Symmetric Poisson geometry

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Poisson geometry

Poisson geometry originates from the mathematical formulation of classical mechanics.

Given a bivector field $\pi \in \Gamma(\wedge^2 T)$, consider the maps:

$$\{\ ,\ \}\colon \times^2 C^\infty(M) \to C^\infty(M), \qquad (f,g) \longmapsto \pi(\mathrm{d}f, \mathrm{d}g)$$

$$\mathrm{Ham}\colon C^\infty(M) \to \Gamma(T). \qquad f \longmapsto \iota_{\mathrm{d}f}\pi = \{f,\ \}$$

We have the following series of equivalences:

$$\mathrm{Jac}_{\{\ ,\ \}} = 0 \quad \Leftrightarrow \quad \mathrm{Ham}\{f,g\} = [\mathrm{Ham}\, f, \mathrm{Ham}\, g]_{\mathrm{Lie}} \quad \Leftrightarrow \quad [\pi,\pi]_{\mathrm{Sc}} = 0.$$

The **Schouten bracket** on $\Gamma(\wedge^\bullet T)$ is the unique map $[\ ,\ ]_{\mathrm{Sc}} : \times^2\Gamma(\wedge^\bullet T) \to \Gamma(\wedge^\bullet T)$ s.t.

1. $[\mathcal{X},\ ]_{\mathrm{Sc}}$ is a degree-$(|\mathcal{X}|-1)$ graded derivation of $\Gamma(\wedge^\bullet T)$,
2. $[X,\ ]_{\mathrm{Sc}} = \pounds_X$,
3. $[\mathcal{X},\mathcal{Y}]_{\mathrm{Sc}} = -(-1)^{(|\mathcal{X}|-1)(|\mathcal{Y}|-1)}[\mathcal{Y},\mathcal{X}]_{\mathrm{Sc}}$.

A **Poisson structure** is $\pi \in \Gamma(\wedge^2 T)$ s.t. $[\pi,\pi]_{\mathrm{Sc}} = 0$.

Equivalently, a **Poisson structure** is an $\mathbb{R}$-bilinear map $\{\ ,\ \} : \times^2 C^\infty(M) \to C^\infty(M)$ s.t.

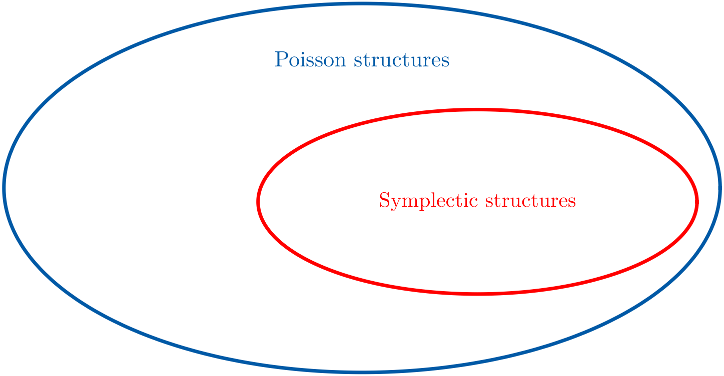
$$\{f,g\} = -\{g,f\}, \qquad \{f,gh\} = g\{f,h\} + \{f,g\}h, \qquad \mathrm{Jac}_{\{\ ,\ \}} = 0.$$

Relation between Poisson and symplectic geometry

If $\pi \in \Gamma(\wedge^2 T)$ is non-degenerate, $\omega := \pi^{-1} \in \Gamma(\wedge^2 T^*)$, then

$$[\pi, \pi]_{\mathrm{Sc}} = 0 \qquad \Leftrightarrow \qquad \mathrm{d}\omega = 0.$$

A **symplectic structure** is a non-degenerate $\omega \in \Gamma(\wedge^2 T^*)$ s.t. $\mathrm{d}\omega = 0$.



Motivation

From a mathematical point of view, there is a very natural question:

What happens when instead of $\pi \in \Gamma(\wedge^2 T)$ one has $\vartheta \in \Gamma(\odot^2 T)$?

