# Generalized geometry with skew-symmetric pairing

Filip Moučka
(joint work with Roberto Rubio)

FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING CTU IN PRAGUE

UAB Universitat Autònoma de Barcelona

Student Colloquium and School on Mathematical Physics,
Stará Lesná

August 29, 2024

## Introduction

Studying geometry of a smooth manifold $M$ via its tangent bundle $T$.

$$\omega \in \Gamma(\wedge^2 T^*), \qquad \text{(2-form)}$$

$$\pi \in \Gamma(\wedge^2 T), \qquad \text{(bivector field)}$$

$$\Delta \leq T, \qquad \text{(distribution)}$$

## Introduction

Studying geometry of a smooth manifold $M$ via its tangent bundle $T$.

$$\omega \in \Gamma(\wedge^2 T^*), \qquad \mathrm{d}\omega = 0, \qquad \text{(pre-symplectic)}$$

$$\pi \in \Gamma(\wedge^2 T), \qquad [\pi, \pi]_{\mathsf{Schouten}} = 0, \qquad \text{(Poisson)}$$

$$\Delta \leq T, \qquad [\Gamma(\Delta), \Gamma(\Delta)]_{\mathsf{Lie}} \subseteq \Gamma(\Delta). \qquad \text{(foliation)}$$

All of these geometrical conditions may be expressed using solely:

$$X \cdot f := Xf, \qquad [X, Y]_{\mathsf{Lie}} := X \circ Y - Y \circ X.$$

## Introduction

Studying geometry of a smooth manifold $M$ via its tangent bundle $T$.

$$\omega \in \Gamma(\wedge^2 T^*), \qquad d\omega = 0, \qquad \text{(pre-symplectic)}$$
$$\pi \in \Gamma(\wedge^2 T), \qquad [\pi,\pi]_{\text{Schouten}} = 0, \qquad \text{(Poisson)}$$
$$\Delta \leq T, \qquad [\Gamma(\Delta),\Gamma(\Delta)]_{\text{Lie}} \subseteq \Gamma(\Delta). \qquad \text{(foliation)}$$

All of these geometrical conditions may be expressed using solely:

$$X \cdot f := Xf, \qquad [X,Y]_{\text{Lie}} := X \circ Y - Y \circ X.$$

---

Generalized geometry proposes to change the perspective: $T \rightsquigarrow T \oplus T^*$.

$$\mathrm{gr}(\omega) \leq T \oplus T^*, \qquad \text{(2-form)}$$
$$\mathrm{gr}(\pi) \leq T \oplus T^*, \qquad \text{(bivector field)}$$
$$\Delta \oplus \mathrm{Ann}\,\Delta \leq T \oplus T^*, \qquad \text{(distribution)}$$

# Introduction

Studying geometry of a smooth manifold $M$ via its tangent bundle $T$.

$$\omega \in \Gamma(\wedge^2 T^*), \qquad \mathrm{d}\omega = 0, \qquad \text{(pre-symplectic)}$$
$$\pi \in \Gamma(\wedge^2 T), \qquad [\pi, \pi]_{\text{Schouten}} = 0, \qquad \text{(Poisson)}$$
$$\Delta \leq T, \qquad [\Gamma(\Delta), \Gamma(\Delta)]_{\text{Lie}} \subseteq \Gamma(\Delta). \qquad \text{(foliation)}$$

All of these geometrical conditions may be expressed using solely:

$$X \cdot f := Xf, \qquad [X, Y]_{\text{Lie}} := X \circ Y - Y \circ X.$$

---

Generalized geometry proposes to change the perspective: $T \rightsquigarrow T \oplus T^*$.

$$\mathrm{gr}(\omega) \leq T \oplus T^*, \qquad \text{(pre-symplectic)}$$
$$\mathrm{gr}(\pi) \leq T \oplus T^*, \qquad [\Gamma(L), \Gamma(L)]_D \subseteq \Gamma(L). \qquad \text{(Poisson)}$$
$$\Delta \oplus \mathrm{Ann}\,\Delta \leq T \oplus T^*, \qquad \text{(foliation)}$$

