Generalized connections, spinors, and integrability of generalized structures on Courant algebroids

Vicente Cortés
Department of Mathematics
University of Hamburg

Seminar - GENTLE
Universitat Autònoma de Barcelona (virtual), May 8, 2024

Talk based on:

V. C. and L. David, *Generalized connections, spinors, and integrability of generalized structures on Courant algebroids*, Moscow Mathematical Journal 21, no. 4 (2021), 695-736.

I. Courant algebroids

Definition

A Courant algebroid (CA) is a vector bundle $E \to M$ with:

1. scalar product $\langle \cdot, \cdot \rangle$,
2. bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and
3. anchor $\pi : E \to TM$,

such that $\forall u, v, w \in \Gamma(E)$:

A1) $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$,

A2) $\pi(u)\langle v, w \rangle = \langle [u, v], w \rangle + \langle v, [u, w] \rangle$,

A3) $\langle [u, v] + [v, u], w \rangle = \pi(w)\langle u, v \rangle$.

Example (Generalized tangent bundle)

$\mathbb{T}M := TM \oplus T^*M$

$$[X + \xi, Y + \eta] = \mathcal{L}_X(Y + \eta) - \mathcal{L}_Y(\xi) + d(\xi(Y)).$$

II. Generalized complex structures

Definition

A generalized almost complex structure (GACS) on a CA E is a skew-symm. $\mathcal{J} \in \Gamma(\mathrm{End}\, E)$ s.t. $\mathcal{J}^2 = -\mathrm{Id}_E$.
... integrable if $E^{1,0}$ is involutive.

Examples

a) Let J be a complex structure on M. Then

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} : \mathbb{T}M \to \mathbb{T}M$$

is a GCS.

b) Let ω be a symplectic structure on M. Then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} : \mathbb{T}M \to \mathbb{T}M$$

is a GCS.

III. Generalized connections

Definition

A generalized connection on a CA $E \to M$ is a linear map

$$D : \Gamma(E) \to \Gamma(E^* \otimes E), \quad v \mapsto Dv,$$

s.t. $\forall u, v, w \in \Gamma(E), f \in C^\infty(M)$:

$$\begin{aligned} D_u(f\,v) &= \pi(u)(f)\,v + f\,D_u v \\ \pi(u)\langle v, w\rangle &= \langle D_u v, w\rangle + \langle v, D_u w\rangle. \end{aligned}$$

Torsion

1. $T^D \in \Gamma(\wedge^2 E^* \otimes E)$:

$$T^D(u, v) := D_u v - D_v u - [u, v] + (Du)^* v.$$

2. $T^D(u, v, w) := \langle T^D(u, v), w\rangle$ defines a section of $\Gamma(\wedge^3 E^*)$.

IV. Geometric structures on Courant algebroids

Let E be a CA of signature (n_1, n_2) and $H \subset \mathrm{O}(n_1, n_2)$ a Lie subgroup.

Definition

An *H-structure* on E is a reduction of the structure group $\mathrm{O}(n_1, n_2)$ to H.

... assume that the structure is defined by a system Q of tensor fields such as:

1. GACS $Q = \mathcal{J}$, $H = \mathrm{U}(m_1, m_2)$, $n_i = 2m_i$,
2. generalized almost hypercomplex structure $Q = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$, $H = \mathrm{Sp}(m_1, m_2)$, $n_i = 4m_i$,
3. generalized Riemannian metric $Q = G$, $H = \mathrm{O}(n_1) \times \mathrm{O}(n_2)$,
4. generalized almost Hermitian structure, $Q = (G, \mathcal{J})$, $H = \mathrm{U}(m_1) \times \mathrm{U}(m_2)$,
5. generalized almost hyper-Hermitian structure, $Q = (G, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$, $H = \mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2)$.

V. Characterization of integrability

Theorem 1 (C.-David)

Let Q be any of the above structures 1, 2, 4 or 5 on a CA E. Then Q is integrable iff $\exists$ torsionfree generalized connection D on E for which Q is parallel.

