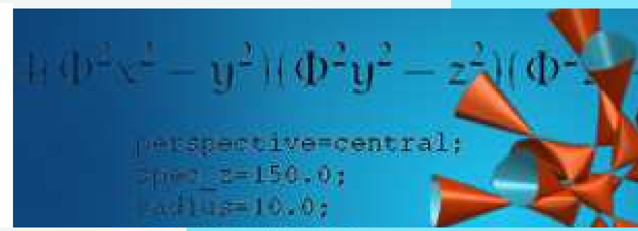
Torsion in differential geometry . . .
. . . and why we should care

Prof. Dr. habil. Ilka Agricola
Philipps-Universität Marburg



GENTLE Seminar, April 2024

Outline:

– Metric connections with torsion, or: why torsion?

– Cartan types of metric connections

– Holonomy as a key technique in differential geometry

– The Ambrose-Singer homogeneity theorem

– The almost complex structure on S^6 and nearly Kähler manifolds

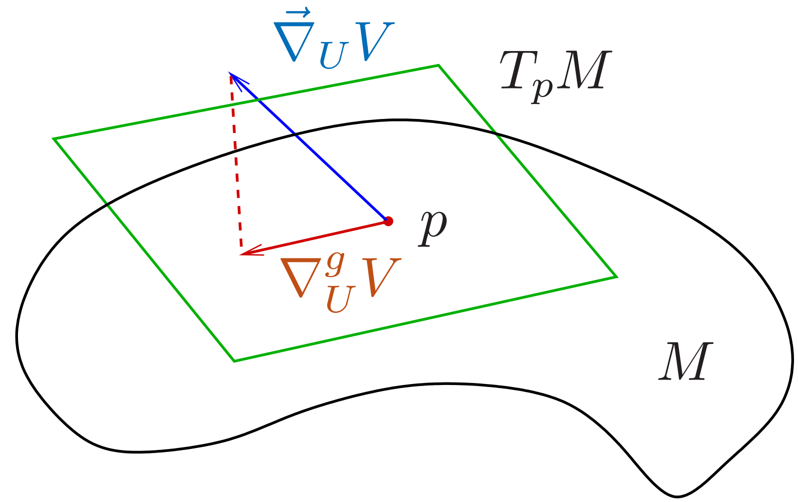
. . .

Élie Cartan (1869–1951)

Given a manifold embedded in affine (or projective or conformal etc.) space, attribute to this manifold the affine (or projective or conformal etc.) connection that reflects in the simplest possible way the relations of this manifold with the ambient space.

[Étant donné une variété plongée dans l'espace affine (ou projectif, ou conforme etc.), attribuer à cette variété la connexion affine (ou projective, ou conforme etc.) qui rende le plus simplement compte des relations de cette variété avec l'espace ambiant.].

Connections



Exa. Projection $\nabla^g_U V$ of dir. derivative $\vec{\nabla}_U V$ to tangent plane
$=$ 'Levi-Civita connection' ∇^g
Compatible with metric, torsion-free

But: not only possibility $\longrightarrow$ connection with torsion [Dfn: Cartan, 1925]

Exa. Electrodynamics: $\nabla_U V := \vec{\nabla}_U V + \frac{ie}{\hbar} A(U) V$ ($\Leftrightarrow$ $\nabla_\mu = \partial_\mu + \frac{ie}{\hbar} A_\mu$)

A: gauge potential $=$ electromagnetic potential

Exa. If $n = 3$: $\nabla_U V := \vec{\nabla}_U V + U \times V$ additional term gives space an 'internal angular momentum', a *torsion*

Fact: $\exists$ 3 types of torsion: vectorial, <u>skew symmetric</u>, and [something else].

Why torsion?

• **General relativity**:

a) Cartan (1929): torsion $\sim$ intrinsic angular momentum, derived a set of gravitational field eqs., but postulated that the energy-momentum tensor should still be divergence-free $\rightarrow$ too restrictive

b) Einstein-Cartan theory ($\geq$ 1950): variation of the scalar curvature <u>and</u> of an additional Lagrangian generating the energy-momentum and the spin tensors: allowed *any torsion and not nec. metric*

• **Superstring theory**:

Classical Yang-Mills theory: curvature $\cong$ field strength,

in superstring theories: torsion $\cong$ higher order field strength

($+$ extra differential eqs.)

• **Differential geometry**: Connections adapted to the geometry useful for 'non-integrable' geometries, like: Hermitian non Kähler mnfds, contact manifolds. . .

Set-up:

(M, g) Riemannian mnfd, ∇ metric conn., ∇^g Levi-Civita conn.

- Torsion as a $(2,1)$-tensor: $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$
- Transform torsion into a $(3,0)$-tensor via the metric:

$$T(X,Y,Z) \ := \ g(T(X,Y),Z)$$

- A metric connection is uniquely determined by its torsion
- Called 'skew torsion' if T is a 3-form

  This implies:

- ∇ may be written as $\quad \nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X,Y,-)$

So vanishing torsion $(T=0)$ reduces ∇ to the Levi-Civita connection

Example: Compact Lie groups

Consider a compact Lie group G, $\mathfrak{g} = T_eG$. A metric g on G is called *biinvariant* if left and right translations are always isometries $\Leftrightarrow$

$$g([V,X],Y) + g(X,[V,Y]) \;=\; 0. \qquad (*)$$

Easy: $\nabla^g_X Y \;=\; \frac{1}{2}[X,Y] \;\forall X,Y \in \mathfrak{g}$. We make the Ansatz that T is proportional to $[,]$, i. e.

$$\nabla^s_X Y \;:= s[X,Y], \quad \forall\, s \in \mathbb{R}, \qquad \text{hence } T^s(X,Y) = (2s-1)[X,Y],$$

This defines an element $T \in \Lambda^3(G)$ iff the metric satisfies $(*)$. The curvature of this connection is

$$\mathcal{R}^s(X,Y)Z = s(1-s)[Z,[X,Y]] = \begin{cases} \frac{1}{4}[Z,[X,Y]] & \text{for the LC conn. } (s=\frac{1}{2}) \\ 0 & \text{for } s = 0,1 \end{cases}$$

The two flat connections are called the $\pm$-connection and were first decribed by Cartan and Schouten (1926).

