Generalized metrics: slice theorem and moduli space

Roberto Rubio

UAB
Universitat Autònoma
de Barcelona

Mathematical supergravity

UNED

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Claire Voisin, Crafoord Prize and BBVA Frontiers 2024

"The problem is, physicists have extraordinary ideas – sort of like magic. But they don't work at the same scale of time as us. We mathematicians need a lot of time to produce the right definitions and to prove theorems. And we are not happy if the statements are not proved rigorously."

Interview to Nature, 9th Feb 2024

Also while finding the right definition and proving theorems,

working with string algebroids and Hull-Strominger with
M. Garcia-Fernandez [GF], C. Tipler [T] and also C. Shahbazi [S],

[GF,R,T] Gauge theory for string algebroids, to appear in JDG
[GF,R,S,T] Canonical metrics on holomorphic Courant algebroids, PLMS 2022
[GF,R,T] Holomorphic string algebroids, TAMS 2020
[GF,R,T] Infinitesimal moduli for the Strominger system and Killing spinors..., MAAN 2017

a basic structural question led to a joint work with Carl Tipler:

The Lie group of automorphisms of a Courant algebroid
and the moduli space of generalized metrics

Rev. Mat. Iberoam. 36 (2020), 485-536

Disclaimer: no latest news, no major surprises, but never out of style.

Plan of the talk

I. Introduction to generalized geometry
(generalized diffeomorphisms and metrics)

II. Infinite-dimensional manifolds and groups

III. Slice theorem and the moduli
space of generalized metrics

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$$TM \ + \ T^*M$$

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ω presymplectic

($\omega \in \Omega^2(M)$, $d\omega = 0$)

$graph(\omega) \subset TM{+}T^*M$

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Pairing $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(Y))$
($\mathrm{O}(n, n)$-bundle)

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Maximally isotropic

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($\mathrm{O}(n,n)$-bundle)

Maximally isotropic

Skew-symmetric, $\mathcal{J}^* + \mathcal{J} = 0$

The Dorfman **bracket** on $\Gamma(TM+T^*M)$

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - i_Y d\alpha$$

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$(\omega \in \Omega^2(M),\ d\omega = 0)$
$graph(\omega) \subset TM+T^*M$

P Poisson
$(P \in \mathfrak{X}^2(M),\ [P,P] = 0)$
$graph(P) \subset TM+T^*M$

Maximally isotropic
Involutive (Dorfman)

Dirac structures
Courant, Weinstein...

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$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$\mathcal{J} \in \mathrm{End}(TM+T^*M),\ \mathcal{J}^2 = -\mathrm{Id}$

Skew-symmetric, $\mathcal{J}^* + \mathcal{J} = 0$
$+i$-eigenbundle involutive

Generalized complex geometry
Hitchin, Gualtieri, Cavalcanti...

The Dorfman **bracket**??

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - i_Y d\alpha$$

$$\begin{aligned}[X+\alpha, X+\alpha] &= [X,X] + \mathcal{L}_X\alpha - i_X d\alpha \\ &= di_X\alpha + i_X d\alpha - i_X d\alpha \\ &= di_X\alpha = d\langle X+\alpha, X+\alpha\rangle\end{aligned}$$

It is not skew-symmetric, but satisfies, for $e, u, v \in \Gamma(TM + T^*M)$,

$$[e,[u,v]] = [[e,u],v] + [u,[e,v]]$$

$$\pi_{TM}(e)\langle u,v\rangle = \langle [e,u],v\rangle + \langle u,[e,v]\rangle$$

Actually, this structure has a name...

The Courant algebroid $(TM+T^*M, \langle\,,\rangle, [\,,], \pi_{TM})$

Definition (Liu-Weinstein-Xu)

A Courant algebroid over M is a tuple $(E, \langle\,,\rangle, [\,,], \pi)$ consisting of

- a vector bundle $E \to M$,
- a nondegenerate symmetric pairing $\langle\,,\rangle$,
- a bilinear bracket $[\,,]$ on $\Gamma(E)$,
- a bundle map $\pi : E \to TM$ covering id_M,

such that, for any $e \in E$,

- the map $[e, \cdot]$ is a derivation of both the bracket and the pairing,
- we have $[e, e] = d\langle e, e\rangle$.

