Dirac products and concurring Dirac structures

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▷ **Discuss two "dual" operations on Dirac structures** (modulo smoothness):

Tangent and cotangent product of L and R:

$$L \star R \qquad\qquad L \circledast R$$

Tangent product (always Dirac):

Description of the characteristic foliation of $L \star R$

When cotangent product Dirac, we say that **L and R concur**:

Concurrence is the natural compatibility condition for Dirac structures

- unifies classical compatibility conditions (Libermann, Magri-Morosi, Frobenius-Nirenberg);
- clarifies constructions (coupling, normal forms);
- produces new results

▷ Generalized tangent bundle $\mathbb{T}M =: TM \oplus T^*M$ with

- non-degenerate pairing

$$\langle u + \xi, v + \eta \rangle := i_v \eta + i_u \xi$$

- Dorfman bracket

$$[u + \xi, v + \eta] := [u, v] + \mathcal{L}_u \eta - i_v \mathrm{d}\xi$$

- anchor pr_{TM}

is a Courant algebroid

- $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ (Leibniz)
- $[x, fy] = f[x, y] + \left(\mathcal{L}_{\mathrm{pr}_T(x)} f\right) y$ (Leibniz property w.r.t. anchor)
- $[x, y] + [y, x] = d\langle x, y \rangle$ (controlled failure of skew-sym)
- $\mathcal{L}_{\mathrm{pr}_T(x)} \langle y, z \rangle = \langle [x, y], z \rangle + \langle y, [x, z] \rangle$ (adjoints derive the pairing)

- $L \subset \mathbb{T}M$ **Dirac** structure if
  - Lagrangian subbundle

$$\langle \Gamma(L), \Gamma(L) \rangle = 0$$

  - Involutive ($\Longleftrightarrow$ trivial Courant tensor)

$$[\Gamma(L), \Gamma(L)] \subset \Gamma(L) \quad (\Longleftrightarrow \langle [\Gamma(L), \Gamma(L)], \Gamma(L) \rangle = 0)$$

▷ Guiding examples of Dirac structures:

i) Foliations $F \subset TM$ $\rightsquigarrow$ $\mathrm{Gr}(F) = F \oplus F^\circ$
ii) Symplectic forms $\omega \in \Omega^2(M)$ $\rightsquigarrow$ $\mathrm{Gr}(\omega) = \{u + i_u\omega \,|\, u \in \mathfrak{X}(M)\}$
iii) Poisson structures $\pi \in \mathfrak{X}^2(M)$ $\rightsquigarrow$ $\mathrm{Gr}(\pi) = \{\xi + i_\xi\pi \,|\, \xi \in \Omega^1(M)\}$

- Dirac structures makes precise smooth partitions of M by presympletic leaves: $L \subset \mathbb{T}M$ Dirac is a Lie algebroid with the induced bracket and anchor
  - characteristic foliation integrating $\mathrm{pr}_T(L)$;
  - leafwise presymplectic forms: $m \in M$, $a, b \in \mathrm{pr}_T(L)$

$$\omega(a,b) := \langle a, z\rangle, \quad z \in L_m,\ \mathrm{pr}_T(z) = b$$

- Reasons to do Dirac geometry:
  - Often constructions in Poisson geometry are better understood when framed in Dirac geometry.
  - Generalized complex geometry.
  - Dirac geometry itself.

- It is often convenient to decouple smoothness and involutivity conditions
  - $L = \coprod L_m$, $L_m \subset \mathbb{T}_m M$ Lagrangian subspace, is a **Lagrangian family**. A **Lagrangian subbundle** is a smooth Lagrangian family.
  - L Lagrangian family is **involutive** if

$$\langle [x, y], z\rangle = 0, \quad \text{for all } x, y, z \in \Gamma(U, L) \text{ smooth local sections}$$

Smooth involutive Lagrangian families = Dirac structures

- Example: Given $f : M \to N$, $a \in \mathbb{T}M$, $b \in \mathbb{T}N$ are f-related ($a \overset{f}{\sim} b$) if

$$f_* \mathrm{pr}_T(a) = \mathrm{pr}_T b, \quad f^* \mathrm{pr}_{T^*}(b) = \mathrm{pr}_{T^*} a$$

  - $R \subset \mathbb{T}N$ Dirac, **pullback**: $f^!(R) = \{a \in T_m\mathbb{M} \mid \exists b \in \mathbb{T}N, a \overset{f}{\sim} b\}$,
  - $L \subset \mathbb{T}M$ Dirac, **pushforward**: $f_!(R) = \{b \in T_n\mathbb{N} \mid \exists a \in \mathbb{T}M, a \overset{f}{\sim} b\}$,

are involutive Lagrangian families (if pushforward f-invariant)

▷ **Gauge transformation by a closed 2-form**

$L \subset \mathbb{T}M$ Dirac, $\omega \in \Omega^2_{\mathrm{cl}}(M) \Rightarrow \mathcal{R}_\omega(L) := \{a + i_{\mathrm{pr}_T(a)}\omega \mid a \in L\}$ is Dirac.

