Symmetric Poisson structures
and where to find them

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Recalling Poisson geometry

In mechanics, we assign dynamics to a Hamiltonian function

$$(\star) \qquad \begin{aligned} \mathcal{C}^\infty(M) &\to \mathfrak{X}(M) \\ f &\mapsto \pi(\mathrm{d}f), \end{aligned}$$

for a skew-symmetric bivector $\pi \in \mathfrak{X}^2(M)$, which also gives the bracket on $\mathcal{C}^\infty(M)$:

$$\{f, \mathrm{g}\} := \pi(\mathrm{d}f, \mathrm{dg}).$$

Jacobi identity for $\{\,,\}$ gives the definition:

> $\pi \in \mathfrak{X}^2(M)$ is a **Poisson structure** if
>
> $$[\pi, \pi] = 0.$$

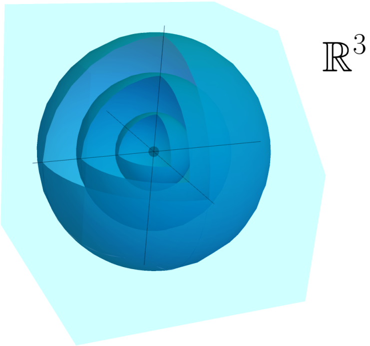
Equivalently, the map $(\star)$ is an algebra morphism $(\mathcal{C}^\infty(M), \{\,,\}) \to (\mathfrak{X}(M), [\,,\,])$.

$[\,,\,]$ is the **Schouten bracket** – a natural extension of the Lie bracket of vector fields.

Recalling Poisson geometry

Geometrically, a Poisson structure π gives a distribution $\operatorname{im}\pi \subseteq TM$ that integrates to a singular partition of M with symplectic leaves.

$$\pi = z\,\partial_x \wedge \partial_y + x\,\partial_y \wedge \partial_z + y\,\partial_z \wedge \partial_x$$



What geometry is encoded by symmetric bivector fields?

Integrability condition for a symmetric bivector field ϑ

If a symmetric bivector field $\vartheta \in \mathfrak{X}^2_{\text{sym}}(M) := \Gamma(\mathrm{Sym}^2\, TM)$ is non-degenerate,

$$g := \vartheta^{-1}$$

is a (pseudo-)Riemannian metric.

A non-degenerate 2-form $\omega \in \Omega^2(M)$ is symplectic if and only if $\pi := \omega^{-1}$ is Poisson.

Following the Poisson geometry approach, every $\vartheta \in \mathfrak{X}^2_{\text{sym}}(M)$ gives

$$\{f, \mathrm{g}\} := \vartheta(\mathrm{d}f, \mathrm{dg}), \qquad \begin{aligned} \mathrm{grad} : \mathcal{C}^\infty(M) &\to \mathfrak{X}(M) \\ f &\mapsto \vartheta(\mathrm{d}f). \end{aligned}$$

Jacobi identity for $\{\,,\}$ gives that $\vartheta = 0$.

Trying the other two:

$$[\vartheta, \vartheta] = 0$$

is a void condition.

$$\mathrm{grad} : (\mathcal{C}^\infty(M), \{\,,\}) \to (\mathfrak{X}(M), [\ ,\ ])$$

is an algebra morphism.

$$\Leftrightarrow \quad \vartheta = 0.$$

We resort to symmetric Cartan calculus!

