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LIE IDEALS IN PROPERLY INFINITE C^*-ALGEBRAS

Hannes Thiel

Abstract: We show that every Lie ideal in a unital, properly infinite C^*-algebra is commutator-equivalent to a unique two-sided ideal. It follows that the Lie ideal structure of such a C^*-algebra is concisely encoded by its lattice of two-sided ideals. This answers a question of Robert in this setting.

We obtain similar structure results for Lie ideals in unital, real rank zero C^*-algebras without characters. As an application, we show that every Lie ideal in a von Neumann algebra is related to a unique two-sided ideal, which solves a problem of Brešar, Kissin, and Shulman.

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1. Introduction

A *Lie ideal* in a C^*-algebra A is a linear subspace $L \subseteq A$ such that for every $a \in A$ and $x \in L$ the commutator $[a,x] := ax - xa$ belongs to L. The study of Lie ideals in C^*-algebras has a long history, and we refer to Section 2 in [**25**] for an overview of available results and directions of active research.

Immediate examples of Lie ideals in a C^*-algebra A are all subspaces contained in the center $Z(A)$ and all subspaces containing the commutator subspace $[A,A]$. If A is unital and simple, then this does in fact describe all Lie ideals—a consequence of Herstein's description of Lie ideals in simple rings [**20**]. Since the center of a unital, simple C^*-algebra A contains only scalar multiples of the identity, a subspace L is a Lie ideal in A if and only if $L = \{0\}$, $L = \mathbb{C}1$, or $[A,A] \subseteq L$.

Pop showed that a unital C^*-algebra satisfies $A = [A,A]$ if and only if A has no tracial states [**30**]. Consequently, a simple, unital C^*-algebra A without tracial states has exactly three Lie ideals: $\{0\}$, $\mathbb{C}1$, and A. Cuntz and Pedersen gave a description of the closure $\overline{[A,A]}$ as the intersection of the kernels of all tracial states on A [**10**]. In particular, a simple, unital C^*-algebra A with a unique tracial state has exactly four closed Lie ideals: $\{0\}$, $\mathbb{C}1$, $\overline{[A,A]}$, and A. This was first observed by Marcoux and Murphy [**26**], generalizing the case of UHF algebras, which was first proved by Marcoux [**24**].

Ng and Robert showed that $[A,A]$ is closed in every pure, exact C^*-algebra [**28**]. It follows that a simple, unital, exact, pure C^*-algebra A with a unique tracial state has exactly four Lie ideals: $\{0\}$, $\mathbb{C}1$, $[A,A]$, and A. This applies for example to UHF algebras, to irrational rotation algebras, and to the Jiang–Su algebra (where it was first shown by Ng [**27**]). It also applies to many, and possibly all, simple, reduced group C^*-algebras. Indeed, if G is a countable, exact, acylindrically hyperbolic group with trivial finite radical and with the rapid decay property, then $C^*_{\mathrm{red}}(G)$ has strict comparison [**1**], and therefore is pure [**3**], and has a unique tracial state [**9**].

Universitat Autònoma de Barcelona

Turning towards Lie ideals in nonsimple C^*-algebras, it is clear that the structure of the lattice of two-sided ideals will play a role. A systematic study of the interplay between Lie and two-sided ideals was initiated by Brešar, Kissin, and Shulman [**8**]. We first introduce the necessary notation. Following the usual convention, given subspaces $M, N \subseteq A$, we use $[M, N]$ to denote the subspace of A generated by the set of commutators $[x, y]$ for $x \in M$ and $y \in N$. Further, given a subspace $K \subseteq A$, we set

$$T(K) := \{a \in A : [A, a] \subseteq K\}.$$

This 'derived subspace' was first considered by Herstein [**20**], and it is denoted by $N(K)$ in [**8**].

Given a two-sided ideal $I \subseteq A$, Brešar, Kissin, and Shulman observed that a subspace $L \subseteq A$ is a Lie ideal whenever

$$[A, I] \subseteq L \subseteq T([A, I]).$$

We say that a Lie ideal L is *embraced by* I if it satisfies these inclusions. The Lie ideals that are embraced by some two-sided ideal are thought of as 'unsurprising'. A major question is if there exist any Lie ideals that do not arise this way:

Problem A (Brešar, Kissin, and Shulman [**8**, p. 74])**.** Given a C^*-algebra A, is every Lie ideal of A embraced by some two-sided ideal of A?

To study this problem, two further notions were introduced in [**8**]: Given a Lie ideal $L \subseteq A$ and a two-sided ideal $I \subseteq A$, we say that L is *commutator-equivalent to* I if

$$[A, I] = [A, L].$$

Further, we say that L is *related to* I if

$$[A, I] \subseteq L \subseteq T(I).$$

It is easy to see that these constitute a sufficient and a necessary conditions for embracement. Indeed, if L is commutator-equivalent to I, then L is embraced by I. Further, if L is embraced by I, then L is related to I.

It was shown in [**8**, Theorem 5.27] that every Lie ideal L in a C^*-algebra A is topologically commutator-equivalent to a two-sided ideal I in the sense that $[A, L]$ and $[A, I]$ have the same closure. This leads to a description of all *closed* Lie ideals in terms of *closed* two-sided ideals. However, this does not show that all closed Lie ideals in a C^*-algebra are commutator-equivalent to (let alone embraced by or related to) a two-sided ideal. For general nonclosed Lie ideals even less is known, and Problem A as well as the following stronger problem remain open:

Problem B (Robert [**31**, Question 1.12])**.** Given a C^*-algebra A, is every Lie ideal of A commutator-equivalent to some two-sided ideal of A?

The results mentioned at the beginning of the introduction show that every Lie ideal in a unital, simple C^*-algebra A is commutator-equivalent to a two-sided ideal, which gives positive solutions to Problems A and B in this setting. Indeed, the only two-sided ideals are $\{0\}$ and A, and a Lie ideal is commutator-equivalent to $\{0\}$ if and only if it is contained in $Z(A)$, while a Lie ideal is commutator-equivalent to A if and only if it contains $[A, A]$. (This uses that $[A, A] = [A, [A, A]]$, which follows for example from [**17**, Corollary 3.4].)

In [**13**], Fong, Miers, and Sourour showed that every Lie ideal in the von Neumann algebra of bounded, linear operators on a separable Hilbert space is embraced by a two-sided ideal. This was substantially extended by Brešar, Kissin, and Shulman, who showed in [**8**, Theorem 5.19] that every Lie ideal in a von Neumann algebra is commutator-equivalent to a two-sided ideal, thereby providing positive solutions to Problems A and B for von Neumann algebras.

Our main result is a positive solution to Problem B for unital, properly infinite C^*-algebras.

Theorem C (5.3)**.** *Let L be a Lie ideal in a unital, properly infinite C^*-algebra A. Then L is commutator-equivalent to the two-sided ideal $I := A[A,L]A$, and hence also embraced by I, and related to I. Further, I is the only two-sided ideal of A to which L is related.*

The proof of Theorem C is inspired by a result of Marcoux [**24**], who showed that if B is a unital C^*-algebra and $n \geq 2$, then every Lie ideal in the matrix algebra $A = M_n(B)$ is related to a two-sided ideal in A. Our main technical advancement is a generalization of this result to unital C^*-algebras A that admit a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$.

The next result summarizes Theorems 2.7, 2.9, 3.8, and 3.10.

Theorem D. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$. The following statements hold:*

(1) *Given a Lie ideal $L \subseteq A$, for the two-sided ideal $I = A[A,L]A$, we have*

$$[A,I] = [A,[A,L]] \subseteq [A,L] \subseteq I \quad \text{and} \quad [A,I] \subseteq L \subseteq T(I).$$

Further, I is the only two-sided ideal of A to which L is related.

(2) *A Lie ideal $L \subseteq A$ is embraced by some two-sided ideal (which then necessarily is $A[A,L]A$) if and only if L is commutator-equivalent to some two-sided ideal, if and only if $[A,L] = [A,[A,L]]$.*

(3) *Given a two-sided ideal $I \subseteq A$, we have*

$$I = A[A,I]A \quad \text{and} \quad [A,I] = [A,[A,I]].$$

(4) *There is a natural bijection between the set $\mathcal{I}$ of two-sided ideals in A and the set*

$$\mathcal{L} = \{L \subseteq A \text{ Lie ideal} \colon L = [A,L]\}$$

given by the maps $I \mapsto [A,I]$ and $L \mapsto A[A,L]A$.

We note that Theorem D applies to all unital, properly infinite C^*-algebras, which then yields Theorem C. It also applies to unital, real rank zero C^*-algebras without characters, and in particular to all von Neumann algebras with zero commutative summand, which we use to recover the solution to Problem B for von Neumann algebras from [**8**]. Our approach also shows that a Lie ideal in a von Neumann algebra without commutative summand is related to a unique two-sided ideal, which solves [**8**, Problem 5.21].

Theorem E (6.1)**.** *Let L be a Lie ideal in a von Neumann algebra M. Then L is commutator-equivalent to the two-sided ideal $I := M[M,L]M$, and hence also embraced by I, and related to I.*

If M has zero commutative summand, then I is the only two-sided ideal of M to which L is related.

Conventions. Given a C^*-algebra, we use A_+ to denote its collection of positive elements, and we use $\widetilde{A}$ to denote the minimal unitization. The absolute value of an element $x \in A$ is defined as $|x| := (x^*x)^{\frac{1}{2}}$.

2. Lie ideals in C^*-algebras with weakly divisible unit

In this section, we prove that every Lie ideal L in a C^*-algebra A that admits a unital $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to A$ is related to a unique two-sided ideal I in A; see Theorem 2.7. We further show that L is commutator-equivalent to I if and only if $[A,L]=[A,[A,L]]$; see Theorem 2.9.

Following [**29**, Definition 5.1], a nonzero projection p in a C^*-algebra is said to be *weakly divisible of degree* n if there exists a unital $*$-homomorphism

$$M_{n_1}(\mathbb{C})\oplus M_{n_2}(\mathbb{C})\oplus\cdots\oplus M_{n_r}(\mathbb{C})\to pAp$$

for some natural numbers $r\geq 1$ and $n_1,\ldots,n_r\geq n$. The $*$-homomorphism may additionally be assumed to be injective.

