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Symmetric Cartan calculus

Filip Moučka[1,2] (joint work with Roberto Rubio[1])

[1]Autonomous University of Barcelona, Faculty of Sciences, Carrer dels Til·lers, 08193 Bellaterra (Cerdanyola del Vallès), Barcelona, Spain
[2]Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Břehová 7, 115 19 Prague 1, Czech Republic

FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING CTU IN PRAGUE

Abstract

We introduce analogues of the exterior derivative, the Lie derivative, and the Lie bracket of vector fields, on the algebra of completely symmetric covariant tensor fields. Then we discuss the basic properties and geometrical interpretation of these objects. Using the correspondence between the Cartan calculus and its symmetric counterpart, we introduce a symmetric version of Poisson geometry and generalized geometry.

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1. Symmetric algebra

Definition: Symmetric algebra

The **symmetric algebra** *of a manifold M is the space of completely symmetric covariant tensor fields*

$$\Gamma(\odot^\bullet T^*) := \bigoplus_{k\in\mathbb{Z}} \Gamma(\odot^k T^*)$$

endowed with the **symmetric product**, *that is the $C^\infty(M)$-bilinear map $\odot : \times^2\Gamma(\odot^\bullet T^*) \to \Gamma(\odot^\bullet T^*)$*

$$\sigma\odot\tau := \frac{|\sigma|!|\tau|!}{(|\sigma|+|\tau|)!}\operatorname{Sym}(\sigma\otimes\tau),$$

where the **symmetrization map** $\operatorname{Sym} : \Gamma(\otimes^\bullet T^*) \to \Gamma(\odot^\bullet T^*)$ *is given by*

$$(\operatorname{Sym} A)(X_1,\dots,X_{|A|}) := \frac{1}{|A|!}\sum_{\kappa\in\mathcal{S}_{|A|}} A(X_{\kappa(1)},\dots,X_{\kappa(|A|)}).$$

- The symmetric algebra $(\Gamma(\odot^\bullet T^*),\odot)$ is graded, unital, associative, and commutative.

2. Graded algebra derivations

Definition: Derivation & Graded derivation

Let $(\mathcal{A} := \bigoplus_{k\in\mathbb{Z}}\mathcal{A}_k,\cdot)$ be a graded algebra and let $D\in\operatorname{End}(\mathcal{A})$ be a vector space $|D|$-degree endomorphism (i.e. $|D|\in\mathbb{Z}$ s.t. $D(\mathcal{A}_k)\subseteq\mathcal{A}_{k+|D|}$ for all $k\in\mathbb{Z}$). Such D is called

- *a* **derivation** *if*
$$D(v\cdot w) = (Dv)\cdot w + v\cdot(Dw);$$
- *a* **graded derivation** *if*
$$D(v\cdot w) = (Dv)\cdot w + (-1)^{|v||D|}v\cdot(Dw).$$

- The space of derivations of a graded algebra $\operatorname{Der}(\mathcal{A},\cdot)$ forms a graded Lie algebra w.r.t. the endomorphism commutator
$$[D_1,D_2] := D_1\circ D_2 - D_2\circ D_1.$$
- The space of graded derivations of a graded algebra $\operatorname{gDer}(\mathcal{A},\cdot)$ forms a graded Lie superalgebra w.r.t. the endomorphism graded commutator
$$[D_1,D_2]_{\mathrm{g}} = D_1\circ D_2 - (-1)^{|D_1||D_2|}D_2\circ D_1.$$

Example: Interior multiplication

Interior multiplication by a vector field $X\in\Gamma(T)$ is a (-1)-degree endomorphism ι_X of the space of covariant tensor fields:

$$(\iota_X A)(X_1,\dots,X_{|A|-1}) := A(X,X_1,\dots,X_{|A|-1}), \qquad \iota_X f = 0.$$

When ι_X is restricted to the symmetric algebra, it is a derivation of $(\Gamma(\odot^\bullet T^*),\odot)$.

Definition: Geometric endomorphism

An endomorphism D of the vector space of covarinat tenosr fields is called **geometric** *if it is of degree 1 and $(Df)(X) = Xf$.*

3. Symmetric derivative

Definition: Symmetric derivative

The **symmetric derivative** *corresponding to a connection ∇ is the vector space endomorphism $\nabla^s\in\operatorname{End}(\Gamma(\odot^\bullet T^*))$:*

$$\nabla^s := \bigoplus_{k\in\mathbb{Z}}(k+1)\operatorname{Sym}\circ\nabla.$$

- Every symmetric derivative is a geometric derivation of $(\Gamma(\odot^\bullet T^*),\odot)$.

