Symmetric Cartan calculus, symmetric Poisson geometry, and C_n-generalized geometry

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Symmetric algebra

Symmetric forms,

$$\Gamma(\odot^\bullet T^*) := \bigoplus_{k\in\mathbb{Z}} \Gamma(\odot^k T^*);$$

$$\odot : \times^2\Gamma(\odot^\bullet T^*) \to \Gamma(\odot^\bullet T^*),$$

$$\sigma \odot \tau := \frac{|\sigma|!|\tau|!}{(|\sigma| + |\tau|)!}\,\mathrm{Sym}(\sigma \otimes \tau);$$

$(\Gamma(\odot^\bullet T^*), \odot)$ is graded,
unital,
associative,
commutative.

Exterior algebra

Exterior forms,

$$\Gamma(\wedge^\bullet T^*) := \Gamma(\bigoplus_{k\in\mathbb{Z}} \wedge^k T^*);$$

$$\wedge : \times^2\Gamma(\wedge^\bullet T^*) \to \Gamma(\wedge^\bullet T^*),$$

$$\varphi\wedge\psi := \frac{|\varphi|!|\psi|!}{(|\varphi| + |\psi|)!}\mathrm{Skew}(\varphi \otimes \psi);$$

$(\Gamma(\wedge^\bullet T^*), \wedge)$ is graded,
unital,
associative,
graded-commutative.

graded-commutative:

$$\varphi \wedge \psi = (-1)^{|\varphi||\psi|}\,\psi \wedge \varphi$$

Motivation

Cartan calculus deals with the operators ι_X, d, $\mathscr{L}_X$, $[\ ,\ ]_{\mathrm{Lie}}$ and their relation to the exterior algebra.

A very natural question is:

Is there a "Cartan calculus" on the symmetric algebra?

The second natural question is:

If so, is it geometrically meaningful?

[HBP] A. Heydari; N. Boroojerdian; E. Peyghan. A description of derivations of the algebra of symmetric tensors. *Archivum Mathematicum*, **42**:175–184, 2006.

Graded algebra derivations

An r-**degree endomorphism** of a graded algebra $(\mathcal{A} := \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k, \cdot)$ is a vector space endomorphism $D \in \mathrm{End}(\mathcal{A})$ s.t.

$$D(\mathcal{A}_k) \subseteq \mathcal{A}_{k+r}.$$

The integer r is called the **degree** of D and is denoted by $|D| := r$.

Such D is called a **derivation** if $D(v \cdot w) = Dv \cdot w + v \cdot Dw$;

a **graded derivation** if $D(v \cdot w) = Dv \cdot w + (-1)^{|v||D|}\, v \cdot Dw$.

The space of derivations of a graded algebra $\mathrm{Der}(\mathcal{A})$ forms a graded Lie algebra w.r.t. the commutator,

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1.$$

The space of graded derivations of a graded algebra $\mathrm{gDer}(\mathcal{A})$ forms a graded Lie superalgebra w.r.t. the graded commutator,

$$[D_1, D_2]_{\mathrm{g}} := D_1 \circ D_2 - (-1)^{|D_1||D_2|}\, D_2 \circ D_1.$$

graded Lie superalgebra:

$$[D_1, D_2]_{\mathrm{g}} = -(-1)^{|D_1||D_2|}[D_2, D_1]_{\mathrm{g}},$$

$$(-1)^{|D_1||D_3|}[D_1, [D_2, D_3]_{\mathrm{g}}]_{\mathrm{g}} + (-1)^{|D_2||D_1|}[D_2, [D_3, D_1]_{\mathrm{g}}]_{\mathrm{g}} + (-1)^{|D_3||D_2|}[D_3, [D_1, D_2]_{\mathrm{g}}]_{\mathrm{g}} = 0$$

Interior multiplication

The **interior multiplication** ι_X is a -1-degree endomorphism of $\Gamma(\otimes^\bullet T^*)$ defined by

$$(\iota_X A)(X_1, \ldots, X_{k-1}) := A(X, X_1, \ldots, X_k), \qquad \iota_X f := 0.$$

When specified to $\Gamma(\odot^\bullet T^*)$:

$$\iota_X(\Gamma(\odot^k T^*)) \subseteq \Gamma(\odot^{k-1} T^*),$$

$$\iota_X(\sigma \odot \tau) = (\iota_X \sigma) \odot \tau + \sigma \odot (\iota_X \tau).$$

$$\Downarrow$$

ι_X is a derivation of degree -1.

