Differential forms as a unifying force for geometric structures

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Erich Kähler's top three **geometric structures**:

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(Smooth category, M manifold)

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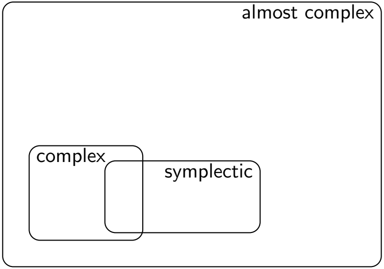

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Riemannian metric	Normal almost contact
	$J \in \mathrm{End}(TM)$
(same)	$\sigma \in \Omega^1(M)$, $Y \in \mathfrak{X}(M)$
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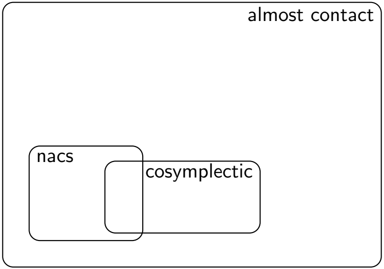

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"**Differential forms** as a unifying force for...

→ Can we do anything about complex ?

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$K \subset \wedge^\bullet T^*_{\mathbb{C}}M$ (canonical bundle of J) whose **local** sections $\zeta \in \Gamma(K \setminus \{0\})$ satisfy:

- ζ decomposable
- $\zeta \wedge \bar{\zeta}$ volume
- $d\zeta = \bar{\partial} f \wedge \zeta$ for some f

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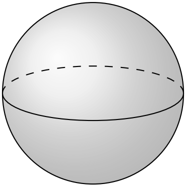

dz on $\mathbb{C}$,
$d(1/z)$ on $\mathbb{C}^* \cup \{\infty\}$

differ on $\mathbb{C}^*$ by
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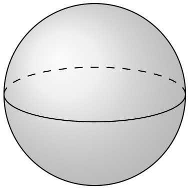

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$\rightarrow$ Can we unify ω and K in some sense?

A recipe for generalized geometry (à la Hitchin, Gualtieri...)

*...Cavalcanti, Alekseev-Bursztyn-Meinrenken'09... and, before, Liu-Weinstein-Xu'97, Courant'90, Chevalley'54!

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Quadratic form on $TM \oplus T^*M$ given by $Q(X + \alpha) = \alpha(X)$

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$$(X + \alpha) \cdot \rho = \iota_X \rho + \alpha \wedge \rho$$

($\wedge^\bullet T^*_{\mathbb{C}} M \simeq$ the spinor representation)

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($\leftrightarrow$ $\mathrm{Ann}(\rho) \subseteq T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ max. isotropic) $B,\omega \in \wedge^2$, $\theta_j \in \wedge^1_{\mathbb{C}}$

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Weakening of $d\rho = 0 \to d\rho = v\cdot\rho$ for $v = X+\alpha$

(or of $d\zeta = \bar\partial f\wedge\zeta$) $\updownarrow$

$\Gamma(\mathrm{Ann}\,\rho)$ involutive for **Dorfman bracket**

$$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - \iota_Y d\alpha$$

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Definition: a **generalized complex structure** is locally given by:

$\rho \in \Omega^\bullet_{\mathbb{C}}$ pure	shape
$(\rho, \overline{\rho}) \sim$ volume	non-degeneracy
$d\rho = v \cdot \rho$	integrability

(the local forms coincide pointwise up to $\mathbb{C}^*$)

The type and ~~another example~~ **the example**

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Pointwise: symplectic subspace
with r-dim complex transversal.
Definition of **type**: r.

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$d\rho = dz = (-\frac{\partial}{\partial w} + 0) \cdot \rho$

Some considerations

$$\rho \sim e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r$$
$$(\rho, \overline{\rho}) = \text{vol}$$
$$d\rho = v \cdot \rho$$