Symmetric Cartan calculus

M., Rubio. Symmetric Cartan calculus, the Patterson-Walker metric and symmetric cohomology, January 2025. arXiv:2501.12442

exterior derivative d on $\Omega^\bullet(M)$	symmetric derivative ∇^s on $\Gamma(\mathrm{Sym}^\bullet T^*M)$
$\mathrm{d}\psi = (r+1)\,\mathrm{skew}\,(\nabla\psi)$	$\nabla^s\varphi = (r+1)\,\mathrm{sym}\,(\nabla\varphi)$
canonical	depending on the choice of ∇

The **symmetric bracket**:

$$[X,Y]_s := \nabla_X Y + \nabla_Y X.$$

It is actually determined by ∇^s:

$$\iota_{[X,Y]_s} = [[\iota_X, \nabla^s], \iota_Y].$$

Compare with: $[X,Y] := X\circ Y - Y\circ X = \nabla_X Y - \nabla_Y X$ and $\iota_{[X,Y]} = [[\iota_X, \mathrm{d}]_\mathrm{g}, \iota_Y]_\mathrm{g}$.

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$\rightsquigarrow$ A natural extension to $\mathfrak{X}^\bullet_{\mathrm{sym}}(M)$
the **symmetric Schouten bracket** $[\ ,\ ]_s$.

Compare with: $[X,Y] := X\circ Y - Y\circ X = \nabla_X Y - \nabla_Y X$ $\rightsquigarrow$ the Schouten bracket $[\ ,\ ]$ on $\mathfrak{X}^\bullet(M)$.

Integrability condition for a pair (ϑ, ∇)

$\pi \in \mathfrak{X}^2(M)$ is a **Poisson structure** if $[\pi, \pi] = 0$.

Equivalently, the map $(\star)$ is an algebra morphism $(\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ])$.

$[\vartheta, \vartheta]_s = 0$.

$$\mathrm{grad} : (\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ]_s)$$

is an algebra morphism.

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(ϑ, ∇) is a **symmetric Poisson structure** if

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Integrability condition for a pair (ϑ, ∇)

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Equivalently, the map $(\star)$ is an algebra morphism $(\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ])$.

(ϑ, ∇) is a **symmetric Poisson structure** if

$$[\vartheta, \vartheta]_s = 0.$$

(ϑ, ∇) is a **strong symmetric Poisson structure** if the map

$$\mathrm{grad} : (\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ]_s)$$

is an algebra morphism.

For ϑ is non-degenerate and $g := \vartheta^{-1}$ we have:

(ϑ, ∇) is symmetric Poisson

$\Leftrightarrow$

$$\nabla^s g = 0,$$

i.e., g is a Killing 2-tensor for ∇.

(ϑ, ∇) is strong symmetric Poisson

$\Leftrightarrow$

∇ is the Levi-Civita connection of g,

i.e., the information is all contained in ϑ.

The characteristic distribution, module and metric

The **characteristic distribution**

$$\operatorname{im}\vartheta := \{\vartheta(\zeta) \,|\, \zeta \in T^*M\} \subseteq TM$$

The **characteristic module**

$$\mathcal{F}_\vartheta := \{\vartheta(\alpha) \,|\, \alpha \in \Omega^1(M)\} \subseteq \mathfrak{X}(M)$$

Extra structure at each point $m \in M$: the **characteristic metric** on $\operatorname{im}\vartheta_m \leq T_m M$

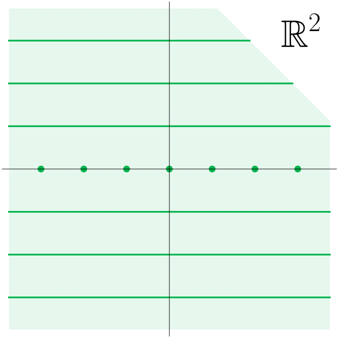
$$g_{\vartheta_m}(\vartheta(\zeta), \vartheta(\eta)) := \vartheta(\zeta, \eta).$$

Example: $\vartheta = y\,\partial_x \otimes \partial_x \in \mathfrak{X}^2_{\mathsf{sym}}(\mathbb{R}^2)$

$$\operatorname{im}\vartheta_m = \begin{cases} \operatorname{span}\{\,\partial_x|_m\} & y \neq 0, \\ \{0\} & y = 0 \end{cases}$$

$$\mathcal{F}_\vartheta = \mathcal{C}^\infty(\mathbb{R}^2)\text{-span}\{y\,\partial_x\}$$

$$g_{\vartheta_m} = \begin{cases} \frac{1}{y}\;\mathrm{d}x|_m \otimes \mathrm{d}x|_m & y \neq 0, \\ 0 & y = 0 \end{cases}$$