Example

For $H \in \Omega^3_{cl}$, define the H-twisted bracket

$$[X + \alpha, Y + \beta]_H = [X, Y] + \mathcal{L}_X\beta - i_Y d\alpha + i_X i_Y H$$

The tuple $(TM+T^*M, \langle\,,\rangle, [\,,]_H, \pi_{TM})$ is a Courant algebroid

Another example

Denote $1 = M \times \mathbb{R}$ and consider

$$TM + 1 + T^*M$$

$$\begin{aligned}
\pi(X + f + \alpha) &= X \\
\langle X + f + \alpha, X + f + \alpha \rangle &= \iota_X \alpha + f^2 \\
[X + f + \alpha, Y + g + \beta] &= [X, Y] + X(g) - Y(f) \\
&\quad + \mathcal{L}_X \beta - \iota_Y d\alpha + 2g df.
\end{aligned}$$

is a Courant algebroid, and moreover an $\mathrm{O}(n+1, n)$-bundle.
As $\mathrm{O}(n+1, n)$ is a real form of $\mathrm{O}(2n+1, \mathbb{C})$, of Lie type B_n:

Generalized geometry of type B_n
(toy model of string or heterotic algebroids)

Automorphisms of Courant algebroids

Definition

The automorphism group $\mathrm{Aut}\, E$ of a Courant algebroid E are the bundle maps $F : E \to E$, covering $f \in \mathrm{Diff}(M)$, such that, for $u, v \in \Gamma(E)$,

- $\langle Fu, Fv \rangle = f_* \langle u, v \rangle$,
- $[Fu, Fv] = f_*[u, v]$,
- $\pi_{TM} \circ F = f_* \circ \pi_{TM}$

Example

On $TM+T^*M$, for any $f \in \mathrm{Diff}(M)$ and $B \in \Omega^2_{cl}(M)$,

$$\left.\begin{array}{ll} f_* = \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}, & X + \alpha \mapsto f_* X + f_* \alpha \\ e^B = \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix}, & X + \alpha \mapsto X + \alpha + i_X B \end{array}\right\} \in \mathrm{Aut}(TM+T^*M)$$

Actually, the so-called **generalized diffeomorphisms** are

$$\mathrm{Aut}(TM+T^*M) = \mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$$

Exact Courant algebroids

For any Courant algebroid we have

$$T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM$$

Definition

An exact Courant algebroid is a Courant algebroid satisfying

$$0 \to T^*M \to E \to TM \to 0$$

As a vector bundle $E \cong TM{+}T^*M$, by choosing a splitting $\lambda' : TM \to E$

The splitting $\lambda : X \mapsto \lambda'(X) - \pi^*\langle \lambda'(X), \cdot \rangle$ is isotropic (isotropic image)
With an isotropic splitting λ, we get a $\langle\, , \rangle$-preserving isomorphism

$$\begin{aligned} \lambda + \pi^* : TM{+}T^*M &\to E \\ X + \alpha &\mapsto \lambda(X) + \pi^*\alpha \end{aligned}$$

Classification of exact Courant algebroids

The isomorphism

$$\lambda + \pi^* : TM{+}T^*M \to E$$
$$X + \alpha \mapsto \lambda(X) + \pi^*\alpha$$

preserves $\langle\,,\rangle$ and π_{TM}, whereas the bracket of E is brought to $[,]_H$,

$$E \simeq_\lambda (TM{+}T^*M)_H$$

$$H(u, v, w) = \langle[\lambda(u), \lambda(v)], \lambda(w)\rangle$$

For any two isotropic splittings of E, $\lambda - \lambda' = \pi^* \circ C$ for $C \in \Omega^2(M)$: **the space of isotropic splittings** Λ is an $\Omega^2(M)$-torsor and

$$E \simeq_{\lambda+\pi^* C} (TM{+}T^*M)_{H+dC}$$

Exact Courant algebroids up to isomorphism: classified by the Ševera class

$$[H] \in H^3(M)$$

Automorphism of exact Courant algebroids

We will use $E \cong_\lambda (TM + T^*M)_H$.

