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DIRECT METHODS FOR A DUAL TEMPERED FRACTIONAL PARABOLIC PROBLEM

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Abstract: In this paper, we consider the dual tempered fractional parabolic problem

$$\partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) = f(t,u(x,t)), \quad \Omega\times\mathbb{R},$$

where Ω may be $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{x\in\mathbb{R}^n \mid x_1>0\}$, $s\in(0,1)$, $\alpha\in(0,2)$, and ρ is a sufficiently small positive constant. We prove that the positive solutions are strictly increasing in the x_1 direction without assuming the solutions to be bounded. We will introduce two methods for dealing with the above dual tempered fractional parabolic problem: the method of moving planes and sliding methods. Unlike previous articles, we investigate the problems that involve both the fractional time derivative ∂_t^s and the tempered fractional Laplacian $-(\Delta+\rho)^{\frac{\alpha}{2}}$. First, by establishing the *narrow region principle* and *averaging effects* for the dual tempered fractional parabolic operators $\partial_t^s-(\Delta+\rho)^{\frac{\alpha}{2}}$, and then developing the direct moving planes to derive the monotonicity of solutions for the dual tempered fractional parabolic problem in $\mathbb{R}^n_+\times\mathbb{R}$. Second, by establishing *maximum principles in unbounded open sets* for problems involving dual tempered fractional parabolic operators, we develop direct sliding methods for the tempered fractional parabolic problem, and derive the one-dimensional symmetry of solutions to the dual tempered fractional parabolic problem in $\mathbb{R}^n\times\mathbb{R}$. As applications, we also prove the Gibbons conjecture for entire solutions to the dual tempered fractional parabolic problem.

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Key words: dual tempered parabolic equations, fractional time diffusion, maximum principles, averaging effects, direct method of moving planes, direct sliding methods.

1. Introduction and results

In recent decades, fractional PDEs have become popular as effective models for characterizing Lévy flights or tempered Lévy flights. In fact, nonlocal (especially fractional-order) pseudo-differential equations derived from microscopic models are the most commonly used and effective models for studying abnormal diffusion in physics. The solution of fractional partial differential equations is usually the probability density function of the particle position where the abnormal dynamics occur; with the deepening of research, it can also be used to describe the functional distribution of particle trajectories (see [**35**]). There are usually two ways to derive fractional partial differential equations from models. One way is based on the Montroll–Weiss equation (see [**32**]) and uses Fourier transforms. Another way is based on the characteristic function of the stable Lévy motion.

This paper is devoted to a study of qualitative properties of solutions for a dual tempered fractional parabolic equation of the form

$$\partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) = f(t,u(x,t)), \quad \Omega\times\mathbb{R}, \tag{1.1}$$

where Ω may be $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_1 > 0\}$, $s \in (0,1)$, $\alpha \in (0,2)$, and ρ is a sufficiently small positive constant, ∂_t^s is the Weyl–Marchaud fractional derivative of order $s \in (0,1)$:

$$\partial_t^s u(x,t) := \frac{s}{\Gamma(1-s)} \int_{-\infty}^{t} \frac{u(x,t)-u(x,\tau)}{(t-\tau)^{1+s}}\,\mathrm{d}\tau, \tag{1.2}$$

with Γ denoting the Gamma function. The tempered fractional Laplacian operator $-(\Delta+\rho)^{\frac{\alpha}{2}}$ is defined as

$$(\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) := -c_{n,\alpha}\ \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x,t)-u(y,t)}{e^{\rho|x-y|}|x-y|^{n+\alpha}}\,\mathrm{d}y,$$

where P.V. stands for Cauchy principal value, and

$$c_{n,\alpha} = \begin{cases} \dfrac{\alpha\Gamma\left(\frac{n+\alpha}{2}\right)}{2^{1-\alpha}\pi^{\frac{n}{2}}\left|\Gamma\left(1-\frac{\alpha}{2}\right)\right|}, & \text{for } \rho = 0 \text{ or } \alpha = 1, \\[2ex] \dfrac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}|\Gamma(-\alpha)|}, & \text{for } \rho > 0 \text{ and } \alpha \neq 1. \end{cases}$$

It is easy to verify, as $\rho \to 0+$, that the tempered fractional operator $-(\Delta+\rho)^{\frac{\alpha}{2}}$ degenerates into the familiar fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$.

In 1695, L'Hôpital wrote to Leibniz asking how to define the derivative $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ when the order $n = \frac{1}{2}$ is not an integer. Since then, various forms of fractional derivatives have been proposed. Later, many physicists discovered that fractional derivatives can be used to describe many physical phenomena and probability theory, such as fluid mechanics, molecular dynamics, relativistic quantum mechanics of stars, conformal geometry, particle systems with sticking and trapping phenomena, magneto-thermoelastic heat conduction, plasma turbulence, and so on (see [**21, 22, 24**]). To this end, more and more mathematicians have studied the qualitative properties of the solutions of fractional partial differential equations.

Next, let us recall the origin of tempered fractional Laplacian $-(\Delta+\rho)^{\frac{\alpha}{2}}$. Microscopically, diffusion is the net movement of particles from a region of higher concentration to a region of lower concentration. And diffusion is divided into normal diffusion (also known as Brownian motion) and anomalous diffusion (or non-Brownian diffusion). In fact, in physics, most diffusion phenomena are anomalous diffusion. The models for describing anomalous diffusion are also varied, such as continuous-time random walks (CTRW), Langevin-type equations, Lévy processes, subordinated Lévy processes, fractional Brownian motion, etc. Two of the most important CTRW models are Lévy flights (Lévy flights are generally regarded as Brownian motions belonging to the Lévy process) and Lévy walks (see [**17, 31, 38**]). From a macroscopic perspective, fractional (nonlocal) partial differential equations derived from microscopic models are the most commonly used and effective models for the study of anomalous diffusion. In fact, the fractional Laplace $\Delta^{\frac{\alpha}{2}}$ is the generator of the α-stable Lévy process, but in this process, all second and higher moments diverge. When it is applied to the physical experiment process, it is believed that there is a defect. Therefore, physicists introduce a small parameter ρ to regulate the Lévy process, and thus the fractional tempered Laplacian comes into being. Compared to the Lévy flight, the tempered fractional Lévy flight has a finite second moment; the tempered Lévy flight exhibits the dynamics of the Lévy flight, while after a sufficient period of time, it turns to normal diffusion.

The last three decades have witnessed the presence of significant advances in the theory of nonlocal elliptic equations. These equations have their root in various branches of pure and applied mathematics. The following methods are commonly used to deal with nonlocal ellipse problems. The earliest method dates back to the early work of Caffarelli and Silvestre [**4**], who first proposed the extension method. Another, more convenient method is to obtain an equivalent integral equation, and then combine the method of moving planes in integral form (see [**5, 10, 11**]), the method of moving spheres in integral form (see [**30, 33, 26, 29**]), and the method of scaling spheres in integral form (see [**19**]) to get some properties of the solution; the advantage of this method is that the calculation results are more accurate. The third commonly used method is the direct method (including the direct method of moving planes, direct sliding methods, etc). The direct method of moving planes was first proposed by Chen, Li, and Li [**9**]. Subsequently, this direct approach was also applied to the fractional p-Laplacian ([**7**]) and a fully nonlinear fractional-order equation [**8**]. Direct sliding methods were first proposed by Wu and Chen ([**36**]) to deal with the fractional p-Laplacian problem. The key to the direct method is to establish various maximum principles (especially unbounded maximum principles). Afterward, such effective direct methods have been widely applied to establish symmetry, monotonicity, nonexistence, and even to obtain estimates in a boundary layer of solutions for various elliptic equations and systems involving other nonlocal operators, such us pseudo-relativistic Schrödinger operators $(-\Delta+m^2)^s$ with $s\in(0,1)$ and mass $m>0$ (see [**20, 27**]), the nonlocal Bellman operators F_s (refer to [**18, 33**]), or the tempered operators $-(\Delta+\rho)^{\frac{\alpha}{2}}$ (see [**28**]).

Above, we mainly introduce the research progress of the fractional-order nonlocal elliptic problem. Recently, many results have been obtained on fractional-order nonlocal parabolic equations. For example, Chen et al. ([**13**]) considered the following nonlocal fractional parabolic equation:

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s u(x,t)=f(t,u),\quad (x,t)\in\mathbb{R}^N\times(0,+\infty). \tag{1.3}$$

They developed a systematic approach in applying an asymptotic method of moving planes to investigate qualitative properties of positive solutions for problems (1.3). Then, Chen et al. ([**16**]) further considered the fractional parabolic equations with indefinite nonlinearities:

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s u(x,t)=x_1u^p(x,t),\quad (x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

They derived nonexistence of solutions to the above problem with $1<p<+\infty$. Next, Chen and Wu ([**15**]) derived Liouville-type theorems and proved that each positive solution $u(x,t)$ is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t\in\mathbb{R}$ to the following fractional parabolic problems:

$$\begin{cases}\partial_t u(x,t)+(-\Delta)^s u(x,t)=f(u(x,t)), & \text{in } \mathbb{R}^n_+\times\mathbb{R},\\ u(x,t)=0, & \text{in } (\mathbb{R}^n\setminus\mathbb{R}^n_+)\times\mathbb{R}.\end{cases}$$

Afterward, Wu and Chen ([**37**]) considered ancient solutions to

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s u(x,t)=f(t,u),\quad (x,t)\in\mathbb{R}^N\times(-\infty,T]. \tag{1.4}$$

They developed a systematic approach in applying the method of moving planes to study qualitative properties of solutions for problem (1.4). In the four references mentioned above, they used the method of moving planes. Recently, Chen and Wu ([**14**]) developed sliding methods and obtained the one-dimensional symmetry and monotonicity of entire positive solutions to fractional reaction-diffusion equations. Not long ago, Chen and Ma ([**12**]) considered the following dual fractional parabolic problem:

$$\begin{cases} \partial_t^\alpha u(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), & \text{in } \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^n \setminus \mathbb{R}^n_+) \times \mathbb{R}. \end{cases} \tag{1.5}$$

They obtained the monotonicity of the unbounded solution of the parabolic equation for the first time. Meanwhile, Caffarelli and his group have made a series of research results on the existence, uniqueness, and regularity of nonlocal parabolic equations (see [**1, 2, 3**]).

Inspired by the above work, in this paper, we mainly study nonlocal parabolic equations involving the fractional time derivative and the fractional tempered Laplace operator (see (1.1)). We have already introduced some recent results on fractional parabolic equations. But the results for tempered fractional parabolic equations are very few. Not long ago, Peng ([**34**]) considered the following parabolic equation involving the tempered fractional operator:

$$\frac{\partial u}{\partial t}(x,t) - (\Delta + \rho)^{\frac{\alpha}{2}} u(x,t) = f(u(x,t)).$$

He derived monotonicity results of solutions to the above tempered fractional parabolic problem in various domains via sliding methods.

Throughout this paper, in order to ensure that the singular integral in (1.2) is well defined, we may assume that $u(x,t) \in \{C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_\alpha(\mathbb{R}^n)\} \times \{C^1(\mathbb{R}) \times \mathcal{L}_s^-(\mathbb{R})\}$, where $\mathcal{L}_s^-(\mathbb{R})$ is a class of slowly increasing functions given by

$$\mathcal{L}_s^-(\mathbb{R}) := \left\{ u(x,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}) \mid \int_{-\infty}^t \frac{|u(x,\tau)|}{1+|\tau|^{1+s}}\, \mathrm{d}\tau < +\infty, \text{ for any } t \in \mathbb{R} \right\},$$

and

$$\mathcal{L}_\alpha(\mathbb{R}^n) := \left\{ u(\cdot,t) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x,t)| e^{-\rho|x|}}{1+|x|^{n+\alpha}}\, \mathrm{d}x < +\infty \right\}.$$

Although many scholars have made many achievements in parabolic equations, there is a lack of research on the geometric properties of solutions to nonlocal parabolic equations with both fractional time derivatives and fractional tempered Laplacian. Inspired by the previous literature, in this article, we will investigate qualitative properties of solutions of the dual fractional parabolic problem (1.1) in various domains.

1.1. Direct method of moving planes and applications. First, we consider Ω to be a half-space $\mathbb{R}^n_+$, that is,

$$\begin{cases} \partial_t^s u(x,t) - (\Delta + \rho)^{\frac{\alpha}{2}} u(x,t) = f(t, u(x,t)), & \mathbb{R}^n_+ \times \mathbb{R}, \\ u(x,t) = 0, & (\mathbb{R}^n \setminus \mathbb{R}^n_+) \times \mathbb{R}. \end{cases} \tag{1.6}$$

Applying the direct method of moving planes, we will explore the monotonicity of possibly *unbounded solutions* for the nonlocal parabolic problem (1.6). The key ingredients are different forms of *maximum principles and averaging effects.*

Recently, under significantly weaker conditions than those in [**15**], specifically, allowing solutions to tend to infinity at some rate, Chen and Ma ([**12**]) have proved that the solution of (1.5) is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t \in \mathbb{R}$. Unlike in [**12**], however, our operators exhibit tempered properties, introducing new challenges. This necessitates novel approaches to overcome difficulties stemming from dual space-time nonlocality. We address these challenges systematically following each theorem's introduction below.

We are now in a position to state the main results of this paper. The first one is the narrow region principle for anti-symmetric functions in unbounded domains.

Theorem 1.1 (Narrow region principle for anti-symmetric functions). *Let Ω be an unbounded narrow region which can be contained in the narrow slab $\{x \in \Sigma_\lambda \mid \lambda - 2l < x_1 < \lambda\}$ with some small l. Suppose that $w(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_\alpha(\mathbb{R}^n)) \times (C^1(\mathbb{R}) \cap \mathcal{L}^-_s(\mathbb{R}))$ is lower semi-continuous with respect to x on $\overline{\Omega}$, satisfying*

$$w(x,t) \geq -C(1+|x|^\gamma), \ \textit{for some } 0<\gamma<\alpha, \tag{1.7}$$

and

$$\begin{cases} \partial_t^s w(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} w(x,t) = c(x,t)w(x,t), & (x,t) \in \Omega \times \mathbb{R}, \\ w(x,t) \geq 0, & (x,t) \in (\Sigma_\lambda \setminus \Omega) \times \mathbb{R}, \\ w(x,t) = -w(x^\lambda, t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases} \tag{1.8}$$

where $c(x,t)$ is bounded from above.

Then

$$w(x,t) \geq 0, \ \textit{in } \Sigma_\lambda \times \mathbb{R} \tag{1.9}$$

for sufficiently small l. Furthermore, if $w(x,t)$ attains zero at some point $(x^0,t_0) \in \Omega \times \mathbb{R}$, then

$$w(x,t) \equiv 0, \ \textit{in } \mathbb{R}^n \times (-\infty, t_0]. \tag{1.10}$$

Remark 1.1. Here we need to point out that the region Ω given in Theorem 1.1 need not be a narrow region (for example, Ω is an unbounded domain in Σ_λ with a finite width in the x_1 direction), so we only assume that the positive part of coefficient $c(x,t)$ is small; to be more precise, $c(x,t) \leq c_0$ in $\Omega \times \mathbb{R}$ for some small positive constant c_0 (where we only need $c_0 < \frac{C_1}{a^\alpha}$, C_1 to be the same as the constant of formula (A.1)), and the same conclusions are still valid.

One of the most important innovations we have made in this paper is that we are thinking about unbounded solutions. The most critical step to obtain the properties of unbounded solutions is to establish the following *averaging effects*.