We still have the action on $C^\infty(M)$ and the bracket (Dorfman bracket):

$$(X + \alpha) \cdot f := Xf, \qquad [X + \alpha, Y + \beta]_D := [X, Y]_{\text{Lie}} + \pounds_X \beta - \iota_Y \mathrm{d}\alpha.$$

## Natural pairings on $T \oplus T^*$

There is the natural symmetric non-degenerate pairing on $T \oplus T^*$:

$$\langle X + \alpha, Y + \beta \rangle_+ := \frac{1}{2}(\alpha(Y) + \beta(X)).$$

All of the subbundles $\mathrm{gr}(\omega)$, $\mathrm{gr}(\pi)$, $\Delta \oplus \mathrm{Ann}\,\Delta$ are maximally isotropic.

A common generalization for pre-symplectic, Poisson, and foliation:

**Definition** (T. Courant, 1990)

$L \leq T \oplus T^*$ is called a **Dirac structure** if it is max. isotropic w.r.t. $\langle\ ,\ \rangle_+$ and

$$[\Gamma(L), \Gamma(L)]_D \subseteq \Gamma(L).$$

There is also the natural skew-symmetric non-degenerate pairing on $T \oplus T^*$:

$$\langle X + \alpha, Y + \beta \rangle_- := \frac{1}{2}(\alpha(Y) - \beta(X)).$$

Natural question:

*Can we do generalized geometry with $\langle\ ,\ \rangle_-$?*

## Q: Is the Dorfman bracket not suitable?

1. Every subbundle of $T \oplus T^*$, which is Lagrangian w.r.t. $\langle\ ,\ \rangle_-$ and involutive w.r.t. $[\ ,\ ]_D$, is of the form

$$\Delta \oplus \mathrm{Ann}\,\Delta, \qquad\qquad [\Gamma(\Delta), \Gamma(\Delta)]_{\text{Lie}} \subseteq \Gamma(\Delta).$$

2. The group of vector bundle automorphisms preserving $\langle\ ,\ \rangle_-$ and $[\ ,\ ]_D$ is isomorphic to $\mathrm{Diff}(M)$.

A: No. And rightly so:

The pairing $\langle\ ,\ \rangle_+$ induces the natural Clifford algebra action $\mathcal{C}l(T \oplus T^*) \circlearrowright \Gamma(\wedge^\bullet T^*)$:

$$(X + \alpha) \cdot \varphi := \iota_X \varphi + \alpha \wedge \varphi.$$

$\Rightarrow$

$$[[(X + \alpha) \cdot , \mathrm{d}]_{\mathrm{g}}, (Y + \beta) \cdot ]_{\mathrm{g}}\, \varphi = [X + \alpha, Y + \beta]_D \cdot \varphi.$$

*We need a new bracket!*

Compare with: $[[\iota_X, \mathrm{d}]_{\mathrm{g}}, \iota_Y]_{\mathrm{g}} = \iota_{[X,Y]_{\text{Lie}}}$.

# New bracket

The pairing $\langle\ ,\ \rangle_-$ induces the natural Weyl algebra action $\mathcal{W}(T \oplus T^*) \circlearrowright \Gamma(\odot^\bullet T^*)$:

$$(X + \alpha) \cdot \sigma := \iota_X \sigma + \alpha \odot \sigma.$$

It means

$$(X+\alpha)\cdot(Y+\beta)\cdot\sigma - (Y+\beta)\cdot(X+\alpha)\cdot\sigma + 2\langle X+\alpha, Y+\beta\rangle_-\sigma = 0.$$

Inspired by the Dorfman bracket:

$$[[(X+\alpha)\cdot\,, \mathbf{?}\,]_g, (Y+\beta)\cdot\,]_g\,\sigma.$$

# New bracket

The pairing $\langle \ , \ \rangle_-$ induces the natural Weyl algebra action $\mathcal{W}(T \oplus T^*) \circlearrowright \Gamma(\odot^\bullet T^*)$:

$$(X+\alpha)\cdot\sigma := \iota_X \sigma + \alpha \odot \sigma.$$

It means

$$(X+\alpha)\cdot(Y+\beta)\cdot\sigma - (Y+\beta)\cdot(X+\alpha)\cdot\sigma + 2\langle X+\alpha, Y+\beta\rangle_- \sigma = 0.$$

