Courant connections

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Goals of the talk

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Introduce new invariants of *Courant algebroids*

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Introduce new invariants of *Courant algebroids*

This will include:

(A) Explicit/hands-on description of Courant differential complex and their associated cohomology (usual descriptions are abstract and hard to work with).

(B) Representations of Courant algebroids (via connections).

(C) Intrinsic Characteristic classes.

Analogous picture for Lie algebroids

(A) Let $(A, \rho, [\cdot, \cdot])$ be a Lie algebroid.

$$C^\infty(M) \to \Gamma A^* \to \Gamma \bigwedge^2 A^* \to \cdots$$

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is a complex with differential:

$$d\omega(a_0, \cdots, a_k) = \sum_{i=0}^{k} (-1)^i \rho(a_i)\omega(a_0, \cdots, \widehat{a}_i, \cdots, a_k) + \sum_{i<j} (-1)^{i+1} \omega(a_0, \cdots, \widehat{a}_i, \cdots, [a_i, a_j], \cdots, a_k)$$

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(B) Representations $\Longleftrightarrow$ vector bundles with a flat $A-$connection.

(C) Characteristic classes are obtained from the adjoint representation up to homotopy, as done by Crainic-Fernandes.

Motivating example of Courant algebroid

$(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], \rho = p_{TM})$ where

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - d i_Y \alpha$$

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In the complex case, this gives rise to Generalized Complex Geometry.

Definition of Courant algebroids

Vector Bundle $E \to M$ with the following structure:

- $\langle \cdot, \cdot \rangle$ nondegenerate symmetric pairing.
- $\rho : E \to TM$ a bundle map.
- $[\![\cdot, \cdot]\!]$ a bracket.

Satisfiyng:

1. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
2. $[\![e_1, fe_2]\!] = \rho(e_1)(f)e_2 + f[\![e_1, e_2]\!]$,
3. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
4. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2 \rangle$,

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is given by $\langle \mathcal{D}f, e \rangle = \rho(e)(f)$.

This is due to Liu-Weinstein-Xu, 1997.

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- Quadratic Lie algebras $(\mathfrak{g}, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho = 0)$

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Other motivations for Courant algebroids:

- Target of Topological Field Theories of AKSZ type: Chern-Simos theory.
- Quasi-Hamiltonian spaces, Poisson-Lie T-duality.
- Manin triples and Drinfield doubles.
- Vertex algebras.
- Higher homotopy structures.

Courant connections

Alekseev-Xu

$(E \to M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], \rho)$ Courant algebroid, $B \to M$ vector bundle.

An E-connection is $\nabla : \Gamma E \times \Gamma B \to \Gamma B$ such that

$$\nabla_{fe} b = f \nabla_e b \qquad \nabla_e (fb) = f \nabla_e b + \rho(e)(f) b.$$

The curvature is

$$F_\nabla(e_1, e_2)(b) = [\nabla_{e_1}, \nabla_{e_2}](b) - \nabla_{[\![e_1, e_2]\!]} b.$$

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Questions:

- Why $F_\nabla(fe_1, e_2) \neq f\ F_\nabla(e_1, e_2)$?
- Where does it lives?
- Cartan calculus for E?

Theorem (Roytenberg, Severa)

Courant algebroids are in one-to-one correspondence with degree 2 symplectic Q-manifolds.

$$(E, \langle \cdot \cdot \rangle, [\![\cdot, \cdot]\!], \rho) \Longleftrightarrow (\mathcal{M}, \{\cdot, \cdot\}, Q)$$

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Examples:

- $TM \oplus T^*M$ corresponds to $T^*[2]T[1]M$
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Questions:

- How to find an easy-to-work description?
- Any Cartan-type formula for the differential?

Keller-Waldmann Algebra

Given a vector bundle with nondegenerate pairing $(E \to M, \langle \cdot, \cdot \rangle)$, define k-cochains as maps

$$\omega : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k} \to C^{\infty}(M)$$

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- $C^\infty(M)$-linear in the last entry
- There exists a map

  $$\sigma_\omega : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k-2} \to \mathfrak{X}(M)$$

  such that

  $$\omega(e_1, \ldots, e_i, e_{i+1}, \ldots, e_k) + \omega(e_1, \ldots, e_{i+1}, e_i, \ldots, e_k)$$
  $$= \sigma_\omega(e_1, \overset{\wedge}{.}.\overset{\wedge}{.}., e_k)(\langle e_i, e_{i+1} \rangle)$$

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$$\begin{aligned} &\omega(e_1, \dots, e_i, e_{i+1}, \dots, e_k) + \omega(e_1, \dots, e_{i+1}, e_i, \dots, e_k) \\ &= \sigma_\omega(e_1, .\hat{.}..\hat{.}., e_k)(\langle e_i, e_{i+1} \rangle) \end{aligned}$$

The space of cochains is denoted $C^\bullet(E)$. It is a graded commutative algebra with a degree -2 Poisson bracket.

