Symmetric Poisson geometry

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1. Symmetric bivector fields

$$\left\{ \begin{matrix} \text{symmetric bivector fields} \\ \vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M) \end{matrix} \right\} \xleftrightarrow[\vartheta(\mathrm{d}f,\mathrm{d}g)=\{f,g\}]{\sim} \left\{ \begin{matrix} \mathbb{R}\text{-bilinear maps } \{\ ,\ \} : \times^2 \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \\ \{f,g\} = \{g,f\}, \quad \{f,gh\} = \{f,g\}h + g\{f,h\} \end{matrix} \right\}$$

Every $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M)$ has associated the **gradient** map

$$\operatorname{grad} : \mathcal{C}^\infty(M) \to \mathfrak{X}(M), \qquad f \mapsto \vartheta(\mathrm{d}f) = \{f,\ \}.$$

What is a natural integrability condition?

1. $[\vartheta,\vartheta] = 0$ (The Schouten bracket on $\mathfrak{X}^\bullet_{\mathsf{sym}}(M)$ is skew-symmetric) $\rightsquigarrow$ void condition
2. The map $\operatorname{grad} : (\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ])$ is an algebra morphism $\rightsquigarrow$ $\{\ ,\ \} = 0$
3. $\operatorname{Jac}(f,g,h) := \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ $\rightsquigarrow$ $\{\ ,\ \} = 0$

2. Symmetric Cartan calculus

For a torsion-free connection ∇ on M, we introduce the **symmetric derivative** on the space of symmetric forms $\Upsilon^\bullet(M)$:

$$\nabla^s : \Upsilon^r(M) \to \Upsilon^{r+1}(M), \qquad \varphi \mapsto (r+1)\operatorname{sym}(\nabla\varphi).$$

Equivalently, using the correspondence

$$\left\{ \begin{matrix} \text{symmetric forms} \\ \varphi \in \Upsilon^r(M) \end{matrix} \right\} \xleftrightarrow[\tilde{\varphi}(u) := \frac{1}{r!}\varphi(u,\dots,u)]{\sim} \left\{ \begin{matrix} \text{degree-}r \text{ polynomials in velocities on } M\text{, that is,} \\ \xi \in \mathcal{C}^\infty(TM) \text{ such that } \xi(\lambda u) = \lambda^r \xi(u) \end{matrix} \right\},$$

the symmetric derivative corresponds to the geodesic spray $X_\nabla \in \mathfrak{X}(TM)$: $\widetilde{\nabla^s\varphi} = X_\nabla\tilde{\varphi}$.

Classical Cartan calculus	Symmetric Cartan calculus
differentials	
exterior derivative d	symmetric derivative ∇^s
$\mathrm{d}\psi = (r+1)\operatorname{skew}(\nabla\psi)$	$\nabla^s\varphi = (r+1)\operatorname{sym}(\nabla\varphi)$
canonical	depending on the choice of ∇
$(\mathrm{d}f)(X) = Xf, \quad \mathrm{d} \in \operatorname{gDer}_1(\Omega^\bullet(M))$	$(\nabla^s f)(X) = Xf, \quad \nabla^s \in \operatorname{Der}_1(\Upsilon^\bullet(M))$
Lie derivatives	
Lie derivative $L_X := [\iota_X, \mathrm{d}]_{\mathsf{g}}$	symmetric Lie derivative $L^s_X := [\iota_X, \nabla^s]$
$L_X = \frac{\mathrm{d}}{\mathrm{d}t}(\Psi^X_t)^*$	$L^s_X = \frac{\mathrm{d}}{\mathrm{d}t}(P^\gamma_{2t,0} \circ (\Psi^X_{-t})^*)$
brackets	
Lie bracket $[X,Y] := X \circ Y - Y \circ X$	symmetric bracket $\langle X : Y \rangle_s := \nabla_X Y + \nabla_Y X$
$\iota_{[X,Y]} = [L_X, \iota_Y]_{\mathsf{g}}$	$\iota_{\langle X:Y \rangle_s} = [L^s_X, \iota_Y]$

Moučka, Rubio. *Symmetric Cartan calculus, the Patterson-Walker metric and symmetric cohomology*. arXiv:2501.12442