Proof (sketch in the case $Q = \mathcal{J}$)

"$\Leftarrow$" The integrability of $\mathcal{J}$ is equivalent to the vanishing of the Nijenhuis tensor $N_{\mathcal{J}} \in \Gamma(\wedge^2 E^* \otimes E)$:

$$N_{\mathcal{J}}(u, v) := [\mathcal{J}u, \mathcal{J}v] - [u, v] - \mathcal{J}([\mathcal{J}u, v] + [u, \mathcal{J}v]).$$

Sketch of proof continued

For any generalized connection D s.t. $D\mathcal{J} = 0$ we show:

$$\begin{aligned}\langle N_{\mathcal{J}}(u,v), w\rangle &= T^D(u,v,w) - T^D(u,\mathcal{J}v,\mathcal{J}w) - T^D(\mathcal{J}u,v,\mathcal{J}w) \\ &\quad - T^D(\mathcal{J}u,\mathcal{J}v,w) \qquad (1)\end{aligned}$$

So $T^D = 0$ implies $N_{\mathcal{J}} = 0$. This proves "$\Leftarrow$".

- For the converse we observe that rhs of (1) can be viewed as $4t_{\mathcal{J}}$, where $t_{\mathcal{J}}$ is the intrinsic torsion of $\mathcal{J}$.

Sketch of proof continued

Definition

The intrinsic torsion t_Q of an H-structure Q on a Courant algebroid E is the image of T^D in $\Gamma(\wedge^3 E^*/\mathrm{im}\,\partial_Q)$, where D is any gen. conn. compatible with Q and

$$\partial_Q : E^* \otimes \wedge^2_Q E^* \to \wedge^3 E^*$$

is the algebraic torsion map (cyclic sum). Here $\wedge^2_Q E^* \subset \wedge^2 E^*$ is the subbundle which corresponds to $\mathfrak{so}(E)_Q \subset \mathfrak{so}(E)$.

For our case $Q = \mathcal{J}$, we exhibit a canonical complement $\wedge^3_Q E^* \subset \wedge^3 E^*$ of $\mathrm{im}\,\partial_Q$ such that

$$\wedge^3 E^*/\mathrm{im}\,\partial_Q \cong \wedge^3_Q E^*$$

and t_Q is viewed as an element of $\wedge^3_Q E^*$.

Sketch of proof continued

To prove the converse "$\Rightarrow$" of Theorem 1, we construct a gen. conn. D s.t. $D\mathcal{J} = 0$ and $T^D = t_{\mathcal{J}}$. Then (1) shows that the integrability of $\mathcal{J}$ implies $T^D = 0$.

- D is constructed as follows. Given any torsionfree gen. conn. D^0, we define

$$D_u v := D^0_u v - \frac{1}{4}\{A_u^{\mathrm{sym}}, \mathcal{J}\}v - \frac{1}{2}\mathcal{J}(D^0_u \mathcal{J})v,$$

  where A_u^{sym} denotes the symmetric part of $A_u = (D^0\mathcal{J})u$.
- Then we prove that D has the claimed properties. □

VI. Generalized first prolongation

Definition

Let V be a pseudo-Euclidean vector space. Given a Lie algebra $\mathfrak{h} \subset \mathfrak{so}(V) \cong \wedge^2 V^*$ we define its generalized first prolongation by

$$\mathfrak{h}^{\langle 1 \rangle} := \ker\left(\partial_{\mathfrak{h}} : V^* \otimes \mathfrak{h} \to \wedge^3 V^*\right),$$

where $\partial_{\mathfrak{h}}$ is again given by cyclic summation.

Proposition (C.-David)

Let Q be a generalized H-structure on a CA E.

- There exists a torsion-free generalized connection compatible with Q iff the intrinsic torsion of Q vanishes. It is unique iff $\mathfrak{h}^{\langle 1 \rangle} = 0$.
- Given a tensor field $T \in \Gamma(\wedge^3 E^*)$, the space of generalized connections compatible with Q which have torsion T is an affine space modelled on $\Gamma((\mathfrak{so}(E)_Q)^{\langle 1 \rangle})$. (Note that $\mathfrak{so}(E)_Q|_p \cong \mathfrak{h}$ for all $p \in M$.)