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$$(\rho, \overline{\rho}) = \text{vol}$$
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- $e^B \wedge$ is a symmetry for B closed (a B-field). E.g., $e^{B+i\omega} \cong e^{i\omega}$
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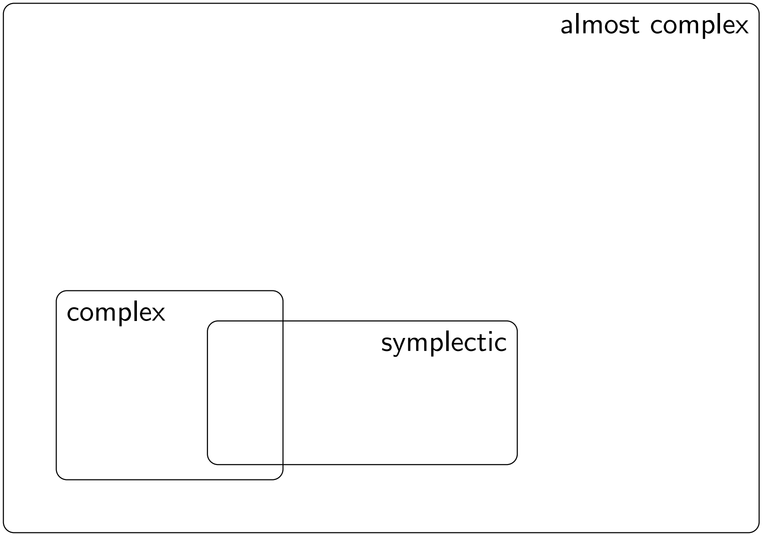
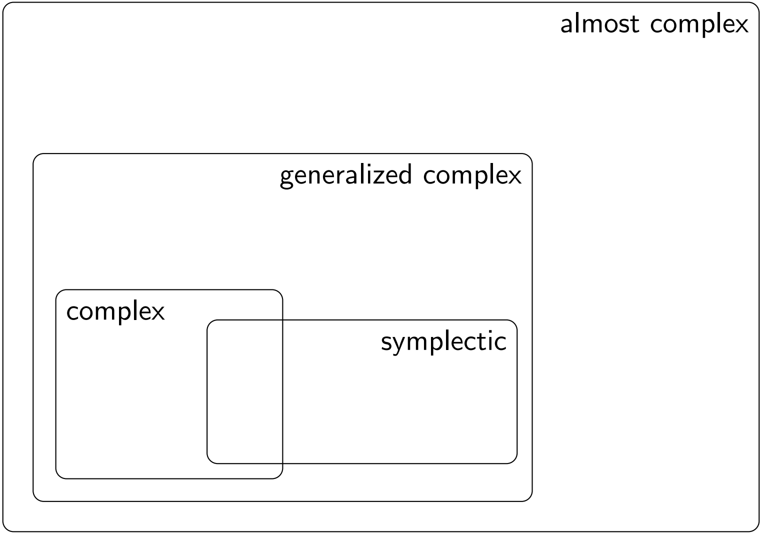
Within generalized complex structures

The interior of the curve is B-equivalent to symplectic structures.

Examples coming from hyperKähler or holomorphic symplectic structures.


almost complex
complex
symplectic


almost complex
generalized complex
complex
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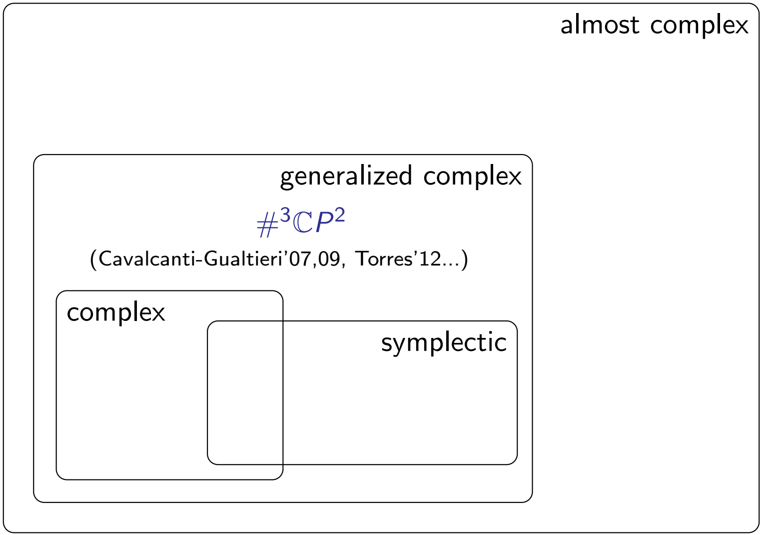
almost complex
generalized complex
$\#^3\mathbb{C}P^2$
(Cavalcanti-Gualtieri'07,09, Torres'12...)
complex
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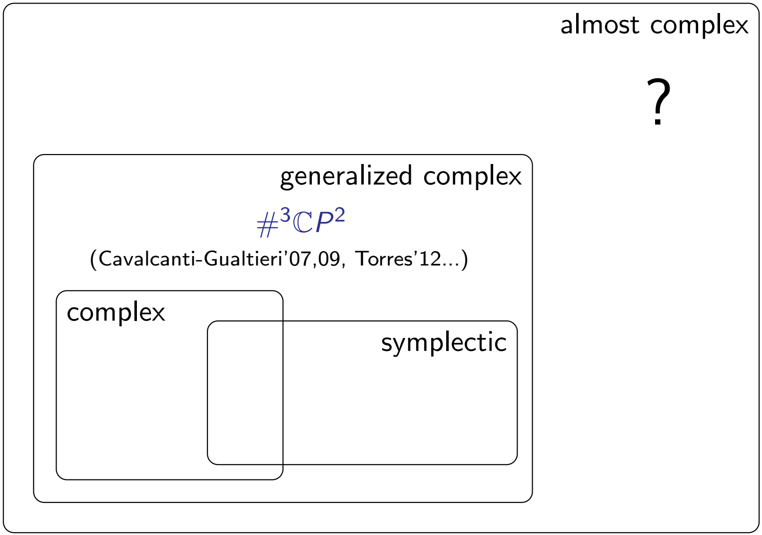
almost complex
?
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So far:

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	symmetric	endomorphism	skew-symmetric
even			
odd			

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	symmetric	endomorphism	skew-symmetric	
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Can we go beyond generalized complex structures?:

A natural variation of the recipe

Quadratic form on $TM \oplus T^*M$ given by $Q(X+\alpha) = \alpha(X)$

$\mathrm{Cl}_{\mathbb{C}}(TM \oplus T^*M)$-module structure on $\wedge^\bullet T^*_{\mathbb{C}}M$

$$(X+\alpha)\cdot\rho = \iota_X\rho + \alpha\wedge\rho$$

($\wedge^\bullet T^*_{\mathbb{C}}M \simeq$ the spinor representation)

Pure spinors are pointwise $\sim e^{B+i\omega}\theta_1\wedge\ldots\wedge\theta_r$

($\leftrightarrow$ $\mathrm{Ann}(\rho)$ max. isotropic) $B,\omega\in\wedge^2$, $\theta_j\in\wedge^1_{\mathbb{C}}$

Chevalley pairing on spinors $(\rho,\psi) = (\rho^T\wedge\psi)_{top}$

($\wedge^{top}T^*_{\mathbb{C}}M$-valued)

$+$

Weakening of $d\rho = 0 \to d\rho = v\cdot\rho$ for $v = X+\alpha$

(or of $d\zeta = \bar{\partial}f\wedge\zeta$) $\updownarrow$

$\Gamma(\mathrm{Ann}\,\rho)$ involutive for **Dorfman bracket**

$$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - \iota_Y d\alpha$$

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Idea: a different generalized geometry

A generalized complex structure is locally given by:

$$\rho \in \Omega^{\bullet}_{\mathbb{C}} \text{ pure}$$
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Example: **any usual generalized complex is B_n-generalized.**

But they also exist in odd dimensions

- Cosymplectic structure: $\omega \in \Omega^2_{cl}$, $\sigma \in \Omega^1_{cl}$ such that $\sigma \wedge \omega^m$ volume

$$\rho = e^{i\sigma + i\omega} = 1 + i\sigma + i\omega - \sigma \wedge \omega \ldots$$

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- **Type-change example**: on $\mathbb{C} \times \mathbb{R}$ with coordinates (z, t),

$$\rho = z + dz + i\, dz \wedge dt$$

So far from the distance:

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	symmetric	endomorphism	skew-symmetric	
even				
odd				

B_n-generalized complex structures

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Do B_n-generalized complex structures reach further?:

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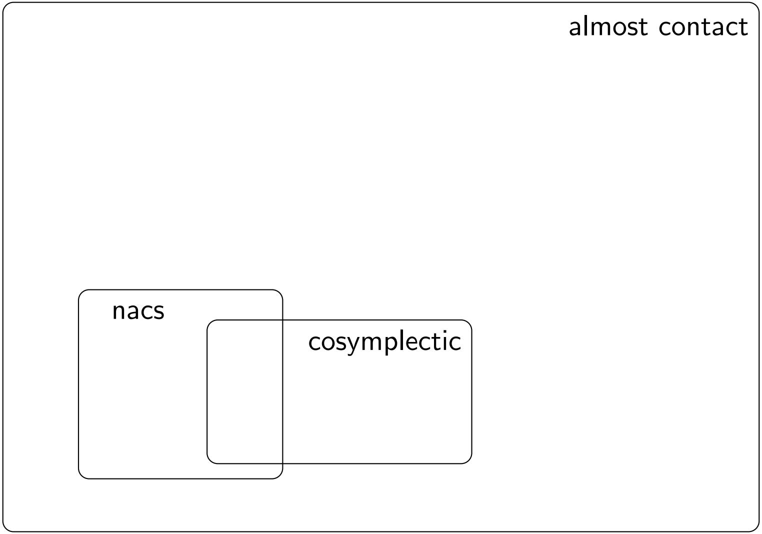
- Type change already possible for $\dim M = 2, 3 \ldots$.


