frac{5}{2}}}\right) \\ &= \frac{1}{3}\left(\int_\Omega \nabla\delta_{(a,\lambda)}\nabla v\,dx - \int_{\Gamma_1}\frac{\partial\delta_{(a,\lambda)}}{\partial\nu}v\,d\sigma\right) + O\left(\frac{\|v\|}{(\lambda\rho)^{\frac{5}{2}}}\right) \\ &= \frac{1}{3}\left(\int_{B(a,\rho)\cap\Omega} \nabla U_{(a,\lambda)}\nabla v\,dx \right. \\ &\qquad \left. + \int_{B(a,\rho)^c\cap\Omega} \nabla\delta_{(a,\lambda)}\nabla v\,dx - \int_{\Gamma_1}\frac{\partial\delta_{(a,\lambda)}}{\partial\nu}v\,d\sigma\right) \\ &\quad + O\left(\frac{\|v\|}{(\lambda\rho)^{\frac{5}{2}}}\right). \end{aligned} \tag{4.4}$$

Using the fact that v satisfies (V_0),

$$\int_{B(a,\rho)\cap\Omega} \nabla U_{(a,\lambda)} \nabla v\, dx = -\int_{B(a,\rho)^c\cap\Omega} \nabla U_{(a,\lambda)} \nabla v\, dx = O\bigg(\frac{\|v\|}{\sqrt{\lambda\rho}}\bigg). \tag{4.5}$$

Moreover, by Hölder's inequalities,

$$\bigg|\int_{\Gamma_1} \frac{\partial \delta_{(a,\lambda)}}{\partial \nu} v\, d\sigma\bigg| \le c\frac{\|v\|}{\lambda}. \tag{4.6}$$

The estimate if $\|\overline{v}\|$ follows from (4.2)–(4.6). □

The result of the above proposition shows that $V_{\Gamma_1}(1,\varepsilon)$ can be parametrized by new variables $(\alpha, a, \lambda, \overline{v})$, since on the V-space we may define a pseudo-gradient, as Bahri did in his seminal paper [**7**], by setting the ordinary differential equation

$$\dot{V} = -\mu V, \quad \text{where } \mu \gg 1.$$

Therefore $\|V(s)\|$ decreases by the action of the pseudo-gradient and for μ large enough, $\|V(s)\|$ will be very small at $s = 1$. This shows that in order to perform our deformations we can work as if $V = 0$ and hence consider $(\alpha, a, \lambda, \overline{v})$ as new parameters of any $u \in V_{\Gamma_1}(1,\varepsilon)$.

We now expand $\lambda\frac{\partial J}{\partial \lambda}$ and $\frac{1}{\lambda}\frac{\partial J}{\partial a}$ in $V_{\Gamma_1}(1,\varepsilon)$.

Proposition 4.2. *Let $u = \alpha U_{(a,\lambda)} \in V_{\Gamma_1}(1,\varepsilon)$. We then have*

$$\bigg\langle \partial J(u), \alpha\lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\bigg\rangle = J(u)\alpha^2\frac{\sigma_1}{2}\mathcal{M}(a)\frac{\log\lambda}{\lambda} + O\bigg(\frac{1}{\lambda\rho}\bigg).$$

Proof: It is straightforward to see that for $u \in \Sigma$ and $h \in V(\Omega)$, we have

$$\langle \partial J(u), h\rangle = 2J(u)\bigg(\langle u, h\rangle - J^3(u)\int_\Omega K u^5 h\, dx\bigg). \tag{4.7}$$

Thus, for $u = \alpha U_{(a,\lambda)}$ and $h = \alpha\lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}$, we have

$$\langle \partial J(u), h\rangle = 2J(u)\alpha^2\bigg(\bigg\langle U_{(a,\lambda)}, \lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\bigg\rangle - J^3(u)\alpha^4\int_\Omega K U^5_{(a,\lambda)}\lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\, dx\bigg). \tag{4.8}$$