Theorem 1.2 (Averaging effects). *Let $D \in \mathbb{R}^n$ be a domain and $t_0 \in \mathbb{R}$ be a real number. For any $x^0 \in \mathbb{R}^n$, suppose that there exists a positive radius $r>0$ such that $B_r(x^0) \cap \overline{D} = \varnothing$, and satisfies*

$$u(x,t) \geq C_0 > 0, \ \textit{in } D \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]. \tag{1.11}$$

Furthermore, suppose that

$$u(x,t) \in (C^{1,1}_{\mathrm{loc}}(B_r(x^0)) \cap \mathcal{L}_\alpha(\mathbb{R}^n)) \times (C^1([t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{\alpha}{s}}]) \cap \mathcal{L}^-_s(\mathbb{R}))$$

is lower semi-continuous in x on $\overline{B_r(x^0)}$, and satisfies

$$\begin{cases} \partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) \geq -\varepsilon, & (x,t) \in B_r(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}], \\ u(x,t) \geq 0, & (x,t) \in B_r^c(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}], \\ u(x,t) \geq 0, & (x,t) \in B_r(x^0) \times (-\infty, t_0 - r^{\frac{\alpha}{s}}], \end{cases} \tag{1.12}$$

for some sufficiently small $\varepsilon > 0$. Then there is a positive constant $\bar{C}(\alpha, n, \alpha, C_0, D, x^0)$ such that

$$u(x^0, t_0) \geq \bar{C}(\alpha, n, \alpha, C_0, D, x^0) > 0.$$

For the convenience of using the method of moving planes, we also need the following *averaging effects for anti-symmetric functions*.

Theorem 1.3 (Averaging effects for anti-symmetric functions)**.** *Let $D \subset \Sigma_\lambda$ be a domain and $t_0 \in \mathbb{R}$ a real number. For any $x^0 \in \Sigma_\lambda$, suppose that there exists a ball $B_r(x^0) \subset \Sigma_\lambda$ such that $B_r(x^0) \cap \overline{D} = \varnothing$ and $r \leq \frac{\mathrm{dist}(x^0, T_\lambda)}{2}$. Suppose that*

$$w(x,t) \in (C^{1,1}_{\mathrm{loc}}(B_r(x^0)) \cap \mathcal{L}_\alpha(\mathbb{R}^n)) \times (C^1([t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]) \cap \mathcal{L}^-_s(\mathbb{R}))$$

is lower semi-continuous in x on $\overline{B_r(x^0)}$, and satisfies

$$\begin{cases} \partial_t^\alpha w(x,t) + (-\Delta)^s w(x,t) \geq -\varepsilon, & (x,t) \in B_r(x^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ w(x,t) \geq 0, & (x,t) \in (\Sigma_\lambda \setminus B_r(x^0)) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ w(x,t) \geq 0, & (x,t) \in B_r(x^0) \times (-\infty, t_0 - r^{\frac{2s}{\alpha}}], \\ w(x,t) = -w(x^\lambda, t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases} \tag{1.13}$$

for some sufficiently small $\varepsilon > 0$, and

$$w(x,t) \geq C_0 > 0, \ \text{ in } D \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]. \tag{1.14}$$

Then there exists a positive constant $C_1 = C_1(\alpha, n, s, C_0, \bar{D}, x^0, T_\lambda)$ such that

$$w(x^0, t_0) \geq C_1 > 0.$$

By using maximum principles and averaging effects as established above, we will apply the direct method of moving planes to investigate monotonicity of solutions to a dual tempered fractional parabolic problem in the half-space $\mathbb{R}^n_+$.

Theorem 1.4 (Monotonicity in a half-space)**.** *Let*

$$u(x,t) \in (C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+) \cap \mathcal{L}_\alpha(\mathbb{R}^n)) \times (C^1(\mathbb{R}) \cap \mathcal{L}^-_s(\mathbb{R}))$$

be a positive solution of (1.1)*. Assume that $u(x,t)$ is uniformly continuous with respect to x, satisfying*

$$u(x,t) \leq C(1 + |x|^\gamma), \ \text{for some } 0 < \gamma < 2s.$$

Assume that the nonlinear term $f(t, \cdot)$ satisfies the following conditions:

- (F_1) *$f(t,u)$ is of class C^1 in u uniformly with respect to t, in other words, functions f and f_u are continuous on $\mathbb{R} \times (0, +\infty)$;*
- (F_2) *$f(t,0) \geq 0$, $f_u(t,0) \leq 0$, in $(-\infty, +\infty)$;*
- (F_3) *$f_u(t,u)$ is bounded from above, that is, there exists a constant C such that*

$$\sup_{t \in \mathbb{R}} \frac{f(t,u_1) - f(t,u_2)}{u_1 - u_2} \leq C.$$

Then we have that $u(x,t)$ is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t \in \mathbb{R}$.

Remark 1.2. In contrast with previous works, we first study unbounded solutions to the nonlocal parabolic equations involving the fractional time derivative ∂_t^s and the tempered fractional Laplacian $(-\Delta + \rho)^{\frac{\alpha}{2}}$. On the other hand, unlike in the elliptic problem, we give the corresponding average effect for the tempered parabolic problem (in general, we consider bounded solutions, where the average effect is mainly to remove the boundedness assumption on solutions). The averaging effects depend on the distance between such two regions.

1.2. Direct sliding methods and applications. Next, we consider Ω to be whole-space $\mathbb{R}^n$, that is,

$$\partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) = f(t,u(x,t)), \quad \mathbb{R}^n \times \mathbb{R}. \tag{1.15}$$

By developing direct sliding methods for the dual tempered fractional parabolic problem (1.15), we will explore the monotonicity of solutions to the dual nonlocal parabolic problem (1.15). The key ingredient is to establish *maximum principles in unbounded open sets* (see Theorem 5.1). After this, turning back to the dual tempered fractional parabolic problem, we will apply the direct sliding method to investigate monotonicity and one-dimensional symmetry (with respect to the space variable x for each fixed t) of solutions to the dual nonlocal parabolic problem (1.15).

The result reads as follows.

Theorem 1.5. *Let $u(x,t) \in \{\mathcal{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)\} \times (C^1(\mathbb{R}) \cap \mathcal{L}_s^-(\mathbb{R}))$ be a bounded solution of*

$$\partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) = f(t,u(x,t)), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.16}$$

with

$$|u(x,t)| \le 1,$$

and

$$u(x', x_n, t) \xrightarrow[x_n \to \pm\infty]{} \pm 1 \text{ uniformly in } x' = (x_1, \dots, x_{n-1}) \text{ and in } t. \tag{1.17}$$

Suppose that $f(t,u)$ is continuous in $\mathbb{R} \times ([-1,1])$ and for any fixed $t \in \mathbb{R}$,

$$f(t,u) \text{ is nonincreasing for } |u| \ge 1-\delta \text{ with some } \delta > 0. \tag{1.18}$$

Then $u(x,t)$ is strictly increasing with respect to x_n, and furthermore it depends on x_n only:

$$u(x,t) = u(x_n,t).$$

Remark 1.3. One should note that the De Giorgi-type nonlinearity $f(t,u) = u - u^3$ and the Zeldovich nonlinearity $f(t,u) = u^2 - u^3$ satisfy the conditions of Theorem 1.5. Our Theorem 1.5 proves the Gibbons conjecture in the fractional parabolic setting.

The remainder of this paper proceeds as follows. In Section 2, we establish the Narrow region principle for anti-symmetric functions involving dual tempered fractional operators, which is Theorem 1.1. And then, we will prove the averaging effects in Section 3, which are Theorem 1.2 and Theorem 1.3. As applications, in Section 4, after extending the direct method of moving planes, we show monotonicity of solutions to the dual tempered fractional parabolic problem in a half-space, which is Theorem 1.4. In Section 5, by developing direct sliding methods, we complete the proof of Theorem 1.5. We put some basic estimates and important lemmas in the appendix.

From now on and in the following of the paper, we always use the same C to denote a constant whose value may be different from line to line, and only the relevant dependence is specified.

2. Maximum principles

In this section, we will establish the narrow region principle (which is Theorem 1.1) in unbounded domains for anti-symmetric functions (without assuming any decay or boundedness conditions on the anti-symmetric function $w(x,t)$ with respect to the space variable x). These maximum principles are key ingredients in applying the direct method of moving planes.

First, we introduce the following two simple maximum principles in bounded domains.

Theorem 2.1 (Maximum principle in bounded domains)**.** *Assume that Ω is a bounded domain in $\mathbb{R}^n$ and $t_1 < t_2$ are two real numbers. Suppose that*

$$u(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_\alpha(\mathbb{R}^n)) \times (C^1([t_1,t_2]) \cap \mathcal{L}^-_s(\mathbb{R}))$$

is lower semi-continuous in x on $\overline{\Omega}$, and satisfies

$$\begin{cases} \partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) \geq 0, & (x,t) \in \Omega \times (t_1,t_2], \\ u(x,t) \geq 0, & (x,t) \in \Omega^c \times (t_1,t_2], \\ u(x,t) \geq 0, & (x,t) \in \Omega \times (-\infty,t_1]. \end{cases} \tag{2.1}$$

Then $u(x,t) \geq 0$ in $\Omega \times (t_1,t_2]$.

Proof: Suppose that the conclusion is not valid; then there exists $(x^0,t_0) \in \Omega \times (t_1,t_2]$ such that

$$u(x^0,t_0) = \min_{\Omega \times (t_1,t_2]} u(x,t) < 0.$$

Moreover, by applying the exterior condition with the initial condition in (2.1), we have

$$\begin{aligned} &\partial_t^s u(x^0,t_0) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x^0,t_0) \\ &\quad = C_\alpha \int_{-\infty}^{t_0} \frac{u(x^0,t_0)-u(x^0,\tau)}{(t_0-\tau)^{1+s}}\,\mathrm{d}\tau + C_{n,\alpha}\ \mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x^0,t_0)-u(y,t_0)}{|x^0-y|^{n+\alpha}e^{\rho|x^0-y|}}\,\mathrm{d}y \\ &\quad = C_\alpha \int_{-\infty}^{t_1} \frac{u(x^0,t_0)-u(x^0,\tau)}{(t_0-\tau)^{1+s}}\,\mathrm{d}\tau + C_s \int_{t_1}^{t_0} \frac{u(x^0,t_0)-u(x^0,\tau)}{(t_0-\tau)^{1+s}}\,\mathrm{d}\tau \\ &\qquad + C_{n,s}\ \mathrm{P.V.}\int_{\Omega} \frac{u(x^0,t_0)-u(y,t_0)}{|x^0-y|^{n+\alpha}e^{\rho|x^0-y|}}\,\mathrm{d}y + C_{n,\alpha}\int_{\Omega^c} \frac{u(x^0,t_0)-u(y,t_0)}{|x^0-y|^{n+\alpha}e^{\rho|x^0-y|}}\,\mathrm{d}y \\ &\quad < 0, \end{aligned}$$

which contradicts the first equation in (2.1). Hence, this finishes the proof of Theorem 2.1. □

Theorem 2.2 (Maximum principle for anti-symmetric functions in bounded domains)**.** *Let $\Omega \subset \Sigma_\lambda$ be a bounded domain and $[t_1,t_2] \subset \mathbb{R}$ be a finite interval. Suppose that*

$$w(x,t) \in (C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathcal{L}_\alpha(\mathbb{R}^n)) \times (C^1([t_1,t_2]) \cap \mathcal{L}^-_s(\mathbb{R}))$$

is lower semi-continuous in x on $\overline{\Omega}$, and satisfies

$$\begin{cases} \partial_t^\alpha w(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} w(x,t) \geq 0, & (x,t) \in \Omega \times (t_1,t_2], \\ w(x,t) \geq 0, & (x,t) \in (\Sigma_\lambda \setminus \Omega) \times (t_1,t_2], \\ w(x,t) \geq 0, & (x,t) \in \Omega \times (-\infty,t_1], \\ w(x,t) = -w(x^\lambda,t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}. \end{cases} \tag{2.2}$$

Then $w(x,t) \geq 0$ in $\Omega \times (t_1,t_2]$.

Proof: Theorem 2.2 can be proved via contradiction arguments. Indeed, suppose on the contrary that there exists $(x^0, t_0) \in \Omega \times (t_1, t_2]$ such that

$$w(x^0, t_0) = \min_{\Sigma_\lambda \times (t_1, t_2]} w(x, t) < 0.$$

The exterior condition and the initial condition, together with the anti-symmetry of $w(x,t)$ in x, yields that

$$\begin{aligned}
&\partial_t^s w(x^0, t_0) - (\Delta + \rho)^{\frac{\alpha}{2}} w(x^0, t_0) \\
&\quad = C_s \int_{-\infty}^{t_0} \frac{w(x^0, t_0) - w(x^0, \tau)}{(t_0 - \tau)^{1+s}} \, \mathrm{d}\tau + C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{w(x^0, t_0) - w(y, t_0)}{|x^0 - y|^{n+\alpha} e^{\rho |x^0 - y|}} \, \mathrm{d}y \\
&\quad = C_s \int_{-\infty}^{t_1} \frac{w(x^0, t_0) - w(x^0, \tau)}{(t_0 - \tau)^{1+s}} \, \mathrm{d}\tau + C_\alpha \int_{t_1}^{t_0} \frac{w(x^0, t_0) - w(x^0, \tau)}{(t_0 - \tau)^{1+s}} \, \mathrm{d}\tau \\
&\qquad + C_{n,\alpha} \text{ P.V.} \int_{\Sigma_\lambda} \frac{w(x^0, t_0) - w(y, t_0)}{|x^0 - y|^{n+\alpha} e^{\rho |x^0 - y|}} \, \mathrm{d}y + C_{n,\alpha} \int_{\Sigma_\lambda^c} \frac{w(x^0, t_0) - w(y, t_0)}{|x^0 - y|^{n+\alpha} e^{\rho |x^0 - y|}} \, \mathrm{d}y \\
&\quad < C_{n,\alpha} \int_{\Sigma_\lambda} \frac{w(x^0, t_0) - w(y, t_0)}{|x^0 - y^\lambda|^{n+\alpha} e^{\rho |x^0 - y|}} \, \mathrm{d}y + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{w(x^0, t_0) + w(y, t_0)}{|x^0 - y^\lambda|^{n+\alpha} e^{\rho |x^0 - y^\lambda|}} \, \mathrm{d}y \\
&\quad = 2C_{n,\alpha} w(x^0, t_0) \int_{\Sigma_\lambda} \frac{1}{|x^0 - y^\lambda|^{n+\alpha} e^{\rho |x^0 - y^\lambda|}} \, \mathrm{d}y < 0,
\end{aligned}$$

where in the last inequality we used the fact that $|x^0 - y| < |x^0 - y^\lambda|$ for $y \in \Sigma_\lambda$. However, the above estimates contradict system (2.2). This completes the proof of Theorem 2.2. □

Proof of Theorem 1.1: Theorem 1.1 can be proved via similar contradiction arguments.