Regard ω as a Dirac structure $\mathrm{Gr}(\omega)$

$$\mathcal{R}_\omega(L) = \{a + \mathrm{pr}_{T^*}(b) \mid a \in L,\ b \in \mathrm{Gr}(\omega),\ \mathrm{pr}_T(a-b) = 0\}$$

▷ **Tangent/tensor product [Gualtieri],[Alekseev,Bursztyn,Meinrenken]**

- Algebraic definition

$$L \star R = \{a + \mathrm{pr}_{T^*}(b) = \mathrm{pr}_{T^*}(a) + b \mid (a,b) \in L \times R,\ \mathrm{pr}_T(a-b) = 0\}.$$

- Geometric definition

$$L \star R := \Delta^!(L \times R), \quad \Delta : M \to M \times M$$

$L \star R$ is an involutive Lagrangian family

Smooth in the open dense where $L \oplus R \to TM$, $(a,b) \mapsto \mathrm{pr}_T(a-b)$ has ct. rank

▷ **Characteristic foliation of $L \star R$**

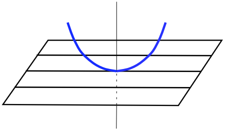
Presymplectic distribution

$$T_m S_{L\star R}(m) = T_m S_L(m) \cap T_m S_R(m), \quad \omega_{L\star R}(m) = \omega_L(m) + \omega_R(m)$$

- Jumping phenomenom in Dirac geometry

Example: $M = \mathbb{R}^3, \quad L = \langle \frac{\partial}{\partial x}, dy + z\frac{\partial}{\partial z}, dz - z\frac{\partial}{\partial y} \rangle$

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (t, 0, t^2) \end{aligned}$$



$f^!(L) = T\mathbb{R} \rightsquigarrow$ The unique induced leaf "jumps" between ambient leaves

Theorem

If $L \star R$ is smooth then

- *its leaves are the clean intersection of leaves of L and R*
- *leaves of one Dirac structure get an induced Dirac structure from the other*

Example: $M = \mathbb{R}^4$

$$L : \pi = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$$

$$R : \mathcal{F} \text{ fibers of } (x_1, x_2, x_3, x_4) \mapsto (x_3 - \tfrac{1}{2}(x_1^2 + x_2^2), x_4 - \tfrac{1}{2}(x_1^2 + x_2^2))$$

- $L \star R$ not Dirac (smooth)
- Leaves of R have induced L-structures, with jumping phenomenom
- Not all leaves of L have induced R-structures
- In $x_3 \neq 0$ dense open subset, $L \star R$ agrees with

$$\pi' = \left(\frac{\partial}{\partial x_1} + x_1 \left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right)\right) \wedge \left(\frac{\partial}{\partial x_2} + x_2 \left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right)\right)$$

▷ Cotangent product

$$L \circledast R = \{a + \mathrm{pr}_T(b) = \mathrm{pr}_T(a) + b \mid (a,b) \in L \times R,\ \mathrm{pr}_{T^*}(a-b) = 0\}$$

Smooth where $L \oplus R \to T^*M$, $(a,b) \mapsto \mathrm{pr}_{T^*}(a-b)$ has ct. rank (open dense)

Definition

If $L \circledast R$ is Dirac (involutive) we say that L and R **concur** (weakly)

Examples:

- $\mathrm{Gr}(\pi) \circledast L = \mathcal{R}_\pi(L) = \{a + i_{\mathrm{pr}_{T^*}(a)}\pi \mid a \in L\}$
- $\mathrm{Gr}(\pi) \circledast \mathrm{Gr}(\pi') = \mathrm{Gr}(\pi + \pi')$
- E, F subbundles $\mathrm{Gr}(E) \circledast \mathrm{Gr}(F) = \mathrm{Gr}(E+F)$

- **Dirac pairs [Dorfman],[Kosmann-Schwarzbach]**

  Torsion of L and R:

$$(u_L, u_R, v_L, v_R, \zeta_L, \zeta_{LR}, \zeta_R) \mapsto \langle [u_L, v_L], \zeta_L\rangle - \langle [u_L, v_R] + [u_R, v_L], \zeta_{LR}\rangle + \langle [u_R, v_R], \zeta_R\rangle,$$

$$(u_L + \xi, u_R + \xi), (v_L + \eta, v_R + \eta), (e + \zeta_L, e + \zeta_{LR}), (f + \zeta_{LR}, f + \zeta_R) \in \Gamma(L \times R)$$