Observe that

$$\begin{aligned}\bigg\langle U_{(a,\lambda)}, \lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\bigg\rangle &= \frac{\lambda}{2}\frac{\partial}{\partial\lambda}\langle U_{(a,\lambda)}, U_{(a,\lambda)}\rangle \\ &= \frac{\sigma_1}{4}\mathcal{M}(a)\frac{\log\lambda}{\lambda} + O\bigg(\frac{1}{\lambda\rho}\bigg),\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\int_\Omega K U^5_{(a,\lambda)}\lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\, dx &= \frac{1}{6}\lambda\frac{\partial}{\partial\lambda}\bigg(\int_\Omega K U^6_{(a,\lambda)}\, dx\bigg) \\ &= O\bigg(\frac{1}{\lambda\rho}\bigg).\end{aligned} \tag{4.10}$$

The desired estimate follows from (4.8)–(4.10). □

Proposition 4.3. *Assume that K is of class C^1 on $\overline{\Omega}$. For $u = \alpha U_{(a,\lambda)} \in V_{\Gamma_1}(1,\varepsilon)$, we have*

$$\bigg\langle \partial J(u), \frac{\alpha}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}\bigg\rangle = -J(u)\alpha^2\frac{S^{\frac{3}{2}}}{3\sqrt{3}}\frac{\nabla K(a)}{K(a)\lambda} + O\bigg(\frac{\log\lambda}{(\lambda\rho)^2}\bigg).$$

Proof: Let $u = \alpha U_{(a,\lambda)}$ and $h = \frac{\alpha}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}$. From (4.7) we have

$$\langle \partial J(u), h\rangle = 2J(u)\alpha^2\left(\left\langle U_{(a,\lambda)}, \frac{1}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}\right\rangle - J^3(u)\alpha^4\int_\Omega KU^5_{(a,\lambda)}\frac{1}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}\,dx\right). \tag{4.11}$$

Observe that

$$\left\langle U_{(a,\lambda)}, \frac{1}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}\right\rangle = \frac{1}{2}\frac{1}{\lambda}\frac{\partial}{\partial a}\langle U_{(a,\lambda)}, U_{(a,\lambda)}\rangle = O\left(\frac{\log\lambda}{(\lambda\rho)^2}\right), \tag{4.12}$$

$$\begin{aligned}\int_\Omega KU^5_{(a,\lambda)}\frac{1}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}\,dx &= \frac{1}{6}\frac{1}{\lambda}\frac{\partial}{\partial a}\left(\int_\Omega KU^6_{(a,\lambda)}\,dx\right)\\ &= \frac{S^{\frac{3}{2}}}{18\sqrt{3}}\frac{\nabla K(a)}{\lambda} + O\left(\frac{1}{(\lambda\rho)^2}\right).\end{aligned} \tag{4.13}$$

Using relation $\alpha^4 J^3(u) = \frac{3}{K(a)} + O(\varepsilon)$, the estimate follows from (4.11)–(4.13). □

Corollary 4.4. *Let $\Omega \in \mathbb{R}^3$ be a bounded domain such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Then for any $u = \alpha U_{(a,\lambda)} \in V_{\Gamma_1}(1,\varepsilon)$, the following two expansions hold:*

$$\left\langle \partial J(u), \alpha\lambda\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\right\rangle = J(u)\alpha^2\frac{\sigma_1}{2}\mathcal{M}(a)\frac{\log\lambda}{\lambda} + o\left(\frac{\log\lambda}{\lambda}\right),$$

$$\left\langle \partial J(u), \frac{\alpha}{\lambda}\frac{\partial U_{(a,\lambda)}}{\partial a}\right\rangle = -J(u)\alpha^2\frac{S^{\frac{3}{2}}}{3\sqrt{3}}\frac{\nabla K(a)}{K(a)\lambda} + o\left(\frac{1}{\lambda}\right).$$

Here $o(f(a,\lambda))$ denotes a function on a and λ such that $|o(f(a,\lambda))| \leq |f(a,\lambda)|\psi(\varepsilon)$, where $\psi(\varepsilon)$ is a positive function independent of a and λ and such that $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof: Under the assumption of the corollary, $\rho = \rho(a)$, $a \in \Gamma_1$, is lower-bounded by a fixed constant, namely $d(\overline{\Gamma}_0, \overline{\Gamma}_1)$. Thus the estimates follow from the expansions of Propositions 4.2 and 4.3. □

We now consider the concentration phenomenon problem in $V_{\Gamma_1}(1,\varepsilon)$. Our aim is to identify the critical points at infinity of J in this set. For this we assume in the next subsection that Ω satisfies the assumption of Corollary 4.4 and K is of class C^1 on $\overline{\Omega}$.

