Riemannian geometry on Courant algebroids

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Outline of talk

I. Courant algebroids

- definition, examples, basic properties,
- weaker versions of Courant algebroid;

II. Riemannian geometry on Courant algebroids

- connections, divergences, torsion, generalized metrics, Levi-Civita connections,
- special emphasis on curvature;

III. Relation to supergravity

- reformulation of supergravity equations,
- Palatini variation.

Courant algebroids

Definition (Liu, Weinstein, Xu – 1997)

A **Courant algebroid** over M is a 4-tuple $(E, \rho, \langle\ ,\ \rangle, [\ ,\ ])$, consisting of

- a vector bundle $E \to M$,
- a vector bundle morphism $(\rho, \mathrm{id}_M) \in \mathrm{Hom}(E, TM)$
- a non-degenerate symmetric pairing $\langle\ ,\ \rangle \in \Gamma(\odot^2 E^*)$,
- an $\mathbb{R}$-bilinear map $[\ ,\ ] : \times^2\Gamma(E) \to \Gamma(E)$,

that is subject to the following axioms:

(Ca1) $[a, a] = \frac{1}{2}\mathcal{D}\langle a, a\rangle$,

(Ca2) $\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$,

(Ca3) $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$,

where

$$\mathcal{D} := \underbrace{\sharp \circ \rho^t}_{=:\rho^*} \circ \mathrm{d} : C^\infty(M) \to \Gamma(E).$$

For $M = \{*\}$, Courant algebroids are just quadratic Lie algebras.

It follows from (Ca1) and (Ca2) that

(Ca4) $[a, fb] = (\rho(a)f)\, b + f[a, b]$,

(Ca5) $[fa, b] = -(\rho(b)f)\, a + f[a, b] + \langle a, b\rangle\mathcal{D}f$.

Courant algebroids with $\rho = 0$ are bundles of quadratic Lie algebras

$$[\ ,\ ] \in \Gamma(\wedge^2 E^* \otimes E).$$

Courant algebroids - basic properties

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0$$

Example: exact Courant algebroids

Given $H \in \Gamma_{\text{closed}}(\wedge^3 T^*)$, the 4-tuple $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$, where

- $\langle X + \alpha, Y + \beta \rangle_+ := \alpha(Y) + \beta(X)$,
- $[X + \alpha, Y + \beta]^H := [X, Y]_{\text{Lie}} + \pounds_X \beta - \iota_Y \mathrm{d}\alpha + \iota_X \iota_Y H$

is a Courant algebroid.

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The axiom (Ca3) (Jacobi-like identity) can be weakened:

Courant algebroids - basic properties

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0$$

Example: pre-Courant algebroids

Given $H \in \Gamma_{\cancel{\text{closed}}}(\wedge^3 T^*)$, the 4-tuple $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$, where

- $\langle X+\alpha, Y+\beta\rangle_+ := \alpha(Y) + \beta(X)$,
- $[X+\alpha, Y+\beta]^H := [X,Y]_{\text{Lie}} + \mathcal{L}_X\beta - \iota_Y \mathrm{d}\alpha + \iota_X\iota_Y H$

is a pre-Courant algebroid.

The axiom (Ca3) (Jacobi-like identity) can be weakened:

(Ca6) $\rho([a,b]) = [\rho(a), \rho(b)]_{\text{Lie}}$ (pre-Courant algebroid),

Courant algebroids - basic properties

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0$$

Example: weak Courant algebroids

Given $H \in \Gamma_{\cancel{\text{closed}}}(\wedge^3 T^*)$, the 4-tuple $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$, where

- $\langle X + \alpha, Y + \beta \rangle_+ := \alpha(Y) + \beta(X)$,
- $[X + \alpha, Y + \beta]^H := \nabla_X Y - \nabla_Y X + \pounds^{\nabla}_X \beta - \iota_Y \mathrm{d}^{\nabla} \alpha + \iota_X \iota_Y H \qquad (T_{\nabla} \neq 0)$

is a weak Courant algebroid.