Interior multiplication on $\Gamma(\wedge^\bullet T^*)$

$$\iota_X(\Gamma(\wedge^k T^*)) \subseteq \Gamma(\wedge^{k-1} T^*),$$

$$\iota_X(\varphi \wedge \psi) = (\iota_X \varphi) \wedge \psi + (-1)^{|\varphi|} \varphi \wedge (\iota_X \psi).$$

$$\Downarrow$$

ι_X is a graded derivation of degree -1.

Symmetric derivative

An endomorphism $D \in \mathrm{End}(\Gamma(\otimes^\bullet T^*))$ is called **geometric** if it is of degree 1 and

$$(Df)(X) = Xf.$$

Exterior derivative

There is a unique geometric graded derivation $\mathrm{d} \in \mathrm{gDer}(\Gamma(\wedge^\bullet T^*))$ s.t. $\mathrm{d} \circ \mathrm{d} = 0$.
It is called the **exterior derivative**.

Def. The **symmetric derivative** corresponding to a connection ∇,

$$\nabla^s := \bigoplus_{k \in \mathbb{Z}} (k+1) \cdot \mathrm{Sym} \circ \nabla,$$

Prop. Every symmetric derivative is a geometric derivation of $\Gamma(\odot^\bullet T^*)$.

Prop. Every geometric derivation of $\Gamma(\odot^\bullet T^*)$ is of the form ∇^s for some ∇.

Prop. Geometric derivations of $\Gamma(\odot^\bullet T^*)$ are in one-to-one correspondence with torsion-free connections.

Prop. There is no geometric derivation $D \in \mathrm{Der}(\Gamma(\odot^\bullet T^*))$ s.t. $D \circ D = 0$.

The covariant gradient $\nabla : \Gamma(\otimes^\bullet T^*) \to \Gamma(\otimes^\bullet T^*)$:

$$\nabla(\Gamma(\otimes^k T^*)) \subseteq \Gamma(\otimes^{k+1} T^*), \qquad (\nabla A)(X, X_1, \ldots, X_k) := (\nabla_X A)(X_1, \ldots, X_k).$$

Symmetric derivative – applications

A **Killing tensor** w.r.t. ∇^s is a symmetric form $K \in \Gamma(\odot^\bullet T^*)$ s.t.

$$\nabla^s K = 0.$$

The concept of Killing tensors is a generalization of **Killing vector fields** of a metric G, these are vector fields satisfying

$$\pounds_X G = 0.$$

A Killing tensor K w.r.t. ∇^s, induces the function $f_K \in C^\infty(TM)$

$$f_K(v) := K_p(v, \dots, v) \qquad \text{for all } p \in M \text{ and } v \in T_pM,$$

that is constant along every geodesic of ∇.

Killing tensors are used in general relativity (Carter tensor in Kerr-Newman spacetime),
integrable systems (separability of Hamilton-Jacobi eq.),
cosmology (FLRW spacetimes),
...

Symmetric derivative – applications

A **conformal Killing tensor** w.r.t. a metric G is a symmetric form $C \in \Gamma(\odot^\bullet T^*)$ s.t.

$$^G\nabla^s C = \sigma \odot G$$

for some $\sigma \in \Gamma(\odot^\bullet T^*)$.

If G is indefinite, the function $f_C \in C^\infty(TM)$, defined by $f_C(v) := C_p(v, \ldots, v)$, is constant along every null geodesic of $^G\nabla$.

A **statistical manifold** – a fundamental notion in information geometry – is a triple (M, G, ∇) consisting of a smooth manifold M, a Riemannian metric G, and a torsion-free connection ∇, s.t.

$$\nabla^s G = 3\nabla G.$$

Symmetric Lie derivative

Lie derivative

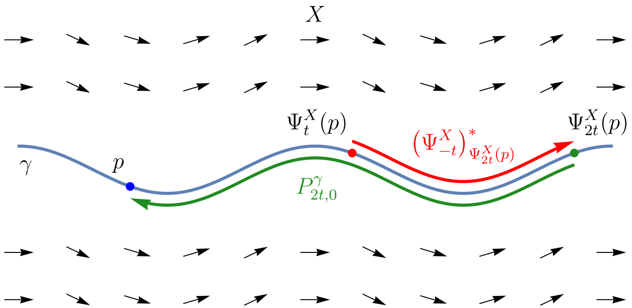
The **Lie derivative** w.r.t. $X \in \Gamma(T)$:

$$\mathcal{L}_X := [\iota_X, \mathrm{d}]_{\mathrm{g}} = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X .$$

