GENERICITY CONJECTURE
IN MAPPING CLASS GROUPS

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Abstract. *In the Cayley graph of the mapping class group of a closed surface, with respect to any generating set, we look at a ball of large radius centered at the identity vertex, and at the proportion among the vertices in this ball representing pseudo-Anosov elements. A well-known conjecture states that this proportion should tend to one as the radius tends to infinity. We prove that it stays bounded away from zero. We also prove similar results for a large class of subgroups of the mapping class group.*

NIELSEN-THURSTON CLASSIFICATION

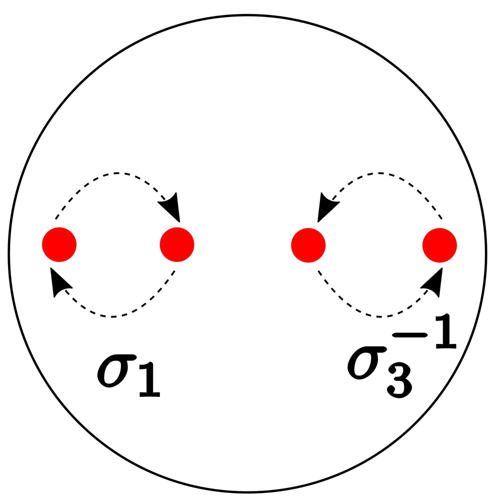
Let G be the mapping class group of a surface S, defined as

$$G = \mathcal{MCG}(S) = \mathrm{Aut}^+(S, \partial S)/\mathrm{Aut}_0(S, \partial S)$$

Then, $g \in G$ fulfils one of the following (up to isotopy):

1. g is periodic, i. e., $g = Id$ for some $n \in Z$.
2. g is reducible, i.e., g preserves a family of isotopy classes of disjoint simple closed non-degenerate curves.
3. g is pseudo-Anosov, i.e., there are two measurable transverse foliations, $(\mathcal{F}^s, \mu_s)$ and $(\mathcal{F}^u, \mu_u)$, and a real number λ such that
$$g(\mathcal{F}^s, \mu_s) = (\mathcal{F}, \lambda^{-1} \cdot \mu_s), \quad g(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda \cdot \mu_u)$$

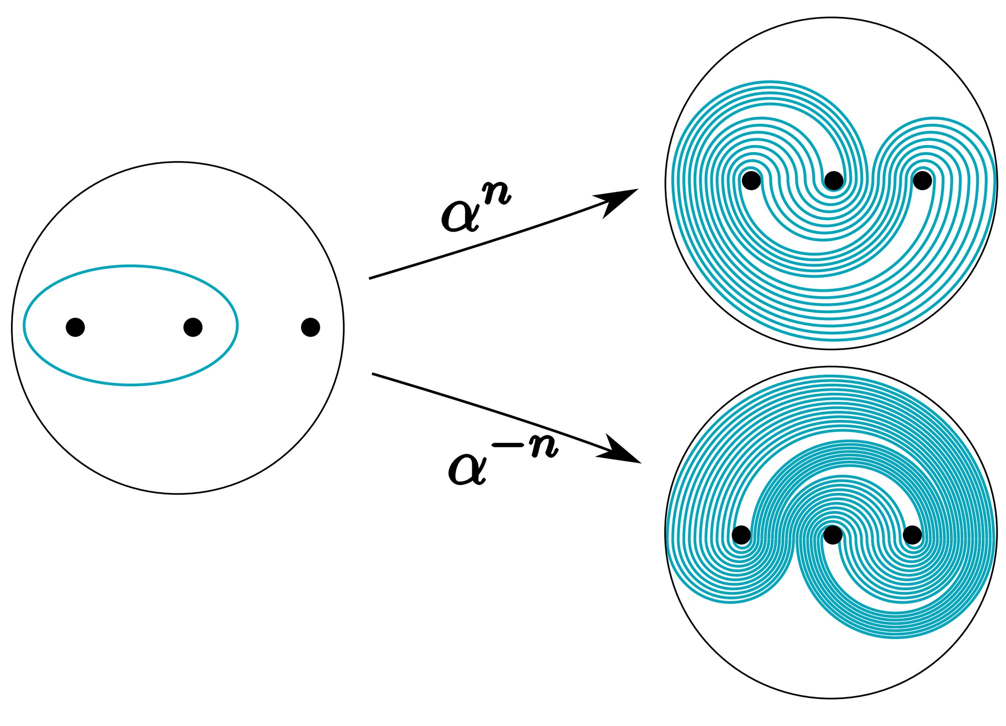
Braid example:



$B_4 = \mathcal{MCG}(D_4)$

The braid group with n strands, B_n, is the mapping class group of the n-punctured disk, D_n. The generators of this group are denoted $\sigma_1, \dots, \sigma_{n-1}$, where each σ_i represents the swap of the punctures in the positions i and $i+1$ by performing a clockwise rotation.