VI. Generalized first prolongation

Examples

1) Consider the case $\mathfrak{h} = \mathfrak{so}(V)$:

$$\mathfrak{so}(V)^{\langle 1 \rangle} \cong \frac{(\mathrm{Sym}^2 V^*) \otimes V^*}{\mathrm{Sym}^3 V^*}.$$

This implies that the space of torsionfree generalized connections on any Courant algebroid E is an affine space modelled on $\Gamma(\frac{(\mathrm{Sym}^2 E^*) \otimes E^*}{\mathrm{Sym}^3 E^*})$.

2) Given a generalized Riemannian metric G on E, it is known that a compatible torsionfree generalized connection exists (García Fernández). It is not unique since

$$(\mathfrak{so}(n_1) \oplus \mathfrak{so}(n_2))^{\langle 1 \rangle} = \mathfrak{so}(n_1)^{\langle 1 \rangle} \oplus \mathfrak{so}(n_2)^{\langle 1 \rangle} \neq 0,$$

provided that $\min(n_1, n_2) > 1$.

VI. Generalized first prolongation

Another example

- The diagonal subalgebra

$$\Delta_{\mathfrak{so}(n)} := \{A \oplus A \mid A \in \mathfrak{so}(n)\} \subset \mathfrak{so}(n) \oplus \mathfrak{so}(n)$$

  has $\Delta_{\mathfrak{so}(n)}^{\langle 1 \rangle} = 0$.
- Conceptional reason for uniqueness of connection ∇ which has appeared in Born geometry (Freidel, Rudolph and Svoboda).
- A Born structure on a mf. M consists of data (η, I, J, K), where η is a pseudo-Riemannian metric of neutral signature, $J, K \in \Gamma(\mathrm{End}(TM))$ are anti-commuting involutions such that $K = IJ$ is η-skew-symmetric, J is η-symmetric and $g = \eta(J\cdot, \cdot) > 0$.
- ∇ compatible with Born structure and $\nabla_X Y - \nabla_Y X - [X, Y]^c + (\nabla X)^* Y = 0$ for all X, Y, where $[X, Y]^c = \nabla^c_X Y - \nabla^c_Y X - (\nabla^c X)^* Y$, $\nabla^c = \nabla^\eta + \frac{1}{2} K(\nabla^\eta K)$.

VII. Spinors over E

Spinor bundles

Let $S \to M$ be a bundle of irreducible $\mathrm{Cl}(E)$-modules:

$$\gamma : \mathrm{Cl}(E) \to \mathrm{End}\, S, \quad a \mapsto \gamma_a = \gamma(a).$$

- We assume for simplicity that E has neutral signature. Then S is $\mathbb{Z}_2$-graded, $S = S^0 + S^1$, and we denote by

  $$[A, B] = AB - (-1)^{\deg A \deg B} BA$$

  the super commutator of homogeneous elements $A, B \in \mathrm{End}\, \Gamma(S) \supset \mathrm{Diff}(S) \supset \Gamma(\mathrm{End}\, S)$.

VII. Dirac generating operators

The theory of Dirac generating operators has been developed by Alexeev and Xu.

Definition

A first order odd differential operator $\not{d}$ on S is called a Dirac generating operator (DGO) if $\forall u, v \in \Gamma(E)$, $f \in C^\infty(M)$:

1. $[[\not{d}, f], \gamma_u] = \pi(u)(f)$,
2. $[[\not{d}, \gamma_u], \gamma_v] = \gamma_{[u,v]}$,
3. $\not{d}^2 \in C^\infty(M)$.

VII. Dirac generating operators continued

Example

Let $(\langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be the standard CA structure on $\mathbb{T}M$.