For $TM+T^*M$, we saw $\mathrm{GDiff} = \mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$

For $(TM+T^*M)_H$, we have

$$\mathrm{GDiff}_H = \{(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^2(M) \mid \varphi^* H - H = dB\}$$

They all lie inside the π-**preserving orthogonal transformations**

$$\mathrm{O}_\pi(TM+T^*M) = \mathrm{Diff}(M) \ltimes \Omega^2(M)$$

Odd exact algebroids: twisted versions of $TM + 1 + T^*M$

$$\begin{tikzcd} & T^* \arrow[d] & & & \\ 0 \arrow[r] & Q \arrow[r] \arrow[d] & E \arrow[r, "\pi"] & T \arrow[r] & 0 \\ & 1 & & & \end{tikzcd}$$

The bracket is twisted by $F \in \Omega^2_{cl}$ and $H \in \Omega^3$ s.t. $dH + F \wedge F = 0$.

$\mathrm{Aut}(TM + 1 + T^*M)$ contains also A-fields

$$e^A = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ A & \mathrm{Id} & 0 \\ -A \otimes A & -2A & \mathrm{Id} \end{pmatrix}, \; X + f + \alpha \mapsto X + f + i_X A + \alpha - 2fA - A(X)A$$

Not abelian! ($e^A e^{A'} = e^{A+A'} e^{-A \wedge A'}$).

So far

Dirac / generalized complex geometry

$$\downarrow$$

Courant algebroids and their automorphisms

$$\Big\downarrow \text{ next}$$

Generalized metrics

Generalized metric (on exact Courant algebroids)

Metric: reduction of the frame $\mathrm{GL}(n)$-bundle to a maximal compact $\mathrm{O}(n)$, so generalized metric: reduction from $\mathrm{O}(n,n)$ to $\mathrm{O}(n)\times\mathrm{O}(n)$.

Definition

A **generalized metric** on an exact Courant algebroid E is a (rank n) subbundle $V_+ \subset E$ such that $\langle\,,\rangle_{|V_+}$ is positive definite

The subbundle $V_- = V_+^\perp$ is negative definite and $E = V_+ + V_-$.

Example

A usual metric g on M defines a generalized metric on $(TM + T^*M)_H$ by its graph
$$V_+ = \{X + i_X g \mid X \in TM\}$$

For $V_+ \subset TM + T^*M$, we have $V_+ \cap T^*M = \{0\}$, so

$$V_+ = gr(g + B : TM \to T^*M)$$

with g symmetric and B skew-symmetric. V_+ positive-definite $\to g$ metric.

So, a generalized metric consists of a metric g and a 2-form B.

Using the notation $\mathcal{M} := \{g \in \Gamma(S^2 T^*M) \mid g \text{ is positive definite}\}$,
the **set $\mathcal{GM}$ of generalized metrics** on E is described by

$$\mathcal{GM} \simeq_\lambda \mathcal{M} \times \Omega^2$$

The action

For an exact Courant algebroid E, denote $\mathrm{Aut}(E)$ by GDiff

The (right) action on $\mathcal{GM}$ is

$$\begin{aligned} \mathrm{GDiff} \times \mathcal{GM} &\to \mathcal{GM} \\ F \cdot V_+ &\mapsto F^{-1}(V_+), \end{aligned}$$

which in terms of $\mathcal{GM} \simeq_\lambda \mathcal{M} \times \Omega^2$ is given by

$$(\varphi, B) \cdot (g, C) \mapsto (\varphi^* g, \varphi^* C - B)$$

We want to study

$$\mathcal{GR} = \frac{\mathcal{GM}}{\mathrm{GDiff}}$$

Why care about generalized metrics? General idea:

They encapsulate a metric
together with some extra fields
(depending on the Courant algebroid).

Equations of motion can be expressed as
the vanishing of the generalized Ricci tensor.

(Coimbra–Strickland-Constable–Waldram...

Garcia-Fernandez, Baraglia–Hekmati...)