A general result for Yamabe-type problems states that any blow-up point of the associated variational structure must be a critical point of the associated prescribed function; see for example [**16**, p. 74] and references [8], [16], and [21] therein. More precisely, let $\delta > 0$ small enough and let

$$V_{\Gamma_1}(1,\varepsilon,\delta) = \left\{u = \alpha U_{(a,\lambda)} + \overline{v} \in V_{\Gamma_1}(1,\varepsilon),\ \text{such that } a \in B_{\Gamma_1}(y,\delta),\ y \in \mathrm{Crit}(K|_{\Gamma_1})\right\},$$

where $B_{\Gamma_1}(y,\delta)$ is the ball in Γ_1 of center y and radius δ. Then the critical points at infinity of J in $V_{\Gamma_1}(1,\varepsilon)$ lie in $V_{\Gamma_1}(1,\varepsilon,\delta)$.

4.2. Concentration phenomena in $V_{\Gamma_1}(1,\varepsilon,\delta)$. We prove the following main result.

Proposition 4.5. *Assume that* $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $K \in C^1(\overline{\Omega})$, *and* $\mathcal{M}(y) \neq 0$, $\forall y \in \mathrm{Crit}(K|_{\Gamma_1})$. *There exist a bounded pseudo-gradient* W *in* $V_{\Gamma_1}(1,\varepsilon,\delta)$ *and a positive constant* c *such that*

(i) $\langle \partial J(u), W(u)\rangle \leq -c\frac{\log\lambda}{\lambda}$,

(ii) $\big\langle \partial J(u+\overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha,a,\lambda)} W(u)\big\rangle \leq -c\frac{\log\lambda}{\lambda}$, *for any* $u = \alpha U_{(a,\lambda)} \in V_{\Gamma_1}(1,\varepsilon,\delta)$.

Moreover,

(iii) *the only case where the speed parameter* $\lambda(s)$ *of* $u(s)$ *increases and tends to* ∞ *under the action of* W *is when the concentration point* $a(s)$ *is close to* $y \in \mathrm{Crit}(K|_{\Gamma_1})$ *with* $\mathcal{M}(y) < 0$.

Proof: Let $\delta > 0$ small enough so that

$$\mathcal{M}(y)\mathcal{M}(a) > 0, \quad \forall a \in B_{\Gamma_1}(y,\delta),\ y \in \mathrm{Crit}(K|_{\Gamma_1}).$$

Let $\chi(t)$, $t \in \mathbb{R}$, be a cut-off function defined by $\chi(t) = 0$, if $|t| < \frac{1}{2}$ and $\chi(t) = 1$, if $|t| > 1$.

Let $u = \alpha U_{(a,\lambda)} \in V_{\Gamma_1}(1,\varepsilon,\delta)$. We move the concentration point a according to the differential equation

$$\dot{a} = \chi(\lambda|\nabla_T K(a)|)\frac{1}{\lambda}\frac{\nabla_T K(a)}{|\nabla_T K(a)|},$$

where $\nabla_T K(a)$ is the projection of $\nabla K(a)$ on the tangent space of Γ_1 at a.

Using the second expansion of Corollary 4.4, we have

$$\left\langle \partial J(u), \alpha\frac{\partial U_{(a,\lambda)}}{\partial a}\dot{a}\right\rangle = -J(u)\alpha^2\frac{S^{\frac{3}{2}}}{3\sqrt{3}}\chi(\lambda|\nabla_T K(a)|)\frac{|\nabla_T K(a)|}{K(a)\lambda} + o\left(\frac{1}{\lambda}\right). \tag{4.14}$$

We now define

$$\dot{\lambda} = -\mathcal{M}(a)\lambda.$$

The first expansion of Corollary 4.4 yields

$$\left\langle \partial J(u), \alpha\frac{\partial U_{(a,\lambda)}}{\partial \lambda}\dot{\lambda}\right\rangle = -J(u)\alpha^2\frac{\sigma_1}{2}\mathcal{M}^2(a)\frac{\log\lambda}{\lambda} + o\left(\frac{\log\lambda}{\lambda}\right). \tag{4.15}$$