- The bracket can be twisted by a closed 3-form H:

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + H(X, Y, \cdot).$$

- $S = \wedge T^*M$ is a bundle of irreducible $\mathrm{Cl}(\mathbb{T}M)$-modules with $\gamma_{X+\xi}\varphi = \iota_X\varphi + \xi\wedge\varphi$ and

$$d_H\varphi = d\varphi - H\wedge\varphi \quad \text{is a DGO.}$$

VIII. Regular Courant algebroids

Definition

A CA $E \to M$ is called regular if the anchor $\pi : E \to TM$ is of constant rank. The CA is called exact if the sequence

$$0 \to T^*M \xrightarrow{\pi^*} E^* \cong E \xrightarrow{\pi} TM \to 0$$

is exact.

Theorem (Severa)

Every exact CA is of the form $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, \pi)$ for some closed 3-form.

Regular Courant algebroids

Severas Theorem has been generalized for regular Courant algebroids (Chen, Stiénon and Xu). These are of the form $E \cong F^* \oplus \mathcal{G} \oplus F$, where $F = \pi(E) \subset TM$ and $\mathcal{G} = (F \oplus F^*)^\perp$ are endowed with the following geometric structures:

VIII. Regular Courant algebroids continued

Data encoding a regular CA $E \cong F^* \oplus \mathcal{G} \oplus F$:

- A fiberwise Lie bracket $[\cdot,\cdot]_\mathcal{G}$ on $\mathcal{G}$ such that $\langle\cdot,\cdot\rangle|_\mathcal{G}$ is ad-invariant,
- an F-connection ∇ on $\mathcal{G}$,
- $R : \wedge^2 F \to \mathcal{G}$ and
- $\mathcal{H} \in \Gamma(\wedge^3 F^*)$ satisfying some compatibility equations.

The bracket

$\forall X, Y \in \Gamma(F), r, s \in \Gamma(\mathcal{G}), \xi, \eta \in \Gamma(F^*)$:

- $[X, Y] = \mathcal{H}(X, Y, \cdot) + R(X, Y) + \mathcal{L}_X Y$,
- $[X, r] = -[r, X] \equiv \nabla_X r \pmod{F^*}$,
- $[X, \xi] = \mathcal{L}_X \xi$, $[\xi, X] = -\mathcal{L}_X \xi + d^F(\xi(X))$,
- $[r, s] \equiv [r, s]_\mathcal{G} \pmod{F^*}$ and $[r, \xi] = [\xi, r] = [\xi, \eta] = 0$.

IX. DGOs on regular Courant algebroids

Facts (Alexeev and Xu, cf. C. and David):

- Local existence: Let E be a regular CA and S a bundle of irreducible $\mathrm{Cl}(E)$-modules. Then S admits locally a DGO $\not{d}$.
- Ambiguity: Given a DGO $\not{d}$, the space of DGOs is an affine space modeled on

$$\{v \in \Gamma(E) \mid [\not{d}, \gamma_v] \in C^\infty(M)\}.$$

- Canonization: $\exists$ line bundle $L \to M$ such that $\mathbb{S} = S \otimes L$ has a canonical DGO.
- $\Longrightarrow$ Global existence: S admits a DGO $\not{d}$.

X. An explicit formula for the canonical DGO

Theorem 2 (C.-David)

The canonical DGO can be expressed as

$$\not{d}(\omega \otimes s \otimes \tau) = (d_{\mathcal{H}}\omega) \otimes s \otimes \tau + \nabla^{\mathcal{S}_{\mathcal{G}}} s \wedge \omega \otimes \tau + \mathcal{L}(\tau) \wedge \omega \otimes s + \text{algebraic},$$

where

- $\omega \in \wedge F^*$,
- s is a section of a certain bundle $\mathcal{S}_{\mathcal{G}}$ of irreducible $\mathrm{Cl}(\mathcal{G})$-modules,
- τ is a section of a certain line bundle with a natural F-connection $\mathcal{L}$ and
- "algebraic" stands for an explicit algebraic operator involving the Cartan 3-form $C \in \Gamma(\wedge^3 \mathcal{G}^*)$ and $R \in \Gamma(\wedge^2 F^* \otimes \mathcal{G})$.