Def. The **symmetric Lie derivative** corresponding to ∇^s w.r.t. $X \in \Gamma(T)$:

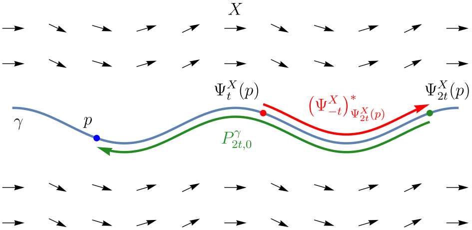
$$\mathcal{L}_X^{\nabla^s} := [\iota_X, \nabla^s] = \iota_X \circ \nabla^s - \nabla^s \circ \iota_X .$$

Thm. $(\mathcal{L}_X^{\nabla^s} \sigma)_p = \lim_{t \to 0} \frac{1}{t} \left(\left(P^{\gamma}_{2t,0} \circ \left(\Psi^X_{-t} \right)^*_{\Psi^X_{2t}(p)} \right) \sigma_{\Psi^X_t(p)} - \sigma_p \right).$

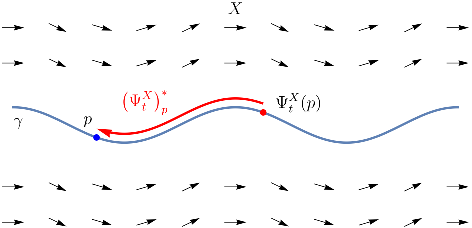


Symmetric Lie derivative

$$(\pounds_X^{\nabla^s}\sigma)_p = \lim_{t\to 0}\frac{1}{t}\left(\left(P^{\gamma}_{2t,0}\circ\left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(p)}\right)\sigma_{\Psi^X_t(p)} - \sigma_p\right)$$

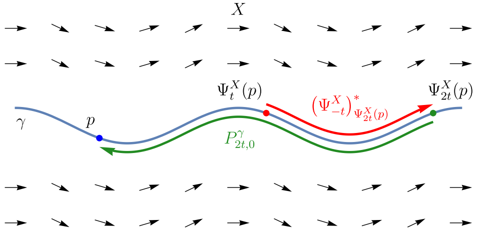


$$(\pounds_X\varphi)_p = \lim_{t\to 0}\frac{1}{t}\left(\left(\Psi^X_t\right)^*_p\varphi_{\Psi^X_t(p)} - \varphi_p\right)$$



Symmetric Lie derivative

$$(\mathcal{L}_X^{\nabla^s}\sigma)_p = \lim_{t\to 0}\frac{1}{t}\left(\left(P^{\gamma}_{2t,0}\circ\left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(p)}\right)\sigma_{\Psi^X_t(p)} - \sigma_p\right)$$



Prop. Let ∇ be a torsion-free connection. Then

$$\nabla_X = \frac{1}{2}(\mathcal{L}_X + \mathcal{L}_X^{\nabla^s}).$$

Symmetric bracket

Lie bracket of vector fields

The **Lie bracket of vector fields** is an $\mathbb{R}$-bilinear map $[\ ,\ ]_{\mathrm{Lie}} : \times^2\Gamma(T) \to \Gamma(T)$ given by

$$\iota_{[X,Y]_{\mathrm{Lie}}} := [\pounds_X, \iota_Y]_{\mathrm{g}} = \pounds_X \circ \iota_Y - \iota_Y \circ \pounds_X .$$

Explicitly: $[X,Y]_{\mathrm{Lie}} = X \circ Y - Y \circ X$.

$$[X,Y]_{\mathrm{Lie}}|_p = (\pounds_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left(\left(\Psi^X_{-t}\right)_{*\Psi^X_t(p)} Y_{\Psi^X_t(p)} - Y_p \right).$$

Def. The **symmetric bracket** corresponding to ∇^s is the $\mathbb{R}$-bilinear map $\langle\ :\ \rangle_{\nabla^s} : \times^2\Gamma(T) \to \Gamma(T)$ given by

$$\iota_{\langle X:Y\rangle_{\nabla^s}} := [\pounds^{\nabla^s}_X, \iota_Y] = \pounds^{\nabla^s}_X \circ \iota_Y - \iota_Y \circ \pounds^{\nabla^s}_X .$$

Explicitly: $\langle X : Y\rangle_{\nabla^s} = \nabla_X Y + \nabla_Y X$.