Involutive symmetric Poisson structures

(ϑ, ∇) is symmetric Poisson

$\Rightarrow \mathcal{F}_\vartheta$ is not necessarily Lie involutive,

but it is preserved by the symmetric braket

$$[\mathcal{F}_\vartheta, \mathcal{F}_\vartheta]_s \subseteq \mathcal{F}_\vartheta.$$

(ϑ, ∇) is strong symmetric Poisson

$\Rightarrow \mathcal{F}_\vartheta$ is Lie involutive

$$[\mathcal{F}_\vartheta, \mathcal{F}_\vartheta] \subseteq \mathcal{F}_\vartheta,$$

$\Rightarrow \operatorname{im}\vartheta$ integrates to a singular partition.

This motivates an a priori intermediate class

$$\left\{ \begin{array}{c} \text{strong symmetric} \\ \text{Poisson structures} \\ \nabla_{\vartheta(\,)}\vartheta = 0 \end{array} \right\} \subseteq \left\{ \begin{array}{c} \textbf{involutive symmetric} \\ \textbf{Poisson structures} \\ [\mathcal{F}_\vartheta, \mathcal{F}_\vartheta] \subseteq \mathcal{F}_\vartheta \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{symmetric} \\ \text{Poisson structures} \\ [\vartheta, \vartheta]_s = 0 \end{array} \right\}.$$

Geometric interpretation: symmetric Poisson structures

Given a connection on M,
we call a distribution $\Delta \subseteq TM$ **locally geodesically invariant**
if for every geodesic $\gamma : I \to M$ satisfying $\dot{\gamma}(t_0) \in \Delta_{\gamma(t_0)}$ for some $t_0 \in I$,
there is a subinterval I' containing t_0 such that $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for all $t \in I'$.

Given $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M)$,
we call a curve $\gamma : I \to M$ **ϑ-admissible** if there is a curve $a : I \to T^*M$ such that

$$\vartheta(a(t)) = \dot{\gamma}(t).$$

Theorem 1. The characteristic distribution of a symmetric Poisson structure (ϑ, ∇) is locally geodesically invariant.

Moreover, $g_\vartheta(\dot{\gamma}, \dot{\gamma})$, is constant along ϑ-admissible geodesics.

Geometric interpretation: involutive and strong symmetric Poisson structures

An involutive symmetric Poisson structure (ϑ, ∇) gives the singular partition of M into leaves such that every leaf $N \subseteq M$ satisfies

$$\operatorname{im} \vartheta|_N = TN.$$

Moreover, every leaf N acquires:

- the **leaf connection** ∇^N given by the restriction of ∇.
- the **leaf metric** g_N given by the metrics g_{ϑ_m}.

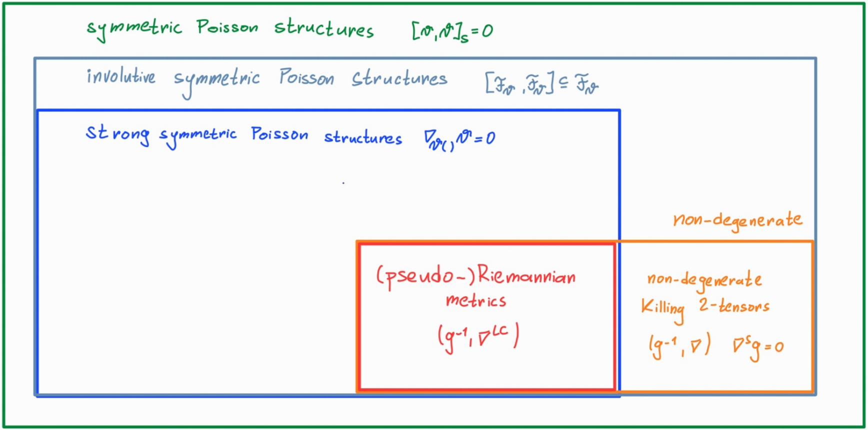
Given a connection on M,
a submanifold $N \subseteq M$ is **totally geodesic**
if for every geodesic $\gamma : I \to M$ satisfying $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}N$ for some $t_0 \in I$,
there is a subinterval I' containing t_0 such that $\dot{\gamma}(t) \in T_{\gamma(t)}N$ for all $t \in I'$.