Types of metric connections

(M^n, g) oriented Riemannian mnfd, ∇ any connection:

$$\nabla_X Y \;=\; \nabla^g_X Y \,+\, A(X,Y)\,.$$

Then: ∇ is metric $\Leftrightarrow\;$ $g(A(X,Y),Z) + g(A(X,Z),Y) = 0$

$$\Leftrightarrow\; A \in \mathcal{A}^g := \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)$$

For metric connections: difference tensor A $\Leftrightarrow$ torsion T via

$$2A(X,Y,Z) = T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y),$$

$$T(X,Y,Z) = A(X,Y,Z) - A(Y,X,Z)$$

So identify $\mathcal{A}^g$ with T: space of possible torsion tensors,

$$\mathcal{A}^g \;\cong\; \mathcal{T} \;\cong\; \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n), \quad \dim = \frac{n^2(n-1)}{2}.$$

Decompose this space under $\mathrm{SO}(n)$ action (E. Cartan, 1925), $n \geq 3$:

$$\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n) \;\cong\; \mathbb{R}^n \,\oplus\, \Lambda^3(\mathbb{R}^n) \,\oplus\, \mathcal{T}'.$$

(For $n = 2$: $\mathbb{R}^2 \otimes \Lambda^2(\mathbb{R}^2) \;\cong\; \mathbb{R}^2$ is irreducible).

- $A \in \Lambda^3(\mathbb{R}^n)$: "Connections with skew (symmetric) torsion":

$$\nabla_X Y \;:=\; \nabla^g_X Y + \frac{1}{2} T(X, Y, -)\,.$$

Lemma. ∇ is metric and geodesics preserving iff its torsion T lies in $\Lambda^3(TM)$. In this case, $2\,A = T$, and the ∇-Killing vector fields coincide with the Riemannian Killing vector fields.

$\longrightarrow$ Connections used in superstring theory

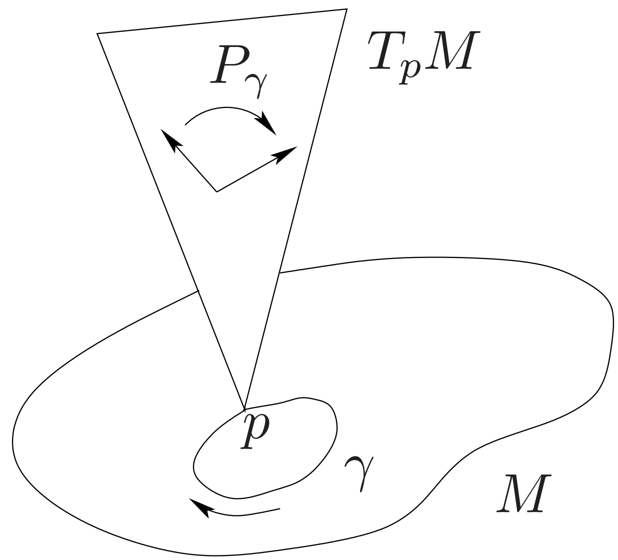
- $A \in \mathbb{R}^n$: "Connections with vectorial torsion", V a vector field:

$$\nabla_X Y \;:=\; \nabla^g_X Y - g(X,Y)\cdot V + g(Y,V)\cdot X\,.$$

In particular, *any metric connection* on a surface is of this type

Holonomy of arbitrary connections

- γ from p to q, ∇ any connection
- $P_\gamma : T_pM \to T_qM$ is the unique map s. t. $V(q) := P_\gamma V(p)$ is parallel along γ, $\nabla V(s)/ds = \nabla_{\dot\gamma} V = 0$.
- $C(p)$: closed loops through p
  $\mathrm{Hol}(p;\nabla) = \{P_\gamma \mid \gamma \in C(p)\}$
- $C_0(p)$: null-homotopic el'ts in $C(p)$
  $\mathrm{Hol}_0(p;\nabla) = \{P_\gamma \mid \gamma \in C_0(p)\}$



Independent of p, so drop p in notation: $\mathrm{Hol}(M;\nabla)$, $\mathrm{Hol}_0(M;\nabla)$.

A priori:

(1) $\mathrm{Hol}(M;\nabla)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$,

(2) $\mathrm{Hol}_0(p)$ is the connected component of the identity of $\mathrm{Hol}(M;\nabla)$.

Holonomy of metric connections

Assume: M carries a Riemannian metric g, ∇ *metric*

$\Rightarrow$ parallel transport is an isometry:

$$\frac{d}{ds} g\big(V(s), W(s)\big) \;=\; g\Big(\frac{\nabla V(s)}{ds}, W(s)\Big) + \Big(V(s), \frac{\nabla W(s)}{ds}\Big) \;=\; 0.$$

and $\mathrm{Hol}(M;\nabla) \subset \mathrm{O}(n,\mathbb{R})$, $\mathrm{Hol}_0(M;\nabla) \subset \mathrm{SO}(n,\mathbb{R})$.

Notation: $\mathrm{Hol}_{(0)}(M;\nabla^g) =$ “Riemannian (restricted) holonomy group”

N.B. (1) $\mathrm{Hol}_{(0)}(M;\nabla)$ needs not to be closed!