  L and R are said to form a **Dirac pair** when their torsion vanishes identically

Theorem

If L and R concur weakly ($L \circledast R$ involutive), then L and R form a Dirac pair

▷ **Transverse Dirac structures**

$$L \pitchfork R \iff L \star -R = \mathrm{Gr}(\pi) \text{ and } L \circledast -R = \mathrm{Gr}(\omega), \quad \pi \in \mathfrak{X}^2(M), \omega \in \Omega^2(M)$$

- **Coupling:** L Dirac is **coupling** for $\mathcal{F}$ if $L \pitchfork \mathrm{Gr}(\mathcal{F})$

$$\mathrm{Gr}(\omega) = L \circledast \mathrm{Gr}(\mathcal{F}) = T\mathcal{F} \oplus C, \quad C = T\mathcal{F}^{\perp} \cap L \quad \rightsquigarrow H = \mathcal{R}_{-\omega} C$$

$$\mathrm{Gr}(\pi) = L \star \mathrm{Gr}(\mathcal{F}) = N^*\mathcal{F} \oplus D, \quad D = N^*\mathcal{F}^{\perp} \cap L \rightsquigarrow H^{\circ} = \mathcal{R}_{-\pi} D$$

$$\mathrm{Gr}(H) = \mathcal{R}_{-\omega}\mathcal{R}_{-\pi}(L) \qquad\qquad L = \mathcal{R}_{\omega}(H) \oplus \mathcal{R}_{\pi}(H^{\circ})$$

▷ **Libermann Theorem**

$$L \text{ Dirac on } M,\ \mathcal{F} \text{ siple foliation} \rightsquigarrow p : M \to M/\mathcal{F}$$

$$\exists \text{ R Dirac on } M/\mathcal{F} \text{ with } p!(L) = R \iff L \text{ and } \mathrm{Gr}(\mathcal{F}) \text{ concur}$$

$$\{ \text{ Dirac st. on } M \text{ containing } T\mathcal{F} \} \xrightarrow{1:1} \{ \text{ pullbacks of Dirac st. on } M/\mathcal{F} \}$$

$$p_!(L) = R \text{ is Dirac} \iff L \circledast \mathrm{Gr}(\mathscr{F}) = p^!(R) \iff L \circledast \mathrm{Gr}(\mathscr{F}) \text{ is Dirac}$$

- **Local normal form:** L Dirac, $m \in M$, $\exists\ U$ around m with

$$L|_U = \mathcal{R}_{\mathrm{d}\alpha} p^! \mathrm{Gr}(\pi) \qquad \alpha \in \Omega^1(U),\ p : U \to N \text{ submersion, } \pi \text{ Poisson in } N$$

$$(\mathcal{R}_{-\mathrm{d}\alpha} L) \cap TU = \begin{cases} \text{trivial} \quad \surd \\ \dim 1 \rightsquigarrow \mathcal{R}_{-\mathrm{d}\alpha}(L) = \mathcal{R}_{-\mathrm{d}\alpha}(L) \circledast \mathrm{Gr}(\mathcal{F}) \overset{Libermann}{=} p^! \mathrm{Gr}(\pi) \end{cases}$$

- **Compatible endomorphisms and closed 2-forms and Poisson bivectors**

- Twisted brackets, concomitants and a-symmetry

$$\begin{cases} a: TM \to TM & \rightsquigarrow [u,v]^a := [au,v]-[u,av]-a[u,v] \\ \pi: \mathbb{T}M \to \mathbb{T}M & \rightsquigarrow [\xi,\eta]^\pi = \mathcal{L}_{\pi\xi}\eta - i_{\pi\eta}\mathrm{d}\xi \\ \omega: \mathbb{T}M \to \mathbb{T}M & \rightsquigarrow [u,v]^\omega = i_u i_v \mathrm{d}\omega \end{cases}$$

$$\begin{cases} a \text{ Nihenhuis} & \overset{def}{\Longleftrightarrow} N(a)=0, \quad N(a)(u,v) := a[\,,\,]^a - [a,a] \\ \pi \text{ Poisson} & \Longleftrightarrow N(\pi)=0, \quad N(\pi)(\xi,\eta) := \pi[\,,\,]^\pi - [\pi,\pi] \\ \omega \text{ closed} & \Longleftrightarrow [\,,\,]^\omega = 0 \end{cases}$$

$$\begin{cases} C(a,\pi)(\xi,\eta) & := a^*[\,,\,]^\pi - ([a^*,\pi]+[\pi,a^*]) \\ C(a,\omega)(u,v) & := [a,\omega]+[\omega,a]-\omega[\,,\,]^a + a^*[\,,\,]^\omega \end{cases}$$

$$\begin{cases} \pi \text{ a} - \text{symmetric} & \overset{def}{\Longleftrightarrow} a\pi = \pi a^* \rightsquigarrow a^i\pi \quad \text{bivector} \\ \omega \text{ a} - \text{symmetric} & \overset{def}{\Longleftrightarrow} \omega a = a^*\omega \rightsquigarrow \omega a^i \quad 2-\text{form} \end{cases}$$