As noted in [**29**, p. 164], every matrix algebra $M_n(\mathbb{C})$ for $n\geq 2$ admits a unital (not necessarily injective) $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to M_n(\mathbb{C})$, and it follows that a projection p weakly divisible of degree 2 if and only if there exists a unital (not necessarily injective) $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to pAp$. We deduce that the unit in a C^*-algebra A is weakly divisible of degree 2 if and only if there exists an injective $*$-homomorphism

$$M_2(\mathbb{C})\to A,\quad M_3(\mathbb{C})\to A,\quad \text{or}\quad M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to A.$$

In the first two cases, we get $A\cong M_2(B)$ and $A\cong M_3(C)$ for suitable unital C^*-algebras B and C, and for such matrix algebras it was shown by Marcoux ([**24**, Theorem 2.6]) that every Lie ideal is related to a two-sided ideal (which is also unique, as shown in [**8**, Corollary 4.18]).

Our main contribution to Theorem 2.7 is therefore in the case that A admits a unital (injective) $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to A$ but no unital $*$-homomorphism $M_n(\mathbb{C})\to A$ for $n=2$ or $n=3$ (or actually any $n\geq 2$). The Cuntz algebra $\mathcal{O}_\infty$ is an important such example. (Using K-theory, we see that there exists no unital $*$-homomorphism $M_n(\mathbb{C})\to\mathcal{O}_\infty$ for any $n\geq 2$. On the other hand, $\mathcal{O}_\infty$ admits a unital $*$-homomorphism from $M_2(\mathbb{C})\oplus M_3(\mathbb{C})$ as explained in the proof of Theorem 5.3.)

Our approach is inspired by the methods developed by Marcoux ([**24**]) to describe Lie ideals in matrix algebras, as well as the techniques of Brešar, Kissin, and Shulman ([**8**, Section 3]) to study Lie ideals in algebras that decompose with respect to an idempotent. However, the result does not simply follow by combining these methods, for example by applying [**8**, Theorem 3.4] for a suitable idempotent in the C^*-algebra. Specifically, given a unital $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to A$, we do not decompose A with respect to the projection p given by the image of the unit of $M_2(\mathbb{C})$, in which case $pAp+(1-p)A(1-p)$ is a direct sum of matrix algebras (whence its Lie ideals are related to two-sided ideals), but it is unclear if (3.17) in [**8**, Theorem 3.4] holds (or if p is locally cyclic in the sense of [**8**, Definition 3.5]). Instead, we use a 'mixed' decomposition with respect to a projection g that is given by the image of the sum of a rank one projection in $M_2(\mathbb{C})$ with a rank one projection in $M_3(\mathbb{C})$, as explained in Paragraph 2.1. In this case, one can show that g is locally cyclic, which is the idea underlying the proof that $A_{12}LA_{12}\subseteq[A,[A,L]]$ in Lemma 2.2. On the other hand, the algebra $gAg+(1-g)A(1-g)$ is not a sum of matrix algebras and it is therefore not immediate that its Lie ideals are related to two-sided ideals.

2.1. Let A be a unital C^*-algebra that admits a unital $*$-homomorphism

$$\varphi\colon M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to A.$$

Let $(e_{ij})_{i,j=1,2}$ denote the image of matrix units in $M_2(\mathbb{C})$ under φ, and similarly let $(f_{ij})_{i,j=1,2,3}$ denote the image of matrix units in $M_3(\mathbb{C})$ under φ. We do not assume that φ is injective, and therefore the e_{ij} or f_{ij} may be zero.

Set

$$g := e_{11} + f_{11} \quad \text{and} \quad h := e_{22} + f_{22} + f_{33}.$$

Then $1 = g + h$, and we view elements in A as operator matrices with respect to the decomposition induced by g and h. More formally, we set

$$A_{11} := gAg, \quad A_{12} := gAh, \quad A_{21} := hAg, \quad \text{and} \quad A_{22} := hAh,$$

and every element a in A can be written uniquely as a sum $a = a_{11} + a_{12} + a_{21} + a_{22}$ with $a_{ij} \in A_{ij}$ for $i, j = 1, 2$.

Lemma 2.2. *Retain the notation from Paragraph 2.1, and let $L \subseteq A$ be a Lie ideal. Then the sets*

$$A_{11}LA_{22}, \quad A_{22}LA_{11}, \quad A_{12}LA_{12}, \quad \textit{and} \quad A_{21}LA_{21}$$

are all contained in $[A, [A, L]]$, and thus also in $[A, L]$ and in L.

Proof: We verify that $A_{11}LA_{22} \subseteq [A, [A, L]]$. We first show that $gLh \subseteq L$. To see this, let $x \in L$. Then

$$gx - xg = [g, x] \in [A, L] \quad \text{and} \quad gx - 2gxg + xg = [g, [g, x]] \in [A, [A, L]] \subseteq [A, L].$$

Adding these expressions, we get

$$2gxh = 2gx - 2gxg \in [A, L] \subseteq L,$$

and using that $hg = 0$ we get

$$gxh = [g, gxh] = \left[\tfrac{1}{2}g, 2gxh\right] \in [A, L] \subseteq L.$$

We have shown that $gLh \subseteq L$.

Now, given $x \in L$ and $a, b \in A$, we have

$$gagxhbh = [[gag, gxh], hbh] \in [[A, L], A] = [A, [A, L]].$$

We have shown that $A_{11}LA_{22} \subseteq [A, [A, L]]$.

Analogously, one verifies that $A_{22}LA_{11} \subseteq [A, [A, L]]$.

We verify that $A_{12}LA_{12} \subseteq [A, [A, L]]$. Given $v \in A_{12}$ and $y \in L$, we have

$$vyv = \left[-\tfrac{1}{2}v, [v, y]\right] \in [A, [A, L]]. \tag{2.1}$$

Set

$$v_1 := e_{12} + f_{12}, \quad w_1 := e_{21} + f_{21}, \quad v_2 := e_{12} + f_{13}, \quad \text{and} \quad w_2 := e_{21} + f_{31}.$$

Then $v_1, v_2 \in A_{12}$ and $w_1, w_2 \in A_{21}$, and we have

$$v_1w_1 = v_2w_2 = e_{11} + f_{11} = g, \quad w_1v_1 = e_{22} + f_{22}, \quad \text{and} \quad w_2v_2 = e_{22} + f_{33}.$$

Now, given $x \in L$ and $a, b \in A_{12} = gAh$, set

$$b' = b\left(\tfrac{1}{2}e_{22} + f_{22} + f_{33}\right).$$

Then b' belongs to gAh and we have

$$b = b'(2e_{22} + f_{22} + f_{33}).$$

We have already proved that $hAhLgAg \subseteq L$, which gives

$$w_1axb'w_1 \in L \quad \text{and} \quad w_2axb'w_2 \in L.$$

Applying (2.1) for $y = w_1axb'w_1$ and $v = v_1$, we get

$$axb'(e_{22} + f_{22}) = v_1(w_1axb'w_1)v_1 \in [A, [A, L]].$$

Analogously, we obtain that

$$axb'(e_{22} + f_{33}) = v_2(w_2axb'w_2)v_2 \in [A, [A, L]],$$

and consequently

$$axb = axb'(2e_{22} + f_{22} + f_{33}) \in [A, [A, L]].$$

We have shown that $A_{12}LA_{12} \subseteq [A, [A, L]]$.

Analogously, one verifies that $A_{21}LA_{21} \subseteq [A, [A, L]]$. □

Lemma 2.3. *Retain the notation from Paragraph 2.1, and let $L \subseteq A$ be a Lie ideal. Then*

$$AgLhA = AhLgA.$$

Proof: Using at the first step that $A = AgA = AhA$ since g and h are full, and applying at the third step that $hAgLhAg \subseteq L$ by Lemma 2.2, we get

$$AgLhA = AhAgLhAgA = Ah(hAgLhAg)gA \subseteq AhLgA.$$

Analogously, we have

$$AhLgA = AgAhLgAhA = Ag(gAhLgAh)hA \subseteq AgLhA,$$

as desired. □

Lemma 2.4. *Retain the notation from Paragraph 2.1, and let $L \subseteq A$ be a Lie ideal. Then*

$$[A, L] \subseteq AgLhA.$$

Proof: To simplify notation, we set $I = AgLhA$. We have

$$[A, gLh] \subseteq AgLh + gLhA \subseteq I.$$

Further, using Lemma 2.3 at the third step, we have

$$[A, hLg] \subseteq AhLg + hLgA \subseteq AhLgA = AgLhA = I.$$

Thus, given $x \in L$, we have $[A, gxh + hxg] \subseteq I$. Note that $gxh, hxg \in L$ by Lemma 2.2. It follows that

$$gxg + hxh = x - gxh - hxg \in L.$$

We need to show that $[A, gxg + hxh] \subseteq I$. Using additivity of the Lie bracket and that I is an additive subgroup, it suffices to verify that $[A_{ij}, gxg + hxh] \subseteq I$ for $i, j = 1, 2$.

We show that $[A_{12}, gxg + hxh] \subseteq I$. Given $a \in A$, we have

$$[gah, gxg + hxh] = gahxh - gxgah.$$

Using that $gxg + hxh \in L$, we have $[gah, gxg + hxh] \in L$, and thus

$$[gah, gxg + hxh] \in L \cap gAh = gLh \subseteq AgLhA = I.$$

We have shown that $[A_{12}, gxg + hxh] \subseteq I$.

Similarly, and using also Lemma 2.3 at the last step, we get

$$[hag, gxg + hxh] = hagxg - hxhag \in L \cap hAg = hLg \subseteq AhLgA = I$$

for every $a \in A$. This shows that $[A_{21}, gxg + hxh] \subseteq I$.