almost contact
nacs
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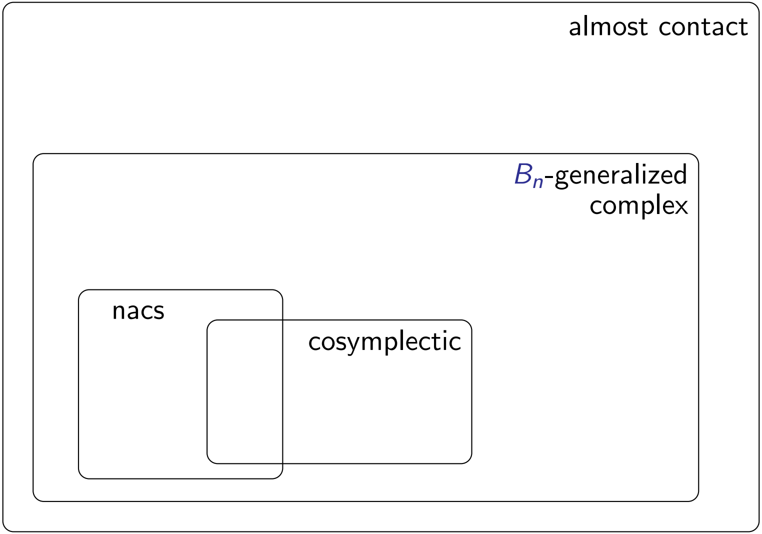
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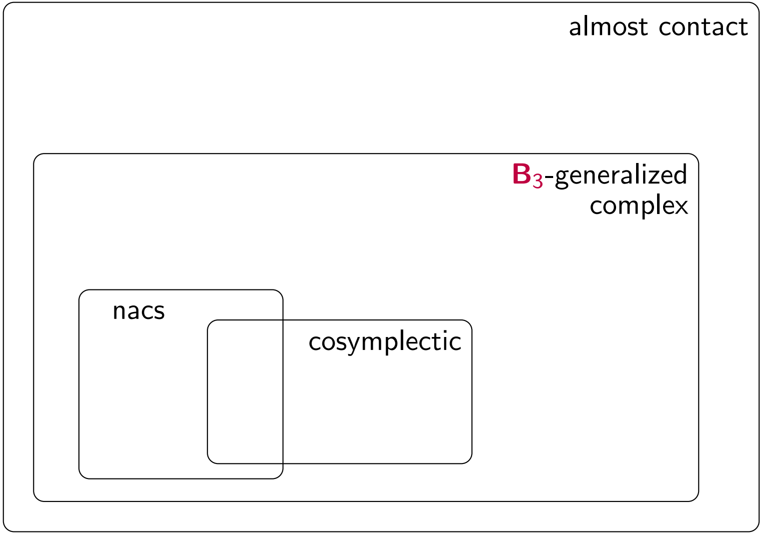
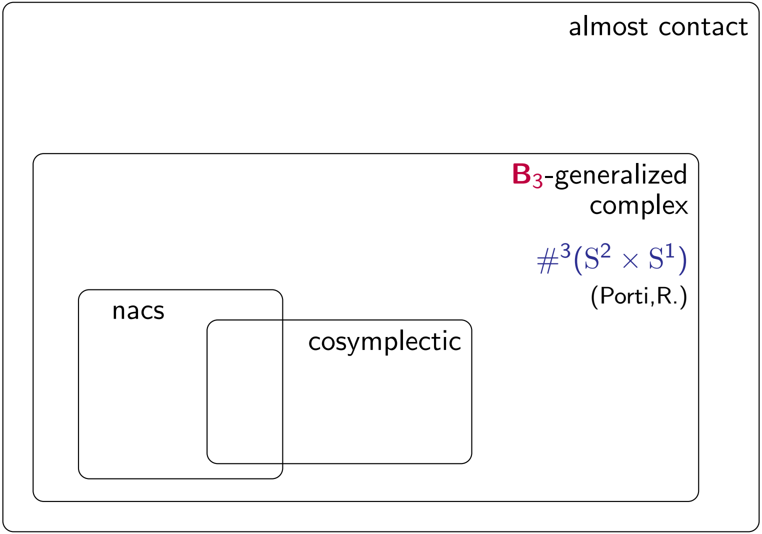