The axiom (Ca3) (Jacobi-like identity) can be weakened:

(Ca6) $\rho([a, b]) = [\rho(a), \rho(b)]_{\mathsf{Lie}}$ (pre-Courant algebroid),

(Ca7) There is $F \in \Gamma(\wedge^2 T^* \otimes T)$ s.t. (weak Courant algebroid)

$$\rho([a, b]) - [\rho(a), \rho(b)]_{\mathsf{Lie}} = F(\rho(a), \rho(b)),$$

Courant algebroids - basic properties

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0$$

Example: ante Courant algebroids

Given $H \in \Gamma_{\cancel{\text{closed}}}(\wedge^3 T^*)$, the 4-tuple $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$, where

- $\langle X + \alpha, Y + \beta \rangle_+ := \alpha(Y) + \beta(X)$,
- $[X + \alpha, Y + \beta]^H := [X, Y]_{\text{Lie}} + \iota_\alpha \iota_\beta \chi + \pounds_X \beta - \iota_Y \mathrm{d}\alpha + \iota_X \iota_Y H \quad (\chi \in \Gamma(\wedge^3 T))$

is an ante Courant algebroid.

The axiom (Ca3) (Jacobi-like identity) can be weakened:

(Ca6) $\rho([a, b]) = [\rho(a), \rho(b)]_{\text{Lie}}$ (pre-Courant algebroid),

(Ca7) There is $F \in \Gamma(\wedge^2 T^* \otimes T)$ s.t. (weak Courant algebroid)

$$\rho([a, b]) - [\rho(a), \rho(b)]_{\text{Lie}} = F(\rho(a), \rho(b)),$$

(Ca8) $\rho \circ \rho^* = 0$ (ante-Courant algebroid).

Courant algebroids - basic properties

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0$$

Example: exact Courant algebroids

Given $H \in \Gamma_{\text{closed}}(\wedge^3 T^*)$, the 4-tuple $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$, where

- $\langle X + \alpha, Y + \beta \rangle_+ := \alpha(Y) + \beta(X)$,
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is a Courant algebroid.

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$$\rho([a, b]) - [\rho(a), \rho(b)]_{\text{Lie}} = F(\rho(a), \rho(b)),$$

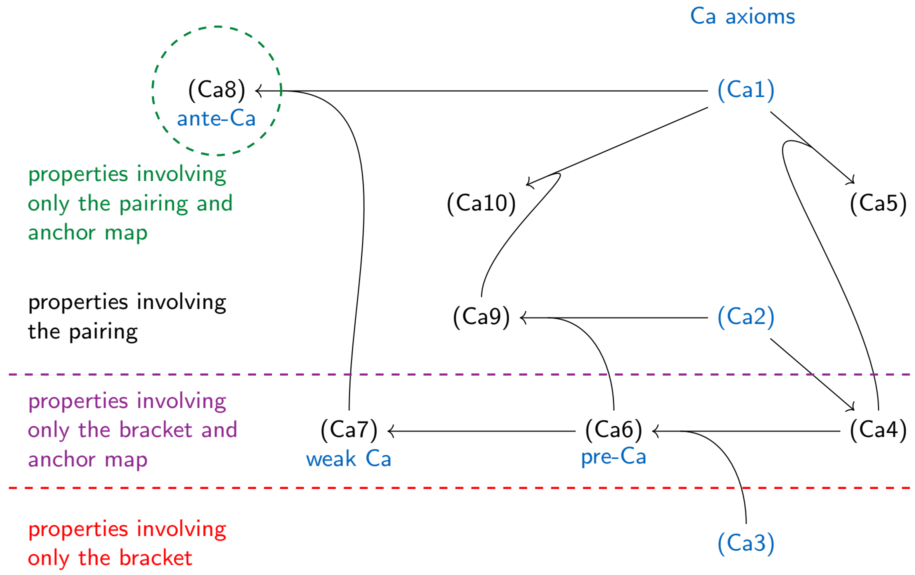
(Ca8) $\rho \circ \rho^* = 0$ (ante-Courant algebroid).

In the framework of a pre-Courant algebroid, the following is true:

(Ca9) $[a, \rho^* \beta] = \rho^* \pounds_{\rho(a)} \beta$,

(Ca10) $[\rho^* \alpha, b] = -\rho^* \iota_{\rho(b)} \mathrm{d}\alpha$.

Courant algebroids - properties diagram



Courant algebroids - examples

Example: Heterotic Courant algebroids

Given a G-bundle $P \to M$, a Ca structure on $TM \oplus \operatorname{ad} P \oplus T^*M$ is induced by

- a non-degenerate invariant symmetric paring $\langle\ ,\ \rangle_{\mathfrak{g}} \in \odot^2 \mathfrak{g}^*$,
- a connection $\mathcal{A}$ on P and $H \in \Gamma(\wedge^3 T^*)$ s.t. $\mathrm{d}H + \frac{1}{2}\langle \mathcal{F} \wedge \mathcal{F} \rangle_{\mathfrak{g}} = 0$.