Indeed, condition (1.7) implies $w(x,t)$ goes to negative infinity as $|x| \to \infty$; hence, the minimizing sequence of $w(x,t)$ in x may leak to infinity and the minimum may not be attained. To overcome this difficulty, we set

$$h(x) := \left[\left(1 - \frac{(x_1 - (\lambda - l))^2}{l^2} \right)_+^{\frac{\alpha}{2}} + 1 \right] (1 + |x'|^2)^{\frac{\beta}{2}}$$

for some $\gamma < \beta < \alpha$ to ensure that

$$\lim_{|x| \to \infty} \bar{w}(x,t) := \lim_{|x| \to \infty} \frac{w(x,t)}{h(x)} \geq 0.$$

It follows that for each fixed $t \in \mathbb{R}$, if there is a point $x \in \Omega$ with $\bar{w}(x,t) < 0$, then there must exist $x(t) \in \Omega$ such that

$$\bar{w}(x(t), t) = \min_{x \in \Omega} \bar{w}(x,t) < 0. \tag{2.3}$$

One can further deduce from (1.7), the definition of $\bar{w}(x,t)$ and $\gamma < \beta$ that $\bar{w}(x(t), t)$ is bounded. Therefore, if (1.9) is not valid, then there exists a positive constant m such that

$$\inf_{\Omega \times \mathbb{R}} \bar{w}(x,t) = \inf_{\mathbb{R}} \bar{w}(x(t), t) =: -m < 0. \tag{2.4}$$

This implies that there exists a sequence $\{(x^k, t_k)\} \subset \Omega \times \mathbb{R}$ such that

$$\bar{w}(x^k, t_k) = -m_k \to -m, \text{ as } k \to \infty.$$

Picking $\varepsilon_k := m - m_k$, then it is obvious that $\varepsilon_k \geq 0$ and $\varepsilon_k \to 0$ as $k \to 0$. Denote

$$v_k(x,t) := \bar{w}(x,t) - \varepsilon_k \eta_k(t)$$

to remedy the scenario that the infimum of $\bar{w}(x(t),t)$ may not be attained due to $t \in \mathbb{R}$, where

$$\eta_k(t) = \eta(t - t_k) \in C_0^\infty((-2 + t_k, 2 + t_k))$$

is a smooth cut-off function satisfying

$$\eta_k(t) \equiv 1 \text{ in } [-1 + t_k, 1 + t_k], \text{ and } \eta_k(t) \in [0,1].$$

On one hand, we have

$$v_k(x^k, t_k) = \bar{w}(x^k, t_k) - \varepsilon_k = -m_k - m + m_k = -m.$$

On the other hand, if $|t - t_k| \geq 2$ and $x \in \Omega$, then it follows from (2.4) that

$$v_k(x,t) = \bar{w}(x,t) \geq -m.$$

Consequently, there exists $(\bar{x}^k, \bar{t}_k) \in \Omega \times (-2 + t_k, 2 + t_k)$ such that

$$-m - \varepsilon_k \leq v_k(\bar{x}^k, \bar{t}_k) = \inf_{\Omega\times\mathbb{R}} v_k(x,t) \leq -m.$$

It is easy to verify that

$$-m \leq \bar{w}(\bar{x}^k, \bar{t}_k) \leq -m + \varepsilon_k = -m_k < 0. \tag{2.5}$$

Through direct computations, we have

$$\partial_t^s v_k(\bar{x}^k, \bar{t}_k) = C_s \int_{-\infty}^{\bar{t}_k} \frac{v_k(\bar{x}^k, \bar{t}_k) - v_k(\bar{x}^k, \tau)}{(\bar{t}_k - \tau)^{1+s}} \, \mathrm{d}\tau \leq 0.$$

Again with the definition of $v_k(x,t)$ and Lemma A.2, we get

$$\partial_t^s \bar{w}(\bar{x}^k, \bar{t}_k) \leq \varepsilon_k \partial_t^s \eta_k(\bar{t}_k) \leq C\varepsilon_k. \tag{2.6}$$

Next, recalling Lemma A.1, and collecting the definition of $\bar{w}(x,t)$, (1.8), (2.3), (2.5), the truth of $|\bar{x}^k - y| < |\bar{x}^k - y^\lambda|$ and $h(y) > h(y^\lambda)$ for $y \in \Sigma_\lambda$, we derive immediately

$$\begin{aligned}
&[-(\Delta+\rho)^{\frac{\alpha}{2}}]w(\bar{x}^k, \bar{t}_k) = [-(\Delta+\rho)^{\frac{\alpha}{2}}](\bar{w}(\bar{x}^k, \bar{t}_k)h(\bar{x}^k)) \\
&\quad = \bar{w}(\bar{x}^k, \bar{t}_k)[-(\Delta+\rho)^{\frac{\alpha}{2}}]h(\bar{x}^k) + C_{n,\alpha}\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{h(y)(\bar{w}(\bar{x}^k, \bar{t}_k) - \bar{w}(y, \bar{t}_k))}{e^{\rho|\bar{x}^k - y^\lambda|}|\bar{x}^k - y|^{n+\alpha}} \, \mathrm{d}y \\
&\quad \leq \bar{w}(\bar{x}^k, \bar{t}_k)[-(\Delta+\rho)^{\frac{\alpha}{2}}]h(\bar{x}^k) + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{h(y)[\bar{w}(\bar{x}^k, \bar{t}_k) - w(y, \bar{t}_k)]}{|\bar{x}^k - y^\lambda|^{n+\alpha} e^{\rho|\bar{x}^k - y^\lambda|}} \, \mathrm{d}y \\
&\qquad + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{h(y^\lambda)[\bar{w}(\bar{x}^k, \bar{t}_k) + w(y, \bar{t}_k)]}{|\bar{x}^k - y^\lambda|^{n+\alpha} e^{\rho|\bar{x}^k - y^\lambda|}} \, \mathrm{d}y \qquad (2.7) \\
&\quad \leq \bar{w}(\bar{x}^k, \bar{t}_k)[-(\Delta+\rho)^{\frac{\alpha}{2}}]h(\bar{x}^k) + C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2h(y^\lambda)\bar{w}(\bar{x}^k, \bar{t}_k)}{|\bar{x}^k - y^\lambda|^{n+\alpha} e^{\rho|\bar{x}^k - y^\lambda|}} \, \mathrm{d}y \\
&\quad \leq \bar{w}(\bar{x}^k, \bar{t}_k)[-(\Delta+\rho)^{\frac{\alpha}{2}}]h(\bar{x}^k) \\
&\quad \leq \frac{C_1}{l^\alpha} h(\bar{x}^k)\bar{w}(\bar{x}^k, \bar{t}_k).
\end{aligned}$$

Then we can deduce from (1.8), (2.5), (2.6), (2.7), and the boundedness of $c(x,t)$ that

$$\begin{aligned}-\frac{C_1}{l^\alpha}m_k h(\bar{x}^k) &\geq \frac{C_1}{l^\alpha}h(\bar{x}^k)\bar{w}(\bar{x}^k,\bar{t}_k)\\ &\geq [-(\Delta+\rho)^{\frac{\alpha}{2}}]w(\bar{x}^k,\bar{t}_k)\\ &= -\partial_t^s w(\bar{x}^k,\bar{t}_k)+c(\bar{x}^k,\bar{t}_k)w(\bar{x}^k,\bar{t}_k)\\ &= -h(\bar{x}^k)\partial_t^s\bar{w}(\bar{x}^k,\bar{t}_k)+c(\bar{x}^k,\bar{t}_k)h(\bar{x}^k)\bar{w}(\bar{x}^k,\bar{t}_k)\\ &\geq -C\varepsilon_k h(\bar{x}^k)-Cmh(\bar{x}^k).\end{aligned}$$

The above formula implies that

$$\frac{C_1}{l^\alpha}\leftarrow\frac{C_1}{l^\alpha}\frac{m_k}{m}\leq\frac{C\varepsilon_k}{m}+C\to C,\ \text{ as } k\to\infty.$$

This will lead to a contradiction if we take the sufficiently small l, which proves that (1.9) is true.

Now we aim to prove (1.10). It follows from (1.9) that

$$w(x^0,t_0)=\min_{\Sigma_\lambda\times\mathbb{R}}w(x,t)=0.$$

If $w(x,t_0)\not\equiv 0$ in Σ_λ, on one hand, through straightforward calculations, we deduce that

$$\partial_t^s w(x^0,t_0)=-C_s\int_{-\infty}^{t_0}\frac{w(x^0,\tau)}{(t_0-\tau)^{1+s}}\,\mathrm{d}\tau\leq 0,$$

and

$$\begin{aligned}&-(\Delta+\rho)^{\frac{\alpha}{2}}w(x^0,t_0)\\ &\quad=C_{n,\alpha}\ \text{P.V.}\int_{\Sigma_\lambda}w(y,t_0)\left(\frac{1}{|x^0-y^\lambda|^{n+\alpha}e^{\rho|x^0-y^\lambda|}}-\frac{1}{|x^0-y|^{n+\alpha}e^{\rho|x^0-y|}}\right)\mathrm{d}y<0.\end{aligned}$$

And therefore

$$\partial_t^s w(x^0,t_0)+[-(\Delta+\rho)^{\frac{\alpha}{2}}]w(x^0,t_0)<0.$$

On the other hand, it follows by (1.8) that

$$\partial_t^s w(x^0,t_0)-(\Delta+\rho)^{\frac{\alpha}{2}}w(x^0,t_0)=0. \tag{2.8}$$

That again implies that $w(x,t_0)\equiv 0$ in Σ_λ.

In addition, thanks to the anti-symmetry of $w(x,t)$ with respect to x, we obtain

$$w(x,t_0)\equiv 0,\ \text{ in } \mathbb{R}^n.$$

Note now that, for any fixed $x\in\Sigma_\lambda$ such that $w(x,t)\not\equiv 0$ in $(-\infty,t_0)$, through a very similar proof to the above, we have

$$\partial_t^s w(x,t_0)=-C_s\int_{-\infty}^{t_0}\frac{w(x,\tau)}{(t_0-\tau)^{1+s}}\,\mathrm{d}\tau<0$$

and

$$-(\Delta+\rho)^{\frac{\alpha}{2}}w(x,t_0)=0,$$

which also means that

$$\partial_t^s w(x,t_0)-(\Delta+\rho)^{\frac{\alpha}{2}}w(x,t_0)<0.$$

This contradicts the equation (2.8), so $w(x,t) \equiv 0$ in $\Sigma_\lambda \times (-\infty, t_0]$. Recalling the anti-symmetry of $w(x,t)$ with respect to x again, we deduce that

$$w(x,t) \equiv 0, \text{ in } \mathbb{R}^n \times (-\infty, t_0].$$

Therefore, we conclude the proof. □

3. Averaging effects

In this section, we develop two averaging effects (Theorem 1.2 and Theorem 1.3) of the dual nonlocal operator $\partial_t^s - (\Delta+\rho)^{\frac{\alpha}{2}}$, which will play crucial roles in the proof of our main results and will become powerful tools in studying fractional problems, particularly in the cases where the solutions are unbounded. We start by presenting the proof of Theorem 1.2.

Proof of Theorem 1.2: We will prove Theorem 1.2 by constructing a subsolution, namely

$$\psi(x,t) := \phi(x)\eta(t) := C\left(1 - \left|\frac{x-x^0}{r}\right|^2\right)_+^{\frac{\alpha}{2}} \eta(t),$$

where $\eta(t)$ is a smooth cut-off function whose compact support is contained in $(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}})$, satisfying

$$0 \le \eta(t) \le 1, \text{ and } \eta(t) \equiv 1 \text{ in } \left[t_0 - \frac{r^{\frac{\alpha}{s}}}{2}, t_0 + \frac{r^{\frac{\alpha}{s}}}{2}\right].$$

In fact, it is well known that

$$-(\Delta+\rho)^{\frac{\alpha}{2}}\phi(x) = \frac{C}{r^\alpha}, \text{ in } B_r(x^0) \tag{3.1}$$

for some positive constant C, where we use the fact of $\phi(x) \equiv 0$ in $B_r^c(x^0)$. More precisely, we consider the following function:

$$\underline{u}(x,t) := u(x,t)\chi_D(x) + \delta\psi(x,t),$$

where δ is a positive constant to be determined later, and χ_D is the characteristic function on D, that is,

$$\chi_D(x) = \begin{cases} 1, & \text{if } x \in D, \\ 0, & \text{if } x \notin D. \end{cases}$$

Next, we want to prove that $\underline{u}(x,t)$ is a subsolution of $u(x,t)$ in $B_r(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$. For $(x,t) \in B_r(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$, thanks to (1.11), (1.12), (3.1), and Corollary A.3 with $B_r(x^0) \cap \overline{D} = \varnothing$, we directly calculate

$$\begin{aligned}
&\partial_t^s(u(x,t) - \underline{u}(x,t)) - (\Delta+\rho)^{\frac{\alpha}{2}}(u(x,t) - \underline{u}(x,t)) \\
&\quad \ge -\varepsilon - \delta\phi(x)\partial_t^\alpha\eta(t) + (\Delta+\rho)^{\frac{\alpha}{2}}(u(x,t)\chi_D(x)) - \delta\eta(t)[-(\Delta+\rho)^{\frac{\alpha}{2}}]\phi(x) \\
&\quad \ge -\varepsilon + C_{n,\alpha}C_0 \int_D \frac{1}{|x-y|^{n+\alpha}e^{\rho|x-y|}}\,\mathrm{d}y - \frac{C\delta}{r^\alpha} \\
&\quad \ge -\varepsilon + C_2 - \frac{C\delta}{r^\alpha}.
\end{aligned}$$

Now choose $\epsilon = \frac{C_2}{2}$ and $\delta = \frac{C_2 r^\alpha}{2C}$, so that the following holds:

$$\partial_t^s(u(x,t)-\underline{u}(x,t))+[-(\Delta+\rho)^{\frac{\alpha}{2}}](u(x,t)-\underline{u}(x,t))\geq 0, \quad (x,t)\in B_r(x^0)\times(t_0-r^{\frac{\alpha}{s}}, t_0+r^{\frac{\alpha}{s}}].$$

For $(x,t) \in B_r^c(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$, by the definition of $\psi(x,t)$ and the exterior condition in (1.12) we conclude

$$u(x,t) - \underline{u}(x,t) = u(x,t) - u(x,t)\chi_D(x) \geq 0.$$

On the other hand, if $(x,t)\in B_r(x^0)\times(-\infty, t_0-r^{\frac{\alpha}{s}}]$, with the initial condition in (1.12), we get

$$u(x,t) - \underline{u}(x,t) = u(x,t) \geq 0.$$

In summary, we have obtained

$$\begin{cases} \partial_t^s(u(x,t)-\underline{u}(x,t))-(\Delta+\rho)^{\frac{\alpha}{2}}(u(x,t)-\underline{u}(x,t))\geq 0, & \text{in } B_r(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}], \\ u(x,t) - \underline{u}(x,t) \geq 0, & \text{in } B_r^c(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}], \\ u(x,t) - \underline{u}(x,t) \geq 0, & \text{in } B_r(x^0) \times (-\infty, t_0 - r^{\frac{\alpha}{s}}]. \end{cases}$$

Applying the maximum principle established in Theorem 2.1 implies that

$$u(x,t) \geq \underline{u}(x,t) \text{ for } (x,t) \in B_r(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}].$$

Hence, we verify that $\underline{u}(x,t)$ is a subsolution of $u(x,t)$ in $B_r(x^0) \times (t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}})$. Finally, it follows that

$$u(x^0, t_0) \geq \underline{u}(x^0, t_0) = \delta\phi(x^0)\eta(t_0) = C\delta =: \bar{C} > 0.$$

Now the proof of Theorem 1.2 is completed. □

Now we turn our attention to the proof of Theorem 1.3 regarding the averaging effects for the anti-symmetric functions.

Proof of Theorem 1.3: This proof argument is very similar to the one in the proof of Theorem 1.2. First, we construct an anti-symmetric subsolution of $w(x,t)$. Denote

$$\phi(x) := \left(1 - \left|\frac{x - x^0}{r}\right|^2\right)_+^{\frac{\alpha}{2}} \text{ and } \phi_\lambda(x) := \left(1 - \left|\frac{x^\lambda - x^0}{r}\right|^2\right)_+^{\frac{\alpha}{2}},$$

and use that

$$\Phi(x) := \phi(x) - \phi_\lambda(x)$$

is an anti-symmetric function with respect to the plane T_λ. We denote

$$\eta(t) \in C_0^\infty((t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}})),$$

which is a smooth cut-off function whose value belongs to $[0,1]$, satisfying

$$\eta(t) \equiv 1, \text{ in } \left[t_0 - \frac{r^{\frac{\alpha}{s}}}{2}, t_0 + \frac{r^{\frac{\alpha}{s}}}{2}\right].$$

Let D^λ be the reflection domain of D with respect to the plane T_λ and

$$\underline{w}(x,t) = w(x,t)\chi_{D\cup D^\lambda}(x) + \delta\Phi(x)\eta(t),$$

where δ is a positive constant to be determined later, and $\chi_{D\cup D^\lambda}(x)$ is the characteristic function in the region $D \cup D^\lambda$.