Proposition

Every geometric derivation of $(\Gamma(\odot^\bullet T^*),\odot)$ is a symmetric derivative ∇^s for some connection ∇.

Proposition

Geometric derivations of $(\Gamma(\odot^\bullet T^*),\odot)$ are in one-to-one correspondence with torsion-free connections.

Proposition

There is no geometric derivation $D\in\operatorname{Der}(\Gamma(\odot^\bullet T^*),\odot)$ s.t.
$$D\circ D = 0.$$

Definition: Killing tensor

A **Killing tensor** *w.r.t. ∇^s is an element $K\in\Gamma(\odot^\bullet T^*)$ s.t. $\nabla^s K = 0$.*

- Killing tensors induce conserved quantities along geodesics, the notion is used in **integrable systems**, **mechanics**, **GR**,...
- The notion of a symmetric derivative can be used for reformulation of the definitions of *Killing vector field*, *conformal Killing tensor*, or a *statistical manifold* – a fundamental notion in **information geometry**.

4. Symmetric Lie derivative

Definition: Symmetric Lie derivative

The **symmetric Lie derivative** *corresponding to ∇^s w.r.t. $X\in\Gamma(T)$ is the 0-degree derivation $\pounds_X^{\nabla^s}\in\operatorname{Der}(\Gamma(\odot^\bullet T^*),\odot)$:*

$$\pounds_X^{\nabla^s} := [\iota_X,\nabla^s] = \iota_X\circ\nabla^s - \nabla^s\circ\iota_X.$$

Theorem: Geometric interpretation of symmetric Lie derivative

Let ∇ be a torsion-free connection. Then

$$(\pounds_X^{\nabla^s}\sigma)_p = \lim_{t\to 0}\frac{1}{t}\left(\left(P^\gamma_{2t,0}\circ\left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(p)}\right)\sigma_{\Psi^X_t(p)} - \sigma_p\right), \qquad (\star)$$

where Ψ^X is the vector field's flow and P^γ is the ∇-parallel transport along the integral curve of X starting at p.

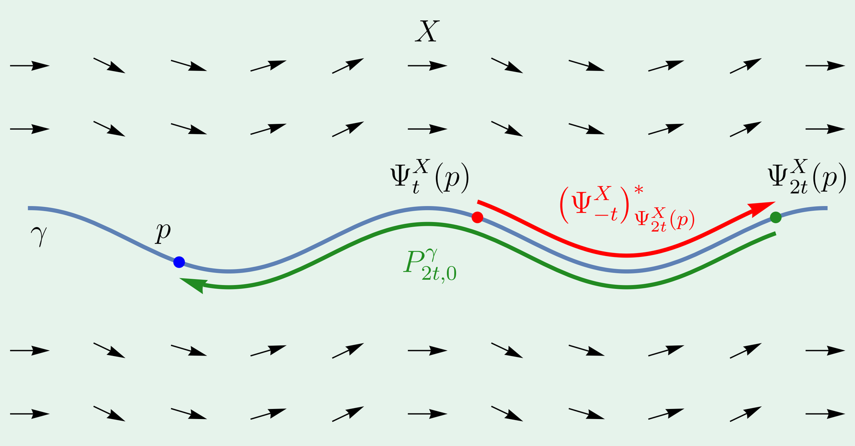

Figure 1. Transport associated with symmetric Lie derivatives.

- The formula $(\star)$ allows us to naturally extend the symmetric Lie derivative from $\Gamma(\odot^\bullet T^*)$ to the entire space of tensor fields.

Proposition

Let ∇ be a torsion-free connection. Then
$$\nabla_X = \frac{1}{2}(\pounds_X + \pounds_X^{\nabla^s}).$$

5. Symmetric bracket

Definition: Symmetric bracket

The **symmetric bracket** *corresponding to ∇^s is the $\mathbb{R}$-bilinear map $\langle\ :\ \rangle_{\nabla^s} : \times^2\Gamma(T)\to\Gamma(T)$:*

$$\iota_{\langle X:Y\rangle_{\nabla^s}} := [\pounds_X^{\nabla^s},\iota_Y] = \pounds_X^{\nabla^s}\circ\iota_Y - \iota_Y\circ\pounds_X^{\nabla^s}.$$

- Explicitly: $\langle X:Y\rangle_{\nabla^s} = \nabla_X Y + \nabla_Y X$.