Example: Courant algebroids induced by ∇

A torsion-free affine connection $\nabla : \times^2 \Gamma(TQ) \to \Gamma(TQ)$ gives us $TT^*Q = H \oplus V$

$$\begin{tikzcd}
TT^*Q = H \oplus V \arrow[d] & \simeq & \pi^*(TQ \oplus T^*Q) \arrow[d] & TQ \oplus T^*Q \arrow[l, "\pi^*"'] \arrow[d] \\
T^*Q & = & T^*Q \arrow[r, "\pi"] & Q
\end{tikzcd}$$

The 4-tuple $(TT^*Q, \mathrm{pr}_H, g, [\ ,\ ])$, where

- $g \in \Gamma(\odot^2 T^*(T^*Q))$ (Patterson-Walker metric, 1952)

$$g|_U = \mathrm{d}p_j \odot \mathrm{d}x^j - p_k \Gamma^k{}_{ij} \mathrm{d}x^i \odot \mathrm{d}x^j, \qquad \boxed{\omega_{\mathsf{can}}|_U = \mathrm{d}p_j \wedge \mathrm{d}x^j},$$

- $[h+v, h'+v'] := \mathrm{pr}_H [h, h']_{\mathsf{Lie}} + \mathrm{pr}_V g^{-1}(\pounds_h g(v') - \iota_{h'} \mathrm{d}g(v))$,

is a weak Ca. It is a Ca iff ∇ is flat. (it can be seen as a Lie bialgebroid)

Courant algebroid connections

Definition (Alekseev, Xu –2001, Gualtieri – 2007)

A **Ca connection** on E is an $\mathbb{R}$-bilinear map $\nabla : \times^2\Gamma(E) \to \Gamma(E)$ s.t.

(Cac1) $\nabla_{fa} b = f\nabla_a b$,

(Cac2) $\nabla_a(fb) = (\rho(a)f)\, b + f\nabla_a b$,

(Cac3) $\rho(a)\langle b, c\rangle = \langle \nabla_a b, c\rangle + \langle b, \nabla_a c\rangle$ (i.e. $\nabla\langle\ ,\ \rangle = 0$).

Given a vb connection $\nabla^{\mathrm{vb}} : \Gamma(T) \times \Gamma(E) \to \Gamma(E)$ compatible with $\langle\ ,\ \rangle$,

$$\nabla : \times^2\Gamma(E) \to \Gamma(E) : (a, b) \mapsto \nabla^{\mathrm{vb}}_{\rho(a)} b,$$

is a Ca connection.

The space of Ca connections is an affine space modeled on $\Gamma(E^* \otimes (\wedge^2 E^*))$.

Example: Ca connections on $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$

For any affine connection $\nabla^{\mathrm{aff}} : \times^2\Gamma(T) \to \Gamma(T)$,

$$\nabla_{X+\alpha}(Y + \beta) := \nabla^{\mathrm{aff}}_X Y + \nabla^{\mathrm{aff}}_X \beta$$

is compatible with $\langle\ ,\ \rangle_+ \quad \Rightarrow \quad \nabla$ is a Ca connection.

Divergences

Definition (Alekseev, Xu – 2001, Garcia-Fernandez – 2019)

A **divergence** on E is an $\mathbb{R}$-linear map $\operatorname{div} : \Gamma(E) \to C^\infty(M)$ s.t.

$$\operatorname{div}(fa) = \rho(a)f + f \operatorname{div} a.$$

A Ca connection ∇ induces the divergence on E:

$$\operatorname{div}_\nabla a := \operatorname{Tr} \nabla a.$$

Conversely (if $\operatorname{Rk} E \neq 1$), for a divergence div on E, there is a Ca connection s.t.

$$\operatorname{div}_\nabla = \operatorname{div}.$$

It is enough to modify an arbitrary Ca connection ∇^0 by $C \in \Gamma(E^* \otimes (\wedge^2 E^*))$,

$$C(a,b,c) := \frac{1}{\operatorname{Rk} E - 1}(\langle a,b\rangle(\operatorname{div}_{\nabla^0} c - \operatorname{div} c) - \langle a,c\rangle(\operatorname{div}_{\nabla^0} b - \operatorname{div} b)).$$

Example: Divergence induced by a volume form

Given a volume form on the base manifold $\mathrm{vol} \in \Gamma(\det T^*)$,

$$\operatorname{div}_{\mathrm{vol}} a := (\pounds_{\rho(a)} \mathrm{vol}) \cdot \mathrm{vol}^{-1}$$

is a divergence on E.