Similarly, we also aim to show that the anti-symmetric function $\underline{w}(x,t)$ is a subsolution of $w(x,t)$ in $B_r(x^0)\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$. For $(x,t)\in B_r(x^0)\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$, thanks to (1.14) with (1.13), (3.1), Corollary A.3, and $r \le \frac{\operatorname{dist}(x^0,T_\lambda)}{3}$, we can choose $\epsilon = \frac{C_2}{2}$ and $\delta = \frac{C_2 r^{2s}}{2C}$, obtaining

$$\begin{aligned}
&\partial_t^s(w(x,t) - \underline{w}(x,t)) - (\Delta+\rho)^{\frac{\alpha}{2}}(w(x,t) - \underline{w}(x,t))\\
&\quad\ge -\varepsilon - \delta\Phi(x)\partial_t^s\eta(t) + (\Delta+\rho)^{\frac{\alpha}{2}}(w(x,t)\chi_{D\cup D^\lambda}(x)) - \delta\eta(t)[-(\Delta+\rho)^{\frac{\alpha}{2}}](\phi(x)-\phi_\lambda(x))\\
&\quad\ge -\varepsilon + C_{n,\alpha}\int_{D\cup D^\lambda}\frac{w(y,t)}{|x-y|^{n+\alpha}e^{\rho|x-y|}}\,\mathrm{d}y - \frac{C\delta}{r^\alpha} - C_{n,\alpha}\delta\int_{B_r((x^0)^\lambda)}\frac{1}{|x-y|^{n+\alpha}e^{\rho|x-y|}}\,\mathrm{d}y\\
&\quad\ge -\varepsilon + C_{n,\alpha}C_0\int_D\left[\frac{1}{|x-y|^{n+\alpha}e^{\rho|x-y|}} - \frac{1}{|x-y^\lambda|^{n+\alpha}e^{\rho|x-y^\lambda|}}\right]\mathrm{d}y - \frac{C\delta}{r^\alpha}\\
&\quad\ge -\varepsilon + C_2 - \frac{C\delta}{r^\alpha} \ge 0.
\end{aligned}$$

For $(x,t)\in(\Sigma_\lambda\setminus B_r(x^0))\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$, one can further deduce from the exterior condition in (1.13) that

$$w(x,t) - \underline{w}(x,t) = w(x,t) - w(x,t)\chi_{D\cup D^\lambda}(x) \ge 0.$$

Finally, taking into account $(x,t)\in B_r(x^0)\times(-\infty, t_0 - r^{\frac{\alpha}{s}}]$, we apply the initial condition in (1.13) to derive

$$w(x,t) - \underline{w}(x,t) = w(x,t) \ge 0.$$

Through the above arguments, we have deduced that

$$\begin{cases}
\partial_t^s(w(x,t) - \underline{w}(x,t)) - (\Delta+\rho)^{\frac{\alpha}{2}}(w(x,t) \\
\qquad\qquad - \underline{w}(x,t)) \ge 0, & \text{in } B_r(x^0)\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}],\\
w(x,t) - \underline{w}(x,t) \ge 0, & \text{in } (\Sigma_\lambda\setminus B_r(x^0))\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}],\\
w(x,t) - \underline{w}(x,t) \ge 0, & \text{in } B_r(x^0)\times(-\infty, t_0 - r^{\frac{\alpha}{s}}],\\
w(x,t) - \underline{w}(x,t) = -(w(x^\lambda,t) - \underline{w}(x^\lambda,t)), & \text{in } \Sigma_\lambda\times\mathbb{R}.
\end{cases}$$

Again with the maximum principle for anti-symmetric functions established in Theorem 2.2, we obtain

$$w(x,t) \ge \underline{w}(x,t),\ \text{ in } B_r(x^0)\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}].$$

Hence, we conclude that $\underline{w}(x,t)$ is a subsolution of $w(x,t)$ in $B_r(x^0)\times(t_0 - r^{\frac{\alpha}{s}}, t_0 + r^{\frac{\alpha}{s}}]$. As a consequence, we have

$$w(x^0,t_0) \ge \underline{w}(x^0,t_0) = \delta\phi(x^0)\eta(t_0) = \delta =: C_1 > 0.$$

This completes the proof of Theorem 1.3. $\square$

4. Monotonicity in a half-space

Based on the maximum principles and the averaging effects introduced in the previous section, we establish the monotonicity of positive solutions for nonlocal parabolic problem (1.1) in a half-space by virtue of the direct method of moving planes. For readers' convenience, we first introduce some notation used throughout the paper.

In what follows, we denote

$$T_\lambda := \{x = (x_1, x') \in \mathbb{R}^n \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

to be the moving planes perpendicular to the x_1-axis,

$$\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \text{ and } \Omega_\lambda := \{x \in \mathbb{R}^n_+ \mid x_1 < \lambda\}$$

to be the region to the left of the hyperplane T_λ in $\mathbb{R}^n$ and in $\mathbb{R}^n_+$ respectively. We denote the reflection of x with respect to the hyperplane T_λ as

$$x^\lambda := (2\lambda - x_1, x_2, \dots, x_n).$$

Let $u(x,t)$ be a solution of (1.1) and $u_\lambda(x,t) := u(x^\lambda, t)$. Define

$$w_\lambda(x,t) := u_\lambda(x,t) - u(x,t),$$

which represents the comparison between the values of $u(x,t)$ and $u(x^\lambda, t)$. It is obvious that $w_\lambda(x,t)$ is an anti-symmetric function of x with respect to the hyperplane T_λ.

Now we show the details.

Proof of Theorem 1.4: Through direct calculations, we consider the following problem:

$$\begin{cases} \partial_t^s w_\lambda(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} w_\lambda(x,t) = C_\lambda(x,t) w_\lambda(x,t), & (x,t) \in \Omega_\lambda \times \mathbb{R}, \\ w_\lambda(x,t) \geq 0, & (x,t) \in (\Sigma_\lambda \setminus \Omega_\lambda) \times \mathbb{R}, \\ w_\lambda(x,t) = -w_\lambda(x^\lambda, t), & (x,t) \in \Sigma_\lambda \times \mathbb{R}, \end{cases} \tag{4.1}$$

where the coefficient function is

$$C_\lambda(x,t) = \int_0^1 f_u(t, \varsigma u_\lambda(x,t) + (1-\varsigma) u(x,t))\, \mathrm{d}\varsigma = \frac{f(t, u_\lambda(x,t)) - f(t, u(x,t))}{u_\lambda(x,t) - u(x,t)}.$$

Recalling the assumption (F_3), we know that there exists a positive constant C such that $C_\lambda(x,t) \leq C$. In order to prove the strict monotonicity of $u(x,t)$ with respect to x_1, our aim is to show that, for any $\lambda > 0$,

$$w_\lambda(x,t) \geq 0, \text{ in } \Omega_\lambda \times \mathbb{R}. \tag{4.2}$$

We divide the proof of Theorem 1.4 into three steps. In Step 1, we show that for $\lambda > 0$ sufficiently close to 0 (which means Ω_λ is a narrow region), equation (4.2) holds. In Step 2, we prove that the plane T_λ moves continuously to the right along the x_1-axis as long as inequality (4.2) is valid, to its limiting position (move to positive infinity). Finally, in Step 3 we prove $u(x,t)$ is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t \in \mathbb{R}$.

Step 1. Start moving the plane T_λ from $x_1 = 0$ to the right along the x_1-axis. For any positive λ small enough (sufficiently close to 0), one can infer from the assumptions in Theorem 1.4 that Ω_λ is a narrow region. Applying Theorem 1.1 to the problem (4.1), we get

$$w_\lambda(x,t) \geq 0, \text{ in } \Omega_\lambda \times \mathbb{R}. \tag{4.3}$$

Inequality (4.3) provides a starting point to move the plane T_λ.

Step 2. In this step, we continue to move the plane T_λ to the right along the x_1-axis, as long as (4.3) holds, to its limiting position. Let

$$\lambda_0 := \sup\{\lambda \mid w_\mu(x,t) \geq 0,\ (x,t) \in \Sigma_\mu \times \mathbb{R} \text{ for any } \mu \leq \lambda\}.$$

We aim to prove that

$$\lambda_0 = +\infty.$$

Otherwise, suppose on the contrary that $0 < \lambda_0 < +\infty$; from the definition of λ_0, there exists a sequence of $\lambda_k > \lambda_0$ such that $\lambda_k \to \lambda_0$ as $k \to \infty$, along which

$$\Sigma_{\lambda_k}^- \times \mathbb{R} := \{(x,t) \in \Sigma_{\lambda_k} \times \mathbb{R} \mid w_{\lambda_k}(x,t) < 0\} \neq \varnothing,$$

and $\inf\limits_{\Sigma_{\lambda_k} \times \mathbb{R}} w_{\lambda_k}(x,t) < 0$. We first claim that

$$\inf_{\Sigma_{\lambda_k} \times \mathbb{R}} w_{\lambda_k}(x,t) \to 0, \text{ as } k \to \infty.$$

If not, then there exists a uniformly positive constant M such that

$$\inf_{\Sigma_{\lambda_k} \times \mathbb{R}} w_{\lambda_k}(x,t) < -M < 0.$$

The above equation implies there exists a sequence $\{(x^k, t_k)\} \subset \Sigma_{\lambda_k} \times \mathbb{R}$ such that

$$w_{\lambda_k}(x^k, t_k) \leq -M < 0. \tag{4.4}$$

If x^k is between T_{λ_0} and T_{λ_k}, then by virtue of $\lambda_k \to \lambda_0$ as $k \to \infty$, we have $|x^k - (x^k)^{\lambda_k}| \to 0$ as $k \to \infty$. Recalling that $u(x,t)$ is uniformly continuous with respect to x, we derive

$$w_{\lambda_k}(x^k, t_k) = u((x^k)^{\lambda_k}, t_k) - u(x^k, t_k) \to 0, \text{ as } k \to \infty,$$

which contradicts (4.4). In the other case, if $x^k \in \Omega_{\lambda_0}$, then together with the uniform continuity of $u(x,t)$ in x with $\lambda_k \to \lambda_0$, we have

$$w_{\lambda_k}(x^k, t_k) - w_{\lambda_0}(x^k, t_k) = u((x^k)^{\lambda_k}, t_k) - u((x^k)^{\lambda_0}, t_k) \to 0, \text{ as } k \to \infty.$$

Using equation (4.4) and the truth of $w_{\lambda_0}(x^k, t_k) \geq 0$, we must derive a contradiction that

$$w_{\lambda_k}(x^k, t_k) - w_{\lambda_0}(x^k, t_k) \leq -M < 0.$$

Hence, we deduce that

$$\inf_{\Sigma_{\lambda_k} \times \mathbb{R}} w_{\lambda_k}(x,t) =: -m_k \to 0, \text{ as } k \to \infty. \tag{4.5}$$

Next, we denote the sequence

$$q_k := \sup_{\Sigma_{\lambda_k}^- \times \mathbb{R}} C_{\lambda_k}(x,t).$$

Hence, one of the following two cases may occur as $k \to \infty$.

Case 1. If $q_k \leq \varepsilon_k \to 0$ as $k \to \infty$, then it infers that $C_{\lambda_k}(x,t) \leq C$ in $\Sigma_{\lambda_k}^- \times \mathbb{R}$ for sufficiently large k and for any given small positive constant C. Hence, thanks to Remark 1.1, we can take $\lambda = \lambda_k$ in the problem (4.1), and we conclude that

$$w_{\lambda_k}(x,t) \geq 0, \text{ in } \Sigma_{\lambda_k} \times \mathbb{R}$$

for sufficiently large k, which is a contradiction with the definition of λ_k.

Case 2. If $q_k \nrightarrow 0$ as $k \to \infty$, then there exist a positive constant δ_0 and a subsequence of $\{q_k\}$ (still denoted by $\{q_k\}$) such that $q_k \geq \delta_0 > 0$. In this case, recalling the assumption (F_2) and (4.5), we can deduce that there exist a positive constant ε_0 and a sequence $\{(x^k, t_k)\} \subset \Sigma_{\lambda_k}^- \times \mathbb{R}$ such that

$$u(x^k, t_k) \geq \varepsilon_0 > 0,$$

and

$$w_{\lambda_k}(x^k, t_k) = -m_k + m_k^2 < 0.$$

Observe that $u(x,t) = 0$ in $(\mathbb{R}^n \setminus \mathbb{R}^n_+) \times \mathbb{R}$ and the continuity of $u(x,t)$ yields that there exists a small radius $r_0 > 0$ independent of k, such that

$$u(x,t) \geq \frac{\varepsilon_0}{2} > 0, \text{ in } B_{r_0}(x^k) \times (t_k - r_0^{\frac{\alpha}{s}}, t_k + r_0^{\frac{\alpha}{s}}] \subset \mathbb{R}^n_+ \times \mathbb{R}. \tag{4.6}$$

Next, we will show via contradiction arguments that $\delta_k := \text{dist}(x^k, T_{\lambda_k}) = \lambda_k - x_1^k$ is bounded away from zero for sufficiently large k. Suppose on the contrary that $\delta_k \to 0$ as $k \to \infty$. Let

$$v_k(x,t) := w_{\lambda_k}(x,t) - m_k^2 \eta_k(x,t),$$

where the sequence of smooth cut-off functions

$$\eta_k(x,t) := \eta\left(\frac{x - x^k}{\delta_k}, \frac{t - t_k}{\delta_k^{\frac{\alpha}{s}}}\right) \in C_0^\infty(B_{\delta_k}(x^k) \times (t_k - \delta_k^{\frac{\alpha}{s}}, t_k + \delta_k^{\frac{\alpha}{s}}))$$

satisfies

$$0 \leq \eta_k \leq 1, \text{ and } \eta_k(x,t) \equiv 1 \text{ in } B_{\frac{\delta_k}{2}}(x^k) \times \left(t_k - \frac{\delta_k^{\frac{\alpha}{s}}}{2}, t_k + \frac{\delta_k^{\frac{\alpha}{s}}}{2}\right).$$

We denote the parabolic cylinder

$$Q_{\delta_k}(x^k, t_k) := B_{\delta_k}(x^k) \times (t_k - \delta_k^{\frac{\alpha}{s}}, t_k + \delta_k^{\frac{\alpha}{s}}),$$

after a direct calculation, we know that

$$v_k(x^k, t_k) = w_{\lambda_k}(x^k, t_k) - m_k^2 \eta_k(x^k, t_k) = -m_k + m_k^2 - m_k^2 = -m_k,$$

and

$$v_k(x,t) = w_{\lambda_k}(x,t) \geq -m_k \text{ for } (x,t) \in Q^c_{\delta_k}(x^k, t_k) \cap (\Sigma_{\lambda_k} \times \mathbb{R}).$$

Hence, there exists a point $(\bar{x}^k, \bar{t}_k) \in Q_{\delta_k}(x^k, t_k)$ such that

$$-m_k - m_k^2 \leq v_k(\bar{x}^k, \bar{t}_k) = \inf_{\Sigma_{\lambda_k} \times \mathbb{R}} v_k(x,t) \leq -m_k.$$