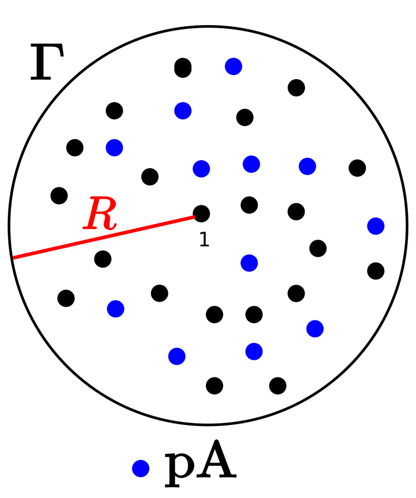
Let α be a pseudo-Anosov braid. To have an idea of how their foliations will be, we can apply several times α and α^{-1} to a round curve in the disk.



Understanding foliations of $\alpha = \sigma_2\sigma_1^{-1}$

GENERICITY CONJECTURE

Let G be the mapping class group of a closed surface S (with $genus(S) \geq 2$). This conjecture establishes that the proportion of pseudo-Anosov elements in a ball on the Cayley graph of G tends to 1 as the radius of the ball tends to infinity.



$$\lim_{R\to\infty} \frac{|\text{pA elements of } G \ \cap \ B_\Gamma(1,R)|}{|B_\Gamma(1,R)|} = 1$$

The following results are from [Cumplido & Wiest].

Main result

Let G be the mapping class group of a closed surface S (with genus$(S) \geq 2$). Then, a positive proportion of elements of G is pseudo-Anosov.

$$\liminf_{R\to\infty} \frac{|\text{pA elements of } G \ \cap \ B_\Gamma(1,R)|}{|B_\Gamma(1,R)|} > 0$$

This result is a corollary of the following theorem:

THEOREM (Positive density of pseudo-Anosovs)

There exists a finite subset F of G such that for any element g of G, at least one of the mapping classes in $\{f \circ g \mid f \in F\}$ is pseudo-Anosov.

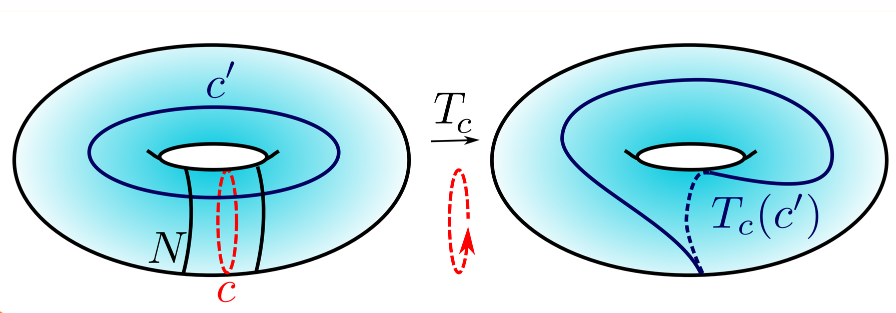
Remark. The proof of this theorem is short and based on two powerful result from [Bowditch] and [Fathi].

GENERICITY FOR SOME SUBGROUPS OF G

Dehn twist

Let c be a simple closed curve on S. Consider a tubular neighbourhood N of c, which will be homeomorphic to an annulus. Performing a twist of 360^o in N and extending to the whole surface S is a self-homeomorfism denoted by T_c and called *Dehn twist* about c.

Example in the torus:



Curve complex

The *curve complex* $\mathcal{C}(S)$ of S is a simplicial complex such that

- The set of vertices is the set of isotopy classes of non-degenerated simple closed curves.
- There is an edge between two vertices if the corresponding curves can be isotoped to be disjoint.

The distance between two curves α and β, $d(\alpha, \beta)$, is the length of the minimal path joining their corresponding vertices in $\mathcal{C}$.

Condition $(*)$ for a subgroup $H \subseteq G$

There are vertices a, b of $\mathcal{C}(S)$ and integers $k_a, k_b \in \mathbb{Z}$ such that

- $d(a, b) \geq R$, where R is certain number associated to G – see [Cumplido & Wiest].
- $T_a^{k_a}, T_b^{k_b} \in H$

$(*)$ is for instance satisfied

- if H is a finite index subgroup of G,
- if H is the Torelli group – see [Farb & Margalit],
- if H is any finite index subgroup of the Torelli group.

THEOREM (Positive density of pseudo-Anosovs in H)

Suppose H is a subgroup of G, the mapping class group of S, satisfying Condition $(*)$. Then, there exists a finite subset F of H such that for any element g of H, at least one of the mapping classes in $\{f \circ g \mid f \in F\}$ is pseudo-Anosov. Moreover,

$$\liminf_{R\to\infty} \frac{|\text{pA elements of } H \ \cap \ B_{\Gamma_H}(1,R)|}{|B_{\Gamma_H}(1,R)|} > 0$$

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