Prop. $\langle X : Y\rangle_{\nabla^s}|_p = (\pounds^{\nabla^s}_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left(\left(P^{\gamma}_{2t,0} \circ \left(\Psi^X_t\right)_{*\Psi^X_t(p)} \right) Y_{\Psi^X_t(p)} - Y_p \right).$

Explicit formulas for ∇^s and $\mathcal{L}_X^{\nabla^s}$

Explicit formulas for d and $\mathcal{L}_X$

$$(\mathrm{d}\varphi)(X_1, \ldots, X_{|\varphi|+1}) = \sum_{j=1}^{|\varphi|+1} (-1)^{j+1} X_j \varphi(X_1, \ldots, X_{|\varphi|+1}) - \sum_{\substack{j,i=1,\\ j<i}}^{|\varphi|+1} (-1)^{j+i} \varphi([X_j, X_i]_{\mathrm{Lie}}, X_1, \ldots, X_{|\varphi|+1}),$$

$$(\mathcal{L}_X \varphi)(X_1, \ldots, X_{|\varphi|}) = X\varphi(X_1, \ldots, X_{|\varphi|}) - \sum_{j=1}^{|\varphi|} (-1)^{j+1} \varphi([X, X_j]_{\mathrm{Lie}}, X_1, \ldots, X_{|\varphi|}).$$

Explicit formulas for ∇^s and $\mathcal{L}_X^{\nabla^s}$:

$$(\nabla^s \sigma)(X_1, \ldots, X_{|\sigma|+1}) = \sum_{j=1}^{|\sigma|+1} X_j \sigma(X_1, \ldots, X_{|\sigma|+1}) - \sum_{\substack{j,i=1,\\ j<i}}^{|\sigma|+1} \sigma(\langle X_j : X_i \rangle_{\nabla^s}, X_1, \ldots, X_{|\sigma|+1}),$$

$$(\mathcal{L}_X^{\nabla^s} \sigma)(X_1, \ldots, X_{|\sigma|}) = X\sigma(X_1, \ldots, X_{|\sigma|}) - \sum_{j=1}^{|\sigma|} \sigma(\langle X : X_j \rangle_{\nabla^s}, X_1, \ldots, X_{|\sigma|}).$$

Symmetric bracket and distributions

Let ∇ be a connection. A distribution $\Delta \subseteq T$ is called ∇-**geodesically invariant** if every geodesic $\gamma : I \to M$ has the property:

$$\exists\, t_0 \in I,\ \dot{\gamma}(t_0) \in \Delta_{\gamma(t_0)} \qquad \Rightarrow \qquad \dot{\gamma}(t) \in \Delta_{\gamma(t)} \text{ for all } t \in I.$$

If a ∇-geodesically invariant distribution is, in addition, integrable, every geodesic is completely contained in a leaf of the associated foliation.

Frobenius theorem

A distribution $\Delta \subseteq T$ is integrable if and only if

$$[\Gamma(\Delta), \Gamma(\Delta)]_{\text{Lie}} \subseteq \Gamma(\Delta).$$

Thm. [Lewis] A distribution $\Delta \subseteq T$ is ∇-geodesically invariant if and only if

$$\langle \Gamma(\Delta) : \Gamma(\Delta) \rangle_{\nabla^s} \subseteq \Gamma(\Delta).$$

Symmetric Cartan calculus and diffeomorphisms

Cartan calculus and diffeomorphisms

Every diffeomorphism $\phi \in \mathrm{Diff}(M)$ satisfies the following equivalent properties:

(a) $\mathrm{d} \circ \phi^* = \phi^* \circ \mathrm{d}$;

(b) $\phi^* \circ \pounds_X = \pounds_{\phi_*^{-1} X} \circ \phi^*$;

(c) $\phi_*[X, Y]_{\mathrm{Lie}} = [\phi_* X, \phi_* Y]_{\mathrm{Lie}}$.

Prop. Let ∇ be a torsion-free connection and $\phi \in \mathrm{Diff}(M)$. The following five claims are equivalent:

(1) $\phi \in \mathrm{Aff}(\nabla)$, i.e. ϕ_* commutes with the parallel transport;

(2) $\phi_* \nabla_X Y = \nabla_{\phi_* X} \phi_* Y$;

(a') $\nabla^s \circ \phi^* = \phi^* \circ \nabla^s$;

(b') $\phi^* \circ \pounds_X^{\nabla^s} = \pounds_{\phi_*^{-1} X}^{\nabla^s} \circ \phi^*$;

(c') $\phi_* \langle X : Y \rangle_{\nabla^s} = \langle \phi_* X : \phi_* Y \rangle_{\nabla^s}$.