Inspired by the Dorfman bracket:

$$[[(X+\alpha)\cdot, \nabla^s]_{\not{s}}, (Y+\beta)\cdot]_{\not{s}}\,\sigma.$$

**Definition**

For any affine connection $\nabla$, we introduce the **symmetric derivative**:

$$\nabla^s : \Gamma(\odot^\bullet T^*) \to \Gamma(\odot^\bullet T^*), \qquad \nabla^s \sigma := (|\sigma| + 1)\,\mathrm{Sym}(\nabla\sigma).$$

Now it works!

$$[[(X+\alpha)\cdot, \nabla^s], (Y+\beta)\cdot] = [[\iota_X, \nabla^s], \iota_Y] + ([\iota_X, \nabla^s]\beta)\odot + (\iota_Y \nabla^s \alpha)\odot$$

*Cartan calculus on the space of symmetric forms?*

Compare with: $\mathrm{d}\varphi := (|\varphi| + 1)\,\mathrm{Skew}(\nabla\varphi)$.

# Symmetric Cartan calculus – symmetric derivatives

**Proposition**

For every $X \in \Gamma(T)$, we have $\iota_X \in \mathrm{Der}_{-1}(\Gamma(\odot^\bullet T^*))$.

$\Rightarrow$ On the space $\Gamma(\odot^\bullet T^*)$, derivations are more natural than graded derivations!

**Proposition**

The assignment $\nabla \mapsto \nabla^s$ gives the one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{torsion-free} \\ \text{affine connections} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} D \in \mathrm{Der}_1(\Gamma(\odot^\bullet T^*)) \text{ s.t.} \\ (Df)(X) = Xf \end{array} \right\}.$$

**Proposition**

If $\dim M > 0$, there is **no** $D \in \mathrm{Der}_1(\Gamma(\odot^\bullet T^*))$ s.t. $(Df)(X) = Xf$ and $D \circ D = 0$.

$\Rightarrow$ A symmetric derivative $\nabla^s$ is the only natural replacement for $\mathrm{d}$.

$\Rightarrow$ The theory depends on the choice of a torsion-free affine connection.

# Symmetric Cartan calculus – symmetric Lie derivatives

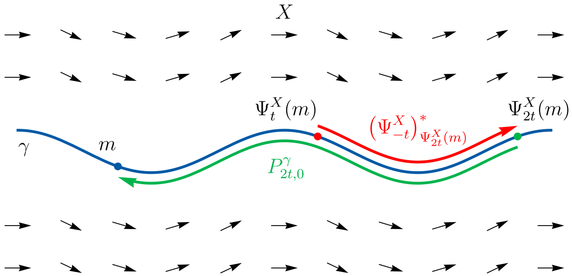
**Definition**

We introduce the **symmetric Lie derivative** w.r.t. $X \in \Gamma(T)$:

$$\pounds_X^{\nabla^s} := [\iota_X, \nabla^s].$$

**Proposition**

$$(\pounds_X^{\nabla^s}\sigma)_m = \lim_{t\to 0}\frac{1}{t}\left(P^\gamma_{2t,0}\left(\Psi^X_{-t}\right)^*_{\Psi^X_{2t}(m)}\sigma_{\Psi^X_t(m)} - \sigma_m\right).$$



## Symmetric Cartan calculus – symmetric brackets

**Definition**

We introduce the **symmetric bracket**

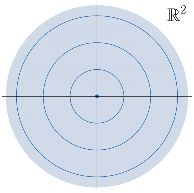
$$\langle\ :\ \rangle_{\nabla^s} : \times^2\Gamma(T) \to \Gamma(T), \qquad \iota_{\langle X:Y\rangle_{\nabla^s}} := [[\iota_X, \nabla^s], \iota_Y].$$

**Proposition**

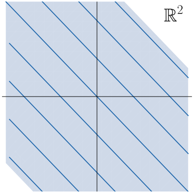
$$\langle X : Y\rangle_{\nabla^s} = \pounds_X^{\nabla^s} Y = \nabla_X Y + \nabla_Y X.$$