Given a symmetric bivector field $\vartheta \in \Gamma(\odot^2 T)$, consider the maps:

$$\begin{aligned} \{\ ,\ \} : \times^2 C^\infty(M) &\to C^\infty(M), \\ (f,g) &\longmapsto \vartheta(\mathrm{d}f, \mathrm{d}g) \end{aligned} \qquad \begin{aligned} \mathrm{grad} : C^\infty(M) &\to \Gamma(T). \\ f &\longmapsto \iota_{\mathrm{d}f}\vartheta = \{f,\ \} \end{aligned}$$

Naively, one can ask

$$\begin{aligned} \mathrm{Jac}_{\{\ ,\ \}} = 0 \quad &\Leftrightarrow \quad \vartheta = 0, \\ \mathrm{grad}\{f,g\} = [\mathrm{grad}\, f, \mathrm{grad}\, g]_{\mathrm{Lie}} \quad &\Leftrightarrow \quad \vartheta = 0, \\ [\vartheta, \vartheta]_{\mathrm{Sc}} = 0 \quad &\Leftrightarrow \quad \vartheta \text{ is arbitrary}. \end{aligned}$$

The **Schouten bracket** on $\Gamma(\odot^\bullet T)$ is the unique map $[\ ,\ ]_{\mathrm{Sc}} : \times^2 \Gamma(\odot^\bullet T) \to \Gamma(\odot^\bullet T)$ s.t.

1. $[\mathcal{X},\ ]_{\mathrm{Sc}}$ is a degree-$(|\mathcal{X}|-1)$ derivation of $\Gamma(\odot^\bullet T)$,
2. $[X,\ ]_{\mathrm{Sc}} = \pounds_X$,
3. $[\mathcal{X}, \mathcal{Y}]_{\mathrm{Sc}} = -[\mathcal{Y}, \mathcal{X}]_{\mathrm{Sc}}$.

Non-degenerate case? The exterior derivative cannot act on the elements of $\Gamma(\odot^\bullet T^*)$.

Any way out?

Recall three ways that lead to the notion of Poisson structure:

1. $[\pi, \pi]_{\mathrm{Sc}} = 0$,
2. $\mathrm{Ham}\{f, g\} = [\mathrm{Ham}\, f, \mathrm{Ham}\, g]_{\mathrm{Lie}}$,
3. non-degenerate case: $\mathrm{d}\omega = 0$, where $\omega := \pi^{-1}$.

The way out is to find analogues of d, $\pounds_X$, *and* $[\ ,\ ]_{\mathrm{Lie}}$!

Symmetric derivative

There is a unique degree-1 graded derivation d of $\Gamma(\wedge^\bullet T^*)$ s.t.

$$(\mathrm{d}f)(X) = Xf, \qquad \mathrm{d}\circ\mathrm{d} = 0.$$

It is called the **exterior derivative**.

Analogue on $\Gamma(\odot^\bullet T^*)$?

Def. The **symmetric derivative** corresponding to a connection ∇,

$$\nabla^s := \bigoplus_{k\in\mathbb{Z}} (k+1)\cdot \mathrm{Sym}\circ\nabla.$$

Prop. There is one-to-one correspondence between torsion-free connections and degree-1 derivations D of $\Gamma(\odot^\bullet T^*)$ s.t.

$$(Df)(X) = Xf.$$

Prop. There is **no** degree-1 derivation D of $\Gamma(\odot^\bullet T^*)$ s.t.

$$(Df)(X) = Xf, \qquad D\circ D = 0.$$

The covariant gradient $\nabla : \Gamma(\otimes^\bullet T^*) \to \Gamma(\otimes^\bullet T^*)$:

$$\nabla(\Gamma(\otimes^k T^*)) \subseteq \Gamma(\otimes^{k+1} T^*), \qquad (\nabla A)(X, X_1, \ldots, X_k) := (\nabla_X A)(X_1, \ldots, X_k).$$

Analogue of closed forms?

A **Killing structure** is a pair (∇, K) consisting of a torsion-free connection ∇ and $K \in \Gamma(\odot^\bullet T^*)$ s.t.

$$\nabla^s K = 0.$$

A Killing structure (∇, K) induces the function $f_K \in C^\infty(TM)$

$$f_K((p,v)) := K_p(v, \dots, v) \qquad \text{for all } (p,v) \in TM,$$

that is constant along every geodesic of ∇.