(2) The holonomy representation needs not to be irreducible on irreducible manifolds!

⟶ Larger variety of holonomy groups, but classification difficult

Curvature & Holonomy

Holonomy can be computed through curvature:

Thm (Ambrose-Singer, 1953). For any connection ∇ on (M, g), the Lie algebra $\mathfrak{hol}(p;\nabla)$ of $\mathrm{Hol}(p;\nabla)$ in $p \in M$ is exactly the subalgebra of $\mathfrak{so}(T_pM)$ generated by the elements

$$P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma \quad V, W \in T_pM, \quad \gamma \in C(p).$$

But only of restricted use:

Thm (Bianchi I). (1) For a metric connection with vectorial torsion $V \in TM^n$: $\overset{X,Y,Z}{\sigma} \mathcal{R}(X,Y)Z = \overset{X,Y,Z}{\sigma} dV(X,Y)Z.$

(2) For a metric connection with skew symmetric torsion $T \in \Lambda^3(M^n)$:

$$\overset{X,Y,Z}{\sigma} \mathcal{R}(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma^T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z),$$

$2\,\sigma^T := \sum_{i=1}^{n} (e_i \lrcorner T) \wedge (e_i \lrcorner T)$ for any orthonormal frame $e_1, \ldots, e_n$.

Thm (Berger, Simons, > 1955). For a non symmetric Riemannian manifold (M, g) and the Levi-Civita connection ∇^g, the possible holonomy groups are $\mathrm{SO}(n)$ or

$4n$	$2n$	$2n$	$4n$	7	8	16
$\mathrm{Sp}_n\mathrm{Sp}_1$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	Sp_n	G_2	$\mathrm{Spin}(7)$	$(\mathrm{Spin}(9))$
quatern. Kähler	Kähler	Calabi-Yau	hyper-Kähler	par.	par.	par.
$\nabla J \neq 0$	$\nabla^g J = 0$	$\nabla^g J = 0$	$\nabla^g J = 0$	$\nabla^g \omega^3 = 0$	$\nabla^g \Omega^4 = 0$	$--$
$\mathrm{Ric} = \lambda g$	$--$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$--$

- Recall – Construction of compact Ricci flat examples difficult!

Q1: What about Riemannian mnfds not covered by this result?

Q2: Good Reformulation / replacement for connections with torsion?

General Holonomy Principle

Thm (General Holonomy Principle). M a manifold, E a (real or complex) vector bundle over M with (any!) connection ∇. Then the following are equivalent:

(1) E has a global section α which is invariant under parallel transport, i. e. $\alpha(q) = P_\gamma(\alpha(p))$ for any path γ from p to q;

(2) E has a parallel global section α, i. e. $\nabla\alpha = 0$;

(3) In some point $p \in M$, there exists an algebraic vector $\alpha_0 \in E_p$ which is invariant under the holonomy representation on the fiber.

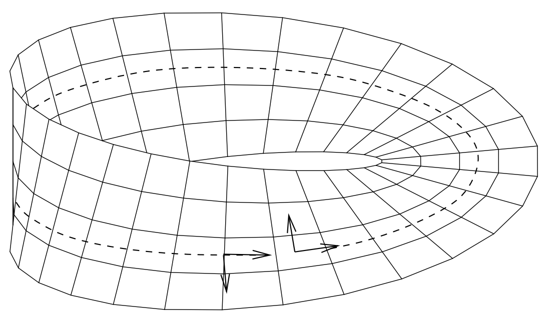
Corollary. The number of parallel global sections of E coincides with the number of trivial representations occuring in the holonomy representation on the fibers.

Example. Orientability from a holonomy point of view:

Lemma. The determinant ist an $\mathrm{SO}(n)$-invariant element in $\Lambda^n(\mathbb{R}^n)$ that is not $\mathrm{O}(n)$-invariant.

Corollary. (M^n, g) is orientable iff $\mathrm{Hol}(M; \nabla) \subset \mathrm{SO}(n)$ for *any metric connection* ∇, and the volume form is then ∇-parallel.

[Take $dM_p := \det = e_1 \wedge \ldots \wedge e_n$ in $p \in M$, then apply holonomy principle to $E = \Lambda^n(TM)$.]



An orthonormal frame that is parallel transported along the drawn curve reverses its orientation.

The Ambrose–Singer homogeneity theorem

Thm. A complete Riemannian manifold (M, g) equipped with a homogeneous structure, i. e. a metric connection ∇ with torsion T and curvature $\mathcal{R}$ such that $\nabla\mathcal{R} = 0$ and $\nabla T = 0$, is locally isometric to a Riemannian homogeneous space. [Ambrose–Singer, 1958]

• Symmetric spaces: Correspond to $T = 0$, the "integrable" case, $\nabla^g\mathcal{R}^g = 0$; intuitively this follows because $\nabla^g\mathcal{R}^g$ would be a $(5, 0)$-tensor, the invariance under reflections then forces it to vanish.

Q. Is the connection unique?

– yes! This will be a non-trivial consequence of the skew torsion holonomy theorem (except on spheres, Lie groups, and their coverings).

$\Rightarrow$ 3 classes of homogeneous spaces according to Cartan type of this torsion!

• Empirical fact: In the non-homogeneous case, metric connections with parallel torsion turn out to be very useful (and natural) as well.

One class: Naturally reductive homogeneous spaces

Traditional approach: $(M = G/H, g)$ a homogeneous space

Dfn. $M = G/H$ is *naturally reductive* if $\mathfrak{h}$ admits a reductive complement $\mathfrak{m}$ in $\mathfrak{g}$ s. t.