Let

$$W_1(u) = \alpha\left(\frac{\partial U_{(a,\lambda)}}{\partial\lambda}\dot{\lambda} + \frac{\partial U_{(a,\lambda)}}{\partial a}\dot{a}\right).$$

From (4.14) and (4.15), we have

$$\langle \partial J(u), W_1(u)\rangle = -J(u)\alpha^2\frac{\sigma_1}{2}\mathcal{M}^2(a)\frac{\log\lambda}{\lambda} + o\left(\frac{\log\lambda}{\lambda}\right) \leq -c\frac{\log\lambda}{\lambda}.$$

Thus W_1 satisfies assertion (i) of Proposition 4.5. Using the fact that $\|\overline{v}\|^2$ is small with respect to the absolute value of the upper bound of (i), see Proposition 4.1, inequality (ii) holds for W_1. Lastly, by construction, $\lambda(s)$ decreases near $y \in \mathrm{Crit}(K|_{\Gamma_1})$ such that $\mathcal{M}(y) > 0$ (it is a deconcentration phenomenon) and increases near $y \in \mathrm{Crit}(K|_{\Gamma_1})$ such that $\mathcal{M}(y) < 0$ (it is a concentration phenomenon). The required pseudo-gradient is

$$W(u) = W_1(u) - \langle u, W_1(u)\rangle u.$$

We have that $W(u)$, $u \in V_{\Gamma_1}(1,\varepsilon,\delta)$, is well defined, since $W(u) \in \langle u\rangle^{\perp}$; the tangent space of $V_{\Gamma_1}(1,\varepsilon,\delta)$ at u moreover satisfies

$$\langle \partial J(u), W(u)\rangle = \langle \partial J(u), W_1(u)\rangle.$$

Thus assertions (i) and (ii) hold for W. Concerning (iii), it follows from the fact that any flow line $\eta(s,u)$ of W is given by $\frac{u(s,u)}{\|u(s,u)\|}$, where $u(s,u)$ is a flow line of W_1.

Therefore $\eta(s,u)$ and $u(s,u)$ have the same concentration points $a(s)$ and the same speeds $\lambda(s)$. This concludes the proof. □

Corollary 4.6. *Under the assumptions of Proposition 4.5 the critical points at infinity of J in $V_{\Gamma_1}(1,\varepsilon)$ are in one-to-one correspondence with the set of critical points y of $K|_{\Gamma_1}$ such that $\mathcal{M}(y)<0$.*

The next section is devoted to proving the existence results of this paper.

5. Proof of theorems

The proofs of Theorems 1.1, 1.2, and 1.3 are consequences of the following lemma. Denote

$$S(K)=\inf_{u\in\Sigma+} J(u), \quad \text{where} \quad J(u)=\frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\left(\displaystyle\int_\Omega Ku^6\,dx\right)^{\frac{1}{3}}},$$

and denote by C_∞ the lower bound of all the critical values at infinity of J.

Lemma 5.1. *If $S(K)<C_\infty$, then $S(K)$ is achieved and hence problem* (1.1) *has a positive solution.*

Proof: Let $u_0\in\Sigma^+$ be such that

$$S(K)<J(u_0)<C_\infty.$$

Let $s\mapsto\eta(s,u_0)$ be the solution of

$$\begin{cases}\dot\eta(s,u_0)=-\partial J(\eta(s,u_0)),\\ \eta(0,u_0)=u_0.\end{cases}$$

It is easy to see that $\eta(s,u_0)$ is defined for any $s\geq 0$, since $(-\partial J)$ is a bounded vector field and J decreases along $\eta(s,u)$. Moreover, Σ^+ remains invariant under the action of $(-\partial J)$. Therefore if J has no positive critical point under the level $J(u_0)$, then by the result of Proposition 2.1, there exists $c=c(u_0)>0$ such that $\|\partial J(\eta(s,u))\|\geq c$, $\forall s\geq 0$, and therefore $J(\eta(s,u_0))\to-\infty$ as $s\to+\infty$. This is impossible and hence J admits a critical point $w_0\in\Sigma^+$ such that

$$S(K)\leq J(w_0)<J(u_0).$$

If $S(K)<J(w_0)$, we proceed by the same argument and we get the existence of a critical point $w_1\in\Sigma^+$ such that

$$S(K)\leq J(w_1)<J(w_0).$$

In this way, we construct a Palais–Smale sequence $(w_k)_k$ in Σ^+ such that

$$\begin{cases}J(w_k)\geq S(K),\\ J(w_k)\longrightarrow S(K),\\ \partial J(w_k)=0.\end{cases}$$