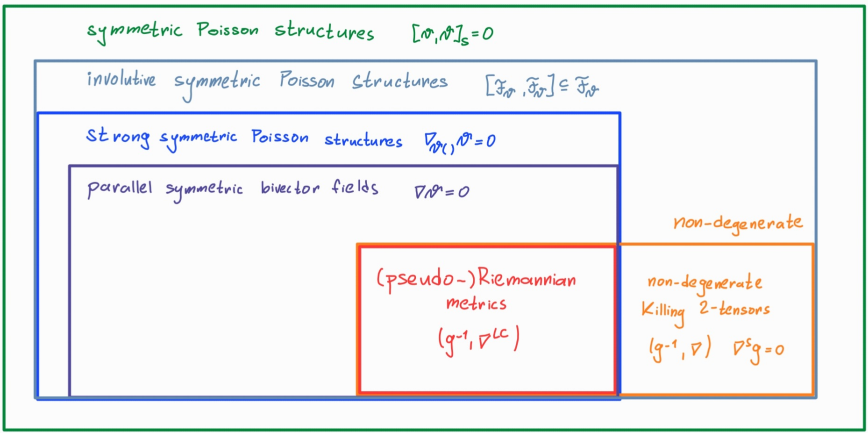
Theorem 2. The characteristic partition of an involutive symmetric Poisson structure (ϑ, ∇) is totally geodesic.

Moreover, on any leaf N, (g_N^{-1}, ∇^N) is non-degenerate symmetric Poisson.

In addition, if (ϑ, ∇) is strong, ∇^N is the Levi-Civita connection of g_N.

Where to find them?