Torsion

"Naive torsion" of a Ca connection ∇:

$$\nabla_a b - \nabla_b a - [a, b]$$

is **not tensorial**.

Definition (Gualtieri – 2007)

Torsion tensor of a Ca connection ∇ is the 3-form $T_\nabla \in \Gamma(\wedge^3 E^*)$:

$$T_\nabla(a, b, c) := \langle \nabla_a b - \nabla_b a - [a, b], c\rangle + \langle \nabla_c a, b\rangle.$$

If $T_\nabla = 0$, ∇ is called **torsion-free** (T-free).

The space of T-free Ca connections is modeled on

$$\Gamma\left({(E^* \otimes (\wedge^2 E^*))}/{\wedge^3 E^*} \right).$$

For a Ca connection ∇^0, there is the associated T-free Ca connection: $\nabla = \nabla^0 - \frac{1}{3} T_{\nabla^0}$.

Example: Torsion of Ca connections on $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$

A T-free affine connection ∇^{aff} **does not** lead to a T-free Ca connection ∇^0.

We have $T_{\nabla^0} = \rho^t H \quad \Rightarrow \quad \nabla_{X+\alpha}(Y+\beta) := \nabla^{\mathrm{aff}}_X Y + \nabla^{\mathrm{aff}}_X \beta - \frac{1}{3}\iota_Y \iota_X H$

(T-free Ca connection).

Curvature

"Naive curvature" of a Ca connection ∇:

$$R^0_\nabla(a,b) := \nabla_a\nabla_b - \nabla_b\nabla_a - \nabla_{[a,b]}$$

is **not tensorial**.

Definition (Jurčo, Vysoký – 2017; motivated by Hohm, Zwiebach – 2013)

Curvature tensor of a Ca connection ∇ is the 4-tensor $R_\nabla \in \Gamma(\otimes^4 E^*)$:

$$R_\nabla(a,b,c,d) := \frac{1}{2}\big(\langle R^0_\nabla(c,d)b,a\rangle + \langle R^0_\nabla(b,a)c,d\rangle + \mathrm{Tr}_{\langle\;,\;\rangle}\langle\nabla_\star c,d\rangle\langle\nabla_\star b,a\rangle\big).$$

The curvature tensor has the following symmetries:

- $R_\nabla(a,b,c,d) = -R_\nabla(a,b,d,c)$,
- $R_\nabla(a,b,c,d) = -R_\nabla(b,a,c,d)$,
- $R_\nabla(a,b,c,d) = R_\nabla(c,d,a,b)$.

An analogue of algebraic Bianchi identity:

$$\begin{aligned}&R_\nabla(a,b,c,d) + \mathrm{cyc}(b,c,d)\\ &\qquad = \frac{1}{2}\big((\nabla_b T_\nabla)(c,d,a) + T_\nabla(a,T_\nabla(b,c),d) + \mathrm{cyc}(b,c,d)\big) - \frac{1}{2}(\nabla_a T_\nabla)(b,c,d).\end{aligned}$$

For ∇ T-free: $R_\nabla(a,b,c,d) + \mathrm{cyc}(b,c,d) = 0$.

Curvature - Ricci tensor and Ricci scalar

Definition

Ricci tensor of a Ca connection ∇ is the 2-tensor $\mathrm{Ric}_\nabla \in \Gamma(\odot^2 E^*)$:

$$\mathrm{Ric}_\nabla(a, b) := \mathrm{Tr}_{\langle\ ,\ \rangle} R_\nabla(\star, a, \star, b).$$

Thanks to the symmetries, Ric_∇ is (up to sign) the unique partial trace of R_∇.

Definition

Ricci scalar of a Ca connection ∇ is the smooth function $\mathcal{R}_\nabla \in C^\infty(M)$:

$$\mathcal{R}_\nabla := \mathrm{Tr}_{\langle\ ,\ \rangle} \mathrm{Ric}_\nabla.$$

Example: Curvature of Ca connecions on $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$

Let ∇ be a Ca connection given by an affine connection ∇^{aff}.