Again by the result of Proposition 2.1, the sequence $(u_k)_k$ has to converge to a positive critical point w of J, which is a minimizer of J on Σ^+. □

Proof of Theorem 1.1: Let $y_0\in\overline{\Omega}$ be such that

$$K(y_0)=\max_{x\in\overline{\Omega}} K(x).$$

By the result of Corollary 3.7, all the critical values at infinity of J are above the level

$$C_1 := \left(K(y_0)^{-\frac{1}{2}}\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}.$$

We then derive that $C_1 \leq C_\infty$. If we assume that $y_0 \in \Gamma_1$, we get for $u = \alpha U_{(y_0,\lambda)}$, $\alpha > 0$, and $\lambda \gg 1$ the following expansion of $J(u)$ (see Proposition 3.1 above).

$$J(u) = \left(K(y_0)^{-\frac{1}{2}}\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}\left[1 - \frac{\sqrt{3}}{S^{\frac{3}{2}}}\sigma_1\mathcal{M}(y_0)\frac{\log\lambda}{\lambda} + O\left(\frac{1}{\lambda\rho}\right) + O(|\alpha^4 J^3(u)K(y_0) - 3|)\right].$$

Here $\rho = \rho(y_0)$ is defined in (2.1). Let $\alpha = (3J^{-3}(u)K(y_0)^{-1})^{\frac{1}{4}}$ and $\lambda \gg \frac{1}{\rho}$. We then have

$$J(u) = \left(K(y_0)^{-\frac{1}{2}}\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}\left[1 - \frac{\sqrt{3}}{S^{\frac{3}{2}}}\sigma_1\mathcal{M}(y_0)\frac{\log\lambda}{\lambda}(1 + o(1))\right],$$

where $o(1) \to 0$ as λ is large enough. Therefore, if we assume that $\mathcal{M}(y_0) > 0$, then $J(u) < c_1$ and the result follows from Lemma 5.1. □

Proof of Theorem 1.2: Let $z_0 \in \overline{\Gamma}_1$ be such that

$$K(z_0) = \max_{x\in\overline{\Gamma}_1} K(x).$$

It follows from the result of Corollary 3.7 that if $|K-1|_{L^\infty(\overline{\Omega})}$ is small enough, then all the critical values at infinity of J are above the level

$$C_1' := \left(K(z_0)^{-\frac{1}{2}}\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}.$$

Thus if we assume that $z_0 \in \Gamma_1$ and $\mathcal{M}(z_0) > 0$, the proof of Theorem 1.2 concludes. □

Proof of Theorem 1.3: Under the assumptions of Theorem 1.3, we have $\overline{\Gamma}_1 = \Gamma_1$ and $\mathcal{M}(x)$ is positive for any $x \in \Gamma_1$. Then Theorem 1.3 follows from Theorem 1.2. □

Proof of Theorem 1.4: To simplify the presentation of the proof, we assure that

$$K = 1 + \gamma K_0,$$

where K_0 is a fixed given function on $\overline{\Omega}$ and $\gamma \in \mathbb{R}\setminus\{0\}$ such that $|\gamma|$ is small. Denote

$$J_\gamma(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\left(\displaystyle\int_\Omega (1+\gamma K_0)u^6\,dx\right)^{\frac{1}{3}}}, \quad u \in \Sigma^+.$$