Theorem: Geometric interpretation of symmetric bracket

Let ∇ be a torsion-free connection. Then $\langle X:Y\rangle_{\nabla^s}|_p = \lim_{t\to 0}\frac{1}{t}\left(\left(P^\gamma_{2t,0}\circ\left(\Psi^X_t\right)_{*\Psi^X_t(p)}\right)Y_{\Psi^X_t(p)} - Y_p\right).$

Definition: Geodesically invariant distribution

*Let ∇ be a connection. A distribution $\Delta\subseteq T$ is called ∇-***geodesically invariant** *if every geodesic $\gamma : I\to M$ has the property: $\exists\, t_0\in I,\ \dot\gamma(t_0)\in\Delta_{\gamma(t_0)},\ \Rightarrow\ \dot\gamma(t)\in\Delta_{\gamma(t)}$ for all $t\in I$. (i.e. geodesics tangent to Δ stay always tangent to Δ)*

Theorem: [Lewis, 1998]

A distribution $\Delta\subseteq T$ of locally constant rank is ∇-geodesically invariant if and only if $\langle\Gamma(\Delta):\Gamma(\Delta)\rangle_{\nabla^s}\subseteq\Gamma(\Delta)$.

- The notion of symmetric bracket is used in **control theory**, in particular, for the formulation of *accessibility criteria*.

Compare with standard Cartan calculus

- The exterior algebra $(\Gamma(\wedge^\bullet T^*),\wedge)$ is graded, unital, associative, and graded-commutative.
- When ι_X is restricted to the exterior algebra, it is a graded derivation of $(\Gamma(\wedge^\bullet T^*),\wedge)$.
- There is a unique geometric graded derivation $\mathrm{d}\in\operatorname{gDer}(\Gamma(\wedge^\bullet T^*),\wedge)$ s.t. $\mathrm{d}\circ\mathrm{d} = 0$. It is called the **exterior derivative**.
- A **closed form** is an element $H\in\Gamma(\wedge^\bullet T^*)$ s.t. $\mathrm{d}H = 0$.
- The **Lie derivative** w.r.t. $X\in\Gamma(T)$ is the 0-degree graded derivation $\pounds_X\in\operatorname{gDer}(\Gamma(\wedge^\bullet T^*),\wedge)$:
$$\pounds_X := [\iota_X,\mathrm{d}]_{\mathrm{g}} = \iota_X\circ\mathrm{d} + \mathrm{d}\circ\iota_X.$$
- Geometric interpretation of the Lie derivative is apparent from the formula
$$(\pounds_X\varphi)_p = \lim_{t\to 0}\frac{1}{t}\left(\left(\Psi^X_t\right)^*_p\varphi_{\Psi^X_t(p)} - \varphi_p\right).$$

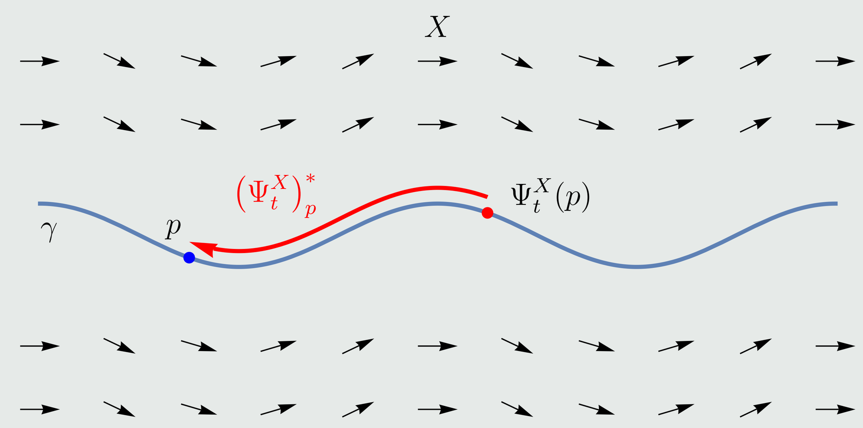

Figure 2. Transport via the vector field's flow.

- The **Lie bracket of vector fields** is the $\mathbb{R}$-bilinear map $[\ ,\ ]_{\mathrm{Lie}} : \times^2\Gamma(T)\to\Gamma(T)$:
$$\iota_{[X,Y]_{\mathrm{Lie}}} := [\pounds_X,\iota_Y]_{\mathrm{g}} = \pounds_X\circ\iota_Y - \iota_Y\circ\pounds_X.$$
- Explicitly: $[X,Y]_{\mathrm{Lie}} = X\circ Y - Y\circ X$.
- **Frobenius theorem**. A distribution $\Delta\subseteq T$ of locally constant rank is integrable if and only if $[\Gamma(\Delta),\Gamma(\Delta)]_{\mathrm{Lie}}\subseteq\Gamma(\Delta)$.