Joint work with J. Porti



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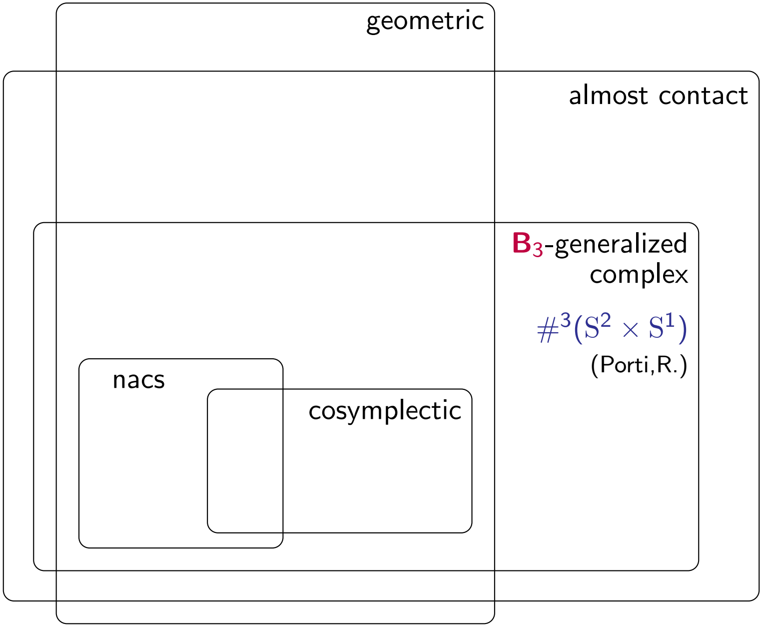

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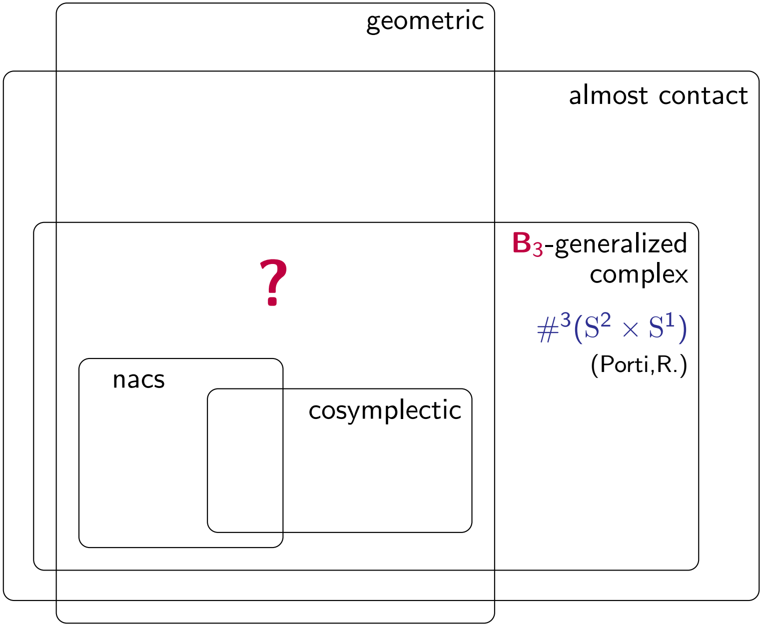

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Main result

Mixing well:

- family of geometric generalized surgeries (topologically Dehn twists)
- open-book decomposition with connected binding, Moser's argument, transitiviy of symplectomorphisms, Dacorogna-Moser theorem