For a given constant c, we also denote

$$J_\gamma^c = \{u \in \Sigma^+,\, J_\gamma(u) < c\}.$$

We then have

Lemma 5.2. *For a given positive constant b there exist $\gamma_0 = \gamma_0(b) > 0$ and $c = c(b) > 0$, such that for $|\gamma| < \gamma_0$,*

$$|J_\gamma(u) - J_0(u)| \leq c|\gamma|, \quad \forall u \in J_\gamma^b.$$

Proof: Let $u \in \Sigma^+$. We have

$$J_0(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\left(\displaystyle\int_\Omega (1+\gamma K_0)u^6\,dx - \int_\Omega \gamma K_0 u^6\,dx\right)^{\frac{1}{3}}}$$

$$= J_\gamma(u)\frac{1}{\left(1-\left(\displaystyle\int_\Omega (1+\gamma K_0)u^6\,dx\right)^{-1}\displaystyle\int_\Omega \gamma K_0 u^6\,dx\right)^{\frac{1}{3}}}.$$

For $u \in J_\gamma^b$, we have

$$\left(\int_\Omega (1+\gamma K_0)u^6\,dx\right)^{-1} = J_\gamma^3(u) \leq b^3,$$

and by Sobolev inequalities,

$$\left|\int_\Omega \gamma K_0 u^6\,dx\right| \leq S^{-3}|\gamma|\,\|K_0\|_{L^\infty(\overline{\Omega})}.$$

It follows that

$$\left|\left(\int_\Omega (1+\gamma K_0)u^6\,dx\right)^{-1}\int_\Omega \gamma K_0 u^6\,dx\right| \leq |\gamma|\frac{b^3\|K_0\|_{L^\infty(\overline{\Omega})}}{S^3}.$$

Let $\gamma_0 = \min\big(\|K_0\|^{-1}_{L^\infty(\Omega)}, \big(\frac{b^3\|K_0\|_{L^\infty(\Omega)}}{S^3}\big)^{-1}\big)$. For any $|\gamma| < \gamma_0$, we have

$$J_0(u) = J_\gamma(u)(1+O(1)),$$

where $|O(\gamma)| \leq |\gamma|\frac{b^3\|K_0\|_{L^\infty(\overline{\Omega})}}{S^3}$ and hence

$$|J_0(u) - J_\gamma(u)| \leq |\gamma|\frac{b^4\|K_0\|_{L^\infty(\overline{\Omega})}}{S^3}, \quad \forall u \in J_\gamma^b. \qquad \square$$

Fix $b > 0$, such that

$$\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}} + 3b < S. \tag{5.1}$$

Denote

$$C^\infty_{\Gamma_1} := \Big\{y \in \Gamma_1,\ \text{such that } \nabla(K_0|_{\Gamma_1})(y) = 0 \text{ and } \mathcal{M}(y) < 0\Big\}.$$

Lemma 5.3. *Let $b > 0$ satisfying (5.1). Assume that (1.1) has no positive solution. There exists $\gamma_0 = \gamma_0(b) > 0$ such that for $0 < |\gamma| < \gamma_0$, J_γ has no critical points at infinity between the levels $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + b$ and $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + 3b$. Moreover, the only critical points at infinity of J_γ under the level $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + b$ are*

$$(y)_\infty := S^{-\frac{3}{4}}U_{(y,\infty)}, \quad y \in C^\infty_{\Gamma_1}.$$

The Morse index of J_γ at $(y)_\infty$ equals

$$i(y)_\infty = 2 - \operatorname{ind}(\gamma K_0|_{\Gamma_1}, y).$$

Proof: Since we have assumed that (1.1) has no positive solution, by Proposition 2.2, the critical points at infinity of J_γ lie in $V(p,\varepsilon)$, $p \geq 1$. Using the result of Corollary 3.7, it follows that there exists $\gamma_0 = \gamma_0(b) > 0$, such that for $|\gamma| < \gamma_0$ the critical points at infinity of J_γ lying in $V(p,\varepsilon)$, $p \geq 1$, and $V(p,\varepsilon) \neq V_{\Gamma_1}(1,\varepsilon)$, are above the level $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + 3b$ and the critical points at infinity of J_γ lying in $V_{\Gamma_1}(1,\varepsilon)$ are below $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + b$.