We show that $[A_{11}, gxg+hxh] \subseteq I$. Given $a \in A$, set $y = gagxg - gxgag$. Then

$$y = gagxg - gxgag = [gag, gxg+hxh] \in [A,L] \subseteq L.$$

Then

$$y(e_{12}+f_{12}) = [y, e_{12}+f_{12}] \in [L,A] \subseteq L$$

and thus

$$y(e_{12}+f_{12}) \in L \cap gAh = gLh \subseteq I.$$

We deduce that

$$y = y(e_{12}+f_{12})(e_{21}+f_{21}) \in IA = I.$$

We show that $[A_{22}, gxg+hxh] \subseteq I$. Given $b \in A$, set $z = hbhxh - hxhbh$. Then

$$z = hbhxh - hxhbh = [hbh, gxg+hxh] \in L \cap A_{22}.$$

Then

$$(e_{12}+f_{12})z = [e_{12}+f_{12}, z] \in L \cap gAh \subseteq I \quad \text{and} \quad f_{13}z = [f_{13}, z] \in I.$$

We deduce that

$$z = (e_{21}+f_{21})(e_{12}+f_{12})z + f_{31}f_{13}z \in AI + AI = I,$$

as desired. ☐

Proposition 2.5. *Retain the notation from Paragraph 2.1, and let $L \subseteq A$ be a Lie ideal. Then*

$$A[A,L] = A[A,L]A = AgLhA = AhLgA.$$

Proof: In general, if $M \subseteq A$ is a Lie ideal, then $AMA = AM$; see, for example, [**15**, Lemma 3.2]. Since $[A,L]$ is a Lie ideal, we get $A[A,L] = A[A,L]A$.

By Lemma 2.4, we have $[A,L] \subseteq AgLhA$, and thus $A[A,L]A \subseteq AgLhA$. By Lemma 2.3, we have $AgLhA = AhLgA$. Finally, by Lemma 2.2, we have $gLh \subseteq [A,L]$, and thus $AgLhA \subseteq A[A,L]A$. ☐

Lemma 2.6. *Retain the notation from Paragraph 2.1, and let $L \subseteq A$ be a Lie ideal, and set $I = A[A,L]A$. Then*

$$[A,I] \subseteq [A,[A,L]].$$

Proof: For each $i,j = 1,2$, set $I_{ij} := I \cap A_{ij}$. Since I is a two-sided ideal, we have $I = I_{11} + I_{12} + I_{21} + I_{22}$. By Proposition 2.5, we have $I = AgLhA$, and thus

$$I = AgLhA = \underbrace{gAgLhAg}_{I_{11}} + \underbrace{gAgLhAh}_{I_{12}} + \underbrace{hAgLhAg}_{I_{21}} + \underbrace{hAgLhAh}_{I_{22}}.$$

By additivity of the Lie bracket and since $[A,L]$ is an additive subgroup, it suffices to show that $[A, I_{ij}] \subseteq [A,[A,L]]$ for each $i,j = 1,2$.

By Lemma 2.2, we have

$$I_{12} = gAgLhAh \subseteq [A,L] \quad \text{and} \quad I_{21} = hAgLhAg \subseteq [A,L],$$

and thus $[A, I_{12}] \subseteq [A,[A,L]]$ and $[A, I_{21}] \subseteq [A,[A,L]]$.

We show that $[A, I_{11}] \subseteq [A,[A,L]]$. It suffices to verify that $[A_{ij}, I_{11}] \subseteq [A,[A,L]]$ for each $i,j = 1,2$. Using that $hg = 0$, and using Lemma 2.2 at the last step, we get

$$[A_{12}, I_{11}] = [gAh, gAgLhAg] = -(gAgLhAg)(gAh) = gAgLhAgAh \subseteq [A,[A,L]].$$

Similarly, we have

$$[A_{21}, I_{11}] = [hAg, gAgLhAg] = (hAg)(gAgLhAg) = hAgAgLhAg \subseteq [A, [A, L]].$$

We also have $[A_{22}, I_{11}] = \{0\} \subseteq [A, [A, L]]$.

Further, using Lemma 2.2 at the last step, we have

$$I_{11} = (gAgLh)Ag \subseteq [gAgLh, Ag] + Ag(gAgLh) \subseteq [A, L] + AgLh.$$

Using again Lemma 2.2 at the last step, we get

$$\begin{aligned}[A_{11}, I_{11}] &\subseteq [A_{11}, [A, L]] + [A_{11}, AgLh] \subseteq [A, [A, L]] + [gAg, AgLh] \\ &\subseteq [A, [A, L]] + gAgAgLh \subseteq [A, [A, L]].\end{aligned}$$

We have verified that $[A, I_{11}] \subseteq [A, [A, L]]$.

Analogously, one shows that $[A, I_{22}] \subseteq [A, [A, L]]$, which completes the proof. ☐

The next result verifies statement (1) in Theorem D.

Theorem 2.7. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, and let $L \subseteq A$ be a (not necessarily closed) Lie ideal. Then, for the two-sided ideal $I := A[A, L]A$, we have*

$$[A, I] = [A, [A, L]] \subseteq [A, L] \subseteq I \quad \text{and} \quad [A, I] \subseteq L \subseteq T(I). \tag{2.2}$$

Further, I is the only two-sided ideal of A to which L is related.

Proof: Using Lemma 2.4 at the first step, and Proposition 2.5 at the second step, we have

$$[A, L] \subseteq AgLhA = I.$$

This implies that $[A, [A, L]] \subseteq [A, I]$, and the converse inclusion $[A, I] \subseteq [A, [A, L]]$ is shown in Lemma 2.6. Using that L is a Lie ideal, we have $[A, L] \subseteq L$, and therefore $[A, [A, L]] \subseteq [A, L]$. This shows the left chain of inclusions in (2.2).

The inclusion $[A, I] \subseteq L$ follows using that $[A, I] \subseteq [A, L]$ and $[A, L] \subseteq L$. Finally, since $[A, L] \subseteq I$, we have $L \subseteq T(I)$.

To show the uniqueness of I, let J be a two-sided ideal that is related to L, that is, such that

$$[A, J] \subseteq L \subseteq T(J).$$

Then $[A, L] \subseteq J$, and since I is the two-sided ideal generated by $[A, L]$ we get $I \subseteq J$.

Applying Lemma 2.2 at the first step for J, considered as a Lie ideal, we get

$$gJh \subseteq [A, J] \subseteq L,$$

and thus

$$gJh \subseteq L \cap gAh = gLh.$$

Since g and h are full, we have $A = AgA$ and $A = AhA$, and consequently

$$J = AJA = AgAJAhA = AgJhA \subseteq AgLhA = I.$$

In conclusion, we get $I = J$, as desired. ☐

Corollary 2.8. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, and let $L \subseteq A$ be a closed Lie ideal. Then, the two-sided ideal $I := A[A, L]A$ is closed.*

Proof: By Theorem 2.7, L is related to I, that is, we have $[A, I] \subseteq L \subseteq T(I)$. Using that L is closed, it follows that the closure $\overline{I}$ satisfies $[A, \overline{I}] \subseteq L \subseteq T(\overline{I})$. Since I is the only two-sided ideal of A to which L is related (Theorem 2.7), we get $I = \overline{I}$. ☐

The next result verifies statement (2) in Theorem D.

Theorem 2.9. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, and let $L \subseteq A$ be a (not necessarily closed) Lie ideal. Then the following statements are equivalent:*

(1) *The Lie ideal L is commutator-equivalent to the two-sided ideal $A[A,L]A$.*

(2) *The Lie ideal L is embraced by some two-sided ideal.*

(3) *We have $[A,L] = [A,[A,L]]$.*

Proof: It is clear that (1) implies (2). Assuming (2), let us show that (3) holds. Set $I := A[A,L]A$, and let $J \subseteq A$ be some two-sided ideal that embraces L. Then L is also related to J. It follows that $I = J$, since I is the unique two-sided ideal to which L is related, by Theorem 2.7. It follows that $L \subseteq T([A,I])$. Using this at the first step, and using Theorem 2.7 at the second step, we get

$$[A,L] \subseteq [A,I] = [A,[A,L]],$$

which verifies (3).

Assuming (3), let us show that (1) holds. Using that assumption at the first step, and Theorem 2.7 at the second step, we get

$$[A,L] = [A,[A,L]] = [A,I],$$

which verifies (1). □

The next result is essentially contained in [**29**, Proposition 5.7]. We only show how to remove the separability assumption. A *character* on a C^*-algebra A is a one-dimensional, irreducible representation, that is, a surjective $*$-homomorphism $A \to \mathbb{C}$.

Proposition 2.10 (Perera, Rørdam [**29**]). *Let A be a unital C^*-algebra of real rank zero. Then A has no characters if and only if A admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$.*

Proof: The backwards implication is clear. To show the forward implication, assume that A has no characters. By [**22**, Lemma 3.5], there exists a *separable* sub-C^*-algebra $A_0 \subseteq A$ containing the unit of A. Using that real rank zero is separably inheritable (see [**6**, Section II.8.5]), we obtain a *separable* sub-C^*-algebra $B \subseteq A$ of real rank zero with $A_0 \subseteq B$. It follows that B has no characters, which allows us to apply [**29**, Proposition 5.7] for B. We obtain a unital $*$-homomorphism from $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ to B, and thus also to A. □

Example 2.11. Let A be a unital C^*-algebra of real rank zero that has no characters. Then Theorem 2.7 applies to A. In particular, every Lie ideal in A is related to a (unique) two-sided ideal.

We expect that the following question has a positive answer.

Problem 2.12. Let A be a unital C^*-algebra that has no characters. Is every Lie ideal in A related to a (unique) two-sided ideal?

3. Two-sided ideals in C^*-algebras with weakly divisible unit

In this section, we prove that every two-sided ideal I in a C^*-algebra A that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$ satisfies $I = A[A,I]A$, and we describe $[A,I]$ as the subspace spanned by certain square-zero elements in I; see Theorem 3.8. If I is a Dixmier ideal (in which case there is a well-defined ideal $I^{\frac{1}{2}}$ satisfying $(I^{\frac{1}{2}})^2 = I$), then $I = [A,I] + [A,I^{\frac{1}{2}}]^2$; see Theorem 3.14. In this case every element in I is a sum of square-zero elements in I and of products of pairs of square-zero elements in $I^{\frac{1}{2}}$.

The next definition generalizes the notion of orthogonally factorizable square-zero elements in a ring to a relative notion with respect to a two-sided ideal; see [**16**, Definition 5.1].