Moreover, it follows from the definition of v_k that

$$-m_k \leq w_{\lambda_k}(\bar{x}^k, \bar{t}_k) \leq -m_k + m_k^2 < 0.$$

Through direct computations, we have

$$\partial_t^s v_k(\bar{x}^k, \bar{t}_k) = C_\alpha \int_{-\infty}^{\bar{t}_k} \frac{v_k(\bar{x}^k, \bar{t}_k) - v_k(\bar{x}^k, \tau)}{(\bar{t}_k - \tau)^{1+s}} \, \mathrm{d}\tau \leq 0,$$

and

$$\begin{aligned}
-(\Delta + \rho)^{\frac{\alpha}{2}} v_k(\bar{x}^k, \bar{t}_k) &= C_{n,s} \text{ P.V.} \int_{\Sigma_{\lambda_k}} \frac{v_k(\bar{x}^k, \bar{t}_k) - v_k(y, \bar{t}_k)}{|\bar{x}^k - y|^{n+\alpha} e^{\rho|\bar{x}^k - y|}} \, \mathrm{d}y \\
&\quad + C_{n,s} \int_{\Sigma_{\lambda_k}^c} \frac{v_k(\bar{x}^k, \bar{t}_k) - v_k(y, \bar{t}_k)}{|\bar{x}^k - y|^{n+\alpha} e^{\rho|\bar{x}^k - y|}} \, \mathrm{d}y \\
&\leq C_{n,s} \int_{\Sigma_{\lambda_k}} \frac{2v_k(\bar{x}^k, \bar{t}_k) + m_k^2 \eta_k(y, \bar{t}_k)}{|\bar{x}^k - y^{\lambda_k}|^{n+\alpha} e^{\rho|\bar{x}^k - y^{\lambda_k}|}} \, \mathrm{d}y \\
&\leq C_{n,s} \int_{\Sigma_{\lambda_k}} \frac{-2m_k + m_k^2}{|\bar{x}^k - y^{\lambda_k}|^{n+\alpha} e^{\rho|\bar{x}^k - y^{\lambda_k}|}} \, \mathrm{d}y \\
&\leq -\frac{Cm_k}{\delta_k^\alpha} + \frac{Cm_k^2}{\delta_k^\alpha},
\end{aligned}$$

where we use the truth of

$$\begin{aligned}
\int_\Sigma \frac{1}{e^{\rho|\bar{x} - \tilde{y}|} |\bar{x} - \tilde{y}|^{n+\alpha}} \, \mathrm{d}y &\geq \int_D \frac{1}{e^{\rho|\bar{x} - y|} |\bar{x} - y|^{n+\alpha}} \, \mathrm{d}y \\
&\geq \int_D \frac{1}{e^{2\rho} |\bar{x} - y|^{n+\alpha}} \, \mathrm{d}y = \int_{\delta_k}^{2\delta_k} \int_0^1 \frac{\omega_{n-1} \tau^{n-2} \, \mathrm{d}\tau}{e^{2\rho} (t^2 + \tau^2)^{\frac{n+\alpha}{2}}} \, \mathrm{d}t \\
&= \int_{\delta_k}^{2\delta_k} \frac{1}{e^{2\rho} t^{1+\alpha}} \int_0^{\frac{1}{t}} \frac{\omega_{n-1} \rho^{n-2} \, \mathrm{d}\rho}{(1 + \rho^2)^{\frac{n+\alpha+1}{2}}} \, \mathrm{d}t \\
&\geq \int_{\delta_k}^{2\delta_k} \frac{1}{e^{2\lambda} t^{1+\alpha}} \int_0^1 \frac{\omega_{n-1} \rho^{n-2} \, \mathrm{d}\rho}{(1 + \rho^2)^{\frac{n+\alpha}{2}}} \, \mathrm{d}t \\
&\geq C_{n,\alpha,\rho} \int_{\delta_k}^{2\delta_k} \frac{1}{t^{1+\alpha}} \, \mathrm{d}t = \frac{C_{n,\alpha,\rho}}{\delta_k^\alpha},
\end{aligned}$$

with $D := \{y = (y_1, y') \in \mathbb{R}^n \mid \delta_k < y_1 - (\bar{x})_1 < 2\delta_k, \, |y' - (\bar{x})'| < 1\}$, $t := y_1 - (\bar{x})_1$, $\tau := |y' - (\bar{x})'|$, $\rho := \tau/t$, and $\omega_{n-1} = |B_1(0)|$ in $\mathbb{R}^{n-1}$.

Recalling the equation in (4.1) and Corollary A.3, and using the truth of f_u having an upper bound, we get

$$\begin{aligned}
\partial_t^s v_k(\bar{x}^k, \bar{t}_k) + -(\Delta + \rho)^{\frac{\alpha}{2}} v_k(\bar{x}^k, \bar{t}_k) &= \partial_t^s w_{\lambda_k}(\bar{x}^k, \bar{t}_k) + [-(\Delta + \rho)]^{\frac{\alpha}{2}} w_{\lambda_k}(\bar{x}^k, \bar{t}_k) \\
&\quad - m_k^2 \partial_t^s \eta_k(\bar{x}^k, \bar{t}_k) - m_k^2 [-(\Delta + \rho)^{\frac{\alpha}{2}}] \eta_k(\bar{x}^k, \bar{t}_k) \\
&\geq C_{\lambda_k}(\bar{x}^k, \bar{t}_k) w_{\lambda_k}(\bar{x}^k, \bar{t}_k) - \frac{Cm_k^2}{\delta_k^\alpha} \\
&\geq -Cm_k - \frac{Cm_k^2}{\delta_k^\alpha}.
\end{aligned}$$

This proves that

$$-\frac{Cm_k}{\delta_k^\alpha} \geq -Cm_k - \frac{Cm_k^2}{\delta_k^\alpha}.$$

We multiply the above equation by $\frac{\delta_k^\alpha}{-m_k}$, and using (4.5) together the assumption $\lim\limits_{k\to\infty} \delta_k = 0$, we obtain

$$0 < C \leq C\delta_k^\alpha + Cm_k \to 0, \text{ as } k \to \infty.$$

This contradiction indicates that δ_k is bounded away from zero for sufficiently large k. Furthermore, since $\lambda_k \to \lambda_0$ as $k \to \infty$, then we deduce that there exists a subsequence of $\{x^k, t_k\}$ (still denoted by $\{x^k, t_k\}$) such that $\{(x^k, t_k)\} \subset \Sigma_{\lambda_0} \times \mathbb{R}$ and $\text{dist}\{x^k, T_{\lambda_0}\} \geq \delta_0 > 0$. As a consequence of (4.6), we further select a radius $r_1 := \min\{r_0, \delta_0\}$ such that

$$u(x,t) \geq \frac{\varepsilon_0}{2} > 0, \text{ in } B_{r_1}(x^k) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}] \subset \Omega_{\lambda_0} \times \mathbb{R}. \tag{4.7}$$

Now we show that

$$w_{\lambda_k}(x^k, t_k) > 0.$$

Setting $\bar{x}^k = (2\lambda_0, (x^k)')$, since

$$\text{dist}(T_{\lambda_0}, T_{2\lambda_0}) = \lambda_0 > 2r_1,$$

then we have $\overline{B_{r_1}(x^k)} \cap B_{2r_1}(\bar{x}^k) = \varnothing$. Next the main purpose is to claim that there exists a positive constant $\varepsilon_1 = \varepsilon_1(\alpha, n, s, \varepsilon_0, \lambda_0, r_1)$ such that

$$u(x,t) \geq \varepsilon_1 > 0, \text{ in } B_{r_1}(\bar{x}^k) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}]. \tag{4.8}$$

Suppose on the contrary that

$$u(x,t) < \varepsilon, \text{ in } B_{r_1}(\bar{x}^k) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}] \text{ for any } \varepsilon > 0. \tag{4.9}$$

Then we conclude from assumptions (F_1) and (F_2) that

$$|f(t, u(x,t)) - f(t, 0)| \leq C|(u(x,t)) - 0| < C\varepsilon, \text{ in } B_{r_1}(\bar{x}^k) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}].$$

Furthermore, recalling the assumption $f(0) \geq 0$, for any $\varepsilon > 0$, we have

$$f(t, u) > -C\varepsilon, \text{ in } B_{r_1}(\bar{x}^k) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}].$$

Combining (1.1), (4.7), and the averaging effects (Theorem 1.2), and using the continuity of $u(x,t)$, we derive

$$u(x,t) \geq \varepsilon_1 > 0, \text{ in } B_{\frac{r_1}{2}}(\bar{x}^k) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right],$$

where the positive constant $\varepsilon_1 = \varepsilon_1(\alpha, n, s, \varepsilon_0, \lambda_0, r_1)$. It apparently contradicts (4.9), and we conclude that (4.8).

Next, let $\hat{x}^k = (0, (x^k)')$, using the exterior condition $u(x,t) \equiv 0$ in $(\mathbb{R}^n \setminus \mathbb{R}^n_+) \times \mathbb{R}$ and the continuity of $u(x,t)$ again, we deduce that there exists some positive small radius $r_2 < \frac{r_1}{2}$ independent of k, such that

$$u(x,t) \leq \frac{\varepsilon_1}{2}, \text{ in } (B_{r_2}(\hat{x}^k) \cap \mathbb{R}^n_+) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}], \tag{4.10}$$

where ε_1 is given in (4.8). Note that for any point $x \in B_{r_2}(\hat{x}^k) \cap \mathbb{R}^n_+$, since $r_2 < \frac{r_1}{2}$, then we have

$$x^{\lambda_0} \in B_{r_2}(\bar{x}^k) \cap \Sigma_{2\lambda_0} \subset B_{\frac{r_1}{2}}(\bar{x}^k).$$

Collecting equation (4.8) and (4.10), for any $(x,t) \in (B_{r_2}(\hat{x}^k) \cap \mathbb{R}^n_+) \times (t_k - r_1^{\frac{\alpha}{s}}, t_k + r_1^{\frac{\alpha}{s}}]$, we have

$$w_{\lambda_0}(x,t) = u(x^{\lambda_0}, t) - u(x,t) \geq \varepsilon_1 - \frac{\varepsilon_1}{2} = \frac{\varepsilon_1}{2} > 0. \tag{4.11}$$

Next, we will show via contradiction arguments that there exists a positive constant $\varepsilon_2 = \varepsilon_2(\alpha, n, s, \varepsilon_0, \lambda_0, r_1)$, such that

$$w_{\lambda_0}(x,t) \geq \varepsilon_2 > 0 \text{ for } (x,t) \in B_{\frac{r_1}{2}}(x^k) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right]. \tag{4.12}$$

Suppose on the contrary that

$$w_{\lambda_0}(x,t) < \varepsilon, \text{ in } B_{\frac{r_1}{2}}(x^k) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right] \text{ for any } \varepsilon > 0. \tag{4.13}$$

Recalling system (4.1) and the definition of λ_0, we obtain

$$\begin{cases} \partial_t^s w_{\lambda_0}(x,t) - (\Delta + \rho)^{\frac{\alpha}{2}} w_{\lambda_0}(x,t) \\ \qquad = C_{\lambda_0}(x,t) w_{\lambda_0}(x,t), & \text{in } B_{\frac{r_1}{2}}(x^k) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right], \\ w_{\lambda_0}(x,t) \geq 0, & \text{in } (\Sigma_{\lambda_0} \setminus B_{\frac{r_1}{2}}(x^k)) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right], \\ w_{\lambda_0}(x,t) \geq 0, & \text{in } B_{\frac{r_1}{2}}(x^k) \times \left(-\infty, t_k - \left(\frac{r_2}{2}\right)^{\frac{\alpha}{s}}\right]. \end{cases}$$

By using (4.13) and $f \in C^1$, one can observe that

$$C_{\lambda_0}(x,t) w_{\lambda_0}(x,t) = f(t, u_{\lambda_0}) - f(t,u) > -C\varepsilon, \text{ in } B_{\frac{r_1}{2}}(x^k) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right],$$

for any $\varepsilon > 0$. Consequently, combining (4.11) and the averaging effects (Theorem 1.3), again with the continuity of $u(x,t)$, we conclude that

$$w_{\lambda_0}(x,t) \geq \varepsilon_2 > 0, \text{ in } B_{\frac{r_1}{4}}(x^k) \times \left(t_k - \left(\frac{r_1}{4}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{4}\right)^{\frac{\alpha}{s}}\right]$$

for some positive constant $\varepsilon_2 = \varepsilon_2(\alpha, n, s, \varepsilon_0, \lambda_0, r_1)$, which contradicts the assumption (4.13). Hence, we verify that (4.12) is valid.

Moreover, as a consequence of (4.12) and the continuity of $w_\lambda(x,t)$ in λ with $\lambda_k \to \lambda_0$ as $k \to \infty$, we arrive at

$$w_{\lambda_k}(x,t) \geq \frac{\varepsilon_2}{2} > 0, \text{ in } B_{\frac{r_1}{2}}(x^k) \times \left(t_k - \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}, t_k + \left(\frac{r_1}{2}\right)^{\frac{\alpha}{s}}\right] \text{ for sufficiently large } k,$$

which implies that $w_{\lambda_k}(x^k, t_k) \geq \frac{\varepsilon_2}{2} > 0$ for sufficiently large k. Hence, it contradicts the assumption that the sequence $\{(x^k, t_k)\} \subset \Sigma^-_{\lambda_k} \times \mathbb{R}$, therefore we must have $\lambda_0 = +\infty$.

Step 3. In this step, we will prove that $u(x,t)$ is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t \in \mathbb{R}$. From Step 1 and Step 2, we have deduced that

$$w_\lambda(x,t) \geq 0, \text{ in } \Sigma_\lambda \times \mathbb{R} \text{ for any } \lambda > 0.$$

In fact, we only have to show that

$$w_\lambda(x,t) > 0, \text{ in } \Omega_\lambda \times \mathbb{R} \text{ for any } \lambda > 0, \tag{4.14}$$

and then we can conclude that $u(x,t)$ is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t \in \mathbb{R}$. Suppose (4.14) is violated; then there exist a fixed $\lambda_0 > 0$ and a point $(x^0, t_0) \in \Omega_{\lambda_0} \times \mathbb{R}$ such that

$$w_{\lambda_0}(x^0, t_0) = \min_{\Sigma_{\lambda_0} \times \mathbb{R}} w_{\lambda_0}(x,t) = 0.$$

Since $w_{\lambda_0}(x,t_0)$ is not identically zero in Σ_{λ_0}, again with the exterior condition $u(x,t) \equiv 0$ in $(\mathbb{R}^n \setminus \mathbb{R}^n_+) \times \mathbb{R}$ and the interior positivity of the solution $u(x,t)$ in $\mathbb{R}^n_+ \times \mathbb{R}$, then through straightforward calculations, we can get the following:

$$\begin{aligned}
&\partial_t^s w_{\lambda_0}(x^0, t_0) - (\Delta + \rho)^{\frac{\alpha}{2}} w_{\lambda_0}(x^0, t_0) \\
&= C_s \int_{-\infty}^{t_0} \frac{w_{\lambda_0}(x^0, t_0) - w_{\lambda_0}(x^0, \tau)}{(t_0 - \tau)^{1+s}} \, d\tau + C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{w_{\lambda_0}(x^0, t_0) - w_{\lambda_0}(y, t_0)}{|x^0 - y|^{n+\alpha} e^{\rho|x^0 - y|}} \, dy \\
&\leq C_{n,\alpha} \text{ P.V.} \int_{\Sigma_{\lambda_0}} w_{\lambda_0}(y, t_0) \left(\frac{1}{|x^0 - y^{\lambda_0}|^{n+\alpha} e^{\rho|x^0 - y^{\lambda_0}|}} - \frac{1}{|x^0 - y|^{n+\alpha} e^{\rho|x^0 - y|}} \right) dy < 0,
\end{aligned}$$

which contradicts

$$\partial_t^s w_{\lambda_0}(x^0, t_0) - (\Delta + \rho)^{\frac{\alpha}{2}} w_{\lambda_0}(x^0, t_0) = f(t_0, u_{\lambda_0}(x^0, t_0)) - f(t_0, u(x^0, t_0)) = 0.$$

Now we have proved (4.14).

Finally, for each fixed $t \in \mathbb{R}$, for any points $\bar{x} = (\bar{x}_1, x')$ and $\hat{x} = (\hat{x}_1, x')$ in $\mathbb{R}^n_+$ with $\bar{x}_1 < \hat{x}_1$, if choosing $\lambda = \frac{\bar{x}_1 + \hat{x}_1}{2}$, then we must have

$$0 < w_\lambda(\bar{x}, t) = u(\bar{x}^\lambda, t) - u(\bar{x}, t) = u(\hat{x}, t) - u(\bar{x}, t).$$

It implies that $u(x,t)$ is strictly increasing with respect to x_1 in $\mathbb{R}^n_+$ for any $t \in \mathbb{R}$. This concludes the proof of Theorem 1.4. □

5. Sliding methods for a dual tempered fractional parabolic problem

In this section, inspired by [**14, 34**], we will establish *maximum principles in unbounded open sets* and develop direct sliding methods for a dual tempered fractional parabolic problem and apply it to investigate monotonicity of solutions to the problem (1.15). The study of Gibbons' conjecture and the sliding method in the context of the dual fractional parabolic equation has witnessed further notable progress, as evidenced by works such as [**6, 25**] and the references therein.