For a torsion-free connection ∇ on M, we introduce the **symmetric Schouten bracket** on the space of symmetric multivector fields $\mathfrak{X}^\bullet_{\mathsf{sym}}(M)$ as the unique map $[\ ,\ ]_s : \times^2\mathfrak{X}^\bullet_{\mathsf{sym}}(M) \to \mathfrak{X}^\bullet_{\mathsf{sym}}(M)$ such that

(i) $[X,f]_s = Xf$ and $[X,Y]_s = \langle X : Y \rangle_s$,
(ii) $[\mathcal{X},\ ]_s \in \operatorname{Der}_{r-1}(\mathfrak{X}^\bullet_{\mathsf{sym}}(M))$ for $\mathcal{X} \in \mathfrak{X}^r_{\mathsf{sym}}(M)$,
(iii) $[\mathcal{X},\mathcal{Y}]_s = [\mathcal{Y},\mathcal{X}]_s$.

It can be also derived:

$$[[\iota_{\mathcal{X}}, \nabla^s], \iota_{\mathcal{Y}}] = \iota_{[\mathcal{X},\mathcal{Y}]_s}.$$

3. Symmetric Poisson structures

A pair (ϑ,∇) is called a **symmetric Poisson structure** if

$$[\vartheta,\vartheta]_s = 0.$$

Equivalently:

$$\operatorname{Jac}(f,g,h) = \mathrm{d}h(\langle f : g \rangle_s) + \operatorname{cyc}(f,g,h).$$

$$\left\{ \begin{matrix} \text{non-degenerate symmetric} \\ \text{Poisson structures } (\vartheta,\nabla) \text{ on } M \end{matrix} \right\} \xleftrightarrow[g=\vartheta^{-1}]{\sim} \left\{ \begin{matrix} \text{non-degenerate Killing } 2\text{-tensors } (g,\nabla) \\ g \in \Upsilon^2(M) \text{ is non-degenerate and } \nabla^s g = 0 \end{matrix} \right\}$$

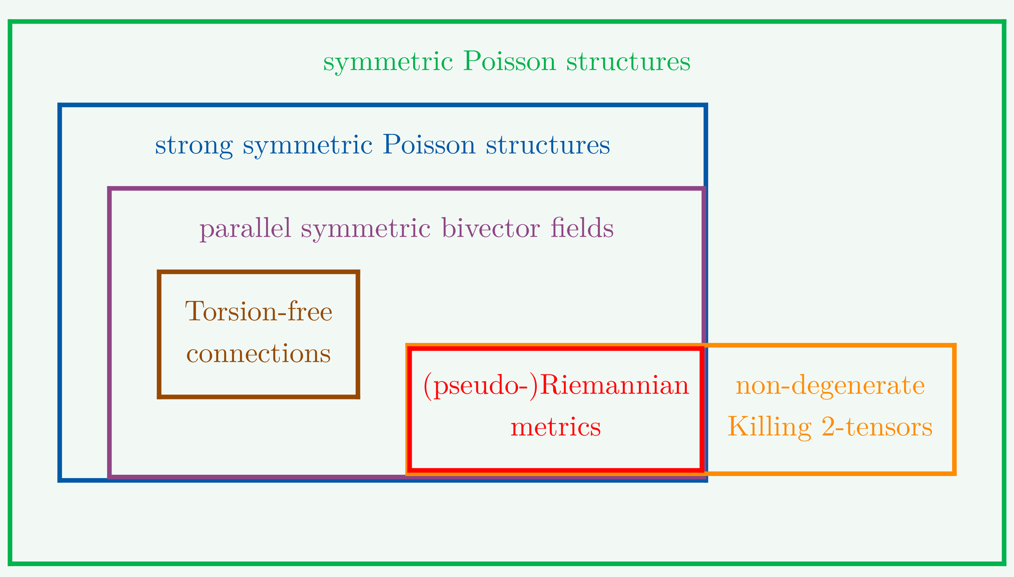
A pair (ϑ,∇) is called a **strong symmetric Poisson structure** if

$$\operatorname{grad} : (\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathfrak{X}(M), \langle\ :\ \rangle_s)$$

is an algebra morphism.