Definition 3.1. Let I be a two-sided ideal in a C^*-algebra A. We let $N_2(I)$ denote the square-zero elements in I, that is,

$$\mathrm{N}_2(I) := \{x \in I : x^2 = 0\}.$$

We further set

$$\mathrm{FN}_2(A, I) := \{x \in I : \text{there exist } a, b \in A,\ y \in I \text{ with } x = ayb \text{ and } ba = 0\}.$$

Lemma 3.2. *Let $I \subseteq A$ be a two-sided ideal in a C^*-algebra, and let $x \in \mathrm{FN}_2(A, I)$. Then there exist $a, b \in A_+$ and $y \in \mathrm{FN}_2(A, I) \subseteq I$ such that*

$$x = ayb \quad \textit{and} \quad ab = ba = 0.$$

Proof: By definition, there are $c, d \in A$ and $z \in I$ such that

$$x = czd \quad \text{and} \quad dc = 0.$$

Let $c = v|c|$ and $d = w|d|$ the polar decompositions in A^{**} and set

$$a := |c^*|^{\frac{1}{2}}, \quad b := |d|^{\frac{1}{2}}, \quad \text{and} \quad y := |c^*|^{\frac{1}{2}} vzw|d|^{\frac{1}{2}}.$$

By properties of the polar decomposition (see, for example, [**14**, Proposition 2.1]), we have $|c^*|^{\frac{1}{2}}v, w|d|^{\frac{1}{2}} \in A$, and thus $y \in A$. Further, we have $x = ayb$.

Since $dc = 0$, we have $(d^*d)^n(cc^*)^m = 0$ for every $m, n \geq 1$. It follows that $p(d^*d)q(cc^*) = 0$ for all polynomials p and q with vanishing constant terms. Using functional calculus, we get $f(d^*d)g(cc^*) = 0$ for all continuous functions $f, g\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = g(0) = 0$. In particular, we get $|d|^{\frac{1}{2}}|c^*|^{\frac{1}{2}} = 0$. Thus, we have $ba = 0$, and then also $ab = 0$.

To show that y belongs to $\mathrm{FN}_2(A, I)$, consider the decomposition

$$y = |c^*|^{\frac{1}{4}}(|c^*|^{\frac{1}{4}} vzw|d|^{\frac{1}{4}})|d|^{\frac{1}{4}}.$$

Then $|c^*|^{\frac{1}{4}} vzw|d|^{\frac{1}{4}}$ belongs to I, and we have also seen above that $|c^*|^{\frac{1}{4}}$ and $|d|^{\frac{1}{4}}$ have zero product, whence $y \in \mathrm{FN}_2(A, I)$. □

The next result is analogous to [**16**, Lemma 5.2].

Lemma 3.3. *Let $I \subseteq A$ be a two-sided ideal in a C^*-algebra. Then*

$$\mathrm{FN}_2(A, I) \subseteq [A, \mathrm{FN}_2(A, I)] \subseteq [A, [A, I]] \subseteq [A, I] \quad \textit{and} \quad \mathrm{FN}_2(A, I) \subseteq \mathrm{N}_2(I).$$

Proof: Let $x \in \mathrm{FN}_2(A, I)$. By Lemma 3.2, we can pick $a, b \in A_+$ and $y \in I$ such that $x = ayb$ and $ba = 0$. As in the proof of Lemma 3.2, we see that $ba^{\frac{1}{2}} = 0$. It follows that $a^{\frac{1}{2}}yb \in \mathrm{FN}_2(A, I)$ and then

$$x = ayb = [a^{\frac{1}{2}}, a^{\frac{1}{2}}yb] \in [A, \mathrm{FN}_2(A, I)].$$

Since $\mathrm{FN}_2(A, I)$ is a subset of I, we have $[A, \mathrm{FN}_2(A, I)] \subseteq [A, I]$. Combining these results, and using at the last step that $[A, I]$ is a Lie ideal, we get

$$\mathrm{FN}_2(A, I) \subseteq [A, \mathrm{FN}_2(A, I)] \subseteq [A, [A, \mathrm{FN}_2(A, I)]] \subseteq [A, [A, I]] \subseteq [A, I].$$

Given $x = ayb \in \mathrm{FN}_2(A, I)$, with $a, b \in A$ and $y \in I$ such that $ba = 0$, we have

$$x^2 = (ayb)(ayb) = 0,$$

and thus $x \in \mathrm{N}_2(I)$. □

Remark 3.4. We note that Definition 3.1 and [**16**, Definition 5.1] agree when considering the ideal $I = A$ in a C^*-algebra A. Indeed, an element x in A belongs to $\mathrm{FN}_2(A)$ in the sense of [**16**, Definition 5.1] if there exist $s, t \in A$ such that $x = st$ and $ts = 0$, while x belongs to $\mathrm{FN}_2(A, A)$ as defined in Definition 3.1 if there exist $a, y, b \in A$ such that $x = ayb$ and $ba = 0$.

Given a factorization $x = ayb$ with $ba = 0$, we can use $s = a$ and $t = yb$. Conversely, given a factorization $x = st$ with $ts = 0$, it follows that $|t|^{\frac{1}{2}}s = 0$. We consider the polar decomposition $t = v|t|$ in A^{**}. Then $v|t|^{\frac{1}{2}}$ belongs to A, and we can use $a = s$, $y = v|t|^{\frac{1}{2}}$, and $b = |t|^{\frac{1}{2}}$.

Given a subset V in a complex vector space, we use $\mathrm{span}_{\mathbb{C}} V$ to denote the linear subspace generated by V. Following [**16**, Definition 2.6], we say that a $\mathbb{C}$-algebra A is *zero-product balanced* if for all $a, b, c \in A$ we have

$$ab \otimes c - a \otimes bc \in \mathrm{span}_{\mathbb{C}}\{u \otimes v \in A \otimes_{\mathbb{C}} A : uv = 0\} \subseteq A \otimes_{\mathbb{C}} A.$$

This is closely related to the concept of 'zero-product determination', for which we refer to [**7**]. By [**16**, Corollary 2.12], a (not necessarily unital) C^*-algebra is zero-product balanced as a $\mathbb{C}$-algebra if and only if it is zero-product determined.

We see that the class of C^*-algebras considered in this section are zero-product balanced:

Lemma 3.5. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$. Then A is zero-product balanced as a $\mathbb{C}$-algebra.*

Proof: As in Paragraph 2.1, let $(e_{ij})_{i,j=1,2}$ and $(f_{ij})_{i,j=1,2,3}$ be the images in A of matrix units in $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$. Then $e_{11}+f_{11}$ and $e_{22}+f_{22}$ are two full orthogonal projections in A. This allows us to apply [**31**, Corollary 3.8], which shows that A is generated by its projections as a $\mathbb{C}$-algebra, and thus is zero-product determined (hence, zero-product balanced) by [**7**, Theorem 2.3]. ☐

A two-sided ideal $I \subseteq A$ in a C^*-algebra is said to be *idempotent* if $I = I^2$, that is, if every element in I is of the form $x_1y_1 + \cdots + x_ny_n$ for some $n \in \mathbb{N}$ and suitable $x_j, y_j \in I$.

Proposition 3.6. *Let A be a C^*-algebra that is zero-product balanced as a $\mathbb{C}$-algebra, and let $I \subseteq A$ be a two-sided ideal such that $I = AIA$ (for example, A is unital or I is idempotent). Then*

$$[A, I] = [A, [A, I]] = \mathrm{span}_{\mathbb{C}} \mathrm{FN}_2(A, I).$$

Proof: The inclusions '$\supseteq$' hold by Lemma 3.3. The proof of the converse inclusion is based on a combination of the methods used in the proofs of Proposition 2.14 and Theorem 5.3 in [**16**], which in turn are inspired by that of [**7**, Theorem 9.1]. To verify that $[A, I] \subseteq \mathrm{span}_{\mathbb{C}} \mathrm{FN}_2(A, I)$, let $x \in A$ and $y \in I$. Set $F := \mathrm{span}_{\mathbb{C}} \mathrm{FN}_2(A, I)$, and define

$$\varphi_y \colon A \times A \to I/F, \quad \varphi_y(a, b) := bya + F \quad (a, b \in A).$$

Note that φ_y is a well-defined $\mathbb{C}$-bilinear map. Further, φ_y preserves zero-products since if $a, b \in A$ satisfy $ab = 0$, then $bya \in \mathrm{FN}_2(A, I) \subseteq F$. Using [**16**, Proposition 2.8] at the second step, we obtain that

$$cyab + F = \varphi_y(ab, c) = \varphi_y(a, bc) = bcya + F$$

and thus

$$bcya - cyab \in F = \mathrm{span}_{\mathbb{C}} \mathrm{FN}_2(A, I)$$

for all $a, b, c \in A$. The assumption that $I = AIA$ allows us to pick n and $c_j, a_j \in A$ and $y_j \in I$ for $j = 1, \ldots, n$ such that $y = \sum_j c_j y_j a_j$. Then

$$[x, y] = \sum_{j=1}^{n} [x, c_j y_j a_j] = \sum_{j=1}^{n} (x c_j y_j a_j - c_j y_j a_j x) \in \operatorname{span}_{\mathbb{C}} \mathrm{FN}_2(A, I),$$

which implies that $[A, I] \subseteq \operatorname{span}_{\mathbb{C}} \mathrm{FN}_2(A, I)$. ☐

Problem 3.7. Does there exist a two-sided ideal I in a C^*-algebra A such that $[A, [A, I]] \neq [A, I]$?

The next result verifies statement (3) in Theorem D.

Theorem 3.8. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, and let $I \subseteq A$ be a (not necessarily closed) two-sided ideal. Then*

$$I = A[A, I]A = A[A, I] \quad \text{and} \quad [A, I] = [A, [A, I]] = \operatorname{span}_{\mathbb{C}} \mathrm{FN}_2(A, I).$$

Proof: Considering I as a Lie ideal in A, it follows from Theorem 2.7 that $A[A, I]A$ is the unique two-sided ideal in A to which I is related. Since I is clearly related to itself, we get $I = A[A, I]A$. The inclusion $A[A, I] \subseteq A[A, I]A$ holds since A is unital. The converse inclusion holds since $[A, I]$ is a Lie ideal in A; see, for example, [**15**, Lemma 3.2].

Now the equality $[A, I] = [A, [A, I]]$ also follows from Theorem 2.7. Further, since A is zero-product balanced by Lemma 3.5, we have $[A, I] = \operatorname{span}_{\mathbb{C}} \mathrm{FN}_2(A, I)$ by Proposition 3.6. ☐

Remark 3.9. Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$. Applying Theorem 3.8 for the ideal A, we get

$$A = A[A, A]A,$$

which means that A is generated by its commutators as an ideal.

In fact, A is even generated by its commutators as a ring. Indeed, by [**18**, Theorem 5.15], every element in A is a sum of a commutator and the product of two commutators, and also the sum of three elements that are products of commutators.

The next result verifies statement (4) in Theorem D.