5.1. Maximum principles in unbounded open sets. In order to apply the sliding method on unbounded open sets, the following maximum principles play an important role.

Theorem 5.1 (Maximum principles in unbounded open sets). *Let Ω be an open set in $\mathbb{R}^n$, possibly unbounded and disconnected, and satisfy*

$$\liminf_{R\to\infty} \frac{|\Omega^c \cap B_R(x)|}{|B_R(x)|} \geq c_0 > 0, \quad \forall x \in \Omega. \tag{5.1}$$

Suppose that $u \in \{\mathcal{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)\} \times (C^1(\mathbb{R}) \cap \mathcal{L}_s^-(\mathbb{R}))$ is bounded from above, and solves

$$\begin{cases} \partial_t^s u - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) + c(x,t)u(x,t) \leq 0, & \text{at points } x\in\Omega \text{ where } u(x,t)>0,\\ u(x,t) \leq 0, & \text{in } \Omega^c\times\mathbb{R}, \end{cases}$$

where $c(x,t)$ is nonnegative in the set $\{x\in\Omega | u(x)>0\}$.

Then we must have

$$u \leq 0, \quad \text{in } \Omega\times\mathbb{R}. \tag{5.2}$$

Proof: Suppose that the conclusion (5.2) is not valid, together with the fact that $u(x,t)$ is bounded from above in $\Omega\times\mathbb{R}$; then there exists a positive constant A such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} u(x,t) := A > 0. \tag{5.3}$$

Note that the domain $\Omega\times\mathbb{R}$ is unbounded and the supremum of $u(x,t)$ may not be attained; then there exists a sequence $\{(x^k,t_k)\}\subset\Omega\times\mathbb{R}$ such that

$$u(x^k,t_k) \to A, \text{ as } k\to\infty.$$

More accurately, there exists a nonnegative sequence $\{\varepsilon_k\}\searrow 0$ such that

$$u(x^k,t_k) = A - \varepsilon_k > 0. \tag{5.4}$$

Recalling the assumption that $u\leq 0$ in $\Omega^c\times\mathbb{R}$ and the continuity of u again, without loss of generality, we may assume that $\mathrm{dist}\{x^k,\Omega^c\}\geq 1$. Now we define the following auxiliary function:

$$v_k(x,t) = u(x,t) + \varepsilon_k\psi_k(x,t),$$

where $\psi_k(x,t) = \psi\big(\frac{x-x^k}{r\cdot e^{\rho r}}, \frac{t-t_k}{r^{\frac{\alpha}{s}}\cdot e^{3\rho r}}\big)$ with any fixed $r>0$ and $\psi(x,t)$ is given by

$$\psi(x,t) = \begin{cases} 1, & \text{if } (x,t)\in B_{\frac{1}{2}}(0)\times\left[-\frac{1}{2},\frac{1}{2}\right],\\ 0, & \text{if } (x,t)\notin B_1(0)\times[-1,1]. \end{cases}$$

It is well known that $\psi\in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$, therefore $|(\Delta+\rho)^{\frac{\alpha}{2}}\psi(x,t)|\leq C_0$ for any $x\in\mathbb{R}^n\times\mathbb{R}$.

Next, for convenience, we denote

$$Q_{re^{\rho r}}(x^k,t_k) := \left\{(x,t) \mid \left|\left(\frac{x-x^k}{r\cdot e^{\rho r}}, \frac{t-t_k}{r^\alpha e^{3\rho r}}\right)\right| < 1\right\}.$$

It is obvious that

$$\max_{x\in Q_{re^{\rho r}}(x^k,t_k)} v_k(x,t) \geq \max_{x\in Q^c_{re^{\rho r}}(x^k,t_k)} v_k(x,t),$$

which implies the maximum value of $v_k(x,t)$ in $\mathbb{R}^n\times\mathbb{R}$ is attained in $Q_{re^{\rho r}}(x^k,t_k)$, along which we will be able to derive a contradiction. More precisely, using the definition of $v_k(x,t)$ and (5.4), we arrive at

$$v_k(x^k,t_k) = A > 0.$$

On the other hand, for any $(x,t)\in(\mathbb{R}^n\times\mathbb{R})\setminus Q_{re^{\rho r}}(x^k,t_k)$, we have

$$v_k(x,t)\le A.$$

Consequently, there exists $(\bar{x}^k,\bar{t}_k)\in Q_{re^{\rho r}}(x^k,t_k)$ such that

$$A+\varepsilon_k\ge v_k(\bar{x}^k,\bar{t}_k)=\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} v_k(x,t)\ge A>0.$$

A computation leads to

$$-(\Delta+\rho)^{\frac{\alpha}{2}}v_k(\bar{x}^k,\bar{t}_k)=c_{n,\alpha}\ \text{P.V.}\int_{\mathbb{R}^n}\frac{v_k(\bar{x}^k,\bar{t}_k)-v_k(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\ge 0,$$

and

$$\partial_t^s v_k(\bar{x}^k,\bar{t}_k)=C_s\int_{-\infty}^{\bar{t}_k}\frac{v_k(\bar{x}^k,\bar{t}_k)-v_k(\bar{x}^k,\tau)}{(t_k-\tau)^{1+s}}\,\mathrm{d}\tau\ge 0.$$

Collecting the above estimates, we conclude that

$$\begin{aligned}
0&\le-(\Delta+\rho)^{\frac{\alpha}{2}}v_k(\bar{x}^k,\bar{t}_k)\\
&=-(\Delta+\rho)^{\frac{\alpha}{2}}u(\bar{x}^k,\bar{t}_k)+\varepsilon_k\cdot[-(\Delta+\rho)^{\frac{\alpha}{2}}]\psi_k(\bar{x}^k,\bar{t}_k)\\
&\le-\partial_t^s u(\bar{x}^k,\bar{t}_k)-c(\bar{x}^k,\bar{t}_k)u(\bar{x}^k,\bar{t}_k)+\frac{C_1\varepsilon_k}{e^{3\rho r}r^{\alpha}}\\
&\le-\partial_t^s u(\bar{x}^k,\bar{t}_k)+\frac{C_1\varepsilon_k}{e^{3\rho r}r^{\alpha}}\\
&=-\partial_t^s v_k(\bar{x}^k,\bar{t}_k)+\partial_t^s\varepsilon_k\psi_k(\bar{x}^k,\bar{t}_k)+\frac{C_1\varepsilon_k}{e^{3\rho r}r^{\alpha}}\\
&\le\partial_t^s\varepsilon_k\psi_k(\bar{x}^k,\bar{t}_k)+\frac{C_1\varepsilon_k}{e^{3\rho r}r^{\alpha}}\le\frac{C\varepsilon_k}{e^{3\rho r}r^{\alpha}},
\end{aligned}\tag{5.5}$$

where the first inequality holds due to

$$\begin{aligned}
|(\Delta+\rho)^{\frac{\alpha}{2}}\psi_k(x)|&=\left|-c_{n,\alpha}\ \text{P.V.}\int_{\mathbb{R}^n}\frac{\psi_k(x)-\psi_k(y)}{e^{\rho|x-y|}|x-y|^{n+\alpha}}\,\mathrm{d}y\right|\\
&=\left|-c_{n,\alpha}\left(\text{P.V.}\int_{B_r(x)}\frac{\psi_k(x)-\psi_k(y)}{e^{\rho|x-y|}|x-y|^{n+\alpha}}\,\mathrm{d}y\right.\right.\\
&\qquad\left.\left.+\ \text{P.V.}\int_{B_r^c(x)}\frac{\psi_k(x)-\psi_k(y)}{e^{\rho|x-y|}|x-y|^{n+\alpha}}\,\mathrm{d}y\right)\right|\\
&\le\left|\int_{B_r(x)}\frac{2c_{n,\alpha}||\psi||_{C^{1,1}(\mathbb{R}^n)}\left|\frac{x}{e^{3\rho r}r}-\frac{y}{e^{3\rho r}r}\right|^2}{|x-y|^{n+\alpha}}\,\mathrm{d}y\right|\\
&\quad+\left|\int_{B_r^c(x)}\frac{2c_{n,\alpha}}{e^{3\rho r}|x-y|^{n+\alpha}}\,\mathrm{d}y\right|\\
&\le\frac{C}{e^{6\rho r}r^{\alpha}}+\frac{C}{e^{3\rho r}r^{\alpha}}\le\frac{C_1}{e^{3\rho r}r^{\alpha}}.
\end{aligned}\tag{5.6}$$

Recalling Lemma A.4, we know that, for any $r > 0$,

$$\frac{[-(\Delta+\rho)^{\frac{\alpha}{2}}]v_k(\bar{x}^k,\bar{t}_k)}{c_{n,\alpha}I(r)} + \frac{1}{I(r)}\int_{B_r^c(\bar{x}^k)} \frac{u(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y \geq u(\bar{x}^k), \tag{5.7}$$

where

$$I(r) := \frac{1}{r^\alpha}\int_{B_1^c(0)} \frac{1}{|y|^{n+\alpha}e^{\rho r|y|}}\,\mathrm{d}y.$$

Next, our goal is to prove that the left-hand side of the inequality (5.7) is strictly less than A. Indeed, direct calculations yield

$$\begin{aligned} I(r) &:= \frac{1}{r^\alpha}\int_{B_1^c(0)} \frac{1}{|y|^{n+\alpha}e^{\rho r|y|}}\,\mathrm{d}y = \frac{1}{r^\alpha}\int_{B_1^c(0)} \frac{1}{|y|^{n-1}|y|^{1+\alpha}e^{\rho r|y|}}\,\mathrm{d}y \\ &\geq \frac{1}{r^\alpha}\int_{B_1^c(0)} \frac{1}{|y|^{n-1}e^{2\rho r|y|}}\,\mathrm{d}y \\ &=: \frac{C_\rho}{r^{\alpha+1}e^{2\rho r}} \geq \frac{C_\rho}{r^\alpha e^{3\rho r}}. \end{aligned} \tag{5.8}$$

Considering this together with (5.5), we derive that

$$\frac{[-(\Delta+\rho)^{\frac{\alpha}{2}}]v_k(\bar{x}^k,\bar{t}_k)}{c_{n,\alpha}I(r)} \leq \frac{C}{c_{n,\alpha}}\frac{r^\alpha e^{3\rho r}}{C_\rho}\frac{\varepsilon_k}{r^\alpha e^{3\rho r}} \leq C\varepsilon_k.$$

Therefore, our goal is to estimate the upper bound of the second term in (5.7). First, since $R > R/\sqrt[n]{2}$, and combining this with assumption (5.1), we know that

$$\varliminf_{R\to\infty} \frac{|(B_R(\bar{x}^k)\setminus B_{R/\sqrt[n]{2}}(\bar{x}^k))\cap\Omega^c|}{|B_R(\bar{x}^k)|} \geq c_0 > 0, \tag{5.9}$$

which indicates that there exist two positive constants $\bar{C}$ and sufficiently large R_k such that

$$\frac{|(B_R(\bar{x}^k)\setminus B_{R/\sqrt[n]{2}}(\bar{x}^k))\cap\Omega^c|}{|B_R(\bar{x}^k)|} \geq \bar{C} > 0, \quad R \geq R_k. \tag{5.10}$$

Combining (5.3) and (5.10), taking $r = R_k/\sqrt[n]{2}$, and noting the fact

$$v_k(y,\bar{t}_k) = u(y,\bar{t}_k) \leq 0, \quad y \in B_r^c(\bar{x}^k)\cap\Omega^c,$$

we obtain

(5.11)

$$
\begin{aligned}
&\frac{1}{I(r)}\int_{B_r^c(\bar{x}^k)}\frac{u(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\quad=\frac{1}{I\big(\big(\frac{R_k}{\sqrt[n]{2}}\big)^{\alpha}\big)}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)}\frac{v_k(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\quad=\frac{1}{I((R_k/\sqrt[n]{2})^{\alpha})}\Bigg(\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega}\frac{v_k(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\qquad\qquad+\int_{B_{R_k/k_2(\bar{x}^k)\cap\Omega^c}}\frac{A+\varepsilon_k}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\qquad\qquad-\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\qquad\qquad+\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{v_k(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\Bigg)\\
&\quad\le\frac{1}{I((R_k/\sqrt[n]{2})^{\alpha})}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)}\frac{A+\varepsilon_k}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\qquad-\frac{1}{I((R_k/\sqrt[n]{2})^{\alpha})}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\quad=A+\varepsilon_k-\frac{1}{I((R_k/\sqrt[n]{2})^{\alpha})}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\quad\le A+\varepsilon_k-\frac{1}{I((R_k/\sqrt[n]{2})^{\alpha})}\int_{[B_{R_k}(\bar{x}^k)\setminus B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)]\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\quad\le A+\varepsilon_k-\frac{A+\varepsilon_k}{I((R_k/\sqrt[n]{2})^{\alpha})}|[B_{R_k}(\bar{x}^k)\setminus B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)]\cap\Omega^c|\frac{1}{e^{\rho\frac{R_k}{\sqrt[n]{2}}}\big|\frac{R_k}{\sqrt[n]{2}}\big|^{n+\alpha}}\\
&\quad\le A+\varepsilon_k-\frac{A+\varepsilon_k}{C_\rho}\left(\frac{R_k}{\sqrt[n]{2}}\right)^{\alpha+1}e^{2\rho\cdot\frac{R_k}{\sqrt[n]{2}}}|[B_{R_k}(\bar{x}^k)\setminus B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)]\cap\Omega^c|\frac{1}{e^{\rho\frac{R_k}{\sqrt[n]{2}}}\big|\frac{R_k}{\sqrt[n]{2}}\big|^{n+\alpha}}\\
&\quad\le(A+\varepsilon_k)(1-C_\rho).
\end{aligned}
$$

Collecting (5.5)–(5.11), we arrive at

$$
\begin{aligned}
A&\le\frac{[-(\Delta+\rho)^{\frac{\alpha}{2}}]v_k(\bar{x}^k,\bar{t}_k)}{c_{n,\alpha}I(r)}+\frac{1}{I(r)}\int_{B_r^c(\bar{x}^k)}\frac{v_k(y,\bar{t}_k)}{e^{\rho|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}\,\mathrm{d}y\\
&\le C\varepsilon_k+(A+\varepsilon_k)(1-C_\rho),
\end{aligned}
$$

and we can immediately get the contradiction as $k\to\infty$. Therefore, conclusion (5.2) must hold. □

5.2. One-dimensional symmetry of solutions via sliding methods. By applying the maximum principles in unbounded open sets (Theorem 5.1) established in Subsection 5.1, and employing the sliding method, we derive the monotonicity of solutions for dual nonlocal parabolic problem (1.15) in whole-space $\mathbb{R}^n$.

First of all, we give some useful notations. For any $x = (x', x_n)$ with $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $\tau \in \mathbb{R}$, we denote

- $x^\tau = x + \tau e_n$, with $e_n = (0^{'}, 1)$,
- $u_\tau(x,t) := u(x', x_n + \tau, t)$,
- $w_\tau(x,t) := u(x,t) - u_\tau(x,t)$.

With the previous result we have the key ingredients to prove our main monotonicity and one-dimensional symmetry with respect to the space variable x for each fixed t.