- $$R_\nabla(X+\alpha, Y+\beta, Z+\gamma, V+\omega) = \frac{1}{2}\big(R_{\nabla^{\mathsf{aff}}}(\alpha, Y, Z, V) - R_{\nabla^{\mathsf{aff}}}(\beta, X, Z, V) + R_{\nabla^{\mathsf{aff}}}(\gamma, V, X, Y) - R_{\nabla^{\mathsf{aff}}}(\omega, Z, X, Y)\big),$$
- $\mathrm{Ric}_\nabla(X+\alpha, Y+\beta) = \dfrac{1}{2}(\mathrm{Ric}_{\nabla^{\mathsf{aff}}}(X, Y) + \mathrm{Ric}_{\nabla^{\mathsf{aff}}}(Y, X)),$
- $\mathcal{R}_\nabla = 0.$

Generalized metrics

Definition (Gualtieri – 2004)

A **generalized metric** is a vec. subbundle $V_+ \leq E$ s.t. $\langle\ ,\ \rangle|_{V_+}$ is non-degenerate.

A **Riemannian generalized metric** is a vec. subbundle $V_+ \leq E$ s.t.

1. $\langle\ ,\ \rangle|_{V_+}$ is positive definite,
2. it is maximal subbundle satisfying 1.

Choosing a generalized metric induces

- the vector bundle decomposition:
$$E = V_+ \oplus V_-,$$
where $V_- := V_+^{\perp}$;
- the non-degenerate (positive definite) symmetric pairing $\mathbf{G} \in \Gamma(\odot^2 E^*)$:
$$\mathbf{G}(a,b) := \langle a_+, b_+\rangle - \langle a_-, b_-\rangle.$$

Example: Generalized metrics on $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$

Consider a (pseudo-)Riemannian metric $g \in \Gamma(\odot^2 T^*)$ and $B \in \Gamma(\wedge^2 T^*)$.

- $\mathrm{gr}(g+B)$ is a generalized metric.
- **Riemannian** generalized metrics $\xleftrightarrow[\mathrm{gr}(g+B)]{1:1}$ (g, B) for g **Riemannian**.

Levi-Civita CA connections

Definition (Garcia-Fernandez – 2014)

A Ca connection ∇ is called **Levi-Civita** w.r.t a generalized metric V_+ if

1. $\nabla_a(\Gamma(V_+)) \subseteq \Gamma(V_+) \quad \Leftrightarrow \quad \nabla \mathbf{G} = 0$ (compatible with V_+),
2. $T_\nabla = 0$ (T-free).

We write $\nabla \in \mathrm{LC}(V_+)$.

Given a generalized metric V_+, there **exists** $\nabla \in \mathrm{LC}(V_+)$.

$\mathrm{LC}(V_+)$ is an affine space modeled on $\Gamma\left({}^{(V_+^* \otimes (\wedge^2 V_+^*))}\!/\!_{\wedge^3 V_+^*} \oplus {}^{(V_-^* \otimes (\wedge^2 V_-^*))}\!/\!_{\wedge^3 V_-^*}\right)$:

$$\mathrm{Rk}\,\mathrm{LC}(V_+) = \frac{1}{3}p(p^2-1) + \frac{1}{3}q(q^2-1), \qquad \text{where } p := \mathrm{Rk}\,V_+,\ q := \mathrm{Rk}\,V_-.$$

Unless $p \in \{0,1\}$ and $q \in \{0,1\}$, the set $\mathrm{LC}(V_+)$ is infinite.

$\Rightarrow$ In general, Levi-Civita Ca connection w.r.t. given V_+ is **not unique**.

We write $\nabla \in \mathrm{LC}(V_+, \mathrm{div})$ for ∇ being Levi-Civita w.r.t. V_+ and $\mathrm{div}_\nabla = \mathrm{div}$.