6. Symmetric Poisson geometry

- Given a symmetric bivector field $\vartheta\in\Gamma(\odot^2 T)$, consider the $\mathbb{R}$-multilinear maps:
$$\{\ ,\ \}_\vartheta : \times^2 C^\infty(M)\to C^\infty(M) : (f,g)\mapsto\vartheta(\mathrm{d}f,\mathrm{d}g), \qquad \operatorname{grad} : C^\infty(M)\to\Gamma(T) : f\mapsto\iota_{\mathrm{d}f}\vartheta.$$

Theorem: ∇^s-Schouten bracket

Let ∇^s be a symmetric derivative. There is a unique $\mathbb{R}$-bilinear map $[\ ,\ ] : \times^2\Gamma(\odot^\bullet T)\to\Gamma(\odot^\bullet T)$ s.t.

1. $[\Gamma(\odot^k T),\Gamma(\odot^l T)]\subseteq\Gamma(\odot^{k+l-1})$,
2. $[\mathcal{X},\mathcal{Y}] = [\mathcal{Y},\mathcal{X}]$,
3. $[\mathcal{X},\mathcal{Y}\odot\mathcal{Z}] = \mathcal{Y}\odot[\mathcal{X},\mathcal{Z}] + [\mathcal{X},\mathcal{Y}]\odot\mathcal{Z}$,
4. $[X,\ ] = \pounds_X^{\nabla^s}$.

*It is called ∇^s-***Schouten bracket** *and it can be expressed explicitly as*

$$[\mathcal{X},\mathcal{Y}]_{\nabla^s\text{-Schouten}} = \frac{(|\mathcal{X}|+|\mathcal{Y}|-1)!}{|\mathcal{X}|!|\mathcal{Y}|!}\operatorname{Sym}\left(|\mathcal{X}|\operatorname{Tr}(\iota_\star\mathcal{X}\otimes\nabla_\star\mathcal{Y}) + |\mathcal{Y}|\operatorname{Tr}(\iota_\star\mathcal{Y}\otimes\nabla_\star\mathcal{X})\right).$$

Proposition

Let ∇^s be a symmetric derivative and $G\in\Gamma(\odot^2 T^*)$ be non-degenerate. Then

$$\nabla^s G = 0 \qquad\Leftrightarrow\qquad [G^{-1},G^{-1}]_{\nabla^s\text{-Schouten}} = 0.$$

Proposition

Let ∇^s be a symmetric derivative and $\vartheta\in\Gamma(\odot^2 T)$. Then

$$[\vartheta,\vartheta]_{\nabla^s\text{-Schouten}} = 0 \qquad\Leftrightarrow\qquad \operatorname{Jac}_{\{\ ,\ \}_\vartheta}(f,g,h) = \langle\operatorname{grad} f:\operatorname{grad} g\rangle_{\nabla^s}h + \operatorname{cyclic}(f,g,h).$$

Definition: Symmetric Poisson structure

A **symmetric Poisson structure** *is a pair (∇,ϑ) consisting of a torsion-free connection ∇ and $\vartheta\in\Gamma(\odot^2 T)$ s.t.*

$$[\vartheta,\vartheta]_{\nabla^s\text{-Schouten}} = 0.$$

- A non-degenerate symmetric Poisson structure is equivalent to a pair (∇,G) consisting of a torsion-free connection ∇ and $G\in\Gamma(\odot^2 T^*)$ that is non-degenerate and $\nabla^s G = 0$. Such pairs (∇,G) are "symmetric analogues" of symplectic structures.

Proposition

Let ∇ be a torsion-free connection and $\vartheta\in\Gamma(\odot^2 T)$. Then

$$[\vartheta,\vartheta]_{\nabla^s\text{-Schouten}} = 0 \quad \underset{\not\Rightarrow}{\Leftarrow} \quad \operatorname{grad}\{f,g\}_\vartheta = \langle\operatorname{grad} f:\operatorname{grad} g\rangle_{\nabla^s} \quad\Leftrightarrow\quad \nabla_{\operatorname{grad} f}\vartheta = 0.$$

