Generalized geometry with skew-symmetric pairing

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1. Standard generalized geometry

- In order to study geometrical structures, generalized geometry suggests to replace T by $T \oplus T^*$.
- It comes with the natural symmetric non-degenerate pairing:

$$\langle X+\alpha, Y+\beta\rangle_+ := \frac{1}{2}(\alpha(Y)+\beta(X)).$$

- Geometry is brought by the Dorfman bracket:

$$[X+\alpha, Y+\beta]_D := [X,Y]_{\mathsf{Lie}} + \pounds_X\beta - \iota_Y \mathrm{d}\alpha.$$

- This framework offers an alternative viewpoint on geometrical structures, e.g. (pre-)symplectic, Poisson, foliation and complex. Moreover, it gives a natural generalization of these structures.

2. Generalized geometry with skew-symmetric pairing

- Besides $\langle\ ,\ \rangle_+$, there is also the natural skew-symmetric non-degenerate pairing:

$$\langle X+\alpha, Y+\beta\rangle_- := \frac{1}{2}(\alpha(Y)-\beta(X)).$$

The Dorfman bracket is not working!

The subgroup of $\mathrm{Aut}(T\oplus T^*)$ preserving $\langle\ ,\ \rangle_-$ and $[\ ,\ ]_D$ is isomorphic to $\mathrm{Diff}(M)$. *(Not generalized!)*

What is behind this?

The symmetric pairing $\langle\ ,\ \rangle_+$ induces the natural Clifford algebra action $\mathcal{C}l(T\oplus T^*) \circlearrowright \Gamma(\wedge^\bullet T^*)$:

$$(X+\alpha)\cdot\varphi := \iota_X\varphi + \alpha\wedge\varphi,$$

which recovers the Dorfman bracket as a derived bracket:

$$[[(X+\alpha)\cdot, \mathrm{d}]_{\mathrm{g}}, (Y+\beta)\cdot]_{\mathrm{g}}\,\varphi = [X+\alpha, Y+\beta]_D\cdot\varphi.$$

- The skew-symmetric pairing $\langle\ ,\ \rangle_-$ induces the natural Weyl algebra action $\mathcal{W}(T\oplus T^*) \circlearrowright \Gamma(\odot^\bullet T^*)$:

$$(X+\alpha)\cdot\sigma := \iota_X\varphi + \alpha\odot\sigma.$$

- Does it lead to a new bracket?

$$[[(X+\alpha)\cdot, ?]_{\mathrm{g}}, (Y+\beta)\cdot]_{\mathrm{g}}\,\sigma.$$

3. Symmetric Cartan calculus

- $\iota_X \in \mathrm{Der}_{-1}(\Gamma(\odot^\bullet T^*))$ for every $X\in\Gamma(T)$, $\Rightarrow$ derivations are more natural than graded derivations for symmetric forms.

Replacement for d?

Definition

For any affine connection ∇, we introduce the **symmetric derivative***:*

$$\nabla^s : \Gamma(\odot^\bullet T^*) \to \Gamma(\odot^\bullet T^*), \qquad \nabla^s\sigma := (|\sigma|+1)\,\mathrm{Sym}(\nabla\sigma).$$

- The assignment $\nabla \mapsto \nabla^s$ gives the one-to-one correspondence:

$$\left\{\text{torsion-free affine connections}\right\} \overset{\sim}{\longleftrightarrow} \left\{ D\in\mathrm{Der}_1(\Gamma(\odot^\bullet T^*)) \text{ such that } D|_{C^\infty(M)} = \mathrm{d}|_{C^\infty(M)}\right\}.$$

- If $\dim M > 0$, there is **no** $D\in\mathrm{Der}_1(\Gamma(\odot^\bullet T^*))$ such that $D|_{C^\infty(M)} = \mathrm{d}|_{C^\infty(M)}$ and $D\circ D = 0$.

Replacement for $\pounds_X$?