Theorem 3.10. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$. There is a natural bijection between the set $\mathcal{I}$ of two-sided ideals in A and the set*

$$\mathcal{L} := \{L \subseteq A \text{ Lie ideal} : L = [A, L]\}$$

given by the maps $I \mapsto [A, I]$ and $L \mapsto A[A, L]A$.

Proof: We denote the maps by $\alpha\colon \mathcal{I} \to \mathcal{L}$ and $\beta\colon \mathcal{L} \to \mathcal{I}$. Given $I \in \mathcal{I}$, using Theorem 3.8 at the second step, we have

$$\alpha(I) = [A, I] = [A, [A, I]] = [A, \alpha(I)]$$

which shows that α is well defined.

Next, we show that α and β are inverses of each other. First, given $I \in \mathcal{I}$, using Theorem 3.8 at the last step, we get

$$\beta(\alpha(I)) = \beta([A, I]) = A[A, I]A = I.$$

Second, given $L \in \mathcal{L}$, it follows from $L = [A, L]$ that $[A, L] = [A, [A, L]]$, and thus $L = [A, [A, L]]$. Using this at the last step, and using Theorem 2.7 at the third step, we get

$$\alpha(\beta(L)) = \alpha(A[A, L]A) = [A, A[A, L]A] = [A, [A, L]] = L,$$

as desired. □

Robert showed in [**31**, Lemma 2.1] that every nilpotent element in a C^*-algebra is a sum of commutators. With a view to Theorem 3.8, we ask:

Problem 3.11. Let I be a (not necessarily closed) two-sided ideal in a C^*-algebra A. Does every nilpotent element in I belong to $[A, I]$?

We give positive answers to Problem 3.11 for arbitrary two-sided ideals in von Neumann algebras (Proposition 6.2), and for semiprime two-sided ideals in C^*-algebras (Proposition 4.5).

Following [**14**, Definition 3.2], a *Dixmier ideal* in a C^*-algebra A is a two-sided ideal $I \subseteq A$ such that I is positively spanned (that is, $I = \mathrm{span}_{\mathbb{C}}(I \cap A_+)$) and hereditary (that is, if $a, b \in A_+$ satisfy $a \leq b$ and $b \in I$, then $a \in I$) and strongly invariant (that is, if $x \in A$ satisfies $x^*x \in I$, then $xx^* \in I$). Two-sided ideals in von Neumann algebras are automatically Dixmier ideals, which was essentially shown by Dixmier (see [**14**, Proposition 3.4]).

Given a Dixmier ideal I and $s \in (0, \infty)$, there is a unique Dixmier ideal whose set of positive elements is $\{a^s : a \in I \cap A_+\}$, and we use I^s to denote this ideal; see Proposition 3.7 and Definition 3.8 in [**14**]. We note that for $n \in \mathbb{N}$ the ring-theoretic definition of I^n as the two-sided ideal generated by $x_1 \cdots x_n$ for $x_1, \ldots, x_n \in I$ agrees with the 'new' definition of I^n as the linear span of elements a^n for $a \in I \cap A_+$. This leads to a well-behaved theory of roots and powers for Dixmier ideals ([**14**, Theorem 3.9]), which we recall for the convenience of the reader.

Theorem 3.12 (Gardella, Kitamura, Thiel [**14**])**.** *Let $I \subseteq A$ be a Dixmier ideal in a C^*-algebra. Then*

$$(I^s)^t = I^{st} = (I^t)^s \quad \textit{and} \quad I^s I^t = I^{s+t} = I^t I^s$$

for all $s, t \in (0, \infty)$.

Lemma 3.13. *Let $I \subseteq A$ be a Dixmier ideal in a C^*-algebra, let $x \in I$ with polar decomposition $x = v|x|$ in A^{**}, and let $s \in (0, \infty)$. Then*

$$v|x|^s, |x|^s, |x^*|^s, |x|^s v^* \in I^s.$$

Proof: Since I is a Dixmier ideal, we have $x^* \in I$, and thus $x^*x, xx^* \in I^2$. Then by Theorem 3.12 we get

$$|x| = (x^*x)^{\frac{1}{2}} \in (I^2)^{\frac{1}{2}} = I \quad \text{and} \quad |x^*| = (xx^*)^{\frac{1}{2}} \in (I^2)^{\frac{1}{2}} = I.$$

It follows that $|x|^s, |x^*|^s \in I^s$.

By [**14**, Proposition 3.10], given a Dixmier ideal $J \subseteq A$, an element $y \in A$ belongs to $J^{\frac{1}{2}}$ if and only if $y^*y \in J$. Applying this for the Dixmier ideal I^{2s}, and using that

$$(v|x|^s)^*(v|x|^s) = |x|^{2s} \in I^{2s},$$

we deduce that $v|x|^s \in (I^{2s})^{\frac{1}{2}} = I^s$. Since I^s is a Dixmier ideal, we further have $|x|^s v^* = (v|x|^s)^* \in I^s$. □

The next result showcases a situation when every element in a Dixmier ideal I is a sum of square-zero elements in I and products of pairs of square-zero elements in $I^{\frac{1}{2}}$. In Theorem 6.4, we strengthen this result for ideals in von Neumann algebras.

Theorem 3.14. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, and let $I \subseteq A$ be a Dixmier ideal. Then*

$$I = [A, I] + [A, I^{\frac{1}{2}}]^2.$$

Proof: We first show that

$$A \cdot \mathrm{FN}_2(A, I) \subseteq [A, I] + [A, I^{\frac{1}{2}}]^2.$$

Let $c \in A$ and $x \in \mathrm{FN}_2(A, I)$. By Lemma 3.2, we can pick $a, b \in A_+$ and $y \in I$ such that $x = ayb$ and $ab = ba = 0$. Set $z := yb \in I$, and let $z = v|z|$ be the polar decomposition in A^{**}. Then

$$cx = [ca^{\frac{1}{2}}, a^{\frac{1}{2}}z] + a^{\frac{1}{2}}zca^{\frac{1}{2}} = [ca^{\frac{1}{2}}, a^{\frac{1}{2}}z] + (a^{\frac{1}{2}}v|z|^{\frac{1}{2}})(|z|^{\frac{1}{2}}ca^{\frac{1}{2}}).$$

By Lemma 3.13, $v|z|^{\frac{1}{2}}$ and $|z|^{\frac{1}{2}}$ belong to $I^{\frac{1}{2}}$. Further, since $za = 0$, we get $|z|^{\frac{1}{2}}a^{\frac{1}{2}} = 0$, and hence

$$a^{\frac{1}{2}}v|z|^{\frac{1}{2}} = [a^{\frac{1}{2}}, v|z|^{\frac{1}{2}}] \in [A, I^{\frac{1}{2}}] \quad \text{and} \quad |z|^{\frac{1}{2}}ca^{\frac{1}{2}} = [-ca^{\frac{1}{2}}, |z|^{\frac{1}{2}}] \in [A, I^{\frac{1}{2}}].$$

It follows that cx belongs to $[A, I] + [A, I^{\frac{1}{2}}]^2$.

Using Theorem 3.8 at the first and second step, and using the above at the last step, we get

$$I = A[A, I] = \mathrm{span}_{\mathbb{C}}(A \cdot \mathrm{FN}_2(A, I)) \subseteq [A, I] + [A, I^{\frac{1}{2}}]^2.$$

On the other hand, we clearly have $[A, I] \subseteq I$. Using Theorem 3.12, we also have $[A, I^{\frac{1}{2}}]^2 \subseteq (I^{\frac{1}{2}})^2 = I$. □

4. Square-zero elements in semiprime ideals in C^*-algebras

In this section, we study the subspace N generated by the square-zero elements in a semiprime ideal in a C^*-algebra A. We show that ANA, the two-sided ideal generated by N, satisfies $ANA = [A, N] + N^2$; see Theorem 4.3. This generalizes results from [**17**, Section 4].

We then characterize semiprimeness for two-sided ideals in a C^*-algebra whose unit is weakly divisible of degree 2; see Theorem 4.6. In particular, such an ideal is semiprime if and only if it is generated by its commutators as an ideal (or as a ring).

We begin with some preparatory results about commutators involving Dixmier ideals, which might be of use in future work.

Lemma 4.1. *Let $I \subseteq A$ be a Dixmier ideal in a C^*-algebra. Then*

$$[A, I] \subseteq [I^{\frac{1}{2}}, I^{\frac{1}{2}}] \subseteq I.$$

Proof: By Theorem 3.12, we have $I = I^{\frac{1}{2}}I^{\frac{1}{2}}$. Using that $[a, xy] = [ax, y] + [ya, x]$ for $a \in A$ and $x, y \in I$, it follows that

$$[A, I] = [A, I^{\frac{1}{2}}I^{\frac{1}{2}}] \subseteq [AI^{\frac{1}{2}}, I^{\frac{1}{2}}] + [I^{\frac{1}{2}}A, I^{\frac{1}{2}}] \subseteq [I^{\frac{1}{2}}, I^{\frac{1}{2}}].$$

Further, we have $[I^{\frac{1}{2}}, I^{\frac{1}{2}}] \subseteq I^{\frac{1}{2}}I^{\frac{1}{2}} = I$. □

Recall that $\mathrm{N}_2(I)$ denotes the set of square-zero elements in an ideal I.