Symmetric Poisson geometry

Poisson geometry

Given a bivector field $\pi \in \Gamma(\wedge^2 T)$, consider the $\mathbb{R}$-multilinear maps:

$$\{\ ,\ \}\colon \times^2 C^\infty(M) \to C^\infty(M), \qquad (f,g) \longmapsto \pi(\mathrm{d}f, \mathrm{d}g)$$

$$\mathrm{Ham}\colon C^\infty(M) \to \Gamma(T). \qquad f \longmapsto \iota_{\mathrm{d}f}\pi = \{f,\ \}$$

We have the following series of equivalences

$$[\pi,\pi]_{\mathrm{Sc}} = 0 \quad \Leftrightarrow \quad \mathrm{Jac}_{\{\ ,\ \}} = 0 \quad \Leftrightarrow \quad \mathrm{Ham}\{f,g\} = [\mathrm{Ham}\, f, \mathrm{Ham}\, g]_{\mathrm{Lie}}.$$

A **Poisson structure** is $\pi \in \Gamma(\wedge^2 T)$ s.t. $[\pi,\pi]_{\mathrm{Sc}} = 0$.

Equivalently a **Poisson structure** is an $\mathbb{R}$-bilinear map $\{\ ,\ \} : \times^2 C^\infty(M) \to C^\infty(M)$ s.t.

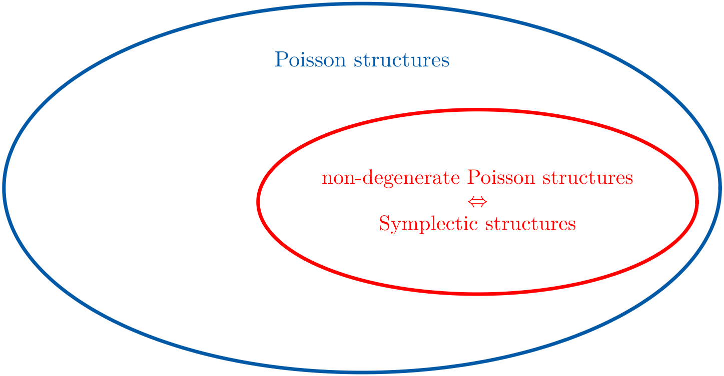
$$\{f,g\} = -\{g,f\}, \qquad \{f,gh\} = g\{f,h\} + \{f,g\}h, \qquad \mathrm{Jac}_{\{\ ,\ \}} = 0.$$

Poisson and symplectic structures

If π is non-degenerate, $\omega := \pi^{-1} \in \Gamma(\wedge^2 T^*)$, then

$$[\pi, \pi]_{\mathrm{Sc}} = 0 \qquad \Leftrightarrow \qquad \mathrm{d}\omega = 0.$$

A **symplectic structure** is a non-degenerate $\omega \in \Gamma(\wedge^2 T^*)$ s.t. $\mathrm{d}\omega = 0$.



Symmetric Poisson geometry – ∇^s-Schouten bracket

Given a symmetric bivector field $\vartheta \in \Gamma(\odot^2 T)$, consider the $\mathbb{R}$-multilinear maps:

$$\{\ ,\ \} : \times^2 C^\infty(M) \to C^\infty(M), \qquad \operatorname{grad} : C^\infty(M) \to \Gamma(T).$$
$$(f,g) \longmapsto \vartheta(\mathrm{d}f, \mathrm{d}g) \qquad f \longmapsto \iota_{\mathrm{d}f}\vartheta = \{f,\ \}$$

Neither the Jacobi identity nor $\operatorname{grad}\{f,g\} = [\operatorname{grad} f, \operatorname{grad} g]_{\mathrm{Lie}}$ work well.

Schouten bracket

The **Schouten bracket** is a unique $\mathbb{R}$-bilinear map $[\ ,\ ]_{\mathrm{Sc}}$:$\times^2\Gamma(\wedge^\bullet T) \to \Gamma(\wedge^\bullet T)$ s.t.