Plan of the talk

I. Introduction to generalized geometry (generalized diffeomorphisms and metrics)

II. Infinite-dimensional manifolds and groups

III. Slice theorem and the moduli space of generalized metrics

Finite-dimensional manifolds and Lie groups are modelled on $\mathbb{R}^n$, finite-dimensional real vector space with the topology given by any norm. Isometries are a Lie group (Myers-Steenrod Thm.). What about $\mathrm{Diff}(M)$?

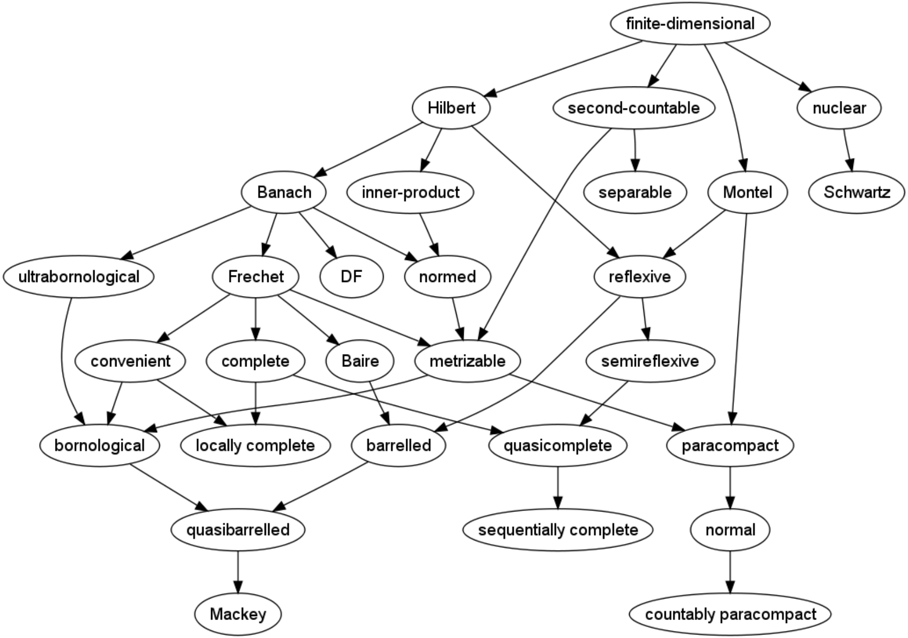
Infinite-dimensional manifolds and Lie groups are modelled on...

some kind of $\mathbb{R}^\infty$,

infinite-dimensional real vector space with... what norm?

The different topologies are the least of our problems: a Banach Lie group acting effectively and transitively on a finite-dimensional compact smooth manifold must be finite-dimensional

Alternatives? Let us look at the magnitude of this issue...



(diagram by Greg Kuperberg)

@Greg Kuperberg: In view of my answer, the place of convenient in this very nice diagram should be: Sequentially complete $\implies$ convenient. – Peter Michor Oct 19, 2012 at 12:13

We said that **Banach** is too restrictive

A Banach Lie group acting effectively and transitively on a finite-dimensional compact smooth manifold must be finite-dimensional

What about Fréchet? Too permisive

Fréchet Lie groups have no local inverse theorem, nor Frobenius' theorem

Let us rather look at a familiar example

Diffeomorphism group

From now on, let M be a **compact** n-dimensional manifold, $n \geq 1$

$\mathrm{Diff}(M)$ is an infinite-dimensional Lie group, how?

Take a riemannian metric on M. For a small neighbourhood U of the zero vector field, the geodesic flow at time 1 gives a chart around the identity:

$$\begin{array}{ccc} U & \to & \mathrm{Diff}(M) \\ X & \mapsto & (p \mapsto \exp_p(X_p)), \end{array}$$

(where $t \mapsto \exp_p(tX_p)$ is the geodesic starting from p in the direction of X_p)

Translate this chart to cover the manifold + independent from the metric

Question: a neighbourhood U, in which topology? Take any Sobolev norm
Issue: $\Gamma(TM)$ is not complete with respect to the k-Sobolev norms... but we can complete and at least say that it is an inverse limit of Hilbert spaces

$$\Gamma(TM)^{n+5} \supset \Gamma(TM)^{n+6} \supset \Gamma(TM)^{n+7} \supset \ldots \supset \Gamma(TM)^{k} \supset \ldots\ldots \supset \Gamma(TM)$$