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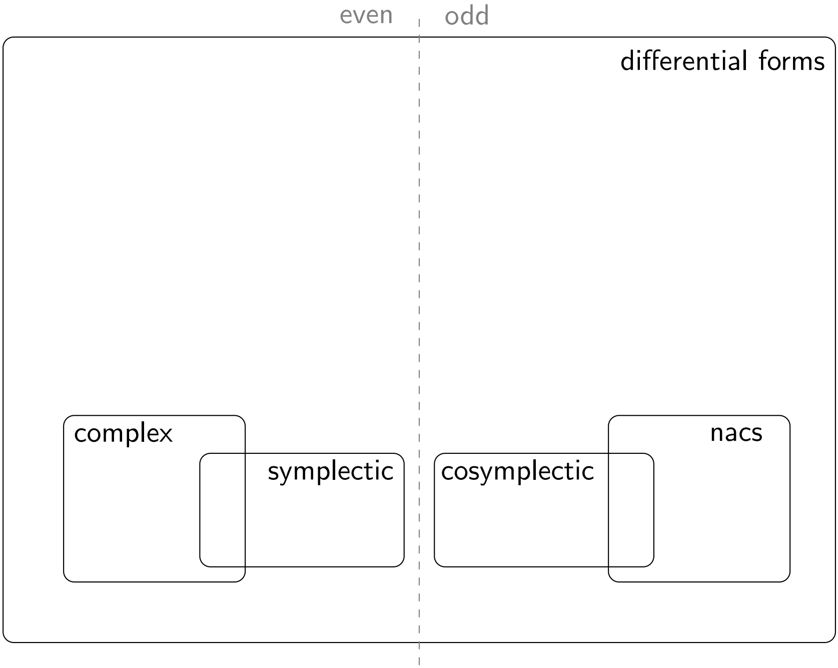
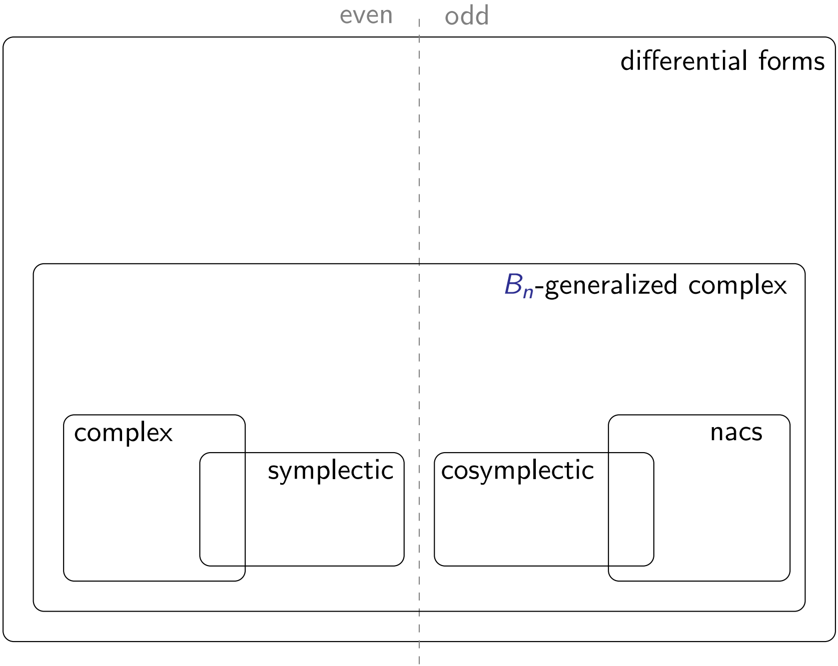
Theorem (Porti,R.)

Any closed oriented 3-manifold admits a B_3-generalized complex structure, which is moreover stable.

New geometric structures on 3-manifolds: surgery and generalized geometry

arXiv:2402.12471


even
odd
differential forms
complex
symplectic
cosymplectic
nacs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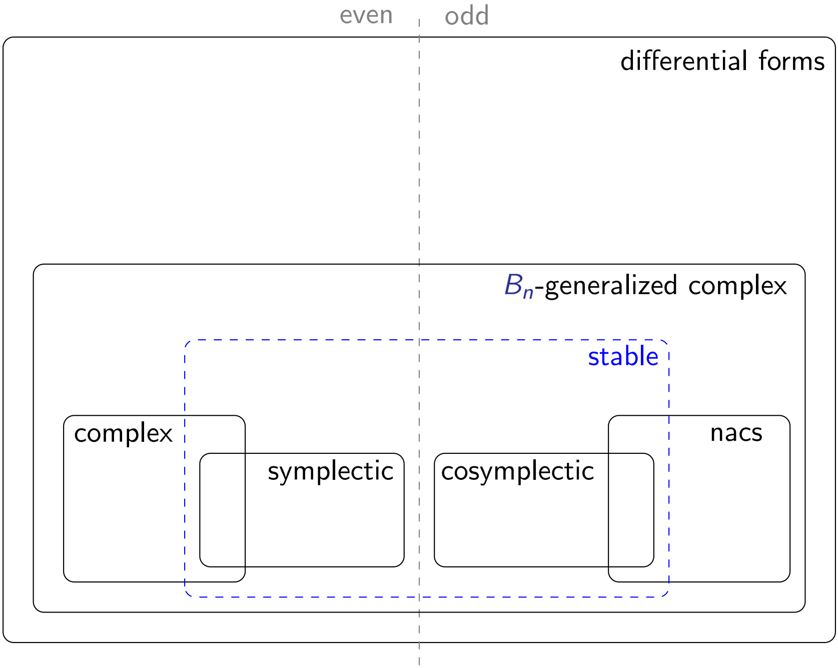
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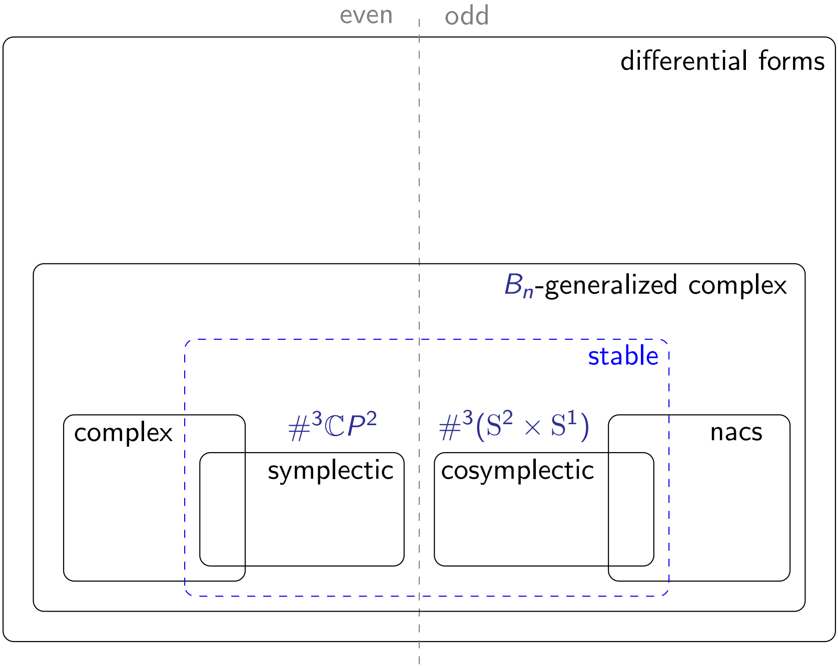
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$\#^3\mathbb{C}P^2$
$\#^3(\mathrm{S}^2 \times \mathrm{S}^1)$
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Local normal models in even dimensions