Equivalently: $\nabla_{\operatorname{grad} f}\,\vartheta = 0.$

$$\left\{ \begin{matrix} \text{non-degenerate strong symmetric} \\ \text{Poisson structures } (\vartheta,\nabla) \text{ on } M \end{matrix} \right\} \xleftrightarrow[g=\vartheta^{-1}]{\sim} \left\{ \begin{matrix} \text{(pseudo-)Riemannian metrics } (g, {}^g\nabla) \\ ({}^g\nabla \text{ is the Levi-Civita connection of } g) \end{matrix} \right\}$$



4. Geometrical interpretation

Every $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M)$ determines:

- the **characteristic distribution** $\operatorname{im}\vartheta \subseteq TM$, a smooth (possibly singular) distribution, defined by
$$(\operatorname{im}\vartheta)_m := \{\vartheta(\alpha) \mid \alpha \in T^*_m M\},$$
- the **characteristic metric** $\{g_\vartheta\}$, a family of linear (pseudo-)Riemannian metrics on $\operatorname{im}\vartheta$, given by
$$g_\vartheta(\vartheta(\alpha),\vartheta(\beta)) := \vartheta(\alpha,\beta).$$

Theorem

The characteristic distribution of a symmetric Poisson structure (ϑ,∇) is locally geodesically invariant.
Moreover, every ϑ-admissible geodesic $\gamma : I \to M$ has constant 'square of the speed' $g_\vartheta(\dot\gamma,\dot\gamma)$.

- We call a distribution Δ **locally geodesically invariant** if for every geodesic $\gamma : I \to M$ satisfying $\dot\gamma(t_0) \in \Delta_{\gamma(t_0)}$ for some $t_0 \in I$, there is a subinterval I' such that $\dot\gamma(t) \in \Delta_{\gamma(t)}$ for all $t \in I'$.
- A curve $\gamma : I \to M$ is called **ϑ-admissible** if there is a curve $a : I \to T^*M$ such that $\vartheta(a(t)) = \dot\gamma(t)$.

Every strong symmetric Poisson structure (ϑ,∇) on M gives the smooth partition of M into leaves such that every leaf $N \subseteq M$ satisfies

$$\operatorname{im}\vartheta|_N = TN.$$

Moreover, every leaf $N \subseteq M$ acquires:

- the **leaf connection** ∇^N, a torsion-free connection on N whose parallel transport coincides with that of ∇.
- the **leaf metric** g_N, a (pseudo-)Riemannian metric on N given by g_ϑ.

Theorem

The smooth partition induced by a strong symmetric Poisson structure (ϑ,∇) is totally geodesic.
Moreover, for every leaf of the partition, the leaf connection is the Levi-Civita connection of the leaf metric.

- A submanifold $N \subseteq M$ is called **totally geodesic** if for every geodesic $\gamma : I \to M$ satisfying $\dot\gamma(t_0) \in T_{\gamma(t_0)}N$ for some $t_0 \in I$, there is a subinterval I' such that $\dot\gamma(t) \in T_{\gamma(t)}N$ for all $t \in I'$.
- A partition is called **totally geodesic** if every leaf is a totally geodesic submanifold.

Example 1: Heisenberg group and $\mathrm{SO}(3)$

The Heisenberg group H_3 is the 3-dimensional non-compact Lie group of 3×3 upper triangular matrices.
The standard basis for the space of left-invariant vector fields (Q,P,I) satisfies

$$[Q,P] = I, \qquad [Q,I] = 0, \qquad [P,I] = 0.$$

The regular distribution generated by Q and P is non-integrable. In fact, it is bracket-generating.
This structure can naturally be described using the non-strong symmetric Poisson structure (ϑ,∇):

$$\vartheta := Q \otimes Q + P \otimes P, \qquad \nabla_Q P := \frac{1}{2}I, \qquad \nabla_Q I := 0, \qquad \nabla_P I := 0.$$

There is a similar compact example of a non-strong symmetric Poisson structure on the Lie group $\mathrm{SO}(3)$.
In the standard basis (X_1,X_2,X_3) for the space of left-invariant vector fields satisfying

$$[X_i,X_j] = \varepsilon_{ijk}X_k,$$

it is described by the pair (ϑ,∇) given by

$$\vartheta := X_1 \otimes X_1 + X_2 \otimes X_2, \qquad \nabla_{X_i}X_j := \frac{1}{2}\varepsilon_{ijk}X_k.$$