symmetric Poisson structures $[\vartheta,\vartheta]_s=0$
involutive symmetric Poisson structures $[\mathcal{F}_\vartheta,\mathcal{F}_\vartheta]\subseteq\mathcal{F}_\vartheta$
strong symmetric Poisson structures $\nabla_{\vartheta()}\vartheta=0$
non-degenerate
(pseudo-)Riemannian metrics
(g^{-1},∇^{LC})
non-degenerate Killing 2-tensors
(g^{-1},∇) $\nabla^s g=0$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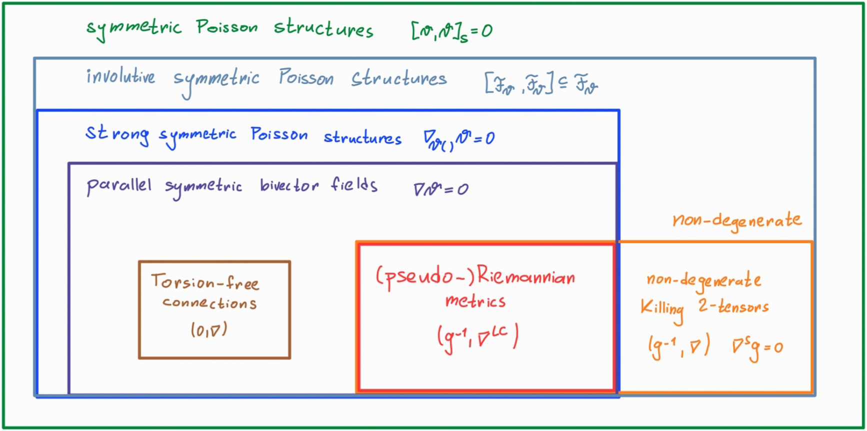
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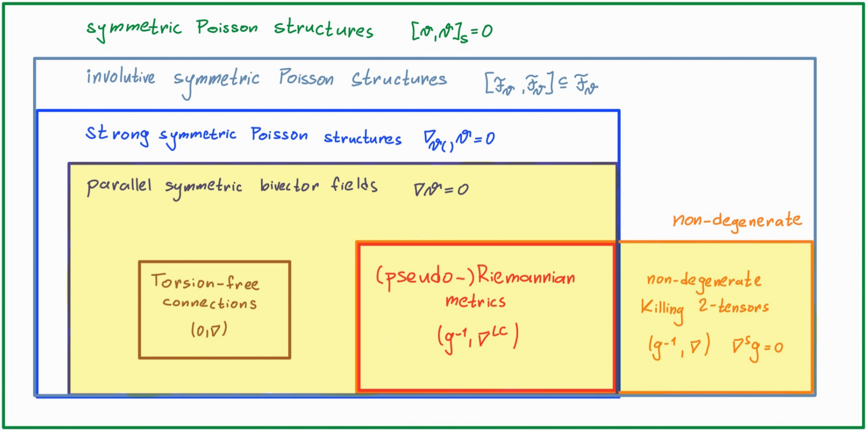
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As autoparallel vector fields

Every $X \in \mathfrak{X}(M)$ gives a symmetric bivector field $X \otimes X$.

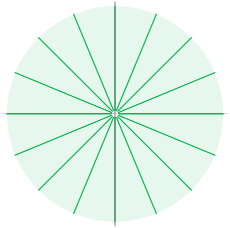
$(X \otimes X, \nabla)$ is symmetric Poisson $\Leftrightarrow$ it is strong symmetric Poisson,
$\Leftrightarrow$ X is autoparallel, that is, $\nabla_X X = 0$,
$\Leftrightarrow$ integral curves of X are geodesics.

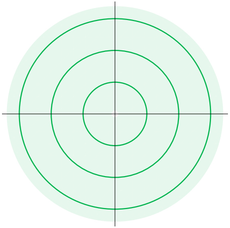
Example: The punctured plane $M := \mathbb{R}^2 \setminus \{0\}$ admits nowhere vanishing vector fields

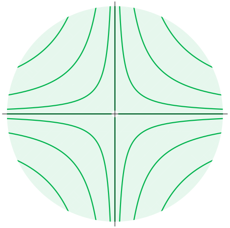
$$R := x\,\partial_x + y\,\partial_y, \qquad S := -y\,\partial_x + x\,\partial_y, \qquad H := x\,\partial_x - y\,\partial_y.$$

We can made them autoparallel by choosing the connection:

$$\nabla_{\partial_x}\partial_x = \nabla_{\partial_y}\partial_y := \pm\frac{1}{x_2 + y^2}R, \qquad \nabla_{\partial_x}\partial_y := 0.$$







On a Lie group

A Lie group G carries the natural connection $\dot\nabla$ given by its values on $X, Y \in \mathfrak{X}_{\mathsf{L}}(G)$:

$$\dot\nabla_X Y = \frac{1}{2}[X, Y].$$

$(\vartheta, \dot\nabla)$ is symmetric Poisson for every $\vartheta \in \mathfrak{X}^2_{\mathsf{sym,L}}(G)$.

Example: For the Heisenberg group H_3,

$$\vartheta = Q \otimes Q + P \otimes P$$

together with $\dot\nabla$ is symmetric Poisson that is not involutive as

$$\mathcal{F}_\vartheta = \mathcal{C}^\infty(H_3)\text{-span}\{Q, P\} \qquad \text{and} \qquad [Q, P] = I.$$

Example: For $G = \mathrm{S}^3$ seen as unit quaternions.