If $p \neq 1$ and $q \neq 1$, there **exists** $\nabla \in \mathrm{LC}(V_+, \mathrm{div})$. $\mathrm{LC}(V_+, \mathrm{div})$ is an affine space,

$$\mathrm{Rk}\,\mathrm{LC}(V_+, \mathrm{div}) = \frac{1}{3}p(p^2-4) + \frac{1}{3}q(q^2-4).$$

Levi-Civita CA connections - example

Starting with a Ca connection ∇^0 compatible with V_+, one can construct $\nabla \in \mathrm{LC}(V_+)$:

$$\langle \nabla_a b, c\rangle := \langle \nabla^0_a b, c\rangle - \frac{1}{3}\big(T_{\nabla^0}(a_+, b_+, c_+) + T_{\nabla^0}(a_-, b_-, c_-)\big) - T_{\nabla^0}(a_+, b_-, c_-) - T_{\nabla^0}(a_-, b_+, c_+)$$

Example: Levi-Civita Ca connection on $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$

Let $g \in \Gamma(\odot^2 T^*)$ be a (pseudo-)Riemannian metric $\rightsquigarrow$ $V_+ := \mathrm{gr}(g) \leq T \oplus T^*$.

- Levi-Civita affine connection ${}^g\nabla$ $\rightsquigarrow$ Ca connection ∇^0 compatible with V_+:

$$\nabla^0_{X+\alpha} = \begin{pmatrix} {}^g\nabla_X & 0 \\ 0 & {}^g\nabla_X \end{pmatrix}, \qquad T_{\nabla^0} = \rho^t H, \qquad \mathrm{div}_{\nabla^0} = \mathrm{div}_g \circ \rho.$$

$$\rightsquigarrow$$

- $\nabla \in \mathrm{LC}(V_+)$:

$$\nabla_{X+\alpha} = \begin{pmatrix} {}^g\nabla_X + \frac{1}{6} g^{-1}(\iota_{(g^{-1}\alpha)} H)(\star,\ ) & -\frac{1}{3} g^{-1}(\iota_X H)(g^{-1}\star,\ ) \\ -\frac{1}{3}(\iota_X H)(\star,\ ) & {}^g\nabla_X + \frac{1}{6}(\iota_{(g^{-1}\alpha)} H)(g^{-1}\star,\ ) \end{pmatrix},$$

$$\mathrm{div}_\nabla = \mathrm{div}_g \circ \rho.$$

Curvature revisited

Consider V_+, vol and $\nabla \in \mathrm{LC}(V_+, \mathrm{div}_{\mathrm{vol}})$. Then

$\mathcal{R}_\nabla \in C^\infty(M)$ **does not depend** on the choice of V_+, vol, and $\nabla \in \mathrm{LC}(V_+, \mathrm{div}_{\mathrm{vol}})$.

$\Rightarrow$ There is a canonical smooth function on every Ca, we denote it by $\mathcal{R} \in C^\infty(M)$.

Example: Canonical Ricci scalars

- q. Lie algebra: $\mathcal{R} = -\frac{1}{6}\dim \mathfrak{g}$,
- exact: $\mathcal{R} = 0$,
- heterotic: $\mathcal{R} = -\frac{1}{6}\dim G$.

Definition

Consider a Ca connection ∇ and a generalized metric V_+.

Metric Ricci scalar is the smooth function $\mathcal{R}_{\nabla, V_+} \in C^\infty(M)$:

$$\mathcal{R}_{\nabla, V_+} := \mathrm{Tr}_{\mathrm{G}} \mathrm{Ric}_\nabla .$$

Metric Ricci tensor is the 2-tensor $\mathrm{Ric}_{\nabla, V_+} \in \Gamma(\odot^2 E^*)$:

$$\mathrm{Ric}_{\nabla, V_+}(a, b) := \mathrm{Tr}_{\mathrm{G}} R_\nabla(\star, a, \star, b).$$

Double metric Ricci scalar is the smooth function $\mathcal{R}_{\nabla, V_+^2} \in C^\infty(M)$:

$$\mathcal{R}_{\nabla, V_+^2} := \mathrm{Tr}_{\mathrm{G}} \mathrm{Ric}_{\nabla, V_+} .$$

Altogether we have 7 curvature quantities: R_∇, Ric_∇, $\mathrm{Ric}_{\nabla, V_+}$, $\mathcal{R}_\nabla$, $\mathcal{R}_{\nabla, V_+}$, $\mathcal{R}_{\nabla, V_+^2}$, $\mathcal{R}$.