Courant algebroid differential

If $E \to M$ is a Courant algebroid, then

$$T(e_1, e_2, e_3) = \langle [\![e_1, e_2]\!], e_3 \rangle \in C^3(E).$$

Courant axioms $\Longleftrightarrow \{T, T\} = 0$

$d_E = \{T, \cdot\}$ is a differential on $C^\bullet(E)$.

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Result 1: Let $E \to M$ a Courant algebroid.
The dg-algebras $(\mathcal{O}_{\mathcal{M}}, Q)$ and $(C^\bullet(E), d_E)$ are ismorphic via

$$\Upsilon(\psi)(e_1, \cdots, e_k) = \{e_k, \{e_{k-1}, \cdots \{e_1, \psi\}, \cdots\}.$$

Keller-Waldmann introduced $C^\bullet(E)$ in an algebraic setting where the correspondence with Q-manifolds doesn't apply.

Cartan calculus

Result 2: The differential satisfies the Cartan formula

$$d_E\omega(e_0, \cdots, e_k) = \sum_{i=0}^{k}(-1)^i \rho(e_i)\omega(e_0, \cdots, \widehat{e_i}, \cdots, e_k) + \sum_{i<j}(-1)^{i+1}\omega(e_0, \cdots, \widehat{e_i}, \cdots, [\![e_i, e_j]\!], \cdots, e_k).$$

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If we introduce contractions and Lie derivatives

$$(\iota_e\omega)(e_1,\cdots,e_{k-1})=\omega(e,e_1,\cdots,e_{k-1}),\quad \mathcal{L}_e\omega=\{\{e,T\},\omega\}.$$

Result 3: The following Cartan relations hold

$$\begin{array}{rclcrcl} d_E^2 &=& 0 & & [\mathcal{L}_e,d_E] &=& 0\\ [\iota_e,d_E] &=& \mathcal{L}_e & & [\mathcal{L}_e,\mathcal{L}_{e'}] &=& \mathcal{L}_{[\![e,e']\!]}\\ [\mathcal{L}_e,\iota_{e'}] &=& \iota_{[\![e,e']\!]} & & & & \end{array}$$

Warning: $[\iota_e,\iota_{e'}]\neq 0$.

E-connections

For ∇ an E-connection on B define $D^{\nabla} : C^{\bullet}(E;B) \to C^{\bullet+1}(E;B)$

$$D^{\nabla}\omega(e_0, \cdots, e_k) = \sum_{i=0}^{k} (-1)^i \nabla_{e_i}\omega(e_0, \cdots, \widehat{e_i}, \cdots, e_k) + \sum_{i<j} (-1)^{i+1}\omega(e_0, \cdots, \widehat{e_i}, \cdots, [\![e_i, e_j]\!], \cdots, e_k)$$

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Result 4: There is a correspondence between:

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- D differentials on $C^\bullet(E;B)$ with $D(\omega\smile\tau)=d_E\omega\smile\tau+(-1)^k\omega\smile D\tau$.

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Result 5: For ∇ an E-conenction on B the curvature F_{∇} satisfy

- $F_{\nabla} \in C^2(E;\mathrm{End}(B))$.
- The Bianchi identity $D^{\widetilde{\nabla}}F_{\nabla} = 0$.

Adjoint connections I

Let $\hat{\nabla}$ be a linear connection on E. Then we can define an E-connection ∇^E on E by:

$$\nabla^E_{e_1} e_2 = [\![e_1, e_2]\!] + \hat{\nabla}_{\rho(e_2)} e_1 - \rho^* \langle D^{\hat{\nabla}} e_1, e_2 \rangle.$$

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∇^E is compatible with the pairing:

$$\langle \nabla^E_{e_1} e_2, e_3 \rangle + \langle e_2, \nabla^E_{e_1} e_3 \rangle = \rho(e_1) \langle e_2, e_3 \rangle \qquad (**)$$

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Examples:

- $E = (\mathfrak{g}, [\cdot,\cdot], \langle \cdot,\cdot \rangle) \rightsquigarrow \nabla^E_{e_1} e_2 = [e_1, e_2] = ad_{e_1}(e_2)$
- $E = TM \oplus_H T^*M \rightsquigarrow$ Pick ∇ Torsion free on TM and

$$\hat{\nabla}_X Y + \beta = \nabla_X Y + \nabla^\dagger_X \beta + \frac{1}{2} i_X i_Y H \quad \text{so} \quad \nabla^E_{X+\alpha} Y + \beta = \nabla_X Y + \nabla^\dagger_X \beta$$

Adjoint connections II

Let $\hat{\nabla}$ be a linear connection on E. We also have E-connections on TM and T^*M by:

$$\nabla_e^{TM} X = [\rho(e), X] + \rho(\hat{\nabla}_X e)$$

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We easily check thath ∇^{T^*M} is the dual connection of ∇^{TM}.

$\nabla^{\mathbb{T}M} = \nabla^{TM} + \nabla^{T^*M}$ is a self-dual E-connection on $TM \oplus T^*M$.