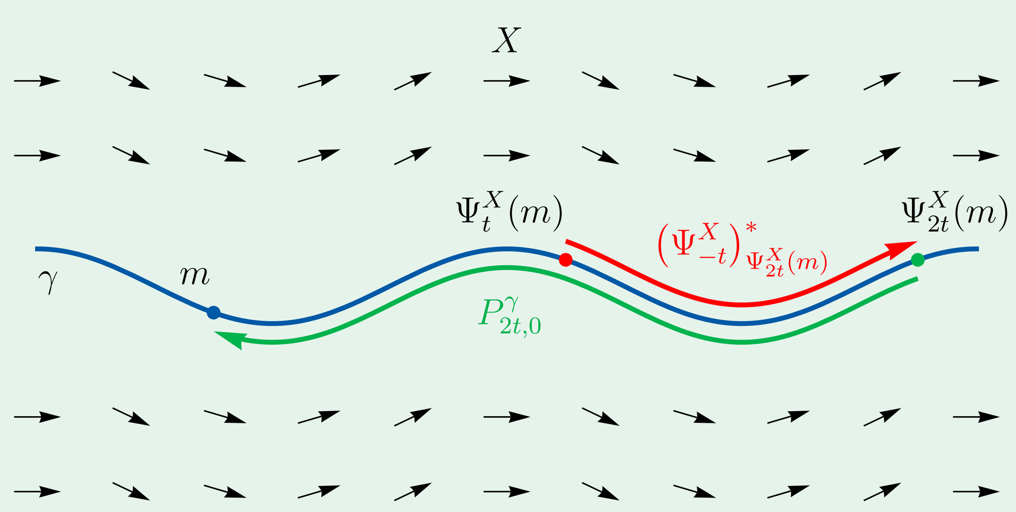
Definition

The **symmetric Lie derivative** *w.r.t. $X\in\Gamma(T)$:*

$$\pounds_X^{\nabla^s} := [\iota_X, \nabla^s].$$

$$(\pounds_X^{\nabla^s}\sigma)_m = \lim_{t\to 0}\frac{1}{t}\left(P^\gamma_{2t,0}\left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(m)}\sigma_{\Psi^X_t(m)} - \sigma_m\right)$$

- The formula is used to extend $\pounds_X^{\nabla^s}$ to arbitrary tensor fields.



Transport generating the symmetric Lie derivative.

Replacement for $[\ ,\ ]_{Lie}$?

Definition

The **symmetric bracket** *is the $\mathbb{R}$-bilinear map:*

$$\langle\ :\ \rangle_{\nabla^s} : \times^2\Gamma(T)\to\Gamma(T), \qquad \iota_{\langle X:Y\rangle_{\nabla^s}} := [[\iota_X, \nabla^s], \iota_Y].$$

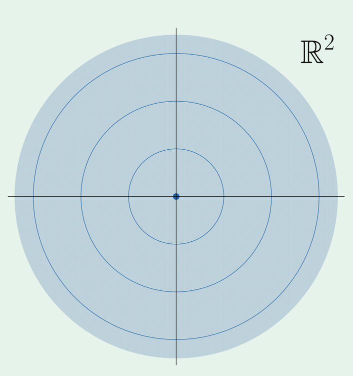
$$\langle X:Y\rangle_{\nabla^s} = \pounds_X^{\nabla^s} Y = \nabla_X Y + \nabla_Y X.$$

Definition

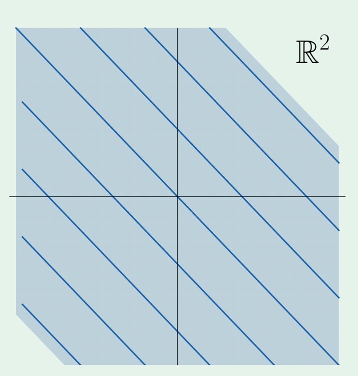
A distribution $\Delta\subseteq T$ is called **geodesically invariant** *if every geodesic $\gamma: I\to M$ has the property:*

$$\exists t_0\in I \text{ such that } \dot\gamma(t_0)\in\Delta_{\gamma(t_0)} \quad\Rightarrow\quad \dot\gamma(t)\in\Delta_{\gamma(t)} \text{ for all } t\in I.$$

- By [Lewis, 1998], a distribution $\Delta\leq T$ is geodesically invariant if and only if $\langle\Gamma(\Delta):\Gamma(\Delta)\rangle_{\nabla^s}\subseteq\Gamma(\Delta)$.



A **non** geodesically invariant distribution.



A geodesically invariant distribution.