Curvature - example

Example: Curvature of $\nabla \in \mathrm{LC}(\mathrm{gr}(g+B))$ on $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$

Every $\nabla \in \mathrm{LC}(\mathrm{gr}(g+B))$ can be uniquely described by a pair of tensor fields

$$K, L \in \Gamma(T^* \otimes (\wedge^2 T^*)) \qquad \text{s.t. } \mathrm{Skew}\, K = \mathrm{Skew}\, L = 0.$$

Denoting

$$\mathcal{K} := \mathrm{Tr}_g\, K(\star, \star,\ ) \in \Gamma(T^*), \qquad \mathcal{L} := \mathrm{Tr}_g\, L(\star, \star,\ ) \in \Gamma(T^*),$$

we have ($H' := H - \mathrm{d}B$):

- $\mathcal{R}_{\nabla, V_+} = \mathcal{R}_{{}^g\nabla, g} - \frac{1}{2}(H', H')_g + 4\,\mathrm{div}_g\, \mathcal{L} - 4\, \|\mathcal{L}\|_g - 4\, \|\mathcal{K}\|_g$,
- $\mathcal{R}_{\nabla, V_+^2} = 4\,\mathrm{div}_g\, \mathcal{K} - 8\, g^{-1}(\mathcal{K}, \mathcal{L})$,
- $\mathrm{Ric}_\nabla|_{V_+ \otimes V_-} = 0$ iff

$$0 = \mathrm{Ric}_{{}^g\nabla}(X, Y) - \frac{1}{2}(\iota_X H', \iota_Y H')_g + ({}^g\nabla^s \mathcal{L})(X, Y),$$

$$0 = \frac{1}{2}(\delta_g H')(X, Y) + H'(X, Y, g^{-1}\mathcal{L}) + (\mathrm{d}\mathcal{K})(X, Y),$$

- $\mathrm{Ric}_{\nabla, V_+}\big|_{V_+ \otimes V_-} = 0$ iff

$$0 = ({}^g\nabla^s \mathcal{K})(X, Y),$$

$$0 = H'(X, Y, g^{-1}\mathcal{K}) + (\mathrm{d}\mathcal{L})(X, Y).$$

Type IIB supergravity equations

Bosonic sector of type IIB supergravity equations without RR–fields ($H' := H - \mathrm{d}B$):

$$0 = \mathrm{Ric}_{^g\nabla}(X,Y) - \frac{1}{2}(\iota_X H', \iota_Y H')_g + ({}^g\nabla^s \mathrm{d}\phi)(X,Y), \tag{1}$$

$$0 = \frac{1}{2}(\delta_g H')(X,Y) + H'(X,Y,\operatorname{grad}\phi), \tag{2}$$

$$0 = \mathcal{R}_{^g\nabla, g} - \frac{1}{2}(H', H')_g + 4\Delta_g\phi - 4\left\|\mathrm{d}\phi\right\|_g. \tag{3}$$

The variables are:

- a Lorentzian metric $g \in \Gamma(\odot^2 T^*)$,
- a 2-form $B \in \Gamma(\wedge^2 T^*)$ (Kalb-Ramond field or B-field),
- a smooth function $\phi \in C^\infty(M)$ (dilaton).

Ca reformulation of (1) - (3):

1. Take the exact Ca $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$.
2. Consider the equations ($\nabla \in \mathrm{LC}(V_+, \mathrm{div}_{\mathrm{vol}})$ is arbitrary):

$$\mathrm{Ric}_\nabla|_{V_+ \oplus V_-} = 0, \qquad \mathcal{R}_{\nabla, V_+} + \mathcal{R} = 0.$$

The variables are:

- a generalized metric $V_+ \leq T \oplus T^*$,
- a volume form $\mathrm{vol} \in \Gamma(\det T^*)$.

3. Look only for physically relevant solutions ($V_+ = \mathrm{gr}(g + B)$ for g Lorentzian).

Coimbra, Strickland-Constable, Waldram; Garcia-Fernandez; Jurčo, Vysoký; Ševera, Valach;...

Palatini variation

For a Courant algebroid $(E, \rho, \langle\ ,\ \rangle, [\ ,\ ])$ over M, we have the action functional:

$$S[V_+, \mathrm{vol}, \nabla] := \int_M (\mathcal{R}_{\nabla, V_+} + \mathcal{R})\, \mathrm{vol}. \tag{4}$$

The dynamical fields are:

- a generalized metric $V_+ \leq E$ s.t. $\mathrm{Rk}\, V_+ \notin \{1, \mathrm{Rk}\, E - 1\}$,
- a volume form $\mathrm{vol} \in \Gamma(\det T^* M)$,
- a Ca connection $\nabla : \times^2 \Gamma(E) \to \Gamma(E)$.