1. The modular class

Modular Class

Let $B \to M$ be a vector bundle with *rank* $B = 1$ and ∇^B a flat $E-$connection. Then B defines a cohomology class in $H^1(E)$. We call it the modular class of B, denoted by $[\psi^B] \in H^1(E)$.

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The section $\psi \in \Gamma E$ is defined via

$$\nabla^B_e \lambda = g_e \lambda, \text{ where } g_e = \langle \psi, e \rangle.$$

∇^B Flat $\Longrightarrow d_E \psi = 0$.

$\bigwedge^{top} E$ has a flat $E-$connection

$$\nabla^{top}_e \omega = \mathcal{L}_e \omega$$

Unimodularity

Proposition 1

Any Courant algebroid is unimodular: $[\psi^{\Lambda^{top} E}] = 0$.

Proof of 1: $\nabla_e^{top}(e_1 \wedge \cdots \wedge e_m) = \sum_i e_1 \wedge \cdots \wedge \nabla_e^E(e_i) \wedge \cdots \wedge e_m$

Because $\langle \nabla^E_{e_1} e_2 - [\![e_1, e_2]\!], e_3 \rangle = \langle \widehat{\nabla}_{\rho(e_2)} e_1, e_3 \rangle - \langle \widehat{\nabla}_{\rho(e_3)} e_1, e_2 \rangle$ is tensorial and vanish when extended to $\wedge^{top} E$.
Therefore $(**) \Rightarrow \nabla^{top}$ self-adjoint so

$$[\psi^{\wedge^{top} E}] = [\psi^{\wedge^{top} E^*}] = -[\psi^{\wedge^{top} E}] \quad \Rightarrow \quad [\psi^{\wedge^{top} E}] = 0.$$

Stiénon-Xu had already defined this class (not noticing it vanishes).

2. Primary classes

Primary characteristic classes

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Proposition

For a Courant algebroid $Ch_k(E) = Ch_k(TM \oplus T^*M)$.

So there is nothing new (just as for Lie algebroids)

3. Secondary classes

Transgressions

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- The 2-form
$$\beta_k|\nabla(\dot{\nabla}_0, \dot{\nabla}_1) = k \sum_{i=0}^{k-2} tr(\dot{\nabla}_0 \cup F_\nabla^i \cup \dot{\nabla}_1 \cup F_\nabla^{k-i-2})$$

They satisfy: $d_E Ch_k = 0, \quad d_E \alpha_k = \delta Ch_k, \quad d_E \beta_k = \delta \alpha_k$

Secondary characteristic classes

Given two E-connections ∇_0, ∇_1, we can produce Chern-Simons-type transgression forms

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As in the classical theory,

$$Ch_k(\nabla_0) - Ch_k(\nabla_1) = \mathrm{tr}(F^k_{\nabla_1}) - \mathrm{tr}(F^k_{\nabla_0}) = d_E \mathrm{cs}_k(\nabla_0, \nabla_1)$$

so, when the primary cocycles vanish, the transgression forms are closed. Therefore if $Ch_k(\nabla_0) = Ch_k(\nabla_1) = 0$ we obtain cohomology classes

$$[cs_k(\nabla_0, \nabla_1)] \in H^{2k-1}(E).$$

Intrinsic secondary characteristic classes

Given $(E \to M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$, make the following choices on E:

- a linear connection $\hat{\nabla}$
- a *positive definite* metric g

Then we can define the adjoint connections ∇^E and $\nabla^{E,g}$.

$$(**) \quad \Rightarrow \quad Ch_l(E) = tr(F^l_{\nabla^E}) = 0 \quad \text{for} \quad l = 2k-1$$

Hence we can define classes

$$[\mathrm{cs}_k(\nabla^E, \nabla^{E,g})] \in H^{4k-3}(E).$$

Theorem

The classes $[\mathrm{cs}_k(\nabla^E, \nabla^{E,g})] \in H^{4k-3}(E)$ *are independent of the choices.*

Examples

- For $E = (\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ we have

$$\mathrm{cs}_k(ad, ad^g)(e_1, \dots, e_{2k-1}) = \sum_{\pi \in S_{2k-1}} (\pi)\,\mathrm{tr}\left(ad_{e_{\pi(1)}} \cdots ad_{e_{\pi(2k-1)}}\right)$$

- For $E = TM \oplus_H T^*M$ we have that for a torsion free connection

$$\nabla^E_{X+\alpha} Y + \beta = \nabla_X Y + \nabla^\dagger_X \beta$$

Therefore, pic g a Riemaniann metric on M and $\nabla = \nabla^{LC}$ then

$$\nabla^E = \nabla^{E,g} \quad \Rightarrow [\mathrm{cs}_k(\nabla^E, \nabla^{E,g})] = 0$$

Rmk: We think that there are E with $[\mathrm{cs}_k(\nabla^E, \nabla^{E,g})] \neq 0$ for some k, so far no examples.

Thanks !!