A brief introduction to deformation quantization

Henrique Bursztyn, IMPA

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Mathematics $\rightleftharpoons$ **Physics**

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R. Dijkgraaf's "Unreasonable effectiveness of quantum physics in modern mathematics".

Outline:

- ◇ Geometry of phase spaces
- ◇ Canonical quantization
- ◇ The first star products
- ◇ Quantization and deformation theory: mathematical set up.
- ◇ Deformation quantization: symplectic case
- ◇ Deformation quantization: Kontsevich's theorem
- ◇ Morita equivalence of star products

Geometry of phase spaces

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In quantum mechanics, there is a drastic change...

Quantum mechanics

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What is "quantization"?

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Differential operators on wave functions:

$\widehat{Q}, \;\; \widehat{P} : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})$,

$$(\widehat{Q}\psi)(q) = q\psi(q), \;\; (\widehat{P}\psi)(q) = -i\hbar \frac{\partial \psi}{\partial q}(q),$$

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Ordering problem: how to quantize $q^k p^l$?

Standard ordering: $q^k p^l \mapsto \widehat{Q}^k \widehat{P}^l = (\hbar/i)^l q^k \frac{\partial^l}{\partial q^l}$

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Standard representation:

$$\varrho_s(f) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{i}\right)^r \frac{\partial^r f}{\partial p^r}\Big|_{p=0} \frac{\partial^r}{\partial q^r}$$

Proposition: $\varrho_s : \mathrm{Pol}(T^*\mathbb{R}) \to \mathrm{DiffOp}(C_0^\infty(\mathbb{R}))$ is bijection.

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Weyl ordering (total symmetrization):

$$q^2 p \mapsto \frac{1}{3}(\widehat{Q}^2\widehat{P} + \widehat{Q}\widehat{P}\widehat{Q} + \widehat{P}\widehat{Q}^2) = -i\hbar q^2 \frac{\partial}{\partial q} - i\hbar q$$

Gives rise to Weyl representation

$$\varrho_w : \mathrm{Pol}(T^*\mathbb{R}) \to \mathrm{DiffOp}(C_0^\infty(\mathbb{R}))$$

Another viewpoint: star products

Pull-back product from $\mathrm{DiffOp}(C_0^\infty(\mathbb{R}))$ to $\mathrm{Pol}(T^*\mathbb{R})$:

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Quantization as a new product of functions! (special properties...)

Quantization and deformation theory

Physics: classical mechanics $\rightsquigarrow$ quantum mechanics

Math: commutative structures $\rightsquigarrow$ noncommutative structures

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Quantization as a deformation (in $\hbar$) :

(Flato et al, 1970's; Gerstenhaber 1960's deformation theory for associative algebras)

$$f \star_\hbar g := f.g + \hbar C_1(f,g) + \hbar^2 C_2(f,g) + \hbar^3 C_3(f,g) + \dots$$

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Questions: Are there C_r's such that $f \star_\hbar (g \star_\hbar h) = (f \star_\hbar g) \star_\hbar h$?
How many ways?

The mathematical set-up for quantization: Classical side

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Classical phase space geometry = Poisson geometry

M manifold, $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that:

1. $\{f, g\} := -\{g, f\}$,
2. $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$
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Local expression in coordinates $(x^1, \ldots, x^n)$:

$$\{f, g\} = \sum_{i,j} \pi^{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad \pi^{ij} = \{x_i, x_j\}$$

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Tensorial description: $\pi \in \Gamma(\wedge^2 TM)$, $[\pi, \pi] = 0$;

$$\{f, g\} = \pi(df, dg).$$

Examples:

- Symplectic (M, ω),

$$\{f, g\} = \omega(X_g, X_f)$$

- Dual of Lie algebras $\mathfrak{g}^*$,

$$\{f, g\}(\xi) = \xi([df|_\xi, dg|_\xi])$$

- compact Lie groups, Poisson homogeneous spaces....

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Star product on M is *associative product* on $C^\infty(M)[[\hbar]]$,

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Associativity: $(f \star g) \star h = f \star (g \star h)$, i.e.,

$$\sum_{r=0}^{k} C_r(f, C_{k-r}(g,h)) = \sum_{r=0}^{k} C_r(C_{k-r}(f,g),h)$$

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Poisson bracket: $\{f, g\} = \frac{1}{\hbar}[f, g]_*\big|_{\hbar=0} = C_1(f, g) - C_1(g, f)$

Equivalence (ordering):

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Moduli of star products: $\mathrm{Def}(M) = \{\star\}/\sim$

Main Questions:

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Do star products exist for any given Poisson bracket $\{\cdot,\cdot\}$*?*

Classification of equivalence classes?

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Both results follow from Kontsevich's formality theorem.

More precise statement....

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such that $\frac{1}{\hbar}[f, g]_\star\big|_{\hbar=0} = \pi_1(df, dg)$.

Formality theorem: There is L_∞-quasi-isomorphism of DGLAs

$$(\mathcal{X}(M)_\hbar, [\cdot,\cdot]_{SN}, d=0) \rightsquigarrow (\mathcal{D}_\hbar(M), [\cdot,\cdot]_G, d_G).$$

In particular, 1-1 correspondence of Maurer-Cartan elements...

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Poisson sigma model; Kontsevich's star product given by semiclassical expansion of suitable path integral (Cattaneo-Felder C.M.P. '2000)

Related developments

- Classical phase space of Poisson sigma model leads to *symplectic groupoids* (Cattaneo-Felder '00)
- Path space method for integration of Lie algebroids. (Crainic-Fernandes, Annals of Math '03)
- Other applications to Lie theory, homotopic algebras...

B-fields and Morita equivalence

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Group of self Morita equivalence of $\mathcal{A}$ is $\mathrm{Pic}(\mathcal{A})$.

Morita theorem: characterization of Morita bimodules.

- $X_{\mathcal{A}}$ is finitely generated, projective
- $\mathcal{B} = \mathrm{End}(X_{\mathcal{A}})$
- full $\quad (X_{\mathcal{A}} = P\mathcal{A}^n, \text{ then } M_n(\mathcal{A})PM_n(\mathcal{A}) = M_n(\mathcal{A}))$

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◇ $E \to M$ vector bundle: $\Gamma(E)$ is Morita bimodule for $\Gamma(\mathrm{End(E)})$ and $C^\infty(M)$

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Morita theorem: characterization of Morita bimodules.

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Morita equivalence (equivalence of categories of representations) has been studied in various other settings (C^*-algebras, Lie groupoids, Poisson manifolds)... key role in noncommutative geometry

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Example: $\omega^{-1} \mapsto \omega^{-1}(1 + B\omega^{-1})^{-1} = (\omega + B)^{-1}$

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Relation with Poisson-geometric Morita equivalence (B., Ortiz, Waldmann, IMRN 2022)

Thanks!