1. $[\Gamma(\wedge^k T), \Gamma(\wedge^l T)]_{\mathrm{Sc}} \subseteq \Gamma(\wedge^{k+l-1} T)$,
2. $[\mathcal{X}, \mathcal{Y}]_{\mathrm{Sc}} = -(-1)^{(|\mathcal{X}|-1)(|\mathcal{Y}|-1)}[\mathcal{Y}, \mathcal{X}]_{\mathrm{Sc}}$,
3. $[\mathcal{X}, \mathcal{Y} \wedge \mathcal{Z}]_{\mathrm{Sc}} = [\mathcal{X}, \mathcal{Y}]_{\mathrm{Sc}} \wedge \mathcal{Z} + (-1)^{(|\mathcal{X}|-1)|\mathcal{Y}|}\mathcal{Y} \wedge [\mathcal{X}, \mathcal{Z}]_{\mathrm{Sc}}$,
4. $[X,\ ]_{\mathrm{Sc}} = \pounds_X$.

Thm. Let ∇^s be a symmetric derivative. There is a unique $\mathbb{R}$-bilinear map $[\ ,\ ] : \times^2\Gamma(\odot^\bullet T) \to \Gamma(\odot^\bullet T)$ s.t.

1. $[\Gamma(\odot^k T), \Gamma(\odot^l T)] \subseteq \Gamma(\odot^{k+l-1} T)$,
2. $[\mathcal{X}, \mathcal{Y}] = [\mathcal{Y}, \mathcal{X}]$,
3. $[\mathcal{X}, \mathcal{Y} \odot \mathcal{Z}] = [\mathcal{X}, \mathcal{Y}] \odot \mathcal{Z} + \mathcal{Y} \odot [\mathcal{X}, \mathcal{Z}]$,
4. $[X,\ ] = \pounds^{\nabla^s}_X$.

We call it the ∇^s-**Schouten bracket** and denote it by $[\ ,\ ]_{\nabla^s\text{-}\mathrm{Sc}}$.

Symmetric Poisson structures

Prop. Let ∇^s be a symmetric derivative and $\vartheta \in \Gamma(\odot^2 T)$. Then

$$[\vartheta, \vartheta]_{\nabla^s\text{-Sc}} = 0 \qquad \Leftrightarrow \qquad \mathrm{Jac}_{\{\ ,\ \}}(f,g,h) = \langle \mathrm{grad}\, f : \mathrm{grad}\, g \rangle_{\nabla^s} h + \mathrm{cyclic}(f,g,h).$$

Def. A **symmetric Poisson structure** is a pair (∇, ϑ) consisting of a torsion-free connection ∇ and $\vartheta \in \Gamma(\odot^2 T)$ s.t. $[\vartheta, \vartheta]_{\nabla^s\text{-Sc}} = 0$.

Equivalently, a **symmetric Poisson structure** is a pair $(\nabla, \{\ ,\ \})$ consisting of a torsion-free connection ∇ and an $\mathbb{R}$-bilinear map $\{\ ,\ \} : \times^2 C^\infty(M) \to C^\infty(M)$ s.t.

$$\{f,g\} = \{g,f\}, \qquad \{f,gh\} = g\{f,h\} + \{f,g\}h,$$

$$\mathrm{Jac}_{\{\ ,\ \}}(f,g,h) = \langle \mathrm{grad}\, f : \mathrm{grad}\, g \rangle_{\nabla^s} h + \mathrm{cyclic}(f,g,h).$$

Def. A **Killing structure** is a pair (∇, G) consisting of a torsion-free connection ∇ and a non-degenerate $G \in \Gamma(\odot^2 T^*)$ s.t. $\nabla^s G = 0$.

Prop. Let ∇^s be a symmetric derivative and $\vartheta \in \Gamma(\odot^2 T)$ be non-degenerate, $G := \vartheta^{-1} \in \Gamma(\odot^2 T^*)$. Then

$$[\vartheta, \vartheta]_{\nabla^s\text{-Sc}} = 0 \qquad \Leftrightarrow \qquad \nabla^s G = 0.$$

Strong symmetric Poisson structures

What about the condition $\mathrm{grad}\{f,g\} = \langle \mathrm{grad}\, f : \mathrm{grad}\, g\rangle_{\nabla^s}$ *?*

Prop. Let ∇ be a torsion-free connection and $\vartheta \in \Gamma(\odot^2 T)$. Then

$$[\vartheta,\vartheta]_{\nabla^s\text{-Sc}} = 0 \quad \underset{\not\Rightarrow}{\Leftarrow} \quad \mathrm{grad}\{f,g\} = \langle \mathrm{grad}\, f : \mathrm{grad}\, g\rangle_{\nabla^s} \quad \Leftrightarrow \quad \nabla_{\mathrm{grad}\, f}\,\vartheta = 0.$$

Def. A **strong symmetric Poisson structure** is a pair (∇,ϑ) consisting of a torsion-free connection ∇ and $\vartheta \in \Gamma(\odot^2 T)$ s.t. $\nabla_{\mathrm{grad}\, f}\,\vartheta = 0$.