Killing tensors are used in general relativity (Carter tensor in Kerr-Newman spacetime),
integrable systems (separability of Hamilton-Jacobi eq.),
cosmology (FLRW spacetimes),
...

Symmetric Lie derivative

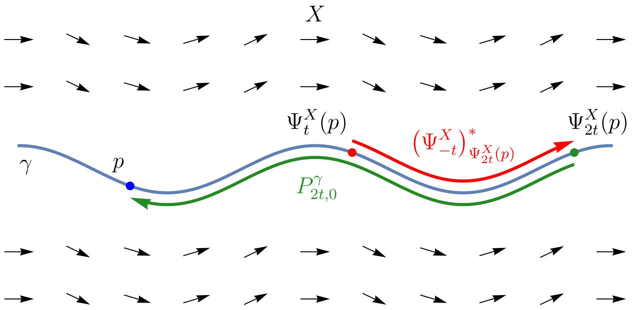
The **Lie derivative** w.r.t. $X \in \Gamma(T)$:

$$\mathcal{L}_X := [\iota_X, \mathrm{d}]_{\mathrm{g}} = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X .$$

Def. The **symmetric Lie derivative** corresponding to ∇^s w.r.t. $X \in \Gamma(T)$:

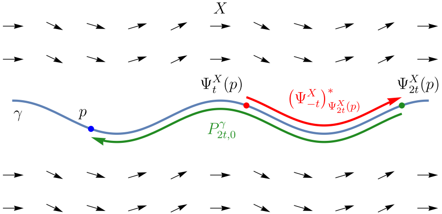
$$\mathcal{L}_X^{\nabla^s} := [\iota_X, \nabla^s] = \iota_X \circ \nabla^s - \nabla^s \circ \iota_X .$$

Prop. $(\mathcal{L}_X^{\nabla^s}\sigma)_p = \lim_{t \to 0} \frac{1}{t} \left(P^{\gamma}_{2t,0} \left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(p)} \sigma_{\Psi^X_t(p)} - \sigma_p \right).$

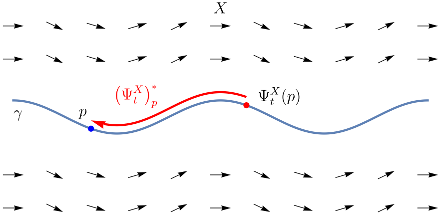


Symmetric Lie derivative

$$(\mathcal{L}_X^{\nabla^s}\sigma)_p = \lim_{t\to 0}\frac{1}{t}\left(P^{\gamma}_{2t,0}\left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(p)}\sigma_{\Psi^X_t(p)} - \sigma_p\right)$$



$$(\mathcal{L}_X\varphi)_p = \lim_{t\to 0}\frac{1}{t}\left(\left(\Psi^X_t\right)^*_p\varphi_{\Psi^X_t(p)} - \varphi_p\right)$$



∇^s-Schouten bracket

Prop. Given a symmetric derivative ∇^s, there is a unique map

$$[\ ,\ ]_{\nabla^s\text{-Sc}} : \times^2\Gamma(\odot^\bullet T) \to \Gamma(\odot^\bullet T)$$

s.t.

1. $[\mathcal{X},\ ]_{\nabla^s\text{-Sc}}$ is a degree-$(|\mathcal{X}|-1)$ derivation of $\Gamma(\odot^\bullet T)$,
2. $[X,\ ]_{\nabla^s\text{-Sc}} = \pounds_X^{\nabla^s}$,
3. $[\mathcal{X},\mathcal{Y}]_{\nabla^s\text{-Sc}} = [\mathcal{Y},\mathcal{X}]_{\nabla^s\text{-Sc}}$.

We call it the ∇^s-**Schouten bracket**.

Prop. Let G be a (pseudo-)Riemannian metric and $\mathcal{X} \in \Gamma(\odot^\bullet T)$. Then

$$[\mathcal{X}, G^{-1}]_{{}^G\nabla^s\text{-Sc}} = 0 \qquad \Leftrightarrow \qquad {}^G\nabla^s G^*\mathcal{X} = 0$$

(i.e. $G^*\mathcal{X} \in \Gamma(\odot^\bullet T^*)$ is a Killing tensor of G).