Lemma 4.2. *Let $I \subseteq A$ be a Dixmier ideal in a C^*-algebra. Then*

$$\mathrm{N}_2(I) \subseteq \mathrm{span}_{\mathbb{C}}[\mathrm{N}_2(I^{\frac{1}{2}}), \mathrm{N}_2(I^{\frac{1}{2}})].$$

Proof: Let $x \in \mathrm{N}_2(I)$, and let $x = v|x|$ be the polar decomposition in A^{**}. We have $x = |x^*|v = |x^*|^{\frac{1}{2}}v|x|^{\frac{1}{2}}$. Since $x^2 = 0$, the elements $|x|$ and $|x^*|$ are orthogonal, and hence so are $|x|^{\frac{1}{2}}$ and $|x^*|^{\frac{1}{2}}$. Therefore

$$x = \left[\tfrac{1}{2}v|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}} - |x^*|^{\frac{1}{2}}\right].$$

By Lemma 3.13, the element $\frac{1}{2}v|x|^{\frac{1}{2}}$ belongs to $I^{\frac{1}{2}}$, and thus $\frac{1}{2}v|x|^{\frac{1}{2}} \in \mathrm{N}_2(I^{\frac{1}{2}})$. Set

$$y := |x|^{\frac{1}{2}} - |x^*|^{\frac{1}{2}} + v|x|^{\frac{1}{2}} - v^*|x^*|^{\frac{1}{2}}.$$

It is elementary (but tedious) to check that $y^2 = 0$. Note that y belongs to $I^{\frac{1}{2}}$ by Lemma 3.13, and thus $y \in \mathrm{N}_2(I^{\frac{1}{2}})$. Using that $v|x|^{\frac{1}{2}}$ and $v^*|x^*|^{\frac{1}{2}}$ also belong to $N_2(I^{\frac{1}{2}})$, it follows that

$$|x|^{\frac{1}{2}} - |x^*|^{\frac{1}{2}} = y - v|x|^{\frac{1}{2}} + v^*|x^*|^{\frac{1}{2}} \in \mathrm{span}_{\mathbb{C}} N_2(I^{\frac{1}{2}}),$$

and thus

$$x = \left[\tfrac{1}{2}v|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}} - |x^*|^{\frac{1}{2}}\right] \in \mathrm{span}_{\mathbb{C}}[N_2(I^{\frac{1}{2}}), N_2(I^{\frac{1}{2}})],$$

as desired. □

A two-sided ideal I in a ring R is said to be *semiprime* if for all two-sided ideals $J \subseteq R$ we have $J \subseteq I$ whenever $J^2 \subseteq I$. We refer to [**23**, Section 10] for details. In [**14**, Section 5], together with Gardella and Kitamura, we showed that a two-sided ideal I in a C^*-algebra is semiprime if and only if it is idempotent (that is, $I = I^2$), if and only if it is a Dixmier ideal and satisfies $I = I^{\frac{1}{2}}$.

The next result is a generalization of [**17**, Theorem 4.2] from the study of square-zero elements in C^*-algebras to square-zero elements in semiprime ideals in C^*-algebras. Recall that $\widetilde{A}$ denotes the minimal unitization of a C^*-algebra A.

Theorem 4.3. *Let $I \subseteq A$ be a semiprime ideal in a C^*-algebra, let N denote the subspace generated by $\mathrm{N}_2(I)$, and let $J := \widetilde{A}N\widetilde{A}$ denote the two-sided ideal of A generated by N. Then J is semiprime, and we have*

$$N = [N,N] \subseteq N^2, \quad J = [J,J]^2 = AN = [A,N]^2 = [A,N] + N^2,$$

and

$$[J,J] = [A,J] = [A,[A,N]] = [[J,J],[J,J]] = [[A,N],[A,N]].$$

Proof: We will use that I is a Dixmier ideal satisfying $I = I^{\frac{1}{2}}$. Applying Lemma 4.2 at the second step, we have

$$N = \mathrm{span}_{\mathbb{C}}\, \mathrm{N}_2(I) \subseteq \mathrm{span}_{\mathbb{C}}[\mathrm{N}_2(I^{\frac{1}{2}}), \mathrm{N}_2(I^{\frac{1}{2}})] = \mathrm{span}_{\mathbb{C}}[\mathrm{N}_2(I), \mathrm{N}_2(I)] = [N,N].$$

The inclusion $[N,N] \subseteq N^2$ follows since every commutator in $[N,N]$ belongs to N^2. To verify that $[N,N] \subseteq N$, let $x, y \in \mathrm{N}_2(I)$. Then

$$[x,y] = (1+x)y(1-x) - y + xyx.$$

Each of the three summands is a square-zero element in I, which shows $[x,y] \in N$.

We have

$$J := \widetilde{A}N\widetilde{A} \subseteq \widetilde{A}N^2\widetilde{A} \subseteq (\widetilde{A}N)(N\widetilde{A}) \subseteq J^2,$$

which shows that J is idempotent and therefore semiprime by [**14**, Theorem A].

The inclusion $AN \subseteq \widetilde{A}N\widetilde{A} = J$ is clear. Conversely, given $a \in \widetilde{A}$, $x \in \mathrm{N}_2(I)$, and a unitary $u \in \widetilde{A}$, using that $u^*xu \in \mathrm{N}_2(I)$ and that $N \subseteq N^2$, we have

$$axu = au(u^*xu) \in \widetilde{A}\,\mathrm{N}_2(I) \subseteq \widetilde{A}N \subseteq \widetilde{A}N^2 \subseteq AN.$$

Since every element in $\widetilde{A}$ is a linear combination of four unitaries, we get $J = AN$.

The inclusion $[A,N] + N^2 \subseteq \widetilde{A}N\widetilde{A} = J$ is clear. For the converse inclusion, we use a similar argument as in the proof of Theorem 3.14 to show that $AN \subseteq [A,N] + N^2$. Given $a \in A$ and $x \in \mathrm{N}_2(I)$, let $x = v|x|$ be the polar decomposition in A^{**}. Since I is semiprime, it is Dixmier and $I = I^{\frac{1}{2}} = I^{\frac{1}{4}}$. By Lemma 3.13, we have $v|x|^{\frac{1}{2}} \in I^{\frac{1}{2}}$ and thus $v|x|^{\frac{1}{2}} \in \mathrm{N}_2(I)$. Similarly, we have $v|x|^{\frac{1}{4}}, |x|^{\frac{1}{4}}a|x^*|^{\frac{1}{2}} \in \mathrm{N}_2(I)$. Using that $x = |x^*|^{\frac{1}{2}}v|x|^{\frac{1}{2}}$, we get

$$ax = [a|x^*|^{\frac{1}{2}}, v|x|^{\frac{1}{2}}] + (v|x|^{\frac{1}{4}})(|x|^{\frac{1}{4}}a|x^*|^{\frac{1}{2}}) \in [A, \mathrm{N}_2(I)] + \mathrm{N}_2(I)^2.$$

Using that $J = AN$, we get $J = [A,N] + N^2$.

The inclusion $[A,N]^2 \subseteq \widetilde{A}N\widetilde{A} = J$ is clear. Conversely, using that $N = [N,N]$, we have

$$[A,N] = [A,[N,N]] \subseteq [N,[A,N]] \subseteq [[A,N],[A,N]] \subseteq [A,N]^2.$$

We also have $N^2 = [N,N]^2 \subseteq [A,N]^2$ and thus

$$J = [A,N] + N^2 \subseteq [A,N]^2.$$

The inclusion $[J,J] \subseteq [A,J]$ is clear. Using that $J = J^2$ and $[a,xy] = [ax,y]+[ya,x]$ for $a \in A$ and $x,y \in J$, we have

$$[A,J] = [A,J^2] \subseteq [AJ,J] + [JA,J] \subseteq [J,J].$$

This shows that $[J,J] = [A,J]$. We get

$$[[A,N],[A,N]] \subseteq [A,[A,N]] \subseteq [A,J] = [J,J].$$

We have

$$[[A,N],N^2] \subseteq [A,N^2] \subseteq [AN,N] + [NA,N] \subseteq [A,N],$$

and similarly $[N^2,N^2] \subseteq [A,N^2] \subseteq [A,N]$. We also have

$$[A,N] = [A,[N,N]] \subseteq [N,[A,N]] = [[N,N],[A,N]] \subseteq [[A,N],[A,N]].$$

Using that $J = [A,N] + N^2$, it follows that

$$\begin{aligned}[J,J] &\subseteq [[A,N],[A,N]] + [N^2,[A,N]] + [[A,N],N^2] + [N^2,N^2] \\ &\subseteq [[A,N],[A,N]].\end{aligned}$$

Using that $[A,N] \subseteq [A,J] = [J,J]$, we also have

$$[J,J] = [[A,N],[A,N]] \subseteq [[J,J],[J,J]].$$

Since the converse inclusion is clear, we get

$$[J,J] = [A,J] = [A,[A,N]] = [[J,J],[J,J]] = [[A,N],[A,N]].$$

Finally, since $[A,N] \subseteq [J,J]$ and $J = [A,N]^2$, we have $J \subseteq [J,J]^2$ and thus $J = [J,J]^2$. ☐

Lemma 4.4. *Let $I \subseteq A$ be a Dixmier ideal in a C^*-algebra, and let $x \in I$ be nilpotent. Then $x \in [I^{\frac{1}{2}}, I^{\frac{1}{2}}]$.*

Proof: As in the proof of [**31**, Lemma 2.1], we proceed by induction on the degree of nilpotency. To start, let $x \in I$ be a square-zero element, and let $x = v|x|$ be the polar decomposition of x in A^{**}. By Lemma 3.13, the elements $v|x|^{\frac{1}{2}}$ and $|x|^{\frac{1}{2}}$ belong to $I^{\frac{1}{2}}$, and thus

$$x = [v|x|^{\frac{1}{2}}, x^{\frac{1}{2}}] \in [I^{\frac{1}{2}}, I^{\frac{1}{2}}].$$

Next, assume that for some $k \geq 2$ we have shown that every element $x \in I$ with $x^k = 0$ belongs to $[I^{\frac{1}{2}}, I^{\frac{1}{2}}]$. Let $x \in I$ satisfy $x^{k+1} = 0$, and let $x = v|x|$ be the polar decomposition of x in A^{**}. Set $y := |x|^{\frac{1}{2}} v|x|^{\frac{1}{2}}$, which belongs to A. As in the proof of [**31**, Lemma 2.1], one shows that $y^k = 0$. By Lemma 3.13, we have $v|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}} \in I^{\frac{1}{2}}$. By assumption of the induction, we have $y \in [I^{\frac{1}{2}}, I^{\frac{1}{2}}]$, and thus

$$x = [v|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}}] + y \in [I^{\frac{1}{2}}, I^{\frac{1}{2}}],$$

as desired. □

Proposition 4.5. *Let $I \subseteq A$ be a semiprime ideal in a C^*-algebra, and let $x \in I$ be nilpotent. Then $x \in [A, I]$.*

Proof: Since I is semiprime, it is a Dixmier ideal and satisfies $I = I^{\frac{1}{2}}$. Applying Lemma 4.4, we have

$$x \in [I^{\frac{1}{2}}, I^{\frac{1}{2}}] \subseteq [A, I^{\frac{1}{2}}] = [A, I],$$

as desired. □

Theorem 4.6. *Let A be a unital C^*-algebra that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, and let $I \subseteq A$ be a two-sided ideal. The following statements are equivalent:*

1. *The ideal $I \subseteq A$ is semiprime.*
2. *We have $I = A[I, I]A$, that is, I is generated by its commutators as an ideal in A.*
3. *We have $I = \widetilde{I}[I, I]\widetilde{I}$, that is, I is generated by its commutators as an ideal.*
4. *We have $I = [I, I]^2$.*
5. *We have $[I, I] = [[I, I], [I, I]]$.*

Proof: To show that (2) implies (1), assume that I is generated by $[I, I]$ as an ideal in A. We clearly have $[I, I] \subseteq I^2$, and therefore

$$I = A[I, I]A \subseteq AI^2A = I^2.$$

It follows that I is idempotent, and thus semiprime by [**14**, Theorem A].