The vector field X given by right-multiplication by i is left-invariant,

hence $(X \otimes X, \dot\nabla)$ is strong symmetric Poisson.

The characteristic partition of S^3 is that of *Hopf fibration* $\mathrm{S}^3 \to \mathrm{S}^2$ into great circles.

On a vector space

In classical Poisson geometry,

$$\left\{ \begin{array}{c} \text{linear Poisson} \\ \text{structures on } V^* \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Lie algebra} \\ \text{structures on } V \end{array} \right\}.$$

$\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(V^*)$ is called **linear** if $\quad \{\mathcal{C}^\infty_{\mathsf{lin}}(V^*), \mathcal{C}^\infty_{\mathsf{lin}}(V^*)\} \subseteq \mathcal{C}^\infty_{\mathsf{lin}}(V^*)$.

As $\mathcal{C}^\infty_{\mathsf{lin}}(V^*) \cong V$, a linear ϑ is equivalent to a commutative algebra structure on V.

V^* is an abelian Lie group, $\dot{\nabla}$ is usually referred to as the **Euclidean connection** ∇^{Euc}.

$$\left\{ \begin{array}{c} \text{linear symmetric Poisson} \\ \text{structures } (\vartheta, \nabla^{\mathsf{Euc}}) \text{ on } V^* \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Jacobi-Jordan algebra} \\ \text{structures } \cdot \text{ on } V \end{array} \right\}.$$

[Burde-Fialowski '14] A commutative algebra $(\mathcal{J}, \cdot)$ is called **Jacobi-Jordan** if

$$u \cdot (v \cdot w) + v \cdot (w \cdot u) + w \cdot (u \cdot v) = 0.$$

In particular, Jacobi-Jordan algebras are Jordan algebras.

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$$\left\{ \begin{array}{c} \text{linear (strong) symmetric Poisson} \\ \text{structures } (\vartheta, \nabla^{\mathsf{Euc}}) \text{ on } V^* \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(associative) Jacobi-Jordan algebra} \\ \text{structures } \cdot \text{ on } V \end{array} \right\}.$$

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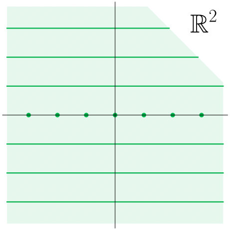
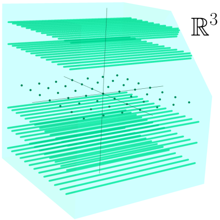
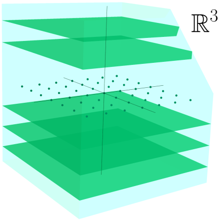
In particular, Jacobi-Jordan algebras are Jordan algebras.

[BF '14]: Jacobi-Jordan algebras are associative if $\dim V \leq 4$.
$\Rightarrow$ Linear symmetric Poisson structures $(\vartheta, \nabla^{\text{Euc}})$ on V^* are strong if $\dim V \leq 4$.

$\dim V$	ϑ	leaf dim.	leaf metric signatures
2	$y\,\partial_x \otimes \partial_x$	$0, 1$	$(1,0), (0,1)$
3	$z\,\partial_x \otimes \partial_x$	$0, 1$	$(1,0), (0,1)$
	$z\,(\partial_x \otimes \partial_x + \partial_y \otimes \partial_y)$	$0, 2$	$(2,0), (0,2)$
4	$t\,\partial_x \otimes \partial_x$	$0, 1$	$(1,0), (0,1)$
	$t\,(\partial_x \otimes \partial_x + \partial_y \otimes \partial_y)$	$0, 2$	$(2,0), (0,2)$
	$t\,\partial_x \otimes \partial_x + z\,\partial_y \otimes \partial_y$	$0, 1, 2$	$(1,0), (0,1), (2,0), (0,2), (1,1)$
	$t\,\partial_x \otimes \partial_x + z\,\partial_x \odot \partial_y$	$0, 1, 2$	$(1,0), (0,1), (1,1)$
	$t\,(\partial_x \otimes \partial_x + \partial_y \odot \partial_z)$	$0, 3$	$(2,1), (1,2)$







