Abstract. Dirac structures are a geometric object generalizing symplectic and Poisson structures. From a physics viewpoint, they describe mechanical systems with both symmetries and constraints. Their deformation theory tells us what nearby Dirac structures look like and can suggest new invariants. The deformation of a Dirac structure involves choosing a complementary 'almost Dirac' structure, which gives an $L_3[1]$ algebra (a 'Lie algebra up to homotopy') controlling the deformations. When the 'almost Dirac' structure is Dirac, the $L_3[1]$ algebra reduces to the simpler structure of a differential grade Lie algebra.

Goal: In a given Courant algebroid, characterize which Dirac structures admit Dirac complements.

Main result: We give topological and algebraic obstructions to the existence of a Dirac complement.

Obstructions to the existence of a Dirac complement

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Why Dirac structures?

So... What is a Dirac structure?

A geometric object which generalizes both Poisson and symplectic structures. Formally, it is given by subbundle of $TX \oplus T^*X$ over a manifold X which is:

- Lagrangian at every point for the pairing:

$$\langle V+\alpha, W+\beta\rangle = \alpha(V)+\beta(W).$$

- Involutive under the Courant bracket:

$$[V+\alpha, W+\beta] = [V,W] + L_V\beta - i_W d\alpha.$$

How does that generalize Poisson and symplectic structures?

Through their graphs! The graphs of Poisson or symplectic structures, as subbundles of $TX \oplus T^*X$, are both Dirac structures.

So how are these more flexible than just using Poisson or symplectic structures?

Well, they have the extra advantage of admitting both pushforwards and pullbacks! For example, in mechanics, this allows us to deal with both symmetries and constraints.

Is there a way to generalize Dirac structures to other settings?

Sure! We can twist the Courant bracket by a closed 3-form H giving us twisted Dirac structures, which encapsulate **twisted Poisson** or **twisted presymplectic** geometry. Complexifying, we can also find **generalized complex structures**, or work in the more general setting of a Courant algebroid.

And you say some of these don't have Dirac complements?

Yes, for example, if H has a nonzero de Rham cohomology class, the Dirac structure T^*X has no Dirac complement.

Deforming Dirac structures

A Courant algebroid over a manifold X, briefly, is given by a vector bundle E, along with:

A pairing $\langle\cdot,\cdot\rangle : \Gamma(E)\otimes\Gamma(E)\to C^\infty(X)$
An anchor map $\rho : E \to TX$
A bracket $[\cdot,\cdot] : \Gamma(E)\otimes\Gamma(E)\to\Gamma(E)$

such that the pairing, bracket, and anchor, all satisfy natural compatibility conditions between them, similar to those satisfied by $TX \oplus T^*X$.

Let $L \subseteq E$ be a Dirac structure. Choosing another Lagrangian subbundle M such that $M \oplus L$ allows us to identify small Lagrangian deformations of L with sections of $\Gamma(\bigwedge^2 M)$.

The decomposition $E = M \oplus L$ endows $\Gamma(\bigwedge^\bullet M)$ with the structure of an $L_3[1]$ algebra. The deformations of L that remain Dirac are given by solutions to the **Maurer-Cartan equation** in that algebra:

$$\mathrm{d}_L\omega + \tfrac{1}{2}[\omega,\omega]_M + \tfrac{1}{3!}l_3(\omega,\omega,\omega) = 0.$$

In the case where M is not only Lagrangian but also Dirac, the above $L_3[1]$ algebra structure reduces to a differential graded Lie algebra. Deformations are now governed by the simpler equation:

$$\mathrm{d}_L\omega + \tfrac{1}{2}[\omega,\omega] = 0.$$

Switching the roles of M and L, we also have the structure of a curved $L_2[1]$ algebra on $\Gamma(\bigwedge^\bullet L)$. Finding a **Dirac complement** for L is equivalent to solving the curved Maurer-Cartan equation:

$$\mathrm{N}_M + d_M\zeta + \tfrac{1}{2}[\zeta,\zeta]_L = 0,$$

where N_M is the Nijenhuis tensor of M, given by $N_M(m_1,m_2,m_3) = \langle[m_1,m_2],m_3\rangle$. Note that M is Dirac $\leftrightarrow N_M = 0$.