Proof: Step 1. We prove that for τ sufficiently large, there hold

$$w_\tau(x,t) \le 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{5.12}$$

Applying assumption (1.17), then there exists a sufficiently large $a > 0$ such that

$$|u(x,t)| \ge 1 - \delta \text{ for } |x_n| \ge a, \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{5.13}$$

Suppose (5.12) is violated; then there exists a constant $A > 0$ such that

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} w_\tau(x,t) = A > 0.$$

In the following, we consider the function

$$\bar{w}_\tau(x,t) = w_\tau(x,t) - \frac{A}{2}.$$

Our aim is to show that

$$\bar{w}_\tau(x,t) \le 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Note that by assumption (1.17), for all $t \in \mathbb{R}$, we can choose a sufficiently large constant $M > a$ such that

$$\bar{w}_\tau(x,t) \le 0, \text{ for } x_n \ge M, \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{5.14}$$

Denote

$$D = \mathbb{R}^{n-1} \times (-\infty, M).$$

Then equation (5.14) leads to

$$\bar{w}_\tau(x,t) \le 0, \quad (x,t) \in D^c \times \mathbb{R}. \tag{5.15}$$

That implies $\bar{w}_\tau(x,t)$ satisfies the exterior condition in Theorem 5.1.

Consequently, equation (1.16) implies that $\bar{w}_\tau(x,t)$ satisfies

$$\begin{aligned}\partial_t^s \bar{w}_\tau(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} \bar{w}_\tau(x,t) &= \partial_t^s w_\tau(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} w_\tau(x,t) \\ &= f(t, u(x,t)) - f(t, u_\tau(x,t)).\end{aligned}$$

Next, we claim that, for any $(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have

$$f(t, u(x,t)) \le f(t, u_\tau(x,t)), \text{ at the points where } w_\tau(x,t) > 0. \tag{5.16}$$

We will prove our claim (5.16) by discussing three different cases.

Case 1. $|x_n| \le a$. For any $\tau \ge 2a$, then $x_n + \tau \ge a$, again with (5.13), at the points where $w_\tau(x,t) > 0$, we have

$$u(x,t) > u_\tau(x,t) \ge 1-\delta.$$

Combining this truth with the monotonicity assumption (1.18) on the function f, one can immediately get claim (5.16).

Case 2. $x_n < -a$. For any $(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, combining with assumption (1.17) and (5.13), we arrive at

$$u(x,t) \le -1+\delta,$$

and therefore at the points where $w_\tau(x,t) > 0$,

$$u_\tau(x,t) < u(x,t) \le -1+\delta,$$

which implies that we can use the monotonicity assumption (1.18) on the function f to derive claim (5.16).

Case 3. $x_n > a$. For any $(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by (5.13), we have, at the points where $w_\tau(x,t) > 0$,

$$u(x,t) > u_\tau(x,t) \ge 1-\delta,$$

and therefore we use the monotonicity assumption (1.18) on the function f again to derive claim (5.16). Thus our claim (5.16) must hold.

One can infer from (5.16) that

$$\partial_t^s \bar{w}_\tau(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} \bar{w}_\tau(x,t) \le 0, \text{ at the points in } D \times \mathbb{R} \text{ where } w_\tau(x,t) > 0.$$

This is also valid at the points in $D \times \mathbb{R}$ where $\bar{w}_\tau(x,t) > 0$, more precisely,

$$\partial_t^s \bar{w}_\tau(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} \bar{w}_\tau(x,t) \le 0, \text{ at the points in } D \times \mathbb{R} \text{ where } \bar{w}_\tau(x,t) > 0,$$

which together with the exterior condition (5.15) on $\bar{w}_\tau(x,t)$ and the maximum principle in unbounded domains (Theorem 5.1) implies

$$\bar{w}_\tau(x,t) \le 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

It follows that

$$w_\tau(x,t) \le \frac{A}{2}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

which contradicts the truth of $\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} w_\tau(x,t) = A > 0$. Therefore, $w_\tau(x,t) \le 0$, for any $\tau \ge 2a$ and $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. This completes the proof of Step 1.

Step 2. Inequality (5.12) provides a starting point for us to carry out the sliding procedure. In this step, we decrease τ from close to $\tau = 2a$ to 0, and prove that for any $0 < \tau < 2a$, we still have

$$w_\tau(x,t) \le 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{5.17}$$

To this end, we define

$$\tau_0 = \inf\{\tau \mid w_\tau(x,t) \le 0,\, (x,t) \in \mathbb{R}^n \times \mathbb{R}\}$$

and prove that $\tau_0 = 0$.

We first show that

$$\sup_{|x_n|\le a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_{\tau_0}(x,t) < 0. \tag{5.18}$$

Suppose (5.18) is false; then

$$\sup_{|x_n|\le a_1 (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_{\tau_0}(x,t)=0,$$

which implies there exists a sequence

$$\{(x^k,t_k)\}\subset(\mathbb{R}^{n-1}\times[-a,a])\times\mathbb{R},\quad k=1,2,\dots,$$

such that

$$w_{\tau_0}(x^k,t_k)\to 0,\ \text{as } k\to\infty.$$

More precisely, there exists a nonnegative sequence $\{\varepsilon_k\}\searrow 0$ such that

$$w_{\tau_0}(x^k,t_k)=-\varepsilon_k. \tag{5.19}$$

In order to obtain more information from the supremum of $w_{\tau_0}(x,t)$, we introduce the following auxiliary function:

$$w_k(x,t)=w_{\tau_0}(x,t)+\varepsilon_k\eta_k(x,t),$$

where

$$\eta_k(x,t)=\eta(x-x^k,t-t_k),$$

with $\eta(x,t)\in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$ and

$$\eta(x,t)=\begin{cases}1, & \text{if } |(x,t)|\le\dfrac{1}{2},\\ 0, & \text{if } |(x,t)|\ge 1.\end{cases}$$

Denote

$$Q_1(x^k,t_k):=\{(x,t)\mid |(x,t)-(x^k,t_k)|<1\}.$$

We can observe that

$$\max_{x\in Q_1(x^k,t_k)} v_k(x,t)\ge\max_{x\in Q_1^c(x^k,t_k)} v_k(x,t).$$

Therefore, the maximum value of $w_k(x,t)$ in $\mathbb{R}^n\times\mathbb{R}$ is attained in $Q_1(x^k,t_k)$, along which we will be able to derive a contradiction. More precisely, from the definition of $w_k(x,t)$ and (5.19), one has

$$w_k(x^k,t_k)=0.$$

On the other hand, for $(x,t)\in(\mathbb{R}^n\times\mathbb{R})\setminus Q_1(x^k,t_k)$, since $\eta_k(x,t)=0$, we get

$$w_k(x,t)\le 0.$$

Therefore, $w_k(x,t)$ attains its maximum value in $Q_1(x^k,t_k)$, that is,

$$\varepsilon_k\ge w_k(\bar{x}^k,\bar{t}_k)=\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} w_k(x,t)\ge 0.$$

To simplify the notion, we write

$$\bar{w}_k(x,t)=w_k(x+\bar{x}^k,t+\bar{t}_k), \tag{5.20}$$

which implies

$$\varepsilon_k\ge\bar{w}_k(0,0)=\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}\bar{w}_k(x,t)\ge 0.$$

Furthermore, we have

$$-(\Delta+\rho)^{\frac{\alpha}{2}}\bar{w}_k(0,0)=C_{n,s}\ \text{P.V.}\int_{\mathbb{R}^n}\frac{\bar{w}_k(0,0)-\bar{w}_k(y,0)}{e^{\rho|y|}|y|^{n+\alpha}}\,\mathrm{d}y\ge 0,$$

and

$$\partial_t^s\bar{w}_k(0,0)=C_s\int_{-\infty}^0\frac{\bar{w}_k(0,0)-\bar{w}_k(0,\tau)}{(0-\tau)^{1+s}}\ge 0.$$

Through direct calculations, together with the truth of $w_{\tau_0}(\bar{x}^k, \bar{t}_k) \to 0$, as $k \to \infty$, we derive that

$$
\begin{aligned}
0 &\le -(\Delta+\rho)^{\frac{\alpha}{2}} \bar{w}_k(0,0) \\
&= -(\Delta+\rho)^{\frac{\alpha}{2}} w_{\tau_0}(\bar{x}^k, \bar{t}_k) + \varepsilon_k[-(\Delta+\rho)^{\frac{\alpha}{2}}]\eta_k(\bar{x}^k, \bar{t}_k) \\
&= -\partial_t^s w_{\tau_0}(\bar{x}^k, \bar{t}_k) + f(\bar{t}_k, u(\bar{x}^k, \bar{t}_k)) \\
&\quad - f(\bar{t}_k, u_{\tau_0}(\bar{x}^k, \bar{t}_k)) + \varepsilon_k[-(\Delta+\rho)^{\frac{\alpha}{2}}]\eta_k(\bar{x}^k, \bar{t}_k) \\
&= -\partial_t^s \bar{w}_k(0,0) - \varepsilon_k \frac{\partial \eta_k}{\partial t}(\bar{x}^k, \bar{t}_k) + f(\bar{t}_k, u(\bar{x}^k, \bar{t}_k)) \\
&\quad - f(\bar{t}_k, u_{\tau_0}(\bar{x}^k, \bar{t}_k)) + \varepsilon_k[-(\Delta+\rho)^{\frac{\alpha}{2}}]\eta_k(\bar{x}^k, \bar{t}_k) \\
&\le C\varepsilon_k + (\bar{t}_k, u(\bar{x}^k, \bar{t}_k)) - f(\bar{t}_k, u_{\tau_0}(\bar{x}^k, \bar{t}_k)) \to 0, \text{ as } k \to \infty.
\end{aligned}
\tag{5.21}
$$

Putting (5.20) and Theorem 5.1 together, we arrive at

$$
\frac{[-(\Delta+\rho)^{\frac{\alpha}{2}}]\bar{w}_k(0,0)}{c_{n,\alpha} I(r)} + \frac{1}{I(r)} \int_{B_r^c(\bar{x})} \frac{\bar{w}_k(y,0)}{e^{\rho|y|}|y|^{n+\alpha}} \,\mathrm{d}y \ge \bar{w}_k(0,0), \text{ for any } r > 0.
\tag{5.22}
$$

Therefore, we can deduce by using (5.21) and the above average inequality (5.22) that for any finite $r > 0$,

$$
\frac{1}{I(r)} \int_{B_r^c(0)} \frac{\bar{w}_k(y,0)}{e^{\rho|y|}|y|^{n+\alpha}} \,\mathrm{d}y \to 0, \text{ as } k \to \infty,
$$

which implies that for any fixed $r > 0$,

$$
\bar{w}_k(y,0) \to 0, \quad y \in B_r^c(0), \text{ as } k \to \infty.
\tag{5.23}
$$

Due to $u(x,t)$ being uniformly continuous, by the Arzelà–Ascoli theorem, up to extraction of a subsequence of $u_k(x,t) := u(x + \bar{x}^k, t + \bar{t}_k)$ (still denoted by itself), we obtain

$$
u_k(x,t) \to u_\infty(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \text{ as } k \to \infty,
$$

which together with (5.23) yields

$$
u_\infty(x,0) - (u_\infty)_{\tau_0}(x,0) \equiv 0, \quad x \in B_r^c(0).
$$

Therefore, for any $j \in \mathbb{N}$ and any fixed $r > 0$, we obtain

$$
\begin{aligned}
u_\infty(x', x_n, 0) &= u_\infty(x', x_n + \tau_0, 0) = u_\infty(x', x_n + 2\tau_0, 0) \\
&= \cdots = u_\infty(x', x_n + j\tau_0, 0), \quad x \in B_r^c(0).
\end{aligned}
\tag{5.24}
$$

Since the n-th variable $\bar{x}_n^k$ of $\bar{x}^k$ is bounded, we deduce from the asymptotic condition of (1.17) that

$$
u_\infty(x', x_n, 0) \xrightarrow[x_n \to \pm\infty]{} \pm 1 \text{ uniformly in } x' = (x_1, \ldots, x_{n-1}).
$$

As a consequence, one can take x_n sufficiently negative to let $u_\infty(x', x_n, 0)$ be close to -1, and then take j sufficiently large to let $u_\infty(x', x_n + j\tau_0, 0)$ be close to 1; this is a contradiction with (5.24). Therefore, conclusion (5.18) must hold.

Now suppose that $\tau_0 > 0$. We next prove there exists an $\varepsilon > 0$ such that

$$
w_\tau(x,t) \le 0, \quad \forall (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0],
$$

which would contradict the definition of τ_0.

First, equation (5.18) implies that there exists a small constant $\varepsilon > 0$ such that

$$\sup_{|x_n| \leq a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}} w_\tau(x,t) \leq 0, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0]. \tag{5.25}$$

Consequently, we only need to show that

$$\sup_{|x_n| > a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}} w_\tau(x,t) \leq 0, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0]. \tag{5.26}$$

Indeed, if (5.26) is not valid, then there exists some $\tau \in (\tau_0 - \varepsilon, \tau_0]$ and a constant $A > 0$ such that

$$\sup_{|x_n| > a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}} w_\tau(x,t) := A > 0. \tag{5.27}$$

Applying the asymptotic condition on u in (1.17), there exists a constant $M > a$ such that

$$w_\tau(x,t) \leq \frac{A}{2}, \quad |x_n| \geq M, \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{5.28}$$

To this end, we define

$$E = \{(x', x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \mid a < |x_n| < M, \, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}\}$$

and the following function

$$v_\tau(x,t) := w_\tau(x,t) - \frac{A}{2}.$$

For any $\tau \in (\tau_0 - \varepsilon, \tau_0]$, $(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, if $a < x_n < M$ at the points in $E \times \mathbb{R}$ where $v_\tau(x,t) > 0$, we have

$$u(x,t) > u_\tau(x,t) + \frac{A}{2} \geq u_\tau(x,t) \geq 1 - \delta,$$

using the monotonicity of f, which implies immediately

$$f(t, u(x,t)) - f(t, u_\tau(x,t)) \leq 0, \quad a < x_n < M, \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{5.29}$$

If $-M < x_n < -a$, at the points in $E \times \mathbb{R}$ where $v_\tau(x,t) > 0$, we have

$$u_\tau(x,t) < u(x,t) \leq -1 + \delta.$$

Again with the assumption (1.18), one has

$$f(u(x,t)) - f(u_\tau(x,t)) \leq 0, \quad -M < x_n < -a, \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{5.30}$$

Therefore, in view of (5.29) and (5.30), at the points in $E \times \mathbb{R}$ where $v_\tau(x,t) > 0$, we have

$$\begin{aligned} \partial_t^s v_\tau(x,t) - (\Delta + \rho)^{\frac{\alpha}{2}} v_\tau(x,t) &= \partial_t^s u_\tau(x,t) - (\Delta + \rho)^{\frac{\alpha}{2}} u_\tau(x,t) \\ &= f(t, u(x,t)) - f(t, u_\tau(x,t)) \leq 0. \end{aligned}$$

Thanks to (5.25) and (5.28), we have the following exterior condition:

$$v_\tau(x,t) \leq 0, \quad (x,t) \in E^c \times \mathbb{R},$$

which together with (5.26) and the maximum principle in unbounded domains (Theorem 5.1), gives

$$v_\tau(x,t) \leq 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0].$$

This contradicts the assumption (5.27). Hence (5.26) is valid, which also yields (5.17). Therefore, we complete the proof of Step 2.