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \;=\; 0 \;\text{ for all } X,Y,Z \in \mathfrak{m}, \qquad (*)$$

where $\langle -, -\rangle$ denotes the inner product on $\mathfrak{m}$ induced from g.
The PFB $G \to G/H$ induces a metric connection ∇ with torsion

$$T(X,Y,Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle,$$

the *canonical connection*. It satisfies $\nabla T = \nabla \mathcal{R} = 0$, so it's just the connection from the AS thm!

- If G/H is symmetric, then $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, hence $T = 0$ and $\nabla = \nabla^g$
- condition $(*)$ $\Leftrightarrow$ T is a 3-form, i. e. $T \in \Lambda^3(M)$.

Approaching a key example: The almost complex structure on S^6

Hopf problem: Does there exist a complex structure on S^6?

Kirchhoff (1947) observes:

• The only sphere (other than S^2) that may carry a complex structure is S^6.

• An almost complex structure on S^6 can easily be defined by interpreting S^6 as the purely imaginary Cayley numbers of norm 1.

This is strongly related to the transitive action of G_2 on S^6 – but it took a while to understand the link

Thm. The only even-dim. sphere with a transitive group G acting that is not orthogonal is S^6 with $G = G_2$. [Borel, 1949]

• Ehresmann, Libermann (1951): Studied the almost complex structure on S^6 in more detail, prove that it's not complex, conclude: 'The structure we considered is therefore locally equivalent to an almost hermitian structure on S^6 admitting G_2 as its group of automorphisms.'

Explicit description I: Hypersurfaces in $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ [Calabi, 1958]

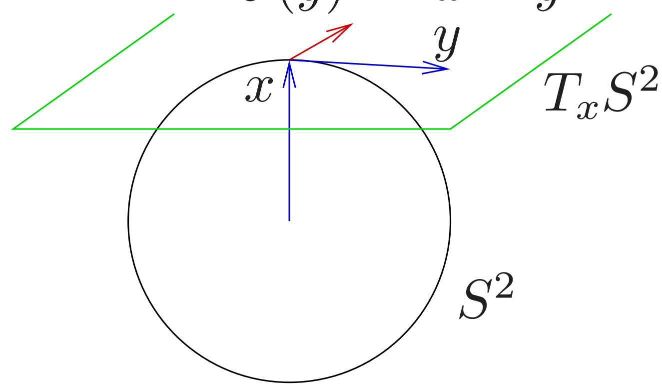
- M^6 a compact hypersurface in $\mathrm{Im}\,\mathbb{O}$
- N: normal vector field
- K: shape operator (Weingarten map)
- Define $J \in \mathrm{End}(TM)$ by

$$J(Y) = N \times Y, \quad Y \in TM$$

- $J^2 = -\mathrm{Id}$ is a non integrable almost complex structure satisfying

for $S^2 \subset \mathbb{R}^3$:

$J(y) := x \times y$



$$\langle (\nabla^g_X J)(Y), Z\rangle \;=\; \langle K(X) \times Y, Z\rangle$$

- For $M^6 = S^6$, $K = \mathrm{Id}$ and J satisfies the simpler eq. $\nabla^g_X J(X) = 0$

Such an almost Hermitian mnfd is called a 'nearly Kähler manifold'.

Sketch of proof: The cross product on $\operatorname{Im}\mathbb{O}$ satisfies:

- $\langle A, (B\times C)\rangle = \langle (A\times B), C\rangle =: (ABC)$ (*scalar triple product identity*),
- $A\times(A\times B) = -|A|^2 B + \langle A, B\rangle A$ (*Malcev identity*),
- $(A\times B)\times C = A\times(B\times C) - \langle A, B\rangle C \quad$ for $C\perp A, B$.

• Prove $J^2 = -\mathrm{Id}$: Malcev id. implies $N\times(N\times X) = -|N|^2 X + \langle N, X\rangle N$ for any $X\in TS$. But $|N| = 1$ and $N\perp X \Rightarrow$ claim.

• Metric is J-compatible: Use scalar triple product and Malcev identity

$\langle J(X), J(Y)\rangle = \langle N\times X, N\times Y\rangle = \langle N\times X)\times N, Y\rangle = -\langle N\times(N\times X), Y\rangle = \langle X, Y\rangle$

• Compute $(\nabla^g_X J)(Y)$:

$(\nabla^g_X J)(Y) = \nabla^g_X(J(Y)) - J(\nabla^g_X Y) = \nabla^g_X(N\times Y) - N\times\nabla^g_X Y = \nabla^g_X N\times Y + N\times\nabla^g_X Y - N\times\nabla^g_X Y = K(X)\times Y.$

In fact, more holds: J is integrable iff $K\circ J = -J\circ K$, which cannot hold on a closed hypersurface of Euclidian space.

Corollary. J is never integrable. For the sphere, $N(X, Y) = 4(\nabla^g_X J)JY$.

Explicit description II: $S^6 = G_2/\mathrm{SU}(3)$ as naturally reductive space

[Fukami, Ishihara, 1955]

Thm. $S^6 = G_2/\mathrm{SU}(3)$ is naturally reductive and the torsion 3-form of the Ambrose-Singer connection is given by the formula

$$T^c(X, Y, Z) \;=\; -\langle J(X \times Y), Z\rangle \;=\; -\langle N, (X \times Y) \times Z\rangle \;.$$

The existence of this connection is no coincidence. Gray started the systematic investigation of nearly Kähler mnfds in the early 70ies and proved:

Thm. Let (M, g, J) be a nearly Kähler mnfd. There exists a unique metric connection preserving J and with skew symmetric torsion, and its torsion 3-form is given by the formula

$$T^c(X, Y, Z) \;=\; \langle (\nabla^g_X J)(JY)\,,\, Z\rangle\,.$$

Furthermore, it satisfies $\nabla T = 0$.