It is clear that (4) implies (3), which in turn implies (2).

Assume that I is semiprime, and let N denote the subspace generated by the square-zero elements in I. Applying Theorem 3.8, we have

$$I = A[A, I]A = \operatorname{span}_{\mathbb{C}} A \operatorname{FN}_2(A, I)A \subseteq ANA.$$

Now it follows from Theorem 4.3 that

$$I = ANA = [ANA, ANA]^2 = [I, I]^2,$$

which verifies (4), and that

$$[I, I] = [ANA, ANA] = [[ANA, ANA], [ANA, ANA]] = [[I, I], [I, I]],$$

which verifies (5).

Finally, assuming (5), let us verify (1). Using that $[a,xy]=[ax,y]+[ya,x]$ for $a\in A$ and $x,y\in I$, we have

$$[A,I^2]\subseteq[AI,I]+[IA,I]\subseteq[I,I].$$

Applying Theorem 3.8 for the ideal I^2, we get

$$I^2=A[A,I^2]A\subseteq A[I,I]A\subseteq A[[I,I],[I,I]]A\subseteq I^4.$$

It follows that $I^2=I^4$, and is thus semiprime by [**14**, Theorem A]. ☐

5. Lie ideals in properly infinite C^*-algebras

Let A be a C^*-algebra with a unit that is weakly divisible of degree 2, that is, that admits a unital $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to A$, and let $L\subseteq A$ be a (not necessarily closed) Lie ideal. We have seen in Theorem 2.7 that L is related to the two-sided ideal $I:=A[A,L]A$. Further, by Theorem 2.9, L is commutator-equivalent to I if and only if $[A,L]=[A,[A,L]]$.

This raises the question of when Lie ideals L in A satisfy $[A,L]=[A,[A,L]]$. A sufficient condition is that $A=Z(A)+[A,A]$, where $Z(A)$ denotes the center of A; see Lemma 5.1. We observe in Proposition 5.2 that this condition is satisfied in a number of situations, in particular for all properly infinite C^*-algebras.

We then prove the main result of the paper: Every Lie ideal in a unital, properly infinite C^*-algebra is commutator-equivalent to a (unique) two-sided ideal; see Theorem 5.3.

Lemma 5.1. *Let A be a unital C^*-algebra such that $A=Z(A)+[A,A]$, and let $L\subseteq A$ be a Lie ideal. Then*

$$[A,L]=[A,[A,L]].$$

Proof: The inclusion '$\supseteq$' is clear. Conversely, using the assumption at the first step, and using the Jacobi identity at the last step, we get

$$[A,L]=[Z(A)+[A,A],L]=[[A,A],L]\subseteq[A,[A,L]],$$

as desired. ☐

Following Winter [**32**], a C^*-algebra A is said to be *pure* if its Cuntz semigroup $\mathrm{Cu}(A)$ is almost unperforated and almost divisible. Pure C^*-algebras form a robust class ([**3**]) that includes every C^*-algebra that tensorially absorbs the Jiang–Su algebra $\mathcal{Z}$, that is, $A\cong A\otimes\mathcal{Z}$. By [**2**, Theorem 7.3.11], a C^*-algebra A is pure if and only if $\mathrm{Cu}(A)\cong\mathrm{Cu}(A)\otimes\mathrm{Cu}(\mathcal{Z})$.

By a *quasitrace* of a C^*-algebra A we mean a lower-semicontinuous, $[0,\infty]$-valued 2-quasitrace. By Haagerup's theorem [**19**], every bounded quasitrace on a unital, exact C^*-algebra is a trace, and this was later extended by Kirchberg [**21**], who showed that arbitrary quasitraces on exact C^*-algebras are traces.

A unital C^*-algebra A is *weakly central* if the center separates maximal ideals: Maximal ideals $M_1,M_2\subseteq A$ satisfy $M_1=M_2$ whenever $M_1\cap Z(A)=M_2\cap Z(A)$. By Vesterstrøm's theorem, A is weakly central if and only if A has the *center-quotient property*: For every closed, two-sided ideal $I\subseteq A$, the quotient map $A\to A/I$ maps $Z(A)$ onto $Z(A/I)$.

A unital C^*-algebra A has the *Dixmier property* if for every element $a\in A$ the closed convex hull of the unitary orbit $\{uau^*: u\in A \text{ unitary}\}$ has a nonempty intersection with the center $Z(A)$. It was shown in [**5**, Theorem 1.1] that this property holds if and only if A is weakly central, every simple quotient of A has at most one tracial state, and every extreme tracial state of A factors through a simple quotient. Dixmier showed that every von Neumann algebra has the property named after him.

Proposition 5.2. *Let A be a unital C^*-algebra. Then $A = Z(A) + [A, A]$ in each of the following cases:*

1. *If A is pure, every quasitrace on A is a trace, and A has the Dixmier property.*
2. *If A has no tracial states (in particular, if A is properly infinite).*
3. *If A is a von Neumann algebra.*

Proof: (1) If A has the Dixmier property, then every element in A is a Dixmier element in the sense of [**4**], and thus it follows from [**4**, Lemma 4.10] that $A = Z(A) + \overline{[A, A]}$.

If A is pure and every quasitrace on A is a trace, then $[A, A]$ is closed by [**28**, Theorem 1.1]. Combining both results, we get $A = Z(A) + [A, A]$.

(2) By [**30**, Theorem 1], we have $A = [A, A]$ if (and only if) A has no tracial states.

(3) It was shown in the proof of [**8**, Theorem 5.19] that every von Neumann algebra A satisfies $A = Z(A) + [A, A]$. □

Theorem 5.3. *Let A be a unital, properly infinite C^*-algebra, and let $L \subseteq A$ be a Lie ideal. Then for the two-sided ideal $I := A[A, L]A$ we have*

$$[A, L] = [A, [A, L]] = [A, I].$$

In particular, L is commutator-equivalent to I, and hence also embraced by I, and related to I.

Further, I is the only two-sided ideal of A to which L is related.

Proof: Since A is unital and properly infinite, there exists a unital $*$-homomorphism from the Cuntz algebra $\mathcal{O}_\infty$ to A; see [**6**, Proposition III.1.3.3]. Further, it is well known that $\mathcal{O}_\infty$ admits a unital $*$-homomorphism from $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$, using for example Proposition 2.10 and that $\mathcal{O}_\infty$ is a unital C^*-algebra of real rank zero that admits no characters. It follows that A admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$.

We may therefore apply Theorem 2.7 and obtain that L is related to I, and that I is the only two-sided ideal to which L is related. Further, since A is unital and properly infinite, we have $A = [A, A]$ by Pop's theorem [**30**], and thus $[A, L] = [A, [A, L]]$ by Lemma 5.1. It therefore follows from Theorem 2.9 that L is commutator-equivalent to I (and thus also embraced by I). □

Corollary 5.4. *Let A be a unital, properly infinite C^*-algebra. Then an additive subgroup $L \subseteq A$ is a Lie ideal if and only if $[A, I] \subseteq L \subseteq T([A, I])$ for some (unique) two-sided ideal $I \subseteq A$. We get a decomposition of the family $\mathcal{L}$ of all Lie ideals in A as:*

$$\mathcal{L} = \bigsqcup_{I \in \mathcal{I}} \{L \subseteq A \text{ additive subgroup} : [A, I] \subseteq L \subseteq T([A, I])\},$$

where $\mathcal{I}$ denotes the lattice of two-sided ideals in A.

Remark 5.5. The proof of Theorem 5.3 can easily be adapted to show the following: In a C^*-algebra A that admits a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$ and such that $A = Z(A) + [A, A]$, every Lie ideal L is commutator-equivalent to the two-sided ideal $A[A, L]A$.

We note that a unital C^*-algebra admits no characters whenever it is pure or has no tracial states. Therefore, with a view to Problem 2.12 and Proposition 5.2, we expect that every Lie ideal L is commutator-equivalent to $A[A, L]A$ whenever A is a unital, pure C^*-algebra with the Dixmier property and such that every quasitrace on A is a trace, or whenever A is unital and has no tracial states.

Problem 5.6. Given Lie ideals $L, M \subseteq A$ in a unital, properly infinite C^*-algebra, what is the relationship between the two-sided ideals generated by $[L, M]$, by $[A, [L, M]]$, and by $[A, L]$ and $[A, M]$?

6. Lie ideals and two-sided ideals in von Neumann algebras

In this section, we show that every (not necessarily closed) Lie ideal L in a von Neumann algebra M is commutator-equivalent to the two-sided ideal $I := M[M, L]M$, that is, $[M, L] = [M, I]$; see Theorem 6.1. This recovers [**8**, Theorem 5.19]. Moreover, if M has zero commutative summand, then we show that I is the only two-sided ideal to which L is related, which solves [**8**, Problem 5.21].

Given a two-sided ideal I in a von Neumann algebra with zero commutative summand, every element in I is a sum of products of pairs of square-zero elements in $I^{\frac{1}{2}}$; see Theorem 6.4.

Theorem 6.1. *Let M be a von Neumann algebra, and let $L \subseteq M$ be a Lie ideal. Then, for the two-sided ideal $I := M[M, L]M$, we have*

$$[M, L] = [M, [M, L]] = [M, I].$$

In particular, L is commutator-equivalent to I, and hence also embraced by I, and related to I.

If M has zero commutative summand, then I is the only two-sided ideal of M to which L is related.