Symmetric bracket

The **Lie bracket of vector fields** is the $\mathbb{R}$-bilinear map

$$[\ ,\ ]_{\text{Lie}} : \times^2\Gamma(T) \to \Gamma(T)$$

given by

$$\iota_{[X,Y]_{\text{Lie}}} := [\pounds_X, \iota_Y]_{\text{g}} = \pounds_X \circ \iota_Y - \iota_Y \circ \pounds_X .$$

Explicitly: $[X,Y]_{\text{Lie}} = X \circ Y - Y \circ X$.

$$[X,Y]_{\text{Lie}}|_p = (\pounds_X Y)_p = \lim_{t\to 0} \frac{1}{t} \Big(\big(\Psi^X_{-t}\big)_{*\Psi^X_t(p)} Y_{\Psi^X_t(p)} - Y_p \Big).$$

Def. The **symmetric bracket** corresponding to ∇^s is the $\mathbb{R}$-bilinear map

$$\langle\ :\ \rangle_{\nabla^s} : \times^2\Gamma(T) \to \Gamma(T)$$

given by

$$\iota_{\langle X:Y\rangle_{\nabla^s}} := [\pounds^{\nabla^s}_X, \iota_Y] = \pounds^{\nabla^s}_X \circ \iota_Y - \iota_Y \circ \pounds^{\nabla^s}_X .$$

Explicitly: $\langle X : Y\rangle_{\nabla^s} = \nabla_X Y + \nabla_Y X$.

Prop. $\langle X : Y\rangle_{\nabla^s}|_p = (\pounds^{\nabla^s}_X Y)_p = \lim_{t\to 0} \frac{1}{t} \Big(P^{\gamma}_{2t,0} \big(\Psi^X_t\big)_{*\Psi^X_t(p)} Y_{\Psi^X_t(p)} - Y_p \Big).$

Back to bivector fields

$$[\pi,\pi]_{\mathsf{Sc}} = 0 \qquad (\mathcal{L}_X \rightsquigarrow \mathcal{L}_X^{\nabla^s}) \qquad [\vartheta,\vartheta]_{\nabla^s\text{-}\mathsf{Sc}} = 0,$$

$$\mathrm{Ham}\{f,g\} = [\mathrm{Ham}\, f, \mathrm{Ham}\, g]_{\mathsf{Lie}} \qquad ([\ ,\ ]_{\mathsf{Lie}} \rightsquigarrow \langle\ ,\ \rangle_{\nabla^s}) \qquad \mathrm{grad}\{f,g\} = \langle \mathrm{grad}\, f, \mathrm{grad}\, g\rangle_{\nabla^s},$$

$$\mathrm{d}\omega = 0 \qquad (\mathrm{d} \rightsquigarrow \nabla^s) \qquad \nabla^s G = 0.$$

Prop. Let ∇ be a torsion-free connection and $\vartheta \in \Gamma(\odot^2 T)$. Then

$$[\vartheta,\vartheta]_{\nabla^s\text{-}\mathsf{Sc}} = 0 \quad \Leftrightarrow \quad (\nabla_{\mathrm{grad}\, f}\vartheta)(\mathrm{d}g, \mathrm{d}h) + \mathrm{cyclic}(f,g,h) = 0,$$

$$\Leftrightarrow \quad \mathrm{Jac}_{\{\ ,\ \}}(f,g,h) = \mathrm{d}h(\langle \mathrm{grad}\, f : \mathrm{grad}\, g\rangle_{\nabla^s}) + \mathrm{cyclic}(f,g,h).$$

Prop. Let ∇ be a torsion-free connection and $\vartheta \in \Gamma(\odot^2 T)$. Then

$$[\vartheta,\vartheta]_{\nabla^s\text{-}\mathsf{Sc}} = 0 \quad \overset{\Leftarrow}{\not\Rightarrow} \quad \mathrm{grad}\{f,g\} = \langle \mathrm{grad}\, f : \mathrm{grad}\, g\rangle_{\nabla^s} \quad \Leftrightarrow \quad \nabla_{\mathrm{grad}\, f}\,\vartheta = 0.$$

Def. A **symmetric Poisson structure** is a pair (∇, ϑ) consisting of a torsion-free connection ∇ and $\vartheta \in \Gamma(\odot^2 T)$ s.t. $[\vartheta,\vartheta]_{\nabla^s\text{-}\mathsf{Sc}} = 0$.