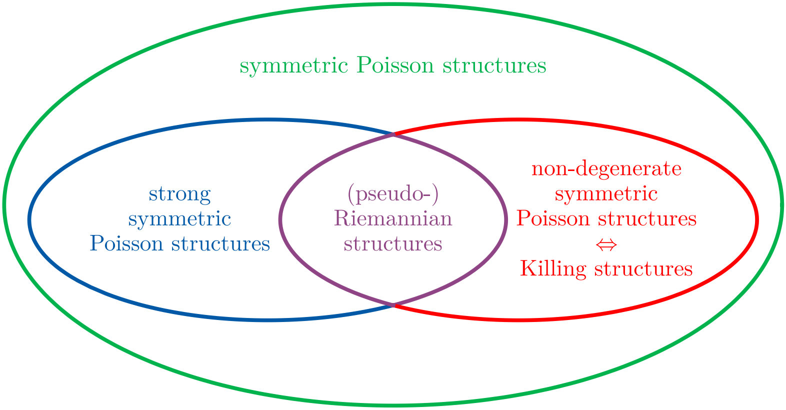
Equivalently, a **strong symmetric Poisson structure** is a pair $(\nabla, \{\ ,\ \})$ consisting of a torsion-free connection ∇ and an $\mathbb{R}$-bilinear map $\{\ ,\ \} : \times^2 C^\infty(M) \to C^\infty(M)$ s.t.

$$\{f,g\} = \{g,f\}, \qquad \{f,gh\} = g\{f,h\} + \{f,g\}h, \qquad \mathrm{grad}\{f,g\} = \langle \mathrm{grad}\, f : \mathrm{grad}\, g\rangle_{\nabla^s}.$$

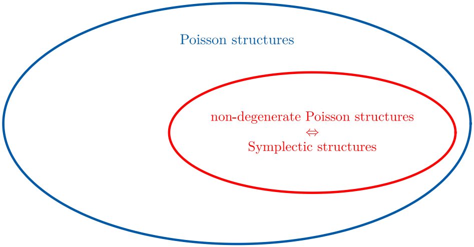
Prop. Let ∇ be a torsion-free connection and $\vartheta \in \Gamma(\odot^2 T)$ be non-degenerate, $G := \vartheta^{-1} \in \Gamma(\odot^2 T^*)$. Then

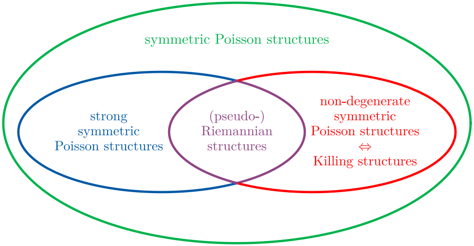
$$\nabla_{\mathrm{grad}\, f}\,\vartheta = 0 \quad \Leftrightarrow \quad \nabla\vartheta = 0 \quad \Leftrightarrow \quad \nabla G = 0 \quad \Leftrightarrow \quad \nabla \text{ is the Levi-Civita connection of } G.$$

(Strong) symmetric Poisson, Killing, and (pseudo-)Riemannian, structures



Comparison of symmetric Poisson and Poisson geometry





C_n-generalized geometry

Standard generalized geometry

Generalized geometry is a novel approach to geometrical structures. It has many applications in both mathematics and physics, so far, in symplectic and complex geometry, mechanics, string theory (supergravity, mirror symmetry), global analysis.