Proof: Let $M = M_0 \oplus M_1$ be the (unique) decomposition of M into a commutative summand M_0 and a summand M_1 that admits no characters. By Proposition 2.10, there is a unital $*$-homomorphism $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to M_1$. Set

$$L_1 := M_1 \cap L \quad \text{and} \quad I_1 := M_1 \cap I.$$

Then L_1 is a Lie ideal in M_1, and we have

$$[M, L] = [M_1, L_1] \quad \text{and} \quad I = I_1 = M_1[M_1, L_1]M_1.$$

We may apply Theorem 2.7 for the Lie ideal L_1 in M_1 and obtain that L_1 is related to I_1, and that I_1 is the only two-sided ideal of M_1 to which L_1 is related. It follows that L is related to I, and if M_0 is zero, then I is also unique with this property. (If M_0 is nonzero, then $M_0 \oplus I$ is another two-sided ideal of M to which L is related.)

As shown in the proof of [**8**, Theorem 5.19], we have $M = Z(M) + [M, M]$. It follows that $[M, L] = [M, [M, L]]$ by Lemma 5.1, and then

$$[M, L] = [M, [M, L]] = [M, I]$$

by Theorem 2.9. $\square$

Recall that $\mathrm{N}_2(I)$ denotes the square-zero elements, and we use $\mathrm{Nil}(I)$ to denote the set of nilpotent elements in an ideal I. Recall the definition of $\mathrm{FN}_2(M, I)$ from Definition 3.1.

Proposition 6.2. *Let $I \subseteq M$ be a two-sided ideal in a von Neumann algebra. Then*

$$[M, I] = \mathrm{span}_{\mathbb{C}} \mathrm{N}_2(I) = \mathrm{span}_{\mathbb{C}} \mathrm{Nil}(I) = \mathrm{span}_{\mathbb{C}} \mathrm{FN}_2(M, I).$$

Proof: Let $M = M_0 \oplus M_1$ denote the (unique) decomposition of M into a commutative summand M_0 and a summand M_1 that admits no characters. Then

$$[M, I] = [M_1, M_1 \cap I], \qquad \mathrm{N}_2(I) = \mathrm{N}_2(M_1 \cap I),$$
$$\mathrm{Nil}(I) = \mathrm{Nil}(M_1 \cap I), \quad \text{and} \quad \mathrm{FN}_2(M, I) = \mathrm{FN}_2(M_1, M_1 \cap I),$$

which allows us to reduce the problem to studying the ideal $M_1 \cap I$ in M_1.

Without loss of generality, we may therefore assume that M has zero commutative summand. Then M admits a unital $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to M$ by Proposition 2.10. Using Theorem 3.8 at the first step, we get

$$[M,I]=\operatorname{span}_{\mathbb{C}}\mathrm{FN}_2(M,I)\subseteq\operatorname{span}_{\mathbb{C}}\mathrm{N}_2(I)\subseteq\operatorname{span}_{\mathbb{C}}\mathrm{Nil}(I).$$

It remains to show that $\mathrm{Nil}(I)\subseteq[M,I]$. As in the proof of Lemma 4.4 (and [**31**, Lemma 2.1]), we proceed by induction on the degree of nilpotency. To start, let $x\in I$ be a square-zero element. Let $x=v|x|$ be the polar decomposition in M. Then $|x|=v^*x\in I$. Since $x^2=0$, we have $|x|v=0$, and thus

$$x=[v,|x|]\in[M,I].$$

Next, assume that for some $k\geq 2$ we have shown that every element $x\in I$ with $x^k=0$ belongs to $[M,I]$. Let $x\in I$ satisfy $x^{k+1}=0$. Again, let $x=v|x|$ be the polar decomposition in M. We will show that the element $y:=|x|v$ satisfies $y^k=0$. Since $x^{k+1}=0$ and $|x|=v^*v|x|$, we have

$$y^k|x|=|x|v\cdots|x|v|x|=v^*v|x|v\cdots|x|v|x|=v^*x^{k+1}=0.$$

For every polynomial p with vanishing constant term it follows that $y^kp(|x|)=0$. Using functional calculus, we get $y^k|x|^{\frac{1}{n}}$ for every $n\geq 1$. The sequence $(|x|^{\frac{1}{n}})_n$ converges in the weak*-topology of M to v^*v, which is the support projection of $|x|$. We deduce that $y^kv^*v=0$. Using that $v=vv^*v$, we get

$$y^k=y^{k-1}|x|v=y^{k-1}|x|vv^*v=y^kv^*v=0.$$

By assumption of the induction, we have $y\in[M,I]$, and thus

$$x=[v,|x|]+|x|v\in[M,I],$$

as desired. □

Remark 6.3. Let H be a separable, infinite-dimensional Hilbert space, and let I be a two-sided ideal in the von Neumann algebra $\mathcal{B}(H)$. Given a nilpotent element $x\in I$, it follows from Proposition 6.2 that $x\in[\mathcal{B}(H),I]$, that is, x is a finite sum of commutators of an element in $\mathcal{B}(H)$ and an element in I. Dykema and Krishnaswamy-Usha showed in [**12**, Proposition 3.1] that one summand suffices, that is, we have $x=[y,z]$ for some $y\in\mathcal{B}(H)$ and $z\in I$. Does their result hold for nilpotent elements in two-sided ideals in arbitrary von Neumann algebras?

Theorem 6.4. *Let M be a von Neumann algebra with zero commutative summand, and let $I\subseteq M$ be a two-sided ideal. Then*

$$I=[M,I^{\frac{1}{2}}]^2=[I^{\frac{1}{4}},I^{\frac{1}{4}}]^2,$$

and

$$[M,I]\subseteq[[M,I^{\frac{1}{2}}],[M,I^{\frac{1}{2}}]]\subseteq[[I^{\frac{1}{4}},I^{\frac{1}{4}}],[I^{\frac{1}{4}},I^{\frac{1}{4}}]]\subseteq[I^{\frac{1}{2}},I^{\frac{1}{2}}].$$

Proof: We will use that I is a Dixmier ideal, since two-sided ideals in von Neumann algebras are automatically Dixmier ideals ([**14**, Proposition 3.4]). Using Proposition 6.2 at the first and last step, and using Lemma 4.2 at the second step, we have

$$[M,I]=\operatorname{span}_{\mathbb{C}}\mathrm{N}_2(I)\subseteq\operatorname{span}_{\mathbb{C}}[\mathrm{N}_2(I^{\frac{1}{2}}),\mathrm{N}_2(I^{\frac{1}{2}})]=[[M,I^{\frac{1}{2}}],[M,I^{\frac{1}{2}}]].$$

Using Lemma 4.1, we then get

$$[M,I]\subseteq[[M,I^{\frac{1}{2}}],[M,I^{\frac{1}{2}}]]\subseteq[[I^{\frac{1}{4}},I^{\frac{1}{4}}],[I^{\frac{1}{4}},I^{\frac{1}{4}}]]\subseteq[I^{\frac{1}{2}},I^{\frac{1}{2}}].$$

By Proposition 2.10, there is a unital $*$-homomorphism $M_2(\mathbb{C})\oplus M_3(\mathbb{C})\to M$. We can therefore apply Theorem 3.14 at the first step, and together with the above we obtain that

$$I=[M,I]+[M,I^{\frac{1}{2}}]^2\subseteq [M,I^{\frac{1}{2}}]^2.$$

Applying Lemma 4.1 for the Dixmier ideal $I^{\frac{1}{2}}$, and using Theorem 3.12 at the last step, we get

$$I\subseteq [M,I^{\frac{1}{2}}]^2\subseteq [I^{\frac{1}{4}},I^{\frac{1}{4}}]^2\subseteq (I^{\frac{1}{2}})^2=I,$$

as desired. ☐

Remark 6.5. It is likely that the inclusions involving $[M,I]$ in Theorem 6.4 are equalities. The key question is whether $[I^{\frac{1}{2}},I^{\frac{1}{2}}]$ is a subset of $[M,I]$. Using that I is a Dixmier ideal and therefore admits a well-behaved theory of roots and powers (Theorem 3.12), we have

$$[I^{\frac{1}{2}},I^{\frac{1}{2}}]=[I^{\frac{1}{4}}I^{\frac{1}{4}},I^{\frac{1}{2}}]\subseteq [I^{\frac{1}{4}},I^{\frac{1}{4}}I^{\frac{1}{2}}]+[I^{\frac{1}{2}},I^{\frac{1}{2}}I^{\frac{1}{4}}]=[I^{\frac{1}{4}},I^{\frac{3}{4}}],$$

and by an analogous argument $[I^{\frac{1}{4}},I^{\frac{3}{4}}]\subseteq [I^{\frac{1}{8}},I^{\frac{7}{8}}]$. Inductively, it follows that $[I^{\frac{1}{2}},I^{\frac{1}{2}}]\subseteq [I^{\frac{1}{2^n}},I^{\frac{2^n-1}{2^n}}]$ for every $n\geq 1$, and thus

$$[I^{\frac{1}{2}},I^{\frac{1}{2}}]\subseteq \bigcap_{n\geq 1}[I^{\frac{1}{2^n}},I^{\frac{2^n-1}{2^n}}]\subseteq \bigcap_{\varepsilon>0}[M,I^{1-\varepsilon}].$$

It is, however, not clear if $\bigcap_{\varepsilon>0}[M,I^{1-\varepsilon}]=[M,I]$.

For two-sided ideals $I,J\subseteq \mathcal{B}(H)$, it was shown in [**11**, Theorem 5.10] that $[\mathcal{B}(H),IJ]=[I,J]$. In particular, we have $[\mathcal{B}(H),I]=[I^{\frac{1}{2}},I^{\frac{1}{2}}]$, and we obtain the following consequence of Theorem 6.4:

Corollary 6.6. *Let H be a separable, infinite-dimensional Hilbert space, and let I be a two-sided ideal in $\mathcal{B}(H)$. Then*

$$[\mathcal{B}(H),I]=[[\mathcal{B}(H),I^{\frac{1}{2}}],[\mathcal{B}(H),I^{\frac{1}{2}}]]=[[I^{\frac{1}{4}},I^{\frac{1}{4}}],[I^{\frac{1}{4}},I^{\frac{1}{4}}]]=[I^{\frac{1}{2}},I^{\frac{1}{2}}].$$

Problem 6.7. Let $I,J\subseteq M$ be two-sided ideals in a von Neumann algebra. Do we have $[M,IJ]=[I,J]$?

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Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Gothenburg SE-412 96, Sweden
E-mail address: `hannes.thiel@chalmers.se`
URL: `www.hannesthiel.org`
ORCID: `0000-0003-0388-6495`

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