Projective to Einstein Correspondence.

- Patterson. E.M & Walker A.G. Riemannian extensions Q.J. Maths 2 1952
- Yano K. 1950s, 60s, ...
- Dunajski & Tod K.P. arXiv 0901.2261
- D & Mettler T. arXiv 1509.04276
- D & Waterhouse A. arXiv 1906.11570

—

(N, ∇) n-dimensional mfd with affine torsion-free connection.

↓

split signature (n,n) metric on $T^*N = M$, dim $2n$

Ω_0 = symplectic form, $\Omega_0 = d\Theta$.

Open set $(x^1, \dots, x^n)$ on N, $J_1, \dots, J_n$ fiber coordinates.

$\Omega_0 = dJ_i \wedge dx^i$, $\Theta = J_i dx^i$.

$h_\nabla = dJ_i \odot dx^i - \Gamma_{ij}^{\ k} J_k \, dx^i \odot dx^j$; Γ = Christoffel symbols of ∇.

Patterson-Walker metric.

- Invariants preserving:

K = vector field on N, $K \lrcorner \Theta = K^i J_i$ (linear function on T^*N).

$K \lrcorner \Theta$ is Hamiltonian vector field on M.

Prop Affine vector fields on N ⟶ Killing vectors of (M, h_∇)

- Projective structure $([\nabla], N)$; equivalence class

$\nabla \sim \hat{\nabla}$ if they share the same unparametrised geodesics,

$$\hat{\nabla}_X(Y) = \nabla_X(Y) + \omega(X)Y + \omega(Y)X, \quad \omega \in \Lambda^1(N).$$

Special connections: preserve a volume form on N.

Special connections: preserve a volume form on N.

allow only $\Omega = df$; $f: N \to \mathbb{R}$.

Prop $\nabla \to \hat{\nabla}$, $\xi_i \to \hat{\xi}_i = e^{2f}\xi_i$

$h_{\hat{\nabla}} = e^{2f} h_\nabla$.

"projective to conformal" lift.

Application. Dim $N = 2$. Projective structure $[\nabla]$ is metrizable iff it contains a Levi-Civita connection of some metric on N

Thm (Dunajski): dim $N = 2$, $[\nabla]$ metrizable iff h_∇ contains a (pseudo) Kähler metric in its conformal class.

Question Is there a way to incorporate the general projective changes of connection with $d\Omega \neq 0$.

(h_∇ from (N, ∇)). Ω_0, Θ, $V: T^*N \to N$

Set $$g_\nabla = h_\nabla + \Lambda\Theta^2 + \frac{1}{\Lambda} V^* \mathrm{Ric}^{(\mathrm{sym})}(\nabla). \quad (*)$$ Λ = constant.

$$\Omega_\nabla = \Omega_0 + \frac{1}{\Lambda(n+1)} \mathrm{Ric}^{(\mathrm{skew})}(\nabla).$$

Thm D-Mettler (2018); [D-Waterhouse 2020].

- $g_{\nabla,\Lambda}$ is Einstein with scalar curvature $4n(n+1)\Lambda$.
- $\Lambda = 1$, $g_{\nabla,1}$ doesn't depend on a choice of $\nabla \in [\nabla]$.
- If $n = 2$ (D-Mettler): Let (M, g) be a $(2,2)$ signature Einstein metric with scalar curvature 24 s.t.
  - $W_+ = 0$ (W_+: anti-self-dual Weyl tensor)

- $W_+ = 0$ (anti-self-dual)
- $\exists$ parallel ASD totally null distribution

Then (M, g) is conformally flat, or it is locally isometric to g_∇ [projective to Einstein correspondence].

Recall: distribution $\mathcal{D} \subset TM$ is parallel w.r.t. g_∇ Levi-Civita if $\nabla_X Y \in \Gamma(\mathcal{D})$ if $Y \in \Gamma(\mathcal{D})$

parallel $\longrightarrow$ Frobenius integrable

In the theorem $\mathcal{D}$ is ASD in that $\mathcal{D} = \ker(F)$ $F \in \Lambda^2_-(M)$.

Examples

Projective vector fields on (N, ∇) $\overset{1:1}{\longleftrightarrow}$ Killing vector of (M, g_∇)

Example 1. $N = \mathbb{RP}^n$, $[\nabla]$ flat; $SL(n+1, \mathbb{R})$.

$M = SL(n+1)/GL(n)$ (n,n) Fubini-study.

$g_\nabla = dY_i \odot dx^i - (Y_i dx^i)^2$. (D-Cover-Westerholm compactification)

Example 2

If $\exists\, \nabla \in [\nabla]$ on N s.t. $Ric^{(sym)}(\nabla) = 0$.

(*) $g_{\nabla,\Lambda} = h_\nabla + \Lambda \Theta^2$ $\longrightarrow$ limit $\Lambda \to 0$ h_∇

Rather-Walker which is also Ricci-flat. and (ASD. dim $N-2$)