Thus J_γ has no critical point at infinity with critical value in $\big(\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + b, \big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + 3b\big)$. Moreover, by the result of Corollary 4.6 the critical points at infinity of J under the level $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}} + b$ are

$$(y)_\infty := S^{-\frac{3}{4}} U_{(y,\infty)}, \quad y \in C^\infty_{\Gamma_1}.$$

The Morse index of J_γ at $(y)_\infty$, $y \in C^\infty_{\Gamma_1}$, is given by expanding $J_\gamma(\alpha U_{(a,\lambda)})$, when a is close to y and $\lambda \gg 1$. It is easy to see that $J_\gamma(\alpha U_{(a,\lambda)})$ does not depend on α, since J_γ is a homogeneous function. Thus by the expansion of Proposition 3.1 we have

$$J_\gamma(\alpha U_{(a,\lambda)}) = J_\gamma(U_{(a,\lambda)}) = \frac{S}{2^{\frac{2}{3}}} \frac{1}{(1+\gamma K_0)^{\frac{1}{3}}(a)} \left(1 - \frac{\sqrt{3}\sigma_1}{S^{\frac{3}{2}}} \mathcal{M}(y) \frac{\log\lambda}{\lambda}(1+o(1))\right).$$

When a is close to y, $a \in \Gamma_1$, we have after recalling that $K|_{\Gamma_1}$ is a Morse function

$$\gamma K_0(a) = \gamma K_0(y) + |y|^2 - |x|^2, \quad \forall \gamma \in \mathbb{R} \setminus \{0\},$$

where $x \in \mathbb{R}^k$, $k = \operatorname{ind}(\gamma K_0|_{\Gamma_1}, y)$, and $y \in \mathbb{R}^{2-k}$. The above expansion is then reduced to

$$J_\gamma(\alpha U_{(a,\lambda)}) = \frac{S}{2^{\frac{2}{3}}} \frac{1}{(1+\gamma K_0)^{\frac{1}{3}}(y)} \left(1 + |x|^2 - |y|^2 + c\frac{\log\lambda}{\lambda}\right),$$

up to some changes of variables. Here c is a positive constant, since $\mathcal{M}(y) < 0$. The Morse index of $(y)_\infty$ follows from the above expansion. □

In the next few lemmas we provide some properties of the functional J_γ, when $\gamma = 0$.

Lemma 5.4. *Under assumption (ii) of Theorem 1.4, the infimum of* J_0 *on* Σ^+ *equals* $\big(\frac{S^{\frac{3}{2}}}{2}\big)^{\frac{2}{3}}$ *and it is not achieved.*

Proof: Let $u = U_{(a,\lambda)}$, $a \in \Gamma_1$, and $\lambda \gg 1$. By the expansion of Proposition 3.1,

$$J_0(u) = \left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}} \left(1 + O\left(\frac{\log\lambda}{\lambda}\right)\right).$$

Letting λ tend to ∞, we get

$$\inf_{\Sigma^+} J_0(u) \leq \left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}.$$

If the above inequality is a strict inequality, we get by Lemma 5.1 the existence of a positive critical point of J_0 which minimizes J_0 on Σ^+. This is impossible, since the infimum of J_0 is never achieved under condition (ii) of Theorem 1.4; see [**30**]. □

Lemma 5.5. *Let* $\varepsilon > 0$ *and small. There exists* $b_1 > 0$*, such that*

$$J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}} + b_1} \subset V_{\Gamma_1}(1,\varepsilon).$$

Proof: If not, for any $k \in \mathbb{N} \setminus \{0\}$, there exists $u_k \in J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+\frac{1}{k}}$ and $u_k \notin V_{\Gamma_1}(1,\varepsilon)$. Therefore, $(u_k)_k$ is a minimizer sequence which is outside $V_{\Gamma_1}(1,\varepsilon)$. By Proposition 2.1, $(u_k)_k$ is a convergent sequence in Σ^+ and this contradicts the fact that the infimum of J_0 is not achieved. □

Lemma 5.6. *Let b_1 be the positive constant subjected to Lemma 5.5. Then, $0<b<b_1$,*

$$J_0^{\left(S^{\frac{3}{2}}\right)^{\frac{2}{3}}+b} \text{ is homotopy-equivalent to } \Gamma_1.$$