ILH spaces

Definition (Omori)

An **ILH chain** is a set of complete locally convex topological vector spaces

$$\{\mathbf{E}, E^k \mid k \in \mathbb{N}_{\geq d}\}$$

- E^k is a Hilbert space
- E^{k+1} embeds continuously in E^k with dense image,
- and $\mathbf{E} = \bigcap_{k \in \mathbb{N}_{\geq d}} E^k$, endowed with the inverse limit topology

Example: the chain $\{\mathbf{\Gamma(TM)}, \Gamma(TM)^k \mid k \in \mathbb{N}_{\geq n+5}\}$

An ILH manifold is a "manifold locally modelled on an ILH chain"

(we shall keep simple the ILH picture in this talk, see the paper for more details)

ILH manifolds

Definition (Omori)

A (strong) ILH manifold M modelled on the ILH chain $\{\mathbf{E}, E^k \mid k \in \mathbb{N}(d)\}$ is a manifold M modelled on $\mathbf{E}$ such that:

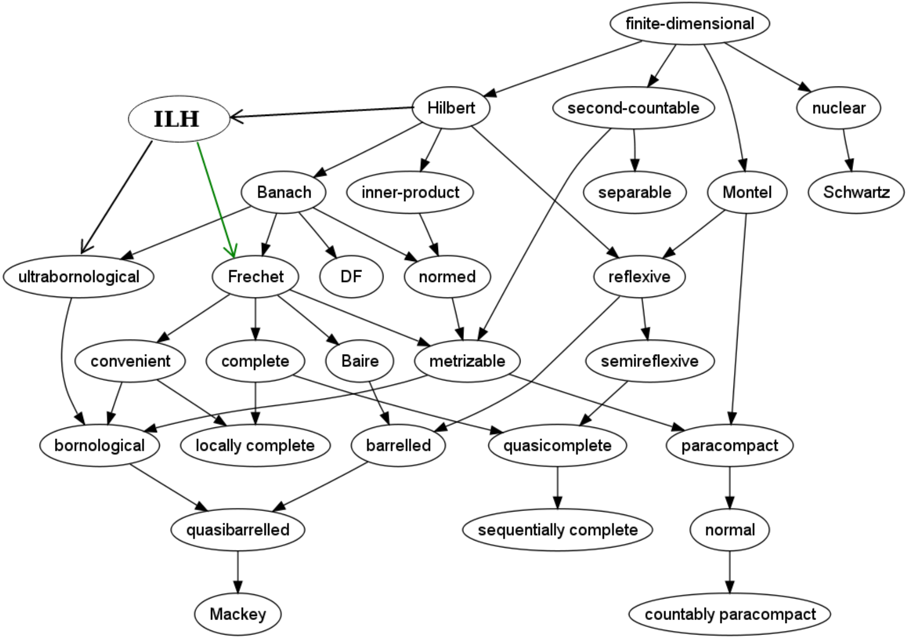
- M is the inverse limit of smooth Hilbert manifolds M^k modelled on E^k
- for any $x \in M$, there exist compatible open charts (U_k, φ_k) of M^k, and the inverse limit of $(U_k)_{k\in\mathbb{N}(d)}$ is an open neighbourhood of x

For $\mathrm{Diff}(M)$, we have $\mathrm{Diff}^{n+5}(M) \supset \mathrm{Diff}^{n+6}(M) \supset \ldots\ldots \supset \mathrm{Diff}(M)$

We can also define ILH maps, (strong) ILH groups, (strong) ILH actions...