Theorem (Jurčo, M., Vysoký – 2022)

The 3-tuple $(V_+, \mathrm{vol}, \nabla)$ extremalizes the action (4) iff

(EoM1) $\mathrm{Ric}_\nabla|_{V_+ \otimes V_-} = 0$,

(EoM2) $\mathcal{R}_{\nabla, V_+} + \mathcal{R} = 0$,

(EoM3) $\nabla \in \mathrm{LC}(V_+, \mathrm{div}_{\mathrm{vol}})$.

- exact Ca: type IIB supergravity,
- heterotic Ca: heterotic supergravity.

Generalized type IIB supergravity equations (Tseytlin; Wulff – 2016)

Bosonic sector of generalized type IIB supergravity equations without RR–fields:

$$0 = \mathrm{Ric}_{{}^g\nabla}(X,Y) - \frac{1}{2}(\iota_X H', \iota_Y H')_g + ({}^g\nabla^s(\mathcal{K}+\mathcal{L}))(X,Y), \tag{5}$$

$$0 = \frac{1}{2}(\delta_g H')(X,Y) + H'(X,Y,g^{-1}(\mathcal{K}+\mathcal{L})) + (\mathrm{d}(\mathcal{K}+\mathcal{L}))(X,Y), \tag{6}$$

$$0 = \mathcal{R}_{{}^g\nabla,g} - \frac{1}{2}(H',H')_g + 4\,\mathrm{div}_g(\mathcal{K}+\mathcal{L}) - 4\,\|\mathcal{K}+\mathcal{L}\|_g\,. \tag{7}$$

The variables are:
- a Lorentzian metric $g \in \Gamma(\odot^2 T^*)$,
- a 2-form $B \in \Gamma(\wedge^2 T^*)$ (Kalb-Ramond field or B-field),
- a pair of 1-forms $\mathcal{K}$, $\mathcal{L} \in \Gamma(T^*)$ (generalized dilaton).

Moreover, the following extra conditions are imposed:

$$({}^g\nabla^s\mathcal{K})(X,Y) = 0, \quad H'(X,Y,g^{-1}\mathcal{K}) + (\mathrm{d}\mathcal{L})(X,Y) = 0, \quad g^{-1}(\mathcal{K},\mathcal{L}) = 0. \tag{8}$$

$\mathcal{K} = 0$ and $\mathcal{L}$ is exact $\Rightarrow$ (8) is trivially satisfied,

$\Rightarrow$ (5) - (7) reduce to ordinary type IIB supergravity.

Ca reformulation of (5) - (8)?

Generalized type IIB supergravity equations - Ca reformulation

1. Take the exact Ca $(T \oplus T^*, \mathrm{pr}_T, \langle\ ,\ \rangle_+, [\ ,\ ]^H)$.
2. Consider the equations ($\nabla \in \mathrm{LC}(V_+, \mathrm{div})$ is arbitrary):

$$\boxed{\left.(\mathrm{Ric}_\nabla + \mathrm{Ric}_{\nabla, V_+})\right|_{V_+ \oplus V_-} = 0, \qquad \mathcal{R}_{\nabla, V_+} + \mathcal{R} + \mathcal{R}_{\nabla, V_+^2} = 0,}$$

   and the extra conditions:

$$\boxed{\left.\mathrm{Ric}_{\nabla, V_+}\right|_{V_+ \otimes V_-} = 0, \qquad \mathcal{R}_{\nabla, V_+^2} = 0.}$$

   The variables are
   - a generalized metric $V_+ \leq T \oplus T^*$,
   - a divergence $\mathrm{div} : \Gamma(T \oplus T^*) \to C^\infty(M)$.
3. Look only for physically relevant solutions ($V_+ = \mathrm{gr}(g + B)$ for g Lorentzian).

Generalized heterotic supergravity?

(Generalized) Supergravity given by $(TT^*Q, \mathrm{pr}_H, g, [\ ,\ ])$?

References

- F. Moučka.
  Generalized geometry and Palatini formalism.
  Master's thesis (supervised by J. Vysoký), 2022.

- B. Jurčo; F. Moučka; J. Vysoký.
  Palatini variation in generalized geometry and string effective actions.
  Journal of Geometry and Physics, **191**, 2023.

Thank you for your attention!

Next talk: March 7, Miquel Cueca,

Courant cohomology, Cartan calculus, connections, curvature, characteristic classes