Step 3. In this step, we will show that $u(x,t)$ is strictly increasing with respect to x_n, and

$$u(x,t) = u(x_n,t). \tag{5.31}$$

From Step 1 and Step 2, we have deduced that

$$w_\tau(x,t) \leq 0, \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \forall \tau > 0.$$

In order to show that $u(x,t)$ is strictly increasing with respect to x_n, we only need to show

$$w_\tau(x,t) < 0, \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \forall \tau > 0. \tag{5.32}$$

In fact, if equation (5.32) is not true, then there exists a point $(x^0,t_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\tau_0 > 0$ such that

$$w_{\tau_0}(x^0,t_0) = 0,$$

which means (x^0,t_0) is a maximum point of $w_{\tau_0}(x,t)$ in $\mathbb{R}^n \times \mathbb{R}$, and

$$\partial_t^s w_{\tau_0}(x^0,t_0) \geq 0.$$

Through a direct calculation, we have

$$\begin{aligned}
0 &= f(t_0, u(x^0,t_0)) - f(t_0, u_{\tau_0}(x^0,t_0)) \\
&= \partial_t^s w_{\tau_0}(x^0,t_0) + [-(\Delta+\rho)]^{\frac{\alpha}{2}} w_{\tau_0}(x^0,t_0) \\
&\geq C_{n,\alpha}\, \text{P.V.} \int_{\mathbb{R}^n} \frac{-w_{\tau_0}(y,t_0)}{e^{\rho|x_0-y|}|x^0-y|^{n+\alpha}}\, \mathrm{d}y,
\end{aligned}$$

which implies immediately that

$$w_{\tau_0}(y,t_0) \equiv 0, \quad y \in \mathbb{R}^n.$$

Therefore, for any $j \in \mathbb{N}$, we have

$$\begin{aligned}
u(y', y_n, t_0) &= u(y', y_n + \tau_0, t_0) = u(y', y_n + 2\tau_0, t_0) \\
&= \cdots = u(y', y_n + j\tau_0, t_0), \quad y \in \mathbb{R}^n.
\end{aligned}$$

Recalling the asymptotic condition, we can take y_n sufficiently negative such that $u(y', y_n, t_0)$ is close to -1, and then take j sufficiently large such that $u(x', x_n + j\tau_0, 0)$ is close to 1, thus we derive a contradiction and obtain (5.32), which yields that $u(x,t)$ is strictly increasing with respect to x_n.

Next, our goal is to prove that $u(x)$ depends on x_n only. Indeed, it can be seen from the above sliding procedure that the methods should still be valid if we replace $u_\tau(x,t)$ with $u(x+\tau\nu, t)$, with $\nu = (\nu_1, \ldots, \nu_n)$ and $\nu_n > 0$. More accurately, applying similar sliding methods as in Step 1 and Step 2, for each ν with $\nu_n > 0$, yields

$$u(x+\tau\nu, t) > u(x,t), \quad \forall \tau > 0, \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Letting $\nu_n \to 0$, again with the continuity of $u(x,t)$, we have

$$u(x+\tau\nu, t) \geq u(x,t)$$

for arbitrary ν with $\nu_n = 0$. Replacing ν with $-\nu$, for arbitrary ν with $\nu_n = 0$, we find that

$$u(x+\tau\nu, t) \leq u(x,t).$$

Therefore,

$$u(x+\tau\nu, t) = u(x,t),$$

which indicates that $u(x', x_n, t)$ is independent of x', hence (5.31) is true. Therefore, we have proved Step 3. This completes the proof of Theorem 1.5. $\square$

5.3. Monotonicity in epigraph E. In this subsection, we consider the following dual tempered fractional parabolic problem on the epigraph E:

$$\partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) = f(t,u(x,t)), \quad E\times\mathbb{R},$$

with the epigraph (refer to [**23**])

$$E := \{x=(x',x_n)\in\mathbb{R}^n \mid x_n > \varphi(x')\},$$

where $\varphi\colon \mathbb{R}^{n-1}\to\mathbb{R}$ is a continuous function. A typical example of epigraph E is the upper half-space $\mathbb{R}^n_+$ ($\varphi\equiv 0$). We will show the following monotonicity result on the epigraph E via the sliding method.

Theorem 5.2. *Let $u\in\{\mathcal{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{\mathrm{loc}}(\Omega)\}\times(C^1(\mathbb{R})\cap\mathcal{L}^-_s(\mathbb{R}))$ be a bounded solution of*

$$\begin{cases}\partial_t^s u(x,t) - (\Delta+\rho)^{\frac{\alpha}{2}} u(x,t) = f(t,u(x,t)), & (x,t)\in E\times\mathbb{R},\\ u(x,t)=0, & (x,t)\notin E\times\mathbb{R},\end{cases}$$

where $f(t,u)$ is nonincreasing in the range of u. Assume that there exists $l>0$ such that

$$u\ge 0,\ \text{in } \{(x,t)=(x',x_n,t)\in E\times\mathbb{R} \mid \varphi(x')<x_n<\varphi(x')+l\}.$$

Then, either $u\equiv 0$ in $\mathbb{R}^n\times\mathbb{R}$ and $f(t,0)=0$, or u is strictly monotone increasing in the x_n direction and hence $u>0$ in $E\times\mathbb{R}$.

Proof: Keep the notations u_τ, w_τ, x^τ defined in Subsection 5.2. Taking advantage of the monotonicity of $f(t,u)$, we arrive at

$$\partial_t^s w(x,t) - (\Delta+\lambda)^{\frac{\alpha}{2}} w_\tau(x,t) = f(t,u(x,t)) - f(t,u_\tau(x,t)) \le 0$$

at points $x\in E$ where $w_\tau(x,t)<0$. In addition, for any $0<\tau<l$, we have

$$w_\tau(x,t)\le 0, \quad \forall (x,t)\in\{\mathbb{R}^n\setminus E\}\times\mathbb{R}.$$

Recalling Theorem 5.1 again, for any $0<\tau<l$, we get

$$w_\tau(x,t)\le 0, \quad \forall (x,t)\in E\times\mathbb{R}.$$

Now, suppose that $u\not\equiv 0$ in $E\times\mathbb{R}$; then there exists a $(\hat{x},\hat{t})\in E\times\mathbb{R}$ such that $u(\hat{x},\hat{t})>0$. Next we need to show that, for any $0<\tau<l$,

$$w_\tau(x,t)<0, \quad \forall x\in E\times\mathbb{R}. \tag{5.33}$$

If not, there exists a point $(x^\tau,t_0)\in E\times\mathbb{R}$ such that

$$w_\tau(x^\tau,t_0) = 0 = \max_{\mathbb{R}^n\times\mathbb{R}} w_\tau(x,t),$$

and by the definition of ∂_t^s, we have

$$\partial_t^s w_\tau(x^\tau,t_0) = C_s\int_{-\infty}^{t_0}\frac{w_\tau(x^\tau,t_0)-w_\tau(x^\tau,y)}{|t_0-y|^{1+s}}\,\mathrm{d}y \ge 0.$$

On one hand, recalling the definition of $-(\Delta+\rho)^{\frac{\alpha}{2}}$, we know that

$$\partial_t^s w_\tau(x^\tau,t_0) - (\Delta+\rho)^{\frac{\alpha}{2}} w_\tau(x^\tau,t_0) \ge C_{n,\alpha}\ \mathrm{P.V.}\int_{\mathbb{R}^n}\frac{-w^\tau(y,t_0)}{e^{\rho|x^\tau-y|}|x^\tau-y|^{n+\alpha}}\,\mathrm{d}y > 0.$$

On the other hand, one also has

$$\partial_t^s w_\tau(x^\tau,t_0) - (\Delta+\lambda)^{\frac{\alpha}{2}} w_\tau(x^\tau,t_0) = f(t_0,u(x^\tau,t_0)) - f(t_0,u_\tau(x^\tau,t_0)) = 0.$$

That is impossible! Therefore, (5.33) holds and hence u is strictly monotone increasing in the x_n direction. In particular, $u>0$ in $E\times\mathbb{R}$, and so we get the desired result. □

Appendix A

In this section, we mainly present several important derivative estimation lemmas, as well as several important inequalities, which are used repeatedly in establishing our main results.

Lemma A.1. *Let $x = (x_1, x')$, and $h(x) = f(x_1)g_\gamma(x')$, $0 < \gamma < \alpha$, where*

$$f(x_1) = \left(1 - \frac{x_1^2}{l^2}\right)_+^{\frac{\alpha}{2}} + 1, \quad \textit{and} \quad g_\gamma(x') = (1 + |x'|^2)^{\frac{\gamma}{2}}, \quad 0 < \gamma < \alpha.$$

Then there exists $C_1 > 0$ such that, for all sufficiently smalll, we have

$$\text{(A.1)} \qquad \frac{-(\Delta + \rho)^{\frac{\alpha}{2}} h(x)}{h(x)} \geq \frac{C_1}{l^{2s}}, \quad \textit{for any } |x_1| < l.$$

Proof: Firstly, if $|x_1| < l$, we have

$$\text{(A.2)} \qquad \begin{aligned} -(\Delta + \rho)^{\frac{\alpha}{2}} f(x_1) &= -(\Delta + \rho)^{\frac{\alpha}{2}} \left[\left(1 - \frac{x_1^2}{l^2}\right)_+^{\frac{\alpha}{2}} + 1\right] = -(\Delta + \rho)^{\frac{\alpha}{2}} \left(1 - \frac{x_1^2}{l^2}\right)_+^{\frac{\alpha}{2}} \\ &= \frac{1}{l^\alpha} \cdot [-(\Delta + \rho)^{\frac{\alpha}{2}} (l^2 - x_1^2)_+^{\frac{\alpha}{2}}] = \frac{C}{l^\alpha}, \end{aligned}$$

and $-(\Delta + \rho)^{\frac{\alpha}{2}} g_\gamma(x')$ is bounded. Indeed, if $|x'| = 0$, then

$$\begin{aligned} |-(\Delta + \rho)^{\frac{\alpha}{2}} g_\gamma(x')| &= C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{(1 + |y'|^2)^{\frac{\gamma}{2}} - 1}{(|y'|^2 + y_1^2)^{\frac{n+\alpha}{2}} e^{\rho(|y'|^2 + y_1^2)^{\frac{1}{2}}}} \, \mathrm{d}y' \, \mathrm{d}y_1 \\ &\leq C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{(1 + |y'|^2)^{\frac{\gamma}{2}} - 1}{(|y'|^2 + |y'|^2 t^2)^{\frac{\alpha}{2}}} |y'| \, \mathrm{d}y' \, \mathrm{d}t \\ &= C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{(1 + |y'|^2)^{\frac{\gamma}{2}} - 1}{|y'|^{n-1+\alpha} (1 + t^2)^{\frac{n+\alpha}{2}}} \, \mathrm{d}y' \, \mathrm{d}t \\ &= C \text{ P.V.} \int_{\mathbb{R}^{n-1}} \frac{(1 + |y'|^2)^{\frac{\gamma}{2}} - 1}{|y'|^{n-1+\alpha}} \, \mathrm{d}y' \\ &= C \int_0^{+\infty} \frac{(1 + r^2)^{\frac{\gamma}{2}} - 1}{r^{1+\alpha}} \, \mathrm{d}r < +\infty. \end{aligned}$$

If $|x'| \to +\infty$, then

$$
\begin{aligned}
|-(\Delta+\rho)^{\frac{\alpha}{2}} g_\gamma(x')| &= C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{|g_\gamma(x') - g_\gamma(y')|}{|x-y|^{n+\alpha} e^{\rho|x-y|}}\,\mathrm{d}y \\
&\leq C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{|g_\gamma(x') - g_\gamma(y')|}{(|x'-y'|^2 + (x_1-y_1)^2)^{\frac{n+\alpha}{2}}}\,\mathrm{d}y \\
&= C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{|g_\gamma(x') - g_\gamma(y')|}{(|x'-y'|^2 + |x'-y'|^2 s^2)^{\frac{n+\alpha}{2}}} |x'-y'|\,\mathrm{d}s\,\mathrm{d}y' \\
&= C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{|g_\gamma(x') - g_\gamma(y')|}{|x'-y'|^{n-1+\alpha}} \cdot \frac{1}{(1+s^2)^{\frac{n+\alpha}{2}}}\,\mathrm{d}s\,\mathrm{d}y' \\
&= C\ \text{P.V.} \int_{\mathbb{R}^{n-1}} \frac{g_\gamma(x') - g_\gamma(y')}{|x'-y'|^{n-1+\alpha}}\,\mathrm{d}y' \\
&= C\ \text{P.V.} \int_{\mathbb{R}^{n-1}} \frac{|(1+|x'|^2)^{\frac{\gamma}{2}} - (1+|y'|^2)^{\frac{\gamma}{2}}|}{|x'-y'|^{n-1+\alpha}}\,\mathrm{d}y' \to 0.
\end{aligned}
$$

Therefore, we have $|-(\Delta+\rho)^{\frac{\alpha}{2}} g_\gamma(x')| \leq C$. Similarly, we have

$$
\left| C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{f(y_1)(g_\gamma(x') - g_\gamma(y'))}{|x-y|^{n+\alpha} e^{\rho|x-y|}}\,\mathrm{d}y \right| \leq C. \tag{A.3}
$$

Secondly, combining (A.2) and (A.3), we derive

$$
\begin{aligned}
-(\Delta+\rho)^{\frac{\alpha}{2}} h(x) &= C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{f(x_1)g_\gamma(x') - f(y_1)g_\gamma(y')}{|x-y|^{n+\alpha} e^{\rho|x-y|}}\,\mathrm{d}y \\
&= C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{f(x_1)g_\gamma(x') - f(y_1)g_\gamma(x') + f(y_1)g_\gamma(x') - f(y_1)g_\gamma(y')}{|x-y|^{n+\alpha} e^{\rho|x-y|}}\,\mathrm{d}y \\
&= C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{(f(x_1) - f(y_1))g_\gamma(x') + f(y_1)(g_\gamma(x') - g_\gamma(y'))}{|x-y|^{n+\alpha} e^{\rho|x-y|}}\,\mathrm{d}y \\
&= g_\gamma(x')[-(\Delta+\rho)^{\frac{\alpha}{2}}] f(x_1) + C_{n,\alpha}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{f(y_1)(g_\gamma(x') - g_\gamma(y'))}{|x-y|^{n+\alpha} e^{\rho|x-y|}}\,\mathrm{d}y \\
&\geq \frac{C}{l^\alpha} g_\gamma(x') - C.
\end{aligned}
$$

Therefore,

$$
\frac{[-(\Delta+\rho)^{\frac{\alpha}{2}}]h(x)}{h(x)} \geq \frac{1}{h(x)} \left(\frac{C}{l^\alpha} g_\gamma(x') - C \right) \geq \frac{C}{l^\alpha}.
$$

This completes the proof of Lemma A.1. □

Lemma A.2 (Lemma 5.1 in [**12**]). *Let $\eta(t) \in C_0^\infty((-2,2))$ be a smooth cut-off function, satisfying $\eta(t) \equiv 1$ in $[-1,1]$ and $0 \leq \eta(t) \leq 1$. Then there exists a positive constant C_0 that depends only on α such that*

$$
|\partial_t^s \eta(t)| \leq C_0, \quad \textit{for } t \in (-2,2).
$$

As a byproduct, we derive the following result by rescaling and translation.

Corollary A.3. *For any $t_0 \in \mathbb{R}$ and $r > 0$, let*

$$\eta_0(t) := \eta\left(\frac{t - t_0}{r^{\frac{\alpha}{s}}}\right) \in C_0^\infty((-2r^{\frac{\alpha}{s}} + t_0, 2r^{\frac{\alpha}{s}} + t_0)),$$

then

$$|\partial_t^s \eta_0(t)| \leq \frac{C_0}{r^\alpha}, \quad \text{for } t \in (-2r^{\frac{\alpha}{s}} + t_0, 2r^{\frac{\alpha}{s}} + t_0),$$

where the smooth cut-off function $\eta(\cdot)$ and the positive constant C_0 are defined in Lemma A.2.

Lemma A.4 (A generalized average inequality; see Lemma 2.1 in [**34**])**.** *Assume $u \in \mathcal{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. For any $r > 0$, if $\bar{x}$ is a maximum point of u in $B_r(\bar{x})$, then we have*

$$\frac{[-(\Delta + \rho)^{\frac{\alpha}{2}}]u(\bar{x}, t)}{c_{n,\alpha} I(r)} + \frac{1}{I(r)} \int_{B_r^c(\bar{x})} \frac{u(y, t)}{e^{\rho|\bar{x} - y|}|\bar{x} - y|^{n+\alpha}}\, \mathrm{d}y \geq u(\bar{x}),$$

where the function $I(r)$ is defined by

$$I(r) := \frac{1}{r^\alpha} \int_{B_1^c(0)} \frac{1}{|y|^{n+\alpha} e^{\rho r|y|}}\, \mathrm{d}y.$$

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