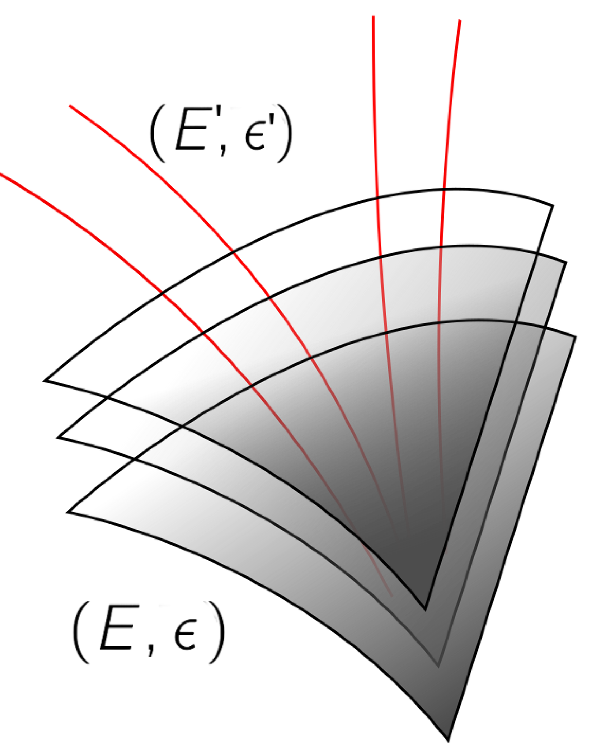

Figure 1: Transverse Dirac structures

A Dirac structure in $TX \oplus T^*X$ also has a representation as a singular foliation E along with a closed 2-form ϵ on the leaves of E. Geometrically, finding a complement is equivalent to finding a transverse singular foliation E' along with a closed 2-form ϵ' on E' such that $i^*_{E\cap E'}(\epsilon-\epsilon')$ is symplectic on the leaves of $E\cap E'$.

A cohomological obstruction

In a given Courant algebroid, for a given Dirac structure L, we construct a cohomology class that vanishes whenever L admits a **Dirac complement**. This class is calculated through the Nijenhuis tensor of any Lagrangian complement. Explicitly, this is the class of an element in:

$$\Gamma(\wedge^\bullet L) \,/\, [\,\Gamma(\wedge^\bullet L)\,,\,\Gamma(\wedge^\bullet L)\,]$$

which carries a canonical differential. We denote this class by $N(L)$.

> **Theorem** • The class $N(L)$ is independent of choice of complement M.
> • If L has a Dirac complement, $N(L) = 0$.

In the case of $TX \oplus T^*X$ twisted by a closed 3-form H, we have the following:

> **Example** Let π be a twisted Poisson structure vanishing on a submanifold $X' \subseteq X$ such that
> $$0 \neq [i_{X'}H] \in H^3_{dR}(X').$$
> Then $N(graph(\pi)) \neq 0$, and hence, $graph(\pi)$ has no Dirac complement.

For a Lie algebra $\mathfrak{g}$, in the case of the Courant algebroid given by the double $\mathfrak{g}\oplus\mathfrak{g}^*$ with bracket twisted by a closed $H \in \bigwedge^3 \mathfrak{g}^*$ and $\rho = 0$, we find:

> **Example** Let $E \leq \mathfrak{g}$ be an ideal such that $H \in \bigwedge^3 E^0$ and the cohomology class of the reduction of H to $\mathfrak{g}/E$ is nonzero. Then $N(E \oplus Ann(E)) \neq 0$, and hence, $E \oplus Ann(E)$ has no Dirac complement.

For example, for $\mathfrak{g}_{4,1} = \langle e_1, e_2, e_3, e_4 \mid [e_2,e_4] = e_1, [e_3,e_4] = e_2\rangle$ and $H = \alpha_2\wedge\alpha_3\wedge\alpha_4$, we have that $\langle e_1\rangle + Ann(\langle e_1\rangle)$ has no Dirac complement.

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