Complex: $dz_1 \wedge \ldots \wedge dz_m$ Symplectic: $\omega = dp_1 \wedge dq_1 + \ldots + dp_m \wedge dq_m$

Proposition (Cavalcanti-Gualtieri'18)

Around the change of type, stable generalized complex structures look like

$$(z_1 + dz_1 \wedge dz_2) \wedge e^{i\omega}$$

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Local models in odd dimensions? (Say, 3, $\rho = \rho_0 + \rho_1 + \rho_2 + \rho_3$)

Definition (Porti-R.)

Around type change, take γ a meridian curve and T a concentric torus.

$$\lambda := \frac{1}{2\pi i} \int_\gamma \iota^*(\rho_1/\rho_0), \qquad \mu := \frac{1}{4\pi^2 i} \int_T \iota^*(\rho_2/\rho_0).$$

Local invariant: λ **Semilocal invariant** (compact case): $arg(\mu)$

Example: $z + \lambda dz + \mu dz \wedge dt$

Condition $\mathrm{Im}(\lambda\overline{\mu}) \neq 0$. **Prop**: stable type-change never a single circle

A type-change example

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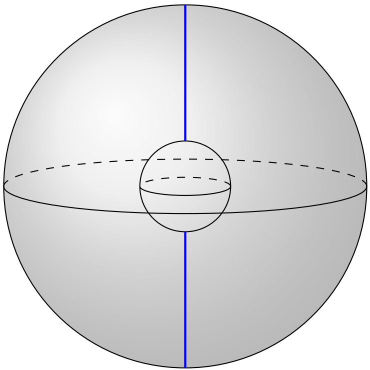

$\mathrm{S}^2 \times \mathrm{S}^1$,
$\lambda, \mu \in \mathbb{C}^\times$ such that
$\mathrm{Im}(\lambda\overline{\mu}) \neq 0$:

On $\mathbb{C} \times \mathrm{S}^1$,
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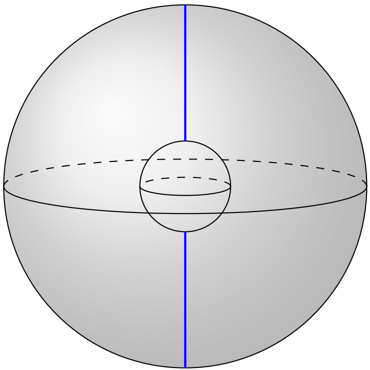

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They differ by z^2 on $\mathbb{C}^* \times \mathrm{S}^1$.

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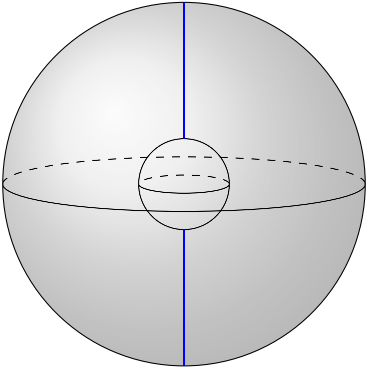

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What is the meaning of λ and μ?