Example 2: Totally geodesic foliation by circles

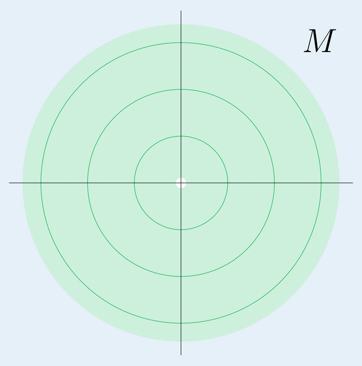
Consider the punctured cartesian plane $M := \mathbb{R}^2 \setminus \{(0,0)\}$ and $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M)$:

$$\vartheta := (-y\,\partial_x + x\,\partial_y) \otimes (-y\,\partial_x + x\,\partial_y).$$

The torsion-free connection ∇ on M, given by

$$\nabla_{\partial_x}\partial_x = \nabla_{\partial_y}\partial_y := \frac{1}{x^2+y^2}(x\,\partial_x + y\,\partial_y), \qquad \nabla_{\partial_x}\partial_y := 0,$$

makes (ϑ,∇) strong symmetric Poisson. Each leaf $\mathbb{S}^1$ inherits the round metric as its leaf metric.



Example 3: Jacobi-Jordan algebras

A symmetric Poisson structure $(\{\ ,\ \},\nabla)$ on V^* is called **linear** if

$$\{\mathcal{C}^\infty_{\mathsf{lin}}(V^*), \mathcal{C}^\infty_{\mathsf{lin}}(V^*)\} \subseteq \mathcal{C}^\infty_{\mathsf{lin}}(V^*).$$

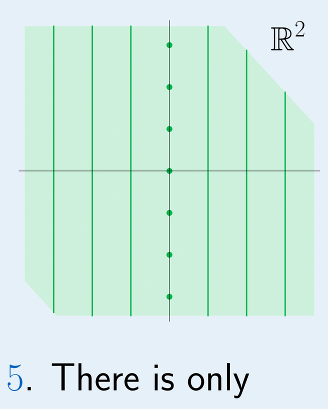
$$\left\{ \begin{matrix} \text{linear (strong) symmetric} \\ \text{Poisson structures} \\ (\{\ ,\ \}, \nabla^{\mathsf{Euc}}) \text{ on } V^* \end{matrix} \right\} \xleftrightarrow[u \cdot v := \{\iota_u, \iota_v\}]{\sim} \left\{ \begin{matrix} \text{(associative) Jacobi-Jordan} \\ \text{algebra structures } \cdot \text{ on } V \end{matrix} \right\}$$

- The **Euclidean connection** on V^* is given by
$$\nabla^{\mathsf{Euc}}_{\partial_{x^i}}\partial_{x^j} = 0,$$
where $\{x^i\}$ is any set of linear global coordinates on V^*.
- A commutative algebra is called **Jacobi-Jordan** if
$$\operatorname{Jac}(u,v,w) = 0.$$

(I) If $\dim V = 2$, there is only one non-trivial Jacobi-Jordan algebra:

$$e_2 \cdot e_2 = e_1 \qquad \xleftrightarrow{\sim} \qquad \vartheta = x\,\partial_y \otimes \partial_y$$

Since it is associative, $(\vartheta, \nabla^{\mathsf{Euc}})$ forms a strong symmetric Poisson structure.



(II) The lowest dimension admitting a non-associative Jacobi-Jordan algebra is $\dim V = 5$. There is only one such algebra. The corresponding non-strong symmetric Poisson structure is:

$$\vartheta := x^2\partial_{x^1} \otimes \partial_{x^1} + x^5\,\partial_{x^1} \odot \partial_{x^4} - \frac{1}{2}x^3\,\partial_{x^1} \odot \partial_{x^5} + x^3\,\partial_{x^2} \odot \partial_{x^4}.$$

Remark: symmetric Poisson structures and Lie algebroids

A pair (ϑ,∇) gives the skew-symmetric bracket on $\Omega^1(M)$:

$$[\alpha,\beta] := \nabla_{\vartheta(\alpha)}\beta - \nabla_{\vartheta(\beta)}\alpha.$$

If (ϑ,∇) is a strong symmetric Poisson structure, the triple $(T^*M, \vartheta, [\ ,\ ])$ is a Lie algebroid if and only if

$$R_\nabla(\vartheta(\alpha),\vartheta(\beta))\eta + \operatorname{cyc}(\alpha,\beta,\eta) = 0.$$

Soon on the arXiv!

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