Unlike the Fréchet category, the ILH category has:

- Frobenius' theorem,
- implicit function theorem


finite-dimensional
ILH
Hilbert
second-countable
nuclear
Banach
inner-product
separable
Montel
Schwartz
ultrabornological
Frechet
DF
normed
reflexive
convenient
complete
Baire
metrizable
semireflexive
bornological
locally complete
barrelled
quasicomplete
paracompact
quasibarrelled
sequentially complete
normal
Mackey
countably paracompact

Generalized diffeomorphisms acting on generalized metrics

Theorem (R-Tipler)

The set of generalized metrics is an ILH manifold
The group of generalized diffeomorphisms is an ILH group
The action of the latter on the former is an ILH action

Some ideas from the proof:

- For the set of generalized metrics $\mathcal{GM} \cong \mathcal{M} \times \Lambda$ consider the ILH chain

$$\{\mathbf{\Gamma}(S^2T^*M) \times \mathbf{\Omega}^2, \Gamma(S^2T^*M)^k \times \Omega^{2,k}, k \geq n+5\},$$

  and $\mathcal{GM}$ is then regarded as an open subspace of $\mathbf{\Gamma}(S^2T^*M) \times \mathbf{\Omega}^2$

- For the generalized diffeomorphisms, prove first that

$$O_\pi(E) \simeq_\lambda \mathrm{Diff}(M) \ltimes \Omega^2(M)$$

  is an ILH group (for which we choose a metric g on M). Note that choosing (g, λ) is actually choosing a generalized metric on E

- Use the ILH implicit function theorem for $\mathrm{Aut}(E) \simeq_\lambda \mathrm{GDiff}_H \subset \mathrm{O}_\pi(TM+T^*M)$
- Check that the ILH structure is independent from the splitting chosen

How cheap is it to be ILH?

Symplectomorphisms and volume-preserving
diffeomorphisms are ILH groups.

The groups of diffeomorphisms preserving
a foliation are **not** generally ILH.

L. Meersseman, M. Nicolau, J. Ribón. On the automorphism group of foliations with geometric transverse structures. Math. Z. 301, 1603–1630 (2022)

When the foliation is Riemannian,
or transversely holomorphic, they are.

Plan of the talk

I. Introduction to generalized geometry
(generalized diffeomorphisms and metrics)

II. Infinite-dimensional manifolds and groups

III. Slice theorem and the moduli
space of generalized metrics
(proved for exact and odd exact Courant algebroids)

The action

For an exact Courant algebroid E, denote $\mathrm{Aut}(E)$ by GDiff

We consider the action

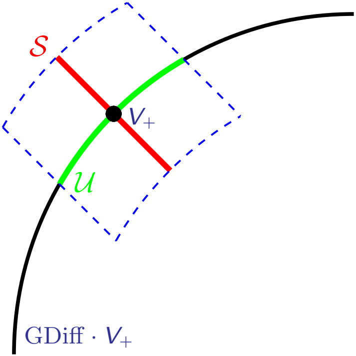
$$\begin{aligned} \mathrm{GDiff} \times \mathcal{GM} &\to \mathcal{GM} \\ F \cdot V_+ &\mapsto F^{-1}(V_+) \end{aligned}$$

For $V_+ \in \mathcal{GM}$, denote its isometries by $\mathsf{Isom}\, V_+ \subset \mathrm{GDiff}$

We want to study

$$\mathcal{GR} = \frac{\mathcal{GM}}{\mathrm{GDiff}}$$

Slice theorem



Theorem (R-Tipler)

For a generalized metric $V_+ \subset E$, there exists an ILH submanifold $\mathcal{S} \subset \mathcal{GM}$ s.t.

a) *For all $F \in \mathsf{Isom}\, V_+$, $F \cdot \mathcal{S} = \mathcal{S}$*

b) *If $(F \cdot \mathcal{S}) \cap \mathcal{S} \neq \emptyset$ for $F \in \mathrm{GDiff}$, then $F \in \mathsf{Isom}\, V_+$*

c) *There is $V_+ \in \mathcal{U} \subset \mathrm{GDiff} \cdot V_+$ and local section $\chi : \mathcal{U} \to \mathrm{GDiff}$ of action, such that $\mathcal{U} \times \mathcal{S} \to \mathcal{GM}$ given by $(V_1, V_2) \mapsto \chi(V_1) \cdot V_2$ is a homeomorphism onto its image*

(proved by Ebin for Diff acting on $\mathcal{M}$)

(but here, **no** elliptic operator theory available!)