A distribution $\Delta \subseteq T$ is called **geodesically inv.** if every geodesic $\gamma : I \to M$ satisfies:

$$\exists t_0 \in I \text{ s.t. } \dot\gamma(t_0) \in \Delta_{\gamma(t_0)} \qquad \Rightarrow \qquad \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ for all } t \in I.$$




a **non** geodesically inv. distribution




a geodesically inv. distribution

By [Lewis, 1998], a distribution $\Delta \leq T$ is geodesically inv. iff $\langle\Gamma(\Delta) : \Gamma(\Delta)\rangle_{\nabla^s} \subseteq \Gamma(\Delta)$.

# Symmetric multivector fields

The **Schouten bracket** is the unique $\mathbb{R}$-bilinear map

$$[\ ,\ ]_{\mathsf{Sch}} : \times^2\Gamma(\odot^\bullet T) \to \Gamma(\odot^\bullet T)$$

s.t.

(Sch1) $[\mathcal{X},\ ]_{\mathsf{Sch}} \in \mathrm{Der}_{|\mathcal{X}|-1}(\Gamma(\odot^\bullet T))$,

(Sch2) $[X,\ ]_{\mathsf{Sch}} = \mathcal{L}_X$,

(Sch3) $[\mathcal{X},\mathcal{Y}]_{\mathsf{Sch}} = -[\mathcal{Y},\mathcal{X}]_{\mathsf{Sch}}$.

For every $k \in \mathbb{N}_0$ there is the one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{degree-}k \text{ symmetric} \\ \text{multivector fields} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{degree-}k \text{ polynomials} \\ \text{in momenta} \end{array} \right\} \subseteq C^\infty(T^*M).$$

This correspondence $\mathcal{X} \mapsto \xi_\mathcal{X}$ is a Poisson algebra morphism, namely

$$\xi_{\mathcal{X}+\mathcal{Y}} = \xi_\mathcal{X} + \xi_\mathcal{Y}, \quad \xi_{\mathcal{X}\odot\mathcal{Y}} = \xi_\mathcal{X}\xi_\mathcal{Y}, \quad \xi_{[\mathcal{X},\mathcal{Y}]_{\mathsf{Sch}}} = -\{\xi_\mathcal{X},\xi_\mathcal{Y}\}_{\mathsf{can}} = \omega_{\mathsf{can}}(\mathrm{Ham}\,\xi_\mathcal{X},\mathrm{Ham}\,\xi_\mathcal{Y}).$$

## Symmetric multivector fields

The $\nabla^s$-**Schouten bracket** is the unique $\mathbb{R}$-bilinear map

$$[\ ,\ ]_{\nabla^s\text{-Sch}} : \times^2\Gamma(\odot^\bullet T) \to \Gamma(\odot^\bullet T)$$

s.t.

($\nabla^s$-Sch1) $[\mathcal{X},\ ]_{\nabla^s\text{-Sch}} \in \mathrm{Der}_{|\mathcal{X}|-1}(\Gamma(\odot^\bullet T))$,

($\nabla^s$-Sch2) $[X,\ ]_{\nabla^s\text{-Sch}} = \pounds_X^{\nabla^s}$,

($\nabla^s$-Sch3) $[\mathcal{X},\mathcal{Y}]_{\nabla^s\text{-Sch}} = +[\mathcal{Y},\mathcal{X}]_{\nabla^s\text{-Sch}}$.

For every $k \in \mathbb{N}_0$ there is the one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{degree-}k\text{ symmetric} \\ \text{multivector fields} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{degree-}k\text{ polynomials} \\ \text{in momenta} \end{array} \right\} \subseteq C^\infty(T^*M).$$

This correspondence $\mathcal{X} \mapsto \xi_\mathcal{X}$ is a Poisson algebra morphism, namely

$$\xi_{\mathcal{X}+\mathcal{Y}} = \xi_\mathcal{X} + \xi_\mathcal{Y}, \quad \xi_{\mathcal{X}\odot\mathcal{Y}} = \xi_\mathcal{X}\xi_\mathcal{Y}, \quad \xi_{[\mathcal{X},\mathcal{Y}]_{\text{Sch}}} = -\{\xi_\mathcal{X},\xi_\mathcal{Y}\}_{\text{can}} = \omega_{\text{can}}(\mathrm{Ham}\,\xi_\mathcal{X},\mathrm{Ham}\,\xi_\mathcal{Y}).$$

We have a new Lie derivative $\Rightarrow$ a new Schouten bracket.