The basic idea: $T \rightsquigarrow (T \oplus T^*,\ \langle\ ,\ \rangle_+)$, where

$$\langle X + \alpha, Y + \beta \rangle_+ := \frac{1}{2}(\alpha(Y) + \beta(X));$$

The associated Clifford algebra $\mathcal{C}l(\langle\ ,\ \rangle_+)$ has the natural representation on $\Gamma(\wedge^\bullet T^*)$:

$$(X + \alpha) \cdot \varphi := \iota_X \varphi + \alpha \wedge \varphi.$$

An $\mathbb{R}$-bilinear map $\times^2 \Gamma(T \oplus T^*) \to \Gamma(T \oplus T^*)$ can be derived:

$$[[(X + \alpha)\cdot, \mathrm{d}]_{\mathrm{g}}, (Y + \beta)\cdot]_{\mathrm{g}}\, \varphi = ([X, Y]_{\mathrm{Lie}} + \pounds_X \beta - \iota_Y \mathrm{d}\alpha) \cdot \varphi.$$

It is called the **Dorfman bracket**: $[X + \alpha, Y + \beta]_D := [X, Y]_{\mathrm{Lie}} + \pounds_X \beta - \iota_Y \mathrm{d}\alpha$.

Mathematically, standard generalized geometry is the study of $(T \oplus T^*, \langle\ ,\ \rangle_+, [\ ,\ ]_D)$.

The **Clifford algebra** of a vector space V endowed with a symmetric pairing $\eta \in \odot^2 V^*$:

$$\mathcal{C}l(\eta) := \frac{\otimes^\bullet V}{\langle \{ v \otimes v - \eta(v, v) \mid v \in V \} \rangle}.$$

C_n-generalized geometry

The basic idea of C_n-generalized geometry is to replace $\langle\ ,\ \rangle_+$ with the canonical skew-symmetric pairing $\langle\ ,\ \rangle_-$:

$$\langle X+\alpha, Y+\beta\rangle_- := \frac{1}{2}(\alpha(Y)-\beta(X));$$

The Dorfman bracket is not natural here and, in fact, does not work well.

The associated Weyl algebra $\mathcal{W}(\langle\ ,\ \rangle_-)$ has the natural representation on $\Gamma(\odot^\bullet T^*)$:

$$(X+\alpha)\cdot\sigma := \iota_X\sigma + \alpha\odot\sigma.$$

An $\mathbb{R}$-bilinear map $\times^2\Gamma(T\oplus T^*)\to\Gamma(T\oplus T^*)$ can be derived:

$$[[(X+\alpha)\cdot,\nabla^s],(Y+\beta)\cdot]\,\sigma = (\langle X:Y\rangle_{\nabla^s} + \pounds_X^{\nabla^s}\beta + \iota_Y\nabla^s\alpha)\cdot\sigma.$$

It is called the ∇^s-**Dorfman bracket**: $[X+\alpha,Y+\beta]_{\nabla^s} := \langle X:Y\rangle_{\nabla^s} + \pounds_X^{\nabla^s}\beta + \iota_Y\nabla^s\alpha$.

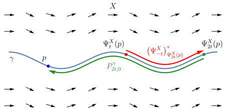
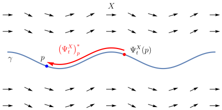
C_n-generalized geometry is the study of $(T\oplus T^*, \langle\ ,\ \rangle_-, [\ ,\ ]_{\nabla^s})$.

Analogues of Dirac structures, generalized complex structures, generalized metrics, Courant algebroids, quadratic Lie algebras, etc. arise.

The **Weyl algebra** of a vector space V endowed with a skew-symmetric pairing $\varepsilon\in\wedge^2V^*$:

$$\mathcal{W}(\varepsilon) := \frac{\otimes^\bullet V}{\langle\{v\otimes w - w\otimes v + 2\varepsilon(v,w)\mid v,w\in V\}\rangle}.$$

Summary

symmetric algebra	exterior algebra
commutative derivations, $[\ ,\ ]$	graded-commutative graded derivations, $[\ ,\ ]_{\mathrm{g}}$
depending on choice of ∇	*canonical*
∇^s, $\pounds_X^{\nabla^s}$, $\langle\ :\ \rangle_{\nabla^s}$ Killing tensors  geodesically invariant distributions affine transformations	d, $\pounds_X$, $[\ ,\ ]_{\mathrm{Lie}}$ closed forms  integrable distributions diffeomorphisms
gradients (strong) symmetric Poisson Killing ((pseudo-)Riemannian).	Hamiltonian vector fields Poisson symplectic
$(T \oplus T^*, \langle\ ,\ \rangle_-, [\ ,\ ]_{\nabla^s})$	$(T \oplus T^*, \langle\ ,\ \rangle_+, [\ ,\ ]_D)$

Thank you for your attention!

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