∇ is the first example of what it now known as a *characteristic connection*!

Thm. The only homogeneous nearly Kähler 6-manifolds are S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold $F_{1,2}$. They are all naturally reductive and their characteristic connection coincides with the Ambrose-Singer connection. [Butruille, 2005]

Thm. Let (M, g, J) be a 6–dimensional nearly Kähler manifold that is not Kähler. Then

- g is an Einstein metric on M,
- the first Chern class of M vanishes and hence it is spin. [Gray, 1976]

Thm. The characteristic torsion of a nearly Kähler 6-manifold is parallel with respect to the characteristic connection, $\nabla^c T^c = 0$. [Kirichenko, 1977]

Philosophically: The LC connection does not 'see' the nearly Kähler structure of these mnfds; the characteristic connection, having non-generic holonomy, carries crucial information and is a tool for proofs even in the non-homogeneous case

Non homogeneous nearly Kähler manifolds

Were widely believed to exist, but their explicit construction was an open problem for many years!

Thm. There exists a non-homogeneous nearly Kähler structure on S^6 and on $S^3 \times S^3$. [Foscolo-Haskins, 2017]

They admit an isometric action of a compact Lie group such that generic orbits of the action are of codimension one.

The Lie group considered in this case is $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and the generic orbits are $S^2 \times S^3$, which is motivated by results of Podesta and Spiro (2012) characterizing all possible groups and orbits for cohomogeneity one nearly Kähler.

Local homogeneous non-homogeneous examples of nearly Kähler manifolds are constructed by Cortés and Vásquez (2015).

Algebraic preliminaries for holonomy theorem

Dfn. For a 3-form T, define [introduced in AFr, 2004]

- kernel: $\ker T = \{X \in R^n \,|\, X \lrcorner T = 0\}$ (for later)
- Lie algebra generated by its image: $\mathfrak{g}_T := \mathrm{Lie}\langle X \lrcorner T \,|\, X \in \mathbb{R}^n\rangle$

isotropy Lie algebra : $\mathfrak{h}_T := \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^*T = 0\}$

$\mathfrak{g}_T$ is *not* related in any obvious way to $\mathfrak{h}_T$!

Thm. $\mathfrak{g}_T$ is a compact semisimple Lie algebra.

Next step: In its original version, Berger's holonomy theorem is not suitable for generalization to connections with skew torsion.

Formulate a holonomy theorem in terms of $\mathfrak{g}_T$!

The skew torsion holonomy theorem

Dfn. Let $0 \neq T \in \Lambda^3(V)$, $\mathfrak{g}_T$ as before, $G_T \subset \mathrm{SO}(n)$ its Lie group. Hence, $X \lrcorner T \in \mathfrak{g}_T \subset \mathfrak{so}(V) \cong \Lambda^2(V)\ \forall\ X \in V$. Then (G_T, V, T) is called a *skew-torsion holonomy system (STHS)*. It is said to be

- *irreducible* if G_T acts irreducibly on V,
- *transitive* if G_T acts transitively on the unit sphere of V,
- and *symmetric* if T is G_T-invariant.

Recall: The only transitive sphere actions are:

$\mathrm{SO}(n)$ on $S^{n-1} \subset \mathbb{R}^n$, $[\mathrm{S}]\mathrm{U}(n)$ on $S^{2n-1} \subset \mathbb{C}^n$, $\mathrm{Sp}(n)[\mathrm{Sp}(1)]$ on $S^{4n-1} \subset \mathbb{H}^n$, G_2 on S^6, $\mathrm{Spin}(7)$ on S^7, $\mathrm{Spin}(9)$ on S^{15}. [Montgomery-Samelson, 1943]

Thm (STHT). Let (G_T, V, T) be an irreducible STHS. If it is transitive, $G_T = \mathrm{SO}(n)$. If it is not transitive, it is symmetric, and

- V is a simple Lie algebra of rank ≥ 2 w. r. t. the bracket $[X, Y] = T(X, Y)$, and G_T acts on V by its adjoint representation,
- T is unique up to a scalar multiple.

[transitive: AFr 2004, general: Olmos-Reggiani, 2012; Nagy 2013]

The newer proofs are based on general holonomy theory. The statement about transitive actions is easily verified on a case by case basis.

Want to apply this to *existence of characteristic connections!*

The characteristic connection of a geometric structure

Fix $G \subset \mathrm{SO}(n)$, $\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, $\mathcal{F}(M^n)$: frame bundle of (M^n, g).

Dfn. A geometric G-structure on M^n is a G-PFB $\mathcal{R}$ which is subbundle of $\mathcal{F}(M^n)$: $\mathcal{R} \subset \mathcal{F}(M^n)$.

Choose a G-adapted local ONF $e_1, \ldots, e_n$ in $\mathcal{R}$ and define *connection 1-forms of ∇^g:*

$$\omega_{ij}(X) := g(\nabla^g_X e_i, e_j), \quad g(e_i, e_j) = \delta_{ij} \;\Rightarrow\; \omega_{ij} + \omega_{ji} = 0.$$

Define a skew symmetric matrix Ω with values in $\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$ by $\Omega(X) := \big(\omega_{ij}(X)\big) \in \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ und set

$$\Gamma := \mathrm{pr}_{\mathfrak{m}}(\Omega).$$

• Γ is a 1-Form on M^n with values in $\mathfrak{m}$, $\Gamma_x \in \mathbb{R}^n \otimes \mathfrak{m}$ $(x \in M^n)$ [“intrinsic torsion”, Swann/Salamon]

Fact: $\Gamma = 0 \Leftrightarrow \nabla^g$ is a G-connection $\Leftrightarrow \mathrm{Hol}(\nabla^g) \subset G$

Via Γ, geometric G-structures $\mathcal{R} \subset \mathcal{F}(M^n)$ correspond to irreducible components of the G-representation $\mathbb{R}^n \otimes \mathfrak{m}$.