π, a is a PN structure if π is a-symmetric and $N(\pi) = N(a) = C(a, \pi) = 0$

ω, a is a ΩN structure if ω is a-symmetric and $\mathrm{d}\omega = N(a) = C(a, \omega) = 0$

π, ω is a $P\Omega$ structure if it is a PN and ΩN structure for $a = \omega\pi$

For a closed 2-form ω and a Poisson bivector π the following are equivalent:

1. π, ω is a $P\Omega$ structure
2. $\mathrm{Gr}(\omega)$ and $\mathrm{Gr}(\pi)$ concur
3. ω is a complementary 2-form for π.

ω is a bivector on $(T^*M, [\cdot,\cdot]^\pi)$. It is **complementary** if it is Poisson:

$$\omega\left([\cdot,\cdot]^\pi\right)^\omega - [\omega,\omega]^\pi = 0$$

▷ **Complex dirac structures:**

Involutive Lagrangian subbundles of $(\mathbb{T}_{\mathbb{C}}M, \langle \, , \, \rangle_{\mathbb{C}}, [\, , \,]_{\mathbb{C}})$

Examples:

- Scalar extensions of (real) Dirac structures
- To L $\mathbb{C}$-Dirac we associate two scalar extensions (modulo smoothness)

$$L \star \overline{L} \quad \text{(involutive)} \qquad\qquad L \circledast \overline{L} \quad \text{(Lagrangian family)}$$

- Involutive structures: $E \subset T_{\mathbb{C}}M$ involutive subbundle $\rightsquigarrow \mathrm{Gr}(E)$ $\mathbb{C}$-Dirac
- Generalized complex structures: L $\mathbb{C}$-Dirac with $L \pitchfork \overline{L}$

- **Involutive structures**
  - Complex structures: $T_{\mathbb{C}}M = E \oplus \overline{E}$
  - CR structures: $E \cap \overline{E} = \{0\}$
  - Holomorphic foliations $\mathcal{F}$: $T_{\mathbb{C}}\mathcal{F} = E \oplus \overline{E}$.
  - Transversaly holomorphic foliations $\mathcal{F}$: $T_{\mathbb{C}}\mathcal{F} = E \cap \overline{E}$, $T_{\mathbb{C}}M = E + \overline{E}$.
  - Nirenberg structures: atlas $\phi_i : U_i \to \mathbb{R}^{\mathrm{dimM}-2n-d} \times \mathbb{C}^n \times \mathbb{R}^d$

$$\phi_{ij}(x,z,y) = (a(x), b(x,z), c(x,z,y)), \quad b(x,\cdot) \text{ holomorphic}$$

For an involutive structure E the following are equivalent:

1. $\mathrm{Gr}(E)$ and $\mathrm{Gr}(\overline{E})$ concur
2. E is a Nirenberg structure of type (n,d), $d = \mathrm{rank}(E \cap \overline{E})$, $n = \mathrm{rank}(E) - d$

- **Generalized complex structures** $\quad L \subset \mathbb{T}_{\mathbb{C}} M$ Lagrangian subbundle

$$L \pitchfork \overline{L} \Longleftrightarrow \begin{cases} L & = (J + i\mathrm{I})(\mathbb{T}M), \quad J \in \mathrm{O}(\langle , \rangle) \\ J^2 & = -\mathrm{I} : \quad J = \begin{pmatrix} a & \pi \\ \omega & -a^* \end{pmatrix}, \quad a^2 + \pi\omega, \ a^*\omega = \omega a, \ a\pi = \pi a^* \end{cases}$$

$$L \text{ generalized cx.} \overset{\text{Crainic}}{\Longleftrightarrow} \begin{cases} N(\pi) = 0, & C(a, \pi) = 0 \\ N(a) = -\pi[\, , \,]^{\omega}, & C(a, \omega) = 0 \end{cases}$$

For a generalized complex structure L the following are equivalent:

1. L concurs with $\overline{L}$ (J concurs with $\overline{J} = -J$)
2. π, a is a PN structure and ω, a is an ΩN structure

$$L \star (-\overline{L}) = \mathrm{Gr}(\tfrac{1}{2i}\pi) \qquad L \circledast (-\overline{L}) = \mathrm{Gr}(\tfrac{1}{2i}\omega).$$

$$L \text{ concurs with } \overline{L} \Longleftrightarrow \mathrm{d}\omega = 0$$