4. Symmetric Poisson structures

Definition

*The ∇^s-***Schouten bracket** *is the unique $\mathbb{R}$-bilinear map $[\ ,\ ]_{\nabla^s\text{-}Sch} : \times^2\Gamma(\odot^\bullet T)\to\Gamma(\odot^\bullet T)$ satisfying*

1. $[\mathcal{X},\ ]_{\nabla^s\text{-}Sch} \in \mathrm{Der}_{|\mathcal{X}|-1}(\Gamma(\odot^\bullet T))$, 2. $[X,\ ]_{\nabla^s\text{-}Sch} = \pounds_X^{\nabla^s}$, 3. $[\mathcal{X},\mathcal{Y}]_{\nabla^s\text{-}Sch} = [\mathcal{Y},\mathcal{X}]_{\nabla^s\text{-}Sch}$.

Definition

A pair (∇,ϑ) consisting of a torsion-free affine connection ∇ and $\vartheta\in\Gamma(\odot^2 T)$ is called a **symmetric Poisson structure** *if $[\vartheta,\vartheta]_{\nabla^s\text{-}Sch} = 0$.*

$$\left\{\begin{matrix}\text{symmetric bivector}\\ \text{fields on } M\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\mathbb{R}\text{-bilinear maps } \{\ ,\ \} : \times^2 C^\infty(M)\to C^\infty(M) \text{ such that}\\ 1.\ \{f,g\}=\{g,f\} \quad 2.\ \{f,gh\} = g\{f,h\}+\{f,g\}h\end{matrix}\right\}.$$

- $\mathrm{grad} := \vartheta\circ\mathrm{d} : C^\infty(M)\to\Gamma(T)$, or in terms of the corresponding bracket $\mathrm{grad}\, f = \{f,\ \}$.

Proposition

(∇,ϑ) *is a symmetric Poisson structure*

$\Leftrightarrow$

$$\mathrm{Jac}_{\{\ ,\ \}}(f,g,h) = \mathrm{d}f(\langle \mathrm{grad}\, g : \mathrm{grad}\, h\rangle_{\nabla^s}) + \mathrm{cyc}(f,g,h),$$

$\Leftrightarrow$ *(non-deg. case)*

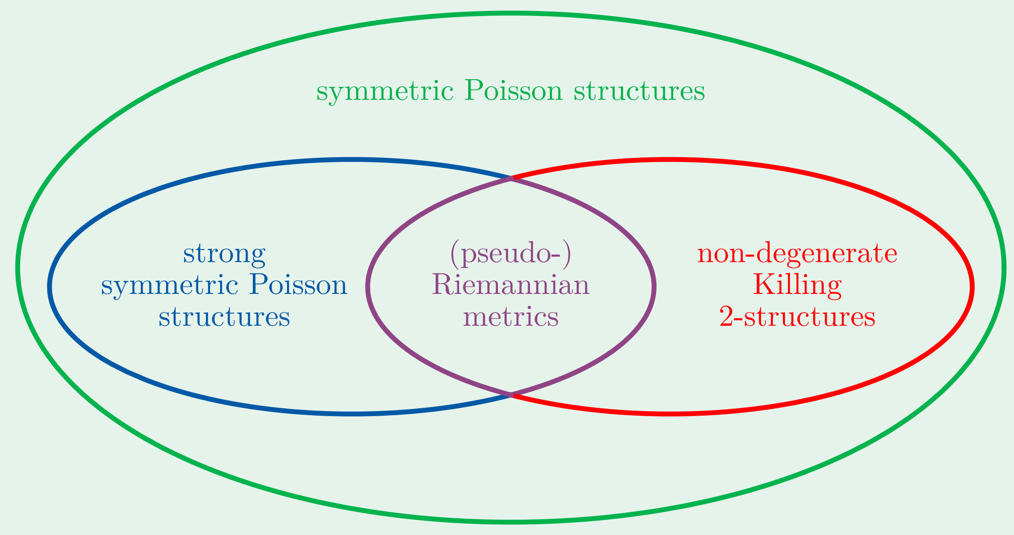
$\nabla^s\vartheta^{-1} = 0$, *i.e.* $\vartheta^{-1}\in\Gamma(\odot^2 T^*)$ *is a Killing 2-tensor.*

Definition

A pair (∇,ϑ) is called a **strong symmetric Poisson structure** *if $\mathrm{grad}\{f,g\} = \langle\mathrm{grad}\, f : \mathrm{grad}\, g\rangle_{\nabla^s}$.*

Proposition

(∇,ϑ) *is a strong symmmetric Poisson structure* $\Leftrightarrow$ $\nabla_{\mathrm{grad}\, f}\vartheta = 0$ $\underset{(non\text{-}deg.\ case)}{\Leftrightarrow}$ ∇ *is Levi-Civita w.r.t.* ϑ^{-1}.