XI. Spinorial characterization of generalized Kähler structures on regular Courant algebroids

- Let $(G, \mathcal{J})$ be a gen. alm. Herm. structure on E and denote by $E_\pm$ the eigenbundles of the involution $G^{\mathrm{end}} \in \Gamma(\mathrm{End}(E))$, $G = \langle G^{\mathrm{end}} \cdot, \cdot \rangle$.
- Assume for simplicity that $\mathrm{rk}\, E_+ = \mathrm{rk}\, E_- \equiv 0 \pmod 8$. Then we can decompose $S = S_+ \hat{\otimes} S_-$ as $\mathbb{Z}_2$-graded tensor product of a $\mathrm{Cl}(E_+)$-module S_+ and a $\mathrm{Cl}(E_-)$-module S_-.
- $\mathcal{J}|_{E_\pm}$ defines pure spinors (up to scale) $\eta_\pm \in \Gamma(S_\pm^{\mathbb{C}})$ s.t.

$$\ker(\gamma\eta_\pm : E_\pm^{\mathbb{C}} \to S_\pm^{\mathbb{C}}) = E_\pm^{1,0}$$

and $\eta = \eta_+ \otimes \eta_-$ is pure spinor in $S^{\mathbb{C}}$ s.t. $E^{1,0} = \ker(\gamma\eta)$.

XI. Spinorial characterization of generalized Kähler structures on regular Courant algebroids

Theorem 3 (C.-David)

$(G, \mathcal{J})$ is generalized Kähler iff $\exists$ torsionfree metric generalized connection D s.t.

$$\not{D}^{S_+}\eta_+ \in \Gamma(\gamma_{E_+^{\mathbb{C}}}\eta_+), \quad \not{D}^{S_-}\eta_- \in \Gamma(\gamma_{E_-^{\mathbb{C}}}\eta_-), \tag{2}$$

$$D^{S_+}_{v_-}\eta_+ \in \Gamma(\mathbb{C}\eta_+), \quad D^{S_-}_{v_+}\eta_- \in \Gamma(\mathbb{C}\eta_-), \tag{3}$$

for all $v_\pm \in \Gamma(E_\pm)$.

- Here $D^{S_\pm}$ stands for an E-connection on $S_\pm$ compatible with the E-connection $D|_{\Gamma(E_\pm)} : \Gamma(E_\pm) \to \Gamma(E^* \otimes E_\pm)$ and $\not{D}^{S_\pm}$ for the corresponding Dirac operator; e.g. $\not{D}^{S_+} = \frac{1}{2}\sum \gamma_{e_i^+} D^{S_+}_{e_i^+}$ in terms of a local ON frame (e_i^+) of E_+.

Sketch of proof

"$\Rightarrow$"

- We use the existence of a torsionfree compatible generalized connection D established in Theorem 1 to check eqs. (2), (3).
- In fact, compatibility implies $D^{S_+}\eta_+ \in \Gamma(E^* \otimes \eta_+)$, $D^{S_-}\eta_- \in \Gamma(E^* \otimes \eta_-)$, hence (2), (3).

"$\Leftarrow$"

- We check that the equations (2), (3) imply that η is projectively closed, i.e. $\not{d}\eta \in \Gamma(\gamma_{E^{\mathbb{C}}}\eta)$, for the Dirac generating operator $\not{d}$ of Theorem 2.
- This property implies the integrability of $\mathcal{J}$ by results of Alexeev and Xu.
- Similarly, (2), (3) do also imply that $\eta_+ \otimes \bar{\eta}_-$ is projectively closed, where $\eta_+ \otimes \bar{\eta}_-$ is the pure spinor associated with $\mathcal{J}G^{\mathrm{end}}$, implying the integrability of $\mathcal{J}G^{\mathrm{end}}$. □

Gràcies !

¡ Gracias !