*Can we say anything about $\xi_{[\mathcal{X},\mathcal{Y}]_{\nabla^s\text{-Sch}}}$?*

## Patterson-Walker metric

A torsion-free affine connection induces the dual connection on $T^*M$, hence

$$T(T^*M) = H_\nabla \oplus V \cong_\nabla \mathrm{pr}^*(TM \oplus T^*M).$$

$\Rightarrow$ Equivalent definition of the canonical 2-form:

$$\omega_{\mathsf{can}} = 2(\langle\ ,\ \rangle_-)_\nabla.$$

**Definition** (Patterson-Walker, 1952)

The **Patterson-Walker metric** is $G_\nabla \in \Gamma(\odot^2 T^*(T^*M))$ given by

$$G_\nabla := 2(\langle\ ,\ \rangle_+)_\nabla.$$

**Proposition**

- $G_\nabla$ is a split signature metric on $T^*M$, and $H_\nabla$, $V$ are maximally isotropic.
- $G_\nabla|_{T^*U} = \mathrm{d}p_j \odot \mathrm{d}x^j - p_l\, \Gamma^l{}_{ij} \mathrm{d}x^i \odot \mathrm{d}x^j$.
- $\xi_{[\mathcal{X},\mathcal{Y}]_{\nabla^s\text{-Sch}}} = G_\nabla(\mathrm{grad}_\nabla\, \xi_\mathcal{X}, \mathrm{grad}_\nabla\, \xi_\mathcal{Y})$.

$\mathrm{grad}_\nabla := G_\nabla^{-1} \circ \mathrm{d} : C^\infty(T^*M) \to \Gamma(T(T^*M))$.

## Patterson-Walker metric

**Proposition**

The Patterson-Walker metric $G_\nabla$ is related to the canonical $1$-form:

$$G_\nabla = \nabla^s \alpha_{\text{can}}.$$

**Remark**: Dorfman bracket $\overset{\nabla}{\rightsquigarrow}$ bracket on $T(T^*M)$.
If $\nabla$ is flat, we get a non-transitive Courant algebroid on $T(T^*M) \to T^*M$.

*Can we do dynamics with $G_\nabla$ instead of $\omega_{can}$?*

The Hamiltonian $h$ is conserved along the trajectories iff $\operatorname{grad}_\nabla h$ is isotropic.

**Proposition**

$$\operatorname{pr}_{H_\nabla} \operatorname{grad}_\nabla h = -\operatorname{pr}_{H_\nabla} \operatorname{Ham} h, \qquad \text{and} \qquad \operatorname{pr}_V \operatorname{grad}_\nabla h = \operatorname{pr}_V \operatorname{Ham} h.$$

If $\operatorname{grad}_\nabla h$ is vertical or horizontal, the trajectories coincide with the classical ones.

Compare with: $\omega_{\text{can}} = \mathrm{d}\alpha_{\text{can}}$.

## Patterson-Walker dynamics

The trajectories are locally governed by the system of ODEs:

$$\dot{x}^j = \frac{\partial h}{\partial p_j}, \qquad \dot{p}_j = \frac{\partial h}{\partial x^j} + 2p_l\, \Gamma^l{}_{ij} \frac{\partial h}{\partial p_i}.$$

EOM of a mechanical system for $\beta : I \to T^*M$ covering $\gamma : I \to M$:

$${}^g\nabla_{\dot\gamma}\dot\gamma = -\operatorname{grad}_g f, \qquad \beta = \frac{1}{2} g(\dot\gamma)$$

is obtained by considering $h(m,\alpha) := \frac{1}{2} g_m^{-1}(\alpha,\alpha) - f(m)$ and $\nabla := {}^g\nabla$.