Proof: To define a homotopy equivalence between $J_0^{\left(S^{\frac{3}{2}}\right)^{\frac{2}{3}}+b}$ and Γ_1, we first define a continuous mapping

$$f_{\lambda_0}\colon \Gamma_1 \longrightarrow V_{\Gamma_1}(1,\varepsilon)$$
$$a \longmapsto \frac{U_{(a,\lambda_0)}}{||U_{(a,\lambda_0)}||},$$

where ε and λ_0 are two fixed positive constants such that $\lambda_0 > \frac{1}{\varepsilon}$. The expansion of Proposition 3.1 yields for ε small enough

$$J_0(f_{\lambda_0}(a)) < (S^{\frac{3}{2}})^{\frac{2}{3}} + b, \quad \forall a \in \Gamma_1.$$

Thus f_{λ_0} is valued in $J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b}$ and induces a continuous mapping denoted again f_{λ_0},

$$f_{\lambda_0}\colon \Gamma_1 \longrightarrow J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b}. \tag{5.2}$$

Observe that by the result of Lemma 5.5 there exists a natural embedding denoted i such that

$$i\colon J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b} \longrightarrow V_{\Gamma_1}(1,\varepsilon). \tag{5.3}$$

Now, let $u = \alpha U_{(a,\lambda)} + v \in V_{\Gamma_1}(1,\varepsilon)$. It is well known by the computations of [**9**, Proposition 7] that if $\alpha U_{(a,\lambda)} + v = \alpha' U_{(a',\lambda')} + v'$, then $\alpha = \alpha'$, $a = a'$, $\lambda = \lambda'$, and $v = v'$. It follows that the following projection is well defined, namely

$$\begin{aligned} p\colon V_{\Gamma_1}(1,\varepsilon) &\longrightarrow \Gamma_1 \\ u = \alpha U_{(a,\lambda)} + v &\longmapsto a. \end{aligned} \tag{5.4}$$

Define

$$g = p \circ i,$$

where i and p are defined in (5.3) and (5.4) respectively, and let f_{λ_0} be the mapping defined in (5.2). We then have

$$g \circ f_{\lambda_0} \sim \mathrm{id}_{\Gamma_1} \quad \text{and} \quad (f_{\lambda_0} \circ g) \sim \mathrm{id}_{J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b}}.$$

Therefore f_{λ_0} defines a homotopy equivalence and hence $J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b_1}$ and Γ_1 have the same type of homotopy. □

The proof of Theorem 1.4 is completed. Let $b > 0$ such that $2b < b_1$ and (5.1) holds. By the result of Lemma 5.1, there exists $\gamma_0 > 0$ such that for $0 < |\gamma| < \gamma_0$ we have

$$J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b} \subset J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+2b} \subset J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+3b}. \tag{5.5}$$

Arguing by contradiction and assuming that (1.1) has no positive solution for $K = 1 + \gamma K_0$, it follows by Lemma 5.2 that

$$J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+3b} \text{ retracts by deformation on } J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b},$$

and

$$J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b} \text{ retracts by deformation on } \bigcup_{y\in C^\infty_{\Gamma_1}} W_u^\infty(y)_\infty. \tag{5.6}$$

Here $W_u^\infty(y)_\infty$ is the unstable manifold at infinity of the critical point at infinity $(y)_\infty$. Therefore by (5.5) we get

$$J_0^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+2b} \text{ retracts by deformation on } J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b},$$

and since $2b < b_1$, we deduce from Lemma 5.6 that

$$J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b} \text{ is homotopy-equivalent to } \Gamma_1. \tag{5.7}$$

For a topological space M, let $\chi(M)$ be the Euler–Poincaré characteristic of M. It follows from (5.7) that

$$\chi\left(J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b}\right) = \chi(\Gamma_1). \tag{5.8}$$

Computing the Euler–Poincaré characteristic in (5.6), we find after recalling that $\dim W_u^\infty(y)_\infty = i(y)_\infty$, where $i(y)_\infty$ is defined in Lemma 5.2, that

$$\chi\left(J_\gamma^{\left(\frac{S^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}+b}\right) = \sum_{y\in C^\infty_{\Gamma_1}} (-1)^{i(y)_\infty}. \tag{5.9}$$

Assertions (5.8) and (5.9) yield

$$\sum_{y\in C^\infty_{\Gamma_1}} (-1)^{i(y)_\infty} = \chi(\Gamma_1).$$

This contradicts the hypothesis (v) of Theorem 1.4. This finishes the proof. □

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