If, in addition, ϑ is non-degenerate and $G := \vartheta^{-1}$, then

$$\nabla_{\operatorname{grad} f}\vartheta = 0 \quad\Leftrightarrow\quad \nabla G = 0 \quad\Leftrightarrow\quad \nabla \text{ is Levi-Civita connection of } G.$$

Definition: Strong symmetric Poisson structure

A **strong symmetric Poisson structure** *is a pair (∇,ϑ) consisting of a torsion-free connection ∇ and $\vartheta\in\Gamma(\odot^2 T)$ s.t.*

$$\nabla_{\operatorname{grad} f}\vartheta = 0.$$

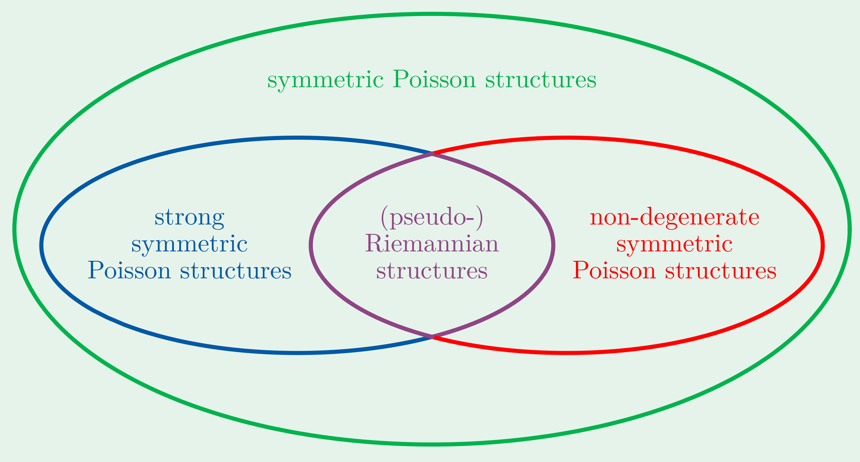

Figure 3. Relations between symmetric Poisson, strong symmetric Poisson, and (pseudo-)Riemannian, structures.

Compare with Poisson geometry

- A **Poisson structure** is a bivector field $\pi\in\Gamma(\wedge^2 T)$ s.t. $[\pi,\pi]_{\mathrm{Schouten}} = 0$.
- Given a bivector field $\pi\in\Gamma(\wedge^2 T)$, consider the map $\operatorname{Ham} : C^\infty(M)\to\Gamma(T) : f\mapsto\iota_{\mathrm{d}f}\pi$. There holds
$$[\pi,\pi]_{\mathrm{Schouetn}} = 0 \quad\Leftrightarrow\quad \operatorname{Jac}_{\{\ ,\ \}_\pi} = 0 \quad\Leftrightarrow\quad \operatorname{Ham}\{f,g\}_\pi = [\operatorname{Ham} f,\operatorname{Ham} g]_{\mathrm{Lie}}.$$
- Let $\omega\in\Gamma(\wedge^2 T^*)$ be non-degenerate. Then
$$\mathrm{d}\omega = 0 \quad\Leftrightarrow\quad [\omega^{-1},\omega^{-1}]_{\mathrm{Schouten}} = 0.$$
- A **symplectic structure** is a non-degenerate $\omega\in\Gamma(\wedge^2 T^*)$ s.t. $\mathrm{d}\omega = 0$.

7. To be continued

Generalized geometry is a novel approach to geometrical structures. It studies the geometry of $T\oplus T^*$.

- In the standard case, the focus is on the **canonical symmetric pairing** $\langle\ ,\ \rangle_+$:
$$\langle X+\alpha,Y+\beta\rangle_+ := \frac{1}{2}(\alpha(Y)+\beta(X)).$$
- The pairing, through its associated **Clifford algebra**, determines a way to derive the so-called **Dorfman bracket**:
$$[X+\alpha,Y+\beta]_D := [X,Y]_{\mathrm{Lie}} + \pounds_X\beta - \iota_Y\mathrm{d}\alpha.$$
- In our new theory, the focus is on the **canonical skew-symmetric pairing** $\langle\ ,\ \rangle_-$:
$$\langle X+\alpha,Y+\beta\rangle_- := \frac{1}{2}(\alpha(Y)-\beta(X)).$$
- The pairing, through its associated **Weyl algebra**, and the symmetric Cartan calculus together give rise to a **new bracket** and a new generalized geometry.

Generalized geometry is nowadays a well-established field of mathematics with many applications, so far, in **symplectic** and **complex geometry**, **mechanics**, **string theory** (**supergravity**, **mirror symmetry**,...), **global analysis**.

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