**Example**: Hamiltonian quadratic in momenta

Let $\vartheta \in \Gamma(\odot^2 TM)$ be possibly degenerate and $h(m,\alpha) := \frac{1}{2}\vartheta_m(\alpha,\alpha)$.

(emerging EOM) $\Rightarrow$ $\nabla_{\dot\gamma}\dot\gamma = \frac{1}{2}\iota_\beta\iota_\beta[\vartheta,\vartheta]_{\nabla^{s\text{-Sch}}}$.

$\Rightarrow$ The tensor $[\vartheta,\vartheta]_{\nabla^{s\text{-Sch}}} \in \Gamma(\odot^3 TM)$ is the obstruction for $\gamma$ to be a geodesic!

Compare with the Hamiltonian equations: $\dot{x}^j = \frac{\partial h}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial h}{\partial x^j}.$

## Symmetric Poisson structures

**Definition**

A pair $(\nabla, \vartheta)$ is called a **symmetric Poisson structure** (sPs) if $[\vartheta, \vartheta]_{\nabla^s\text{-Sch}} = 0$.

$$\left\{ \begin{array}{c} \text{symmetric} \\ \text{bivector fields} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{symmetric maps } \{\ ,\ \} : \times^2 C^\infty(M) \to C^\infty(M) \text{ s.t.} \\ \{f_1, f_2 f_3\} = f_2\{f_1, f_3\} + \{f_1, f_2\} f_3 \end{array} \right\}.$$

$\operatorname{grad} := \vartheta \circ \mathrm{d} : C^\infty(M) \to \Gamma(T)$ or in terms of the bracket $\operatorname{grad} f = \{f, \ \}$.

**Proposition**

$(\nabla, \vartheta)$ is a sPs $\Leftrightarrow$ $\mathrm{Jac}_{\{\ ,\ \}}(f_1, f_2, f_3) = \mathrm{d}f_1(\langle \operatorname{grad} f_2 : \operatorname{grad} f_3 \rangle_{\nabla^s}) + \mathrm{cyc}(1,2,3)$,

$\underset{\text{non-deg.}}{\Leftrightarrow}$ $\nabla^s \vartheta^{-1} = 0$, i.e. $\vartheta^{-1}$ is a Killing 2-tensor.

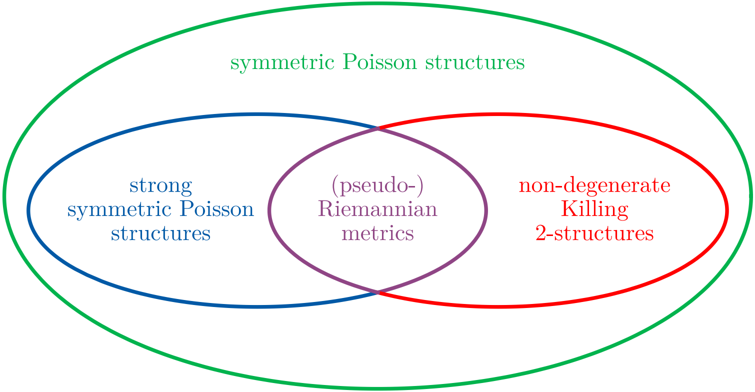
**Definition**

A pair $(\nabla, \vartheta)$ is called a **strong sPs** if $\operatorname{grad}\{f_1, f_2\} = \langle \operatorname{grad} f_1 : \operatorname{grad} f_2 \rangle_{\nabla^s}$.

**Proposition**

$(\nabla, \vartheta)$ is a strong sPs $\Leftrightarrow$ $\nabla_{\operatorname{grad} f} \vartheta = 0$ $\underset{\text{non-deg.}}{\Leftrightarrow}$ $\nabla$ is Levi-Civita w.r.t. $\vartheta^{-1}$.