Def. A **strong symmetric Poisson structure** is a pair (∇, ϑ) consisting of a torsion-free connection ∇ and $\vartheta \in \Gamma(\odot^2 T)$ s.t. $\nabla_{\mathrm{grad}\, f}\,\vartheta = 0$.

Non-degenerate symmetric Poisson structures

Prop. Let ∇ be a torsion-free connection and $\vartheta \in \Gamma(\odot^2 T)$ be non-degenerate, $G := \vartheta^{-1} \in \Gamma(\odot^2 T^*)$. Then

$$\nabla^s G = 0 \quad \Leftrightarrow \quad [\vartheta, \vartheta]_{\nabla^s\text{-Sc}} = 0.$$

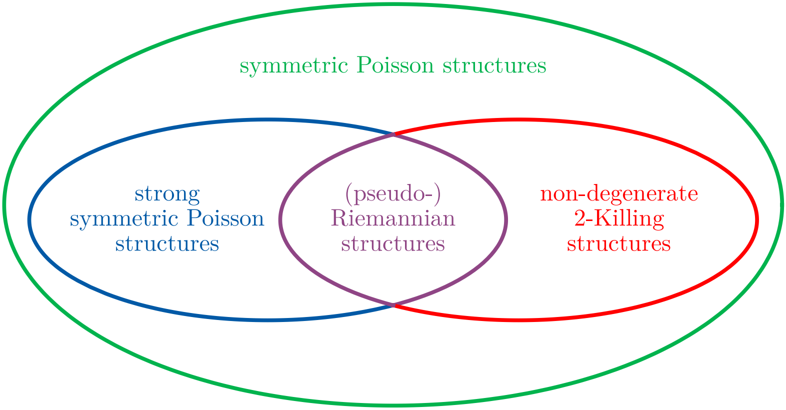
> non-degenerate symmetric Poisson structures
>
> $\overset{1:1}{\longleftrightarrow}$
>
> non-degenerate 2-Killing structures

Prop. Let ∇ be a torsion-free connection and $\vartheta \in \Gamma(\odot^2 T)$ be non-degenerate, $G := \vartheta^{-1} \in \Gamma(\odot^2 T^*)$. Then

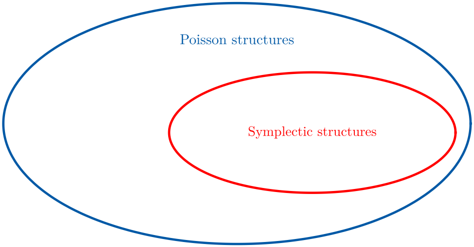
$\nabla_{\mathrm{grad}\, f}\, \vartheta = 0 \quad \Leftrightarrow \quad \nabla\vartheta = 0 \quad \Leftrightarrow \quad \nabla G = 0 \quad \Leftrightarrow \quad \nabla$ is the Levi-Civita connection of G.

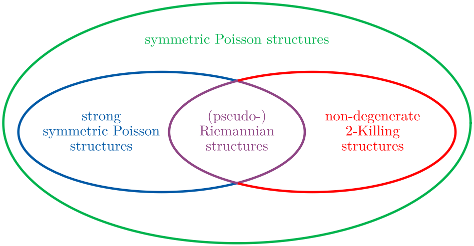
> non-degenerate strong symmetric Poisson structures
>
> $\overset{1:1}{\longleftrightarrow}$
>
> (pseudo-)Riemannian structures

(Strong) symmetric Poisson, Killing, and (pseudo-)Riemannian structures



Comparison of symmetric Poisson and Poisson structures





Patterson-Walker metric

Given a torsion-free connection ∇ on M, one can construct (pseudo-)Riemannian metric $G_\nabla \in \Gamma(\odot^2 T^*(T^*M))$, the so-called **Patterson-Walker metric**.