5. Back to generalized geometry!

Definition

*The ∇^s-***Dorfman bracket***:*

$$[X+\alpha, Y+\beta]_{\nabla^s} = \langle X:Y\rangle_{\nabla^s} + \pounds_X^{\nabla^s}\beta + \iota_Y\nabla^s\alpha.$$

$$[[(X+\alpha)\cdot, \nabla^s]_{\mathrm{g}}, (Y+\beta)\cdot]_{\mathrm{g}}\,\sigma = [[\iota_X,\nabla^s],\iota_Y]\,\sigma + ([\iota_X,\nabla^s]\beta + \iota_Y\nabla^s\alpha)\odot\sigma.$$

Theorem

The subgroup of $\mathrm{Aut}(T\oplus T^*)$ preserving $\langle\ ,\ \rangle_-$ and $[\ ,\ ]_{\nabla^s}$ is isomorphic to

$$\mathrm{Aff}(M,\nabla)\ltimes\mathrm{Kill}^2_\nabla(M).$$

- $\mathrm{Aff}(M,\nabla) := \{\phi\in\mathrm{Diff}(M)\,|\,\phi_*\nabla_X Y = \nabla_{\phi_* X}\phi_* Y\}$ is the group of affine transformations of (M,∇).
- $\mathrm{Kill}^2_\nabla(M) := \ker\nabla^s|_{\Gamma(\odot^2 T^*)}$ is the abelian group of Killing 2-tensors.

Definition

∇^s-**Dirac structure***: a Lagrangian subbundle $L\leq(T\oplus T^*,\langle\ ,\ \rangle_-)$ such that $[\Gamma(L),\Gamma(L)]_{\nabla^s}\subseteq\Gamma(L)$.*

Examples

- $\mathrm{gr}(K)\leq T\oplus T^*$ is a ∇^s-Dirac structure if and only if K is a Killing 2-tensor.
- $\mathrm{gr}(\vartheta)\leq T\oplus T^*$ is a ∇^s-Dirac structure if and only if (∇,ϑ) is a symmetric Poisson structure.
- $\Delta\oplus\mathrm{Ann}\,\Delta\leq T\oplus T^*$ is a ∇^s-Dirac structure if and only if Δ is geodesically invariant.

6. Comparison with standard generalized geometry

Standard generalized geometry	Skew-symmetric generalized geometry
$\frac{1}{2}(\alpha(Y)+\beta(X))$	$\frac{1}{2}(\alpha(Y)-\beta(X))$
$\mathcal{C}l(T\oplus T^*)\circlearrowright\Gamma(\wedge^\bullet T^*)$	$\mathcal{W}(T\oplus T^*)\circlearrowright\Gamma(\odot^\bullet T^*)$
$[X,Y]_{\mathsf{Lie}}+\pounds_X\beta-\iota_Y\mathrm{d}\alpha$	$\langle X:Y\rangle_{\nabla^s}+\pounds_X^{\nabla^s}\beta+\iota_Y\nabla^s\alpha$
$\mathrm{Diff}(M)\ltimes\Gamma_{\mathsf{closed}}(\wedge^2 T^*)$	$\mathrm{Aff}(M,\nabla)\ltimes\mathrm{Kill}^2_\nabla(M)$
$[a,b]_D+[b,a]_D = 2\mathrm{d}\langle a,b\rangle_+$	$[a,b]_{\nabla^s}-[b,a]_{\nabla^s} = 2\mathrm{d}\langle a,b\rangle_-$
$\mathrm{pr}_T(a)\langle b,c\rangle_+ = \langle[a,b]_D,c\rangle_+ + \langle b,[a,c]_D\rangle_+$	$\mathrm{pr}_T(a)\langle b,c\rangle_- = \langle[a,b]_{\nabla^s},c\rangle_- + \langle b,[a,c]_{\nabla^s}\rangle_-$
$[a,[b,c]_D]_D = [[a,b]_D,c]_D + [b,[a,c]_D]_D$	–
Dirac structures	∇^s-Dirac structures
pre-symplectic structures	Killing 2-tensors
Poisson structures	symmetric Poisson structures
foliations	geodesically invariant distributions

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