Diagram of symmetric Poisson strcutures



# Back to generalized geometry!

The point where we stopped...

$$[[(X+\alpha)\cdot, \nabla^s], (Y+\beta)\cdot] = [[\iota_X, \nabla^s], \iota_Y] + ([\iota_X, \nabla^s]\beta)\odot + (\iota_Y \nabla^s \alpha)\odot$$

**Definition**

We introduce the **$\nabla^s$-Dorfman bracket**:

$$[X+\alpha, Y+\beta]_{\nabla^s} := \langle X : Y \rangle_{\nabla^s} + \pounds_X^{\nabla^s} \beta + \iota_Y \nabla^s \alpha$$

**Theorem**

The group of automorphisms preserving $\langle\ ,\ \rangle_-$ and $[\ ,\ ]_{\nabla^s}$ is isomorphic to

$$\mathrm{Aff}(M, \nabla) \ltimes \mathrm{Kill}^2_{\nabla}(M).$$

Compare with the result of standard generalized geometry:

*The group of automorphisms preserving $\langle\ ,\ \rangle_+$ and $[\ ,\ ]_D$ is isomorphic to*

$$\mathrm{Diff}(M) \ltimes \Gamma_{\text{closed}}(\wedge^2 T^*).$$

Compare with the Dorfman bracket: $[X+\alpha, Y+\beta]_D := [X,Y]_{\text{Lie}} + \pounds_X \beta - \iota_Y \mathrm{d}\alpha$.

## Lagrangian subbundles

**Definition**

$L \leq T \oplus T^*$ is called a **$\nabla^s$-Dirac structure** if it is Lagrangian w.r.t. $\langle\ ,\ \rangle_-$ and

$$[\Gamma(L), \Gamma(L)]_{\nabla^s} \subseteq \Gamma(L).$$

**Example**: Killing 2-tensors

Let $K \in \Gamma(\odot^2 T^*)$, $\mathrm{gr}(K)$ is a $\nabla^s$-Dirac iff $K$ is a Killing 2-tensor.

**Example**: symmetric Poisson structures

Let $\vartheta \in \Gamma(\odot^2 T)$, $\mathrm{gr}(\vartheta)$ is a $\nabla^s$-Dirac iff $(\nabla, \vartheta)$ is a symmetric Poisson structure.

**Example**: geodesically invariant distributions

Let $\Delta \leq T$, $\Delta \oplus \mathrm{Ann}\,\Delta$ is a $\nabla^s$-Dirac iff $\Delta$ is a geodesically invariant distribution.

| Standard generalized geometry | Skew-symmetric generalized geometry |
| --- | --- |
| $\frac{1}{2}(\alpha(Y)+\beta(X))$ | $\frac{1}{2}(\alpha(Y)-\beta(X))$ |
| $\mathcal{C}l(T\oplus T^*) \circlearrowright \Gamma(\wedge^\bullet T^*)$ | $\mathcal{W}(T\oplus T^*) \circlearrowright \Gamma(\odot^\bullet T^*)$ |
| $[X,Y]_{\text{Lie}} + \pounds_X\beta - \iota_Y \mathrm{d}\alpha$ | $\langle X:Y\rangle_{\nabla^s} + \pounds_X^{\nabla^s}\beta + \iota_Y\nabla^s\alpha$ |
| $\mathrm{Diff}(M)\ltimes\Gamma_{\text{closed}}(\wedge^2T^*)$ | $\mathrm{Aff}(M,\nabla)\ltimes\mathrm{Kill}^2_\nabla(M)$ |
| $[a,b]_D+[b,a]_D=2\mathrm{d}\langle a,b\rangle_+$ | $[a,b]_{\nabla^s}-[b,a]_{\nabla^s}=2\mathrm{d}\langle a,b\rangle_-$ |
| $\mathrm{pr}_T(a)\langle b,c\rangle_+ = \langle [a,b]_D,c\rangle_+ + \langle b,[a,c]_D\rangle_+$ | $\mathrm{pr}_T(a)\langle b,c\rangle_- = \langle [a,b]_{\nabla^s},c\rangle_- + \langle b,[a,c]_{\nabla^s}\rangle_-$ |
| $[a,[b,c]_D]_D = [[a,b]_D,c]_D + [b,[a,c]_D]_D$ | – |
| Dirac structures | $\nabla^s$-Dirac structures |
| pre-symplectic structures | Killing 2-tensors |
| Poisson structures | symmetric Poisson structures |
| foliations | geodesically invariant distributions |

Thank you for your attention!