Meaning of invariants (work in progress)

Have in mind $z + \lambda dz + \mu dz \wedge dt$ on $\mathrm{S}^2 \times \mathrm{S}^1$:

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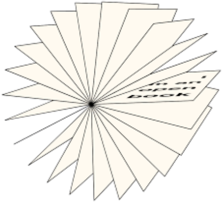

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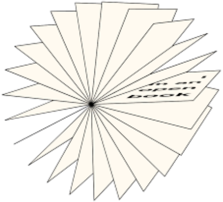

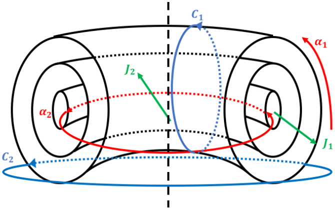

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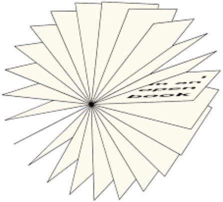

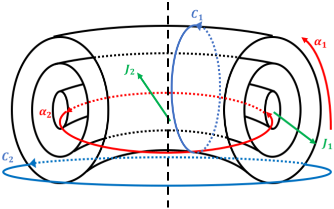

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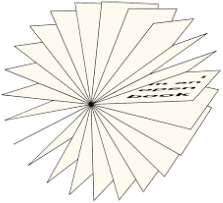

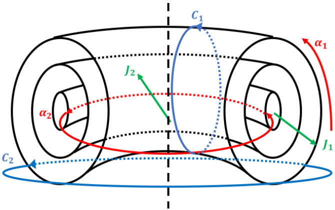

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$arg(\mu)$ gives an invariant related to symplectic structure of leaves?

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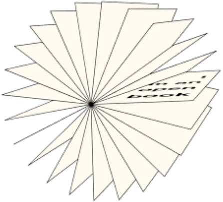

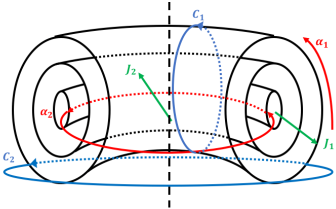

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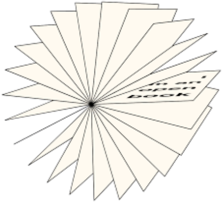

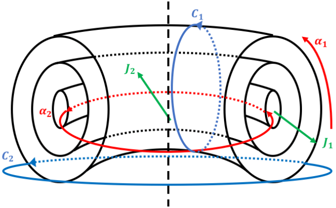

Just as Dirac and generalized complex structures give a smooth structure to foliations and geometric structures on and transverse to them...

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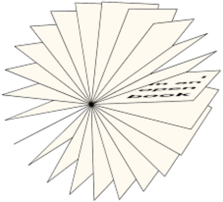

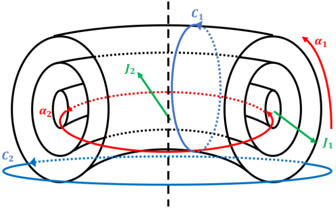

Just as Dirac and generalized complex structures give a smooth structure to foliations and geometric structures on and transverse to them…

…B_3-generalized complex structures do it for open-book decompositions, genus 1 Heegard splittings and related foliations (λ tells which!).

Take-home message: B_n-generalized complex structures

	symmetric	endomorphism	skew-symmetric	
even				
odd				

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$\rho \in \Omega^\bullet_{\mathbb{C}}$ pure
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$d\rho = v \cdot \rho$

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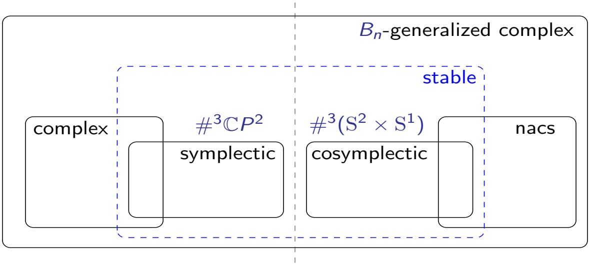

Half-full glass

	symmetric	endomorphism	skew-symmetric	
even				
odd				

Half-empty glass

	symmetric	endomorphism	skew-symmetric	
even				
odd				

Half-empty glass

	symmetric	endomorphism	skew-symmetric	
even				
odd				

It all comes from one choice:

Quadratic form on $TM \oplus T^*M$ given by $Q(X+\alpha) = \alpha(X)$
that is,
symmetric pairing $\langle X+\alpha, Y+\beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$

Half-empty glass

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But we could have done:

skew-symmetric pairing $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) - \beta(X))$

Half-empty glass?

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Within so-called C_n-generalized complex structures

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...a curve going from a metric to a complex structure!

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...a curve going from a metric to a complex structure!

But that's a different talk!

Thank you for your attention!

MINISTERIO DE CIENCIA E INNOVACIÓN
Cofinanciado por la Unión Europea
AGENCIA ESTATAL DE INVESTIGACIÓN

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Slides will be available at
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