The moduli of metrics $\mathcal{GR} = \mathcal{GM}/\mathrm{GDiff}$

Free and proper action of a compact group G on M gives manifold M/G

Possible to quotient by the isotropy group, but these are different in $\mathcal{GM}$
For some $W_+ \subset E$, denote $G = \mathrm{Isom}\, W_+$ and (G) its conjugacy class,

$$\mathcal{GM}_G := \{V_+ \in \mathcal{GM} \mid \mathrm{Isom}\, V_+ = G\}$$
$$\mathcal{GM}_{(G)} := \{V_+ \in \mathcal{GM} \mid \mathrm{Isom}\, V_+ \in (G)\}$$

We have a proper action

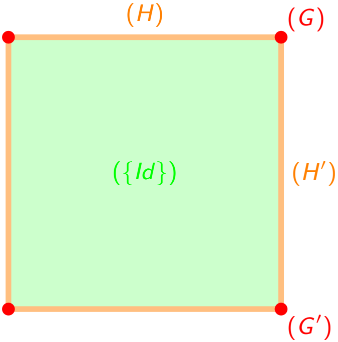
$$\mathrm{GDiff} \times \mathcal{GM}_{(G)} \to \mathcal{GM}_{(G)}$$

which has the same orbit space as the free and proper ILH action

$$G \backslash N_{\mathrm{GDiff}}(G) \times \mathcal{GM}_G \to \mathcal{GM}_G$$

So, by the slice theorem, each $\mathcal{GR}_{(G)} := \mathcal{GM}_{(G)}/\mathrm{GDiff}$ is an ILH manifold

Stratification of $\mathcal{GR} = \mathcal{GM}/\mathrm{GDiff}$



Theorem (R-Tipler)

For an exact Courant algebroid, there is a stratification of $\mathcal{GR}$ by ILH submanifolds

$$\mathcal{GR} = \bigcup_{(G)} \mathcal{GR}_{(G)},$$

which is a countable union, such that

$$\mathcal{GR}_{(G)} \cap \overline{\mathcal{GR}_{(H)}} \neq \emptyset \iff (H) \subseteq (G)$$
$$\iff \mathcal{GR}_{(G)} \subseteq \overline{\mathcal{GR}_{(H)}}$$

Moreover, there is a map

$$\mathcal{GR} = \frac{\mathcal{GM}}{\mathrm{GDiff}} \to \frac{\mathcal{GM}}{\mathrm{O}_\pi} \cong \frac{\mathcal{M}}{\mathrm{Diff}} = \mathcal{R},$$

preimage of a stratum is a union of strata

(stratification for $\mathcal{R}$ proved by Bourguignon)

Do we need ILH?

Bunk-Muñoz-Shahbazi: tame Fréchet category
(The local moduli space of the Einstein-Yang-Mills system, arXiv:2311.07572)
(Tame Fréchet plus extra condition is ILB?)

For Tame Fréchet is, in principle, easier to prove results **but** ILH gives us more structure. So once we have ILH, it is better to keep it!

Has ILH been any useful?

Kuan-Hui Lee: slice theorem for an f-twisted L^2-inner product.
He proved that any 3-dimensional generalized Einstein manifold (M, g, H) is linearly stable plus results about generalized Ricci solitons.

This led him to the stability of non-Kähler Calabi-Yau metrics (**next talk!**)

Claire Voisin, Crafoord Prize and BBVA Frontiers 2024

"When you start doing that, and you come back three years later, and tell the physicists, 'now I have proved your formula rigorously', they already went in another direction. Some mathematicians have stayed in contact with physics and have done extraordinary things. But for me it was not good, because I like to work alone and to ask my own questions."

Interview to Nature, 9th Feb 2024

Dirac / generalized complex geometry

$$\downarrow$$

Exact Courant algebroids

$$\downarrow$$

Odd exact Courant algebroids

$$\downarrow$$

B_n-generalized complex geometry
(in odd dimensions!)

New geometric structures on 3-manifolds:
surgery and generalized geometry

Joint work with Joan Porti, on the arXiv today

I hope to tell you about it soon.

Thank you for your attention!

These slides will be available on
mat.uab.es/$\sim$rubio