In natural coordinates, it is given by

$$G_\nabla|_U = \mathrm{d}x^j \odot \mathrm{d}p_j - p_k \Gamma^k{}_{lj} \mathrm{d}x^l \odot \mathrm{d}x^j .$$

It gives us the bracket $\{\ ,\ \}_\nabla : \times^2 C^\infty(T^*M) \to C^\infty(T^*M)$,

$$\{f, g\}_\nabla|_U = \frac{\partial f}{\partial x^j}\frac{\partial g}{\partial p_j} + \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x^j} + 2p_k \Gamma^k{}_{lj} \frac{\partial f}{\partial p_l}\frac{\partial g}{\partial p_j} .$$

Compare with the canonical symplectic structure $\omega_{\text{can}} \in \Gamma(\wedge^2 T^*(T^*M))$,

$$\omega_{\text{can}}|_U = \mathrm{d}x^j \wedge \mathrm{d}p_j ,$$

and the canonical Poisson bracket $\{\ ,\ \}_{\text{can}} : \times^2 C^\infty(T^*M) \to C^\infty(T^*M)$,

$$\{f, g\}_{\text{can}}|_U = \frac{\partial f}{\partial x^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x^j} .$$

How does it relate to symmetric Poisson structures?

Every $\mathcal{X} \in \Gamma(\odot^k T)$ induces a smooth function $\Phi_{\mathcal{X}} \in C^\infty(T^*M)$,

$$\Phi_{\mathcal{X}}((p,\alpha)) := \frac{1}{k!}\mathcal{X}_p(\alpha,\dots,\alpha) \qquad \text{for all } (p,\alpha) \in T^*M.$$

Prop. The map $\Phi : \Gamma(\odot^\bullet T) \to C^\infty(T^*M)$ is a $C^\infty(M)$-module morphism and satisfies

1. $\Phi_{\mathcal{X}\odot\mathcal{Y}} = \Phi_{\mathcal{X}}\Phi_{\mathcal{Y}}$,
2. $\Phi_{[\mathcal{X},\mathcal{Y}]_{\nabla^{s\text{-Sc}}}} = \{\Phi_{\mathcal{X}}, \Phi_{\mathcal{Y}}\}_\nabla$.

Given a symmetric Poisson structure (∇, ϑ) it follows that

$$\{\Phi_\vartheta, \Phi_\vartheta\}_\nabla = \Phi_{[\vartheta,\vartheta]_{\nabla^{s\text{-Sc}}}} = 0 \qquad \Leftrightarrow \qquad \Phi_\vartheta \text{ is constant along integral curves of } \operatorname{grad}_\nabla \Phi_\vartheta = \{\Phi_\vartheta, \ \}_\nabla.$$

In natural coordinates, we have

$$\operatorname{grad}_\nabla \Phi_\vartheta|_U = \frac{\partial \Phi_\vartheta}{\partial p_j}\frac{\partial}{\partial x^j} + \left(\frac{\partial \Phi_\vartheta}{\partial x^j} + 2p_k\Gamma^k{}_{lj}\frac{\partial \Phi_\vartheta}{\partial p_l}\right)\frac{\partial}{\partial p_j}.$$

Therefore, the integral curves are given by ODEs

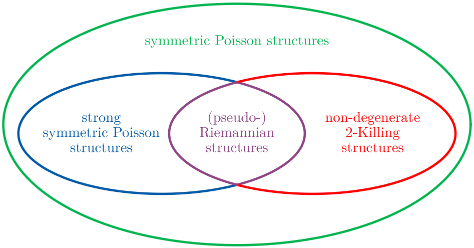
$$\dot{x}^j = \vartheta^{jk}p_k$$

$$\dot{p}_j = \left(\frac{1}{2}\frac{\partial\,\vartheta^{kl}}{\partial x^j} + 2\Gamma^k{}_{mj}\vartheta^{ml}\right)p_kp_l \qquad \Rightarrow \qquad (\nabla_{\dot{x}}\dot{x})^j = \frac{1}{2}([\vartheta,\vartheta]_{\nabla^{s\text{-Sc}}})^{jkl}p_kp_l = 0.$$

(geodesic equation)

Outlook

Symmetric Poisson geometry extends (pseudo-)Riemannian geometry while bringing in features of Poisson geometry. It has the potential to blend these two areas.



- Analogue of Weinstein's splitting theorem?
- Symmetric Poisson cohomology?
- Flat symmetric Poisson structures?

Thank you for your attention!

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