Gugenheim's A_∞ de-Rham theorem and higher holonomies

GENTLE Seminar
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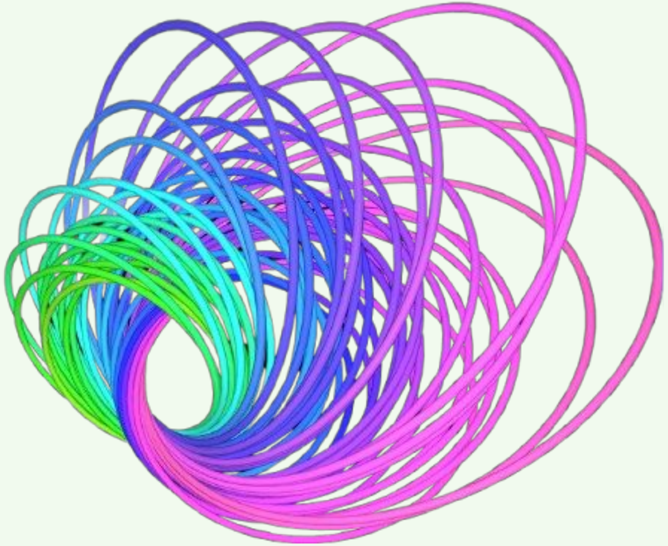


The signature of a path

Given a path $\gamma : [0,1] \to \mathbb{R}^n$

The signature is the formal noncommutative power series

$$S(\gamma) = \sum_I \int_{0 \le t_1 \le \cdots \le t_k \le 1} dx_{t_1}^{i_1} \wedge \cdots \wedge dx_{t_k}^{i_k} \, e^{i_1} \otimes \cdots \otimes e^{i_k}.$$

Goal Explain how Chen's iterated integrals arise in algebraic topology, in relation to differential forms on path spaces higher dimensional holonomy

The signature as a holonomy

Any vector space V comes with a tautological connection on the free Lie algebra on V:

$$\mathrm{id}_V \in \mathrm{Hom}(V, V) \simeq V^* \otimes V \subseteq \Omega(V, \mathrm{FreeLie}(V))$$

This connection is universal with respect to translation invariant connections on any Lie algebra.

The holonomy of this tautological connection is the signature.

The coefficients of the signature

The coefficients of the signature are given by an iterated integration of one forms on V.

The iterated iteration produces a function on the path space associated to a sequence of one forms on V.

Goal: Show that this construction that assigns functions on path spaces to sequences of one forms is very tautological and has natural and interesting generalizations