A non-strong involutive symmetric Poisson structure

[BF '14]: There is a unique non-associative Jacobi-Jordan algebra for $\dim V = 5$.
$\Rightarrow$ The unique linear symmetric Poisson structure for $\dim V = 5$ and it is non-strong:

$$\vartheta = x_2\,\partial_{x_1} \otimes \partial_{x_1} + x_5\,\partial_{x_1} \odot \partial_{x_4} - \frac{1}{2}x_3\,\partial_{x_1} \odot \partial_{x_5} + x_3\,\partial_{x_2} \odot \partial_{x_4}.$$

Its characteristic module $\mathcal{F}_\vartheta$ is generated by

$$X_1 := x_2\,\partial_{x_1} + x_5\,\partial_{x_4} - \frac{1}{2}x_3\,\partial_{x_5}, \qquad X_2 := x_3\,\partial_{x_4},$$
$$X_3 := x_5\,\partial_{x_1} + x_3\,\partial_{x_2}, \qquad X_4 := x_3\,\partial_{x_1}.$$

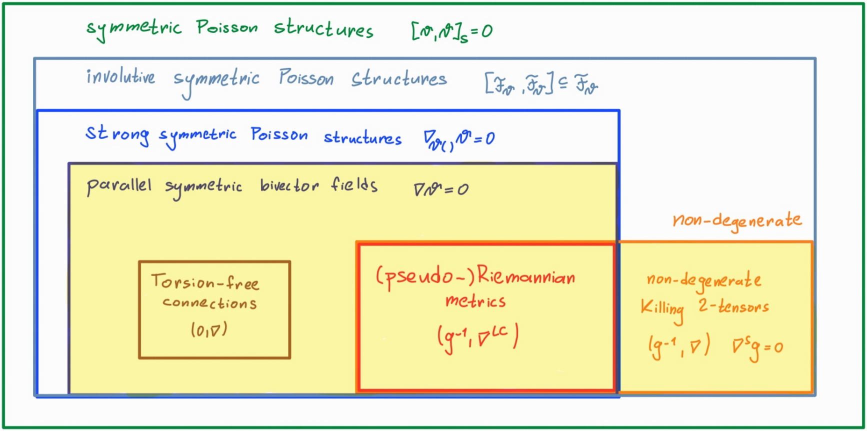
The only non-trivial commutator is $[X_1, X_3] = \frac{1}{2}X_4$, hence it is **involutive**!

Dimensions of the leaves are $0, 1, 2, 4$ with the signatures: $(1,0), (0,1), (1,1), (2,2)$.

All of the leaves (except for the 4-dimensional ones) inherit a strong symmetric Poisson structure, that is, ∇^N is the Levi-Civita connection of g_N.

The leaf metric on a 4-dimensional leaf N_c given by $x_3 = c$, $c \neq 0$:

$$g_{N_c} = -\frac{2}{c}\mathrm{d}x_1 \odot \mathrm{d}x_5 + \frac{1}{c}\mathrm{d}x_2 \odot \mathrm{d}x_4 + \frac{2x_5}{c^2}\mathrm{d}x_2 \odot \mathrm{d}x_1 - \frac{4x_2}{c^2}\mathrm{d}x_5 \otimes \mathrm{d}x_5.$$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It just appeared on arXiv this morning:

M., Rubio
Symmetric Poisson geometry,
totally geodesic foliations
and Jacobi-Jordan algebras.
arXiv: 2508.15890

The third part of the story is coming...

Thank you for your attention!