Thm. A geometric G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a metric G-connection with antisymmetric torsion iff Γ lies in the image of Θ,

$$\Theta : \Lambda^3(M^n) \to T^*(M^n) \otimes \mathfrak{m}, \quad \Theta(T) \ := \ \sum_{i=1}^n e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner T).$$

[Fr, 2003]

If such a connection exists, it is called the *characteristic connection* $\nabla^c \to$ replace the (unadapted) LC connection by ∇^c.

Thm. If $G \not\subset \mathrm{SO}(n)$ acts irreducibly and not by its adjoint rep. on $\mathbb{R}^n \cong T_pM^n$, then $\ker \Theta = \{0\}$, and hence the characteristic connection of a G-structure on a Riemannian manifold (M^n, g) is, if existent, unique.

[A-Fr-Höll, 2013]

Uniqueness of characteristic connections

This is a consequence of the STHT:

Proof. $T \in \ker \Theta$ iff all $X \lrcorner T \in \mathfrak{g} \subset \mathfrak{so}(n)$, that is,

$$\ker \Theta = \{T \in \Lambda^3(\mathbb{R}^n) \,|\, \mathfrak{g}_T \subset \mathfrak{g}\},$$

so $(T, G, \mathbb{R}^n)$ defines an irreducible STHS, which by assumption is non transitive (because $G \not\subset \mathrm{SO}(n)$). By the STHT, it has to be a Lie algebra with the adjoint representation. Since this was excluded as well, it follows that $\ker \Theta = \{0\}$. $\square$

For many G-structures, uniqueness can be proved directly case by case – including a few cases where the G-action is not irreducible.

For example, a large class of almost metric contact manifolds admits a characteristic connection ∇, and for these: $\mathrm{Hol}_0(\nabla) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n+1)$.

We cite only one class of examples:

Thm. An almost hermitian manifold (M^{2n}, g, J) admits a characteristic connection ∇ if and only if the Nijenhuis tensor

$$N(X,Y,Z) \;:=\; g\big(N(X,Y),\, Z\big)$$

is skew-symmetric. Its torsion is then

$$T(X,Y,Z) \;= -d\Omega(JX,JY,JZ) + N(X,Y,Z)$$

and it satisfies: $\nabla\Omega = 0$, $\mathrm{Hol}(\nabla) \subset \mathrm{U}(n)$. [Fr-Ivanov, 2002]

'Trivial case': If (M^{2n}, g, J) is Kähler ($N = 0$ and $d\Omega = 0$), then $T = 0$, the LC connection ∇^g *is* the characteristic connection.

In particular for $\underline{n = 3}$: [Gray-Hervella]

- $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}^6$, $\Gamma \in \mathbb{R}^6 \otimes \mathfrak{m}^6\big|_{\mathrm{U}(3)} \cong W_1^2 \oplus W_2^{16} \oplus W_3^{12} \oplus W_4^6$
- N is skew-symmetric $\Leftrightarrow$ Γ has no W_2-part
- $\Gamma \in W_1$: nearly Kähler manifolds $(S^6, S^3 \times S^3, F(1,2), \mathbb{CP}^3)$
- $\Gamma \in W_3 \oplus W_4$: hermitian manifolds $(N = 0)$

Flat metric connections with skew torsion

Suppose ∇ is metric and has antisymmetric torsion $T \in \Lambda^3(M)$,

$$\nabla_X Y = \nabla^g_X Y + \tfrac{1}{2} T(X, Y).$$

Q: What are the manifolds with a flat metric connection with skew torsion?

[this section: A-Fr, 2010]

- We know that compact Lie groups are examples,
- We show that S^7 is another example.

Assume simply connected where needed.

Flat connections

Dfn. ∇ is called *flat*, if $\mathcal{R}(X,Y)=0$ for all X,Y

$\Leftrightarrow$ $\nabla : TM \to \mathrm{End}(TM),\ X \mapsto \nabla_X$ is Lie algebra homomorphism

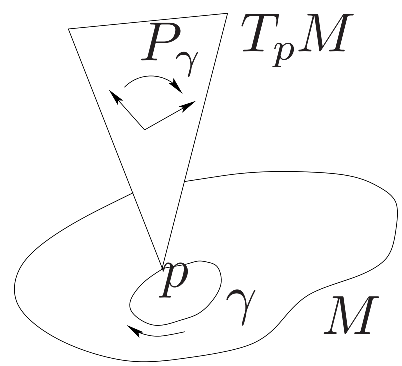
$\Leftrightarrow$ By Ambrose-Singer Thm ($\gamma \in C(p), P_\gamma : T_pM \to T_pM$ par.tr.):

$$0 = \mathfrak{hol}(\nabla, p) = \langle P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma \rangle \subset \mathfrak{so}(T_pM),$$

i. e. $\mathrm{Hol}(p;\nabla)$ is a discrete group

$\Leftrightarrow$ parallel transport is path-independent

$\Rightarrow$ (M,g) is parallelisable and therefore spin



Example 2: S^7

- only parallelisable sphere that is not a Lie group (but almost. . .)

Consider spin representation $\kappa^{\mathbb{C}}: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7^{\mathbb{C}}), \quad \Delta_7^{\mathbb{C}} \cong \mathbb{C}^8$.

In dim.7, this turns out to be complexification of 8-dim. real rep.,

$$\kappa: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7), \quad \Delta_7 \cong \mathbb{R}^8.$$

κ is in fact a repr. of the Clifford algebra over $\mathbb{R}^7$ ($\mathrm{Spin}(7) \subset \mathrm{Cl}(\mathbb{R}^7)$!),

$$\kappa: \ \mathbb{R}^7 \subset \mathrm{Cl}(\mathbb{R}^7) \to \mathrm{End}(\Delta_7).$$

Choose $e_1, \ldots, e_7$ an ON basis of $\mathbb{R}^7$, and set $\kappa_i = \kappa(e_i)$.

- Embed $S^7 \subset \Delta_7$ as spinors of length 1,
- define VFs on S^7 by $V_i(x) = \kappa_i \cdot x$ for all $x \in S^7 \subset \Delta^7$

Properties of the VFs $V_i(x) = \kappa(e_i) \cdot x$

Thm. (1) These vector fields realize a ON trivialization of S^7,

[computation rules for Clifford multipl.]

(2) the connection ∇ defined by $\nabla V_i = 0$ is metric, flat, and with torsion

$$T(V_i, V_j, V_k)(x) \;=\; -\langle [V_i, V_j], V_k \rangle \;=\; 2\langle \kappa_i \kappa_j \kappa_k x, x \rangle \in \Lambda^3(S^7),$$

(3) $\nabla T \neq 0$ (check that T does not have constant coefficients), $\sigma_T \neq 0$

(4) ∇ is a G_2 connection of Fernandez-Gray type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$.

Classification of flat skew torsion manifolds

Thm Any irr.,c.s.c. M with a flat, metric connection with skew torsion $T \in \Lambda^3(M)$ is either a compact Lie group or S^7.

• 1926: Cartan-Schouten "On manifolds with absolute parallelism" – wrong proof.

• 1968: d'Atri-Nickerson "On the existence of special orthonormal frames" – when does (M, g) admit an ONF of Killing vectors?

This is mainly an equivalent problem:

$$V \text{ is Killing VF} \Leftrightarrow g(\nabla^g_X V, Y) + g(X, \nabla_Y V) = 0 \qquad (*)$$

If V is parallel for ∇ with torsion T, then $\nabla^g_X V = -\frac{1}{2}T(X, V)$, hence

$$(*) \Leftrightarrow g(T(X, V), Y) + g(X, T(Y, V)) = 0 \Leftrightarrow T \in \Lambda^3(M)$$

• 1972: J. Wolf "On the geometry and classification of absolute parallelisms" – 2 long papers in J. Diff.Geom.

Both proofs rely on classification of symmetric spaces.

• First classification free proof uses STHT [A-Friedrich, 2010]

Towards a classification of naturally reductive spaces

Main pb: $\nexists$ invariant theory for $\Lambda^3(\mathbb{R}^n)$ under $\mathrm{SO}(n)$ for $n \geq 6$, i. e. normal forms for the $\mathrm{SO}(n)$-orbits of 3-forms!

- Use the recent progress on *metric connections with [parallel] skew torsion*
- Use *torsion* (instead of curvature) as basic geometric quantity, *find a G-structure (contact str., almost hermitian str. etc.)* inducing the nat. red. structure

Obviously: $\boxed{\text{nat.red.homog. Riemannian mnfds}} \subset \boxed{\text{(homogeneous) Riemannian mnfds with parallel skew torsion}}$

Review of some classical results

• all isotropy irreducible homogeneous manifolds are naturally reductive

• the $\pm$-connections on any Lie group with a biinvariant metric are naturally reductive (and, by the way, flat) [Cartan-Schouten, 1926]

• construction / classification (under some assumptions) of left-invariant naturally reductive metrics on compact Lie groups [D'Atri-Ziller, 1979]

• All 6-dim. homog. nearly Kähler mnfds (w. r. t. their canonical almost Hermitian structure) are naturally reductive. These are precisely: $S^3 \times S^3$, $\mathbb{CP}^3$, the flag manifold $F(1,2) = \mathrm{U}(3)/\mathrm{U}(1)^3$, and $S^6 = G_2/\mathrm{SU}(3)$.

• Known classifications:

- dimension 3 [Tricerri-Vanhecke, 1983], dimension 4 [Kowalski-Vanhecke, 1983], dimension 5 [Kowalski-Vanhecke, 1985]

These proceeded by finding normal forms for the curvature operator.

An important tool: the 4-form σ_T

Dfn. For any $T \in \Lambda^3(M)$, define ($e_1, \ldots, e_n$ a local ONF)

$$\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) \quad (= 0 \text{ if } n \leq 4)$$

Exa: For $T = \alpha\, e_{123} + \beta\, e_{456}$, $\sigma_T = 0$;
for $T = (e_{12} + e_{34})e_5$, $\sigma_T = -e_{1234}$

• **σ_T measures the 'degeneracy' of T and, if non degenerate, induces the geometric structure on M**

[σ_T appears in many important relations:

* 1st Bianchi identity: $\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X, Y, Z, V) = \sigma_T(X, Y, Z, V)$
* $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra
* If $\nabla T = 0$: $dT = 2\sigma_T$ and $\nabla^g T = \frac{1}{2}\sigma_T$]

σ_T and the Nomizu construction

Idea: for $M = G/H$, reconstruct $\mathfrak{g}$ from $\mathfrak{h}$, T, $\mathcal{R}$ and $V \cong T_x M$

Set-up: $\mathfrak{h}$ a real Lie algebra, V a real f. d. $\mathfrak{h}$-module with $\mathfrak{h}$-invariant pos. def. scalar product $\langle , \rangle$, i. e. $\mathfrak{h} \subset \mathfrak{so}(V) \cong \Lambda^2 V$

$\mathcal{R} : \Lambda^2 V \to \mathfrak{h}$ an $\mathfrak{h}$-equivariant map, $T \in (\Lambda^3 V)^{\mathfrak{h}}$ an $\mathfrak{h}$-invariant 3-form,

Define a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus V$ by $(A, B \in \mathfrak{h}, X, Y \in V)$:

$$[A + X, B + Y] := ([A, B]_{\mathfrak{h}} - \mathcal{R}(X, Y)) + (AY - BX - T(X, Y))$$

Jacobi identity for $\mathfrak{g}$ $\Leftrightarrow$

- $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X, Y, Z, V) = \sigma_T(X, Y, Z, V)$ (1st Bianchi condition)
- $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(T(X, Y), Z) = 0$ (2nd Bianchi condition)

Observation: If (M, g, T) satisfies $\nabla T = 0$, then $\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M)$ is symmetric (as in the Riemannian case).

Consider $\mathcal{C}(V) := \mathcal{C}(V, -\langle , \rangle)$: Clifford algebra, (recall: $T^2 = -2\sigma_T + \|T\|^2$)

Thm. If $\mathcal{R} : \ \Lambda^2 V \to \mathfrak{h} \subset \Lambda^2 V$ is symmetric, the first Bianchi condition is equivalent to $T^2 + \mathcal{R} \in \mathbb{R} \subset \mathcal{C}(V)$ $(\Leftrightarrow \ 2\sigma_T = \mathcal{R} \subset \mathcal{C}(V))$, and the second Bianchi condition holds automatically.

Exists in the literature in various formulations: based on an algebraic identity (Kostant); crucial step in a formula of Parthasarathy type for the square of the Dirac operator (A, '03); previously used by Schoemann 2007 and Fr. 2007, but without a clear statement nor a proof.

Practical relevance: allows to evaluate the 1st Bianchi identity in one condition, good for implementation on a computer!

Splitting theorems

Dfn. For T 3-form, define [introduced in AFr, 2004]

- kernel: $\ker T = \{X \in TM \mid X \lrcorner T = 0\}$
- Lie algebra generated by its image: $\mathfrak{g}_T := \mathrm{Lie}\langle X \lrcorner T \mid X \in V\rangle$

  $\mathfrak{g}_T$ is *not* related in any obvious way to the isotropy algebra of T!

Thm 1. Let (M, g, T) be a c. s. c. Riemannian mfld with parallel skew torsion T. Then $\ker T$ and $(\ker T)^\perp$ are ∇-parallel and ∇^g-parallel integrable distributions, M is a Riemannian product s. t.

$$(M, g, T) = (M_1, g_1, T_1 = 0) \times (M_2, g_2, T_2), \quad \ker T_2 = \{0\}$$

Thm 2. Let (M, g, T) be a c. s. c. Riemannian mfld with parallel skew torsion T s. t. $\sigma_T = 0$, $TM = \mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_q$ the decomposition of TM in $\mathfrak{g}_T$-irreducible, ∇-par. distributions. Then all $\mathcal{T}_i$ are ∇^g-par. and integrable, M is a Riemannian product, and the torsion T splits accordingly

$$(M, g, T) = (M_1, g_1, T_1) \times \ldots \times (M_q, g_q, T_q)$$

A structure theorem for vanishing σ_T

Thm. Let (M^n, g) be an *irreducible*, c. s. c. Riemannian mnfld with parallel skew torsion $T \neq 0$ s. t. $\sigma_T = 0$, $n \geq 5$. Then M^n is a simple compact Lie group with biinvariant metric or its dual noncompact symmetric space. [A-Ferreira-Friedrich, 2015]

Key ideas: $\sigma_T = 0 \Rightarrow$ Nomizu construction yields Lie algebra structure on TM

use $\mathfrak{g}_T$; use the Skew Torsion Holonomy Theorem to show that G_T is simple and acts on TM by its adjoint rep.

prove that $\mathfrak{g}_T = \mathfrak{iso}(T) = \mathfrak{hol}^g$, hence acts irreducibly on TM, hence M is an irred. symmetric space by Berger's Thm

Classification of nat. red. spaces in $n = 3$

[Tricerri-Vanhecke, 1983]

Then $\sigma_T = 0$, and the Nomizu construction can be applied directly to obtain in a few lines:

Thm. Let $(M^3, g, T \neq 0)$ be a 3-dim. c. s. c. Riemannian mnfld with a naturally reductive structure. Then (M^3, g) is one of the following:

- $\mathbb{R}^3, S^3$ or $\mathbb{H}^3$;

- isometric to one of the following Lie groups with a suitable left-invariant metric:

$$SU(2), \quad \widetilde{SL}(2,\mathbb{R}), \quad \text{or the 3-dim. Heisenberg group } H^3$$

N.B. A general classification of mnfds with par. skew torsion is meaningless – any 3-dim. volume form of a metric connection is parallel.

Classification of nat. red. spaces in $n = 4$

Thm. $(M^4, g, T \neq 0)$ a c. s. c. Riem. 4-mnfld with parallel skew torsion. Then

1) $V := *T$ is a ∇^g-parallel vector field.

2) $\mathrm{Hol}(\nabla^g) \subset \mathrm{SO}(3)$, hence M^4 is isometric to a product $N^3 \times \mathbb{R}$, where (N^3, g) is a 3-manifold with a parallel 3-form T.

• T has normal form $T = e_{123}$, so $\dim \ker T = 1$ and 2) follows at once from our 1st splitting thm: but the existence of V explains directly & *geometrically* the result in a few lines.

Corollary. A 4-dim. naturally reductive Riemannian manifold with $T \neq 0$ is locally isometric to a Riemannian product $N^3 \times \mathbb{R}$, where N^3 is a 3-dimensional naturally reductive Riemannian manifold. [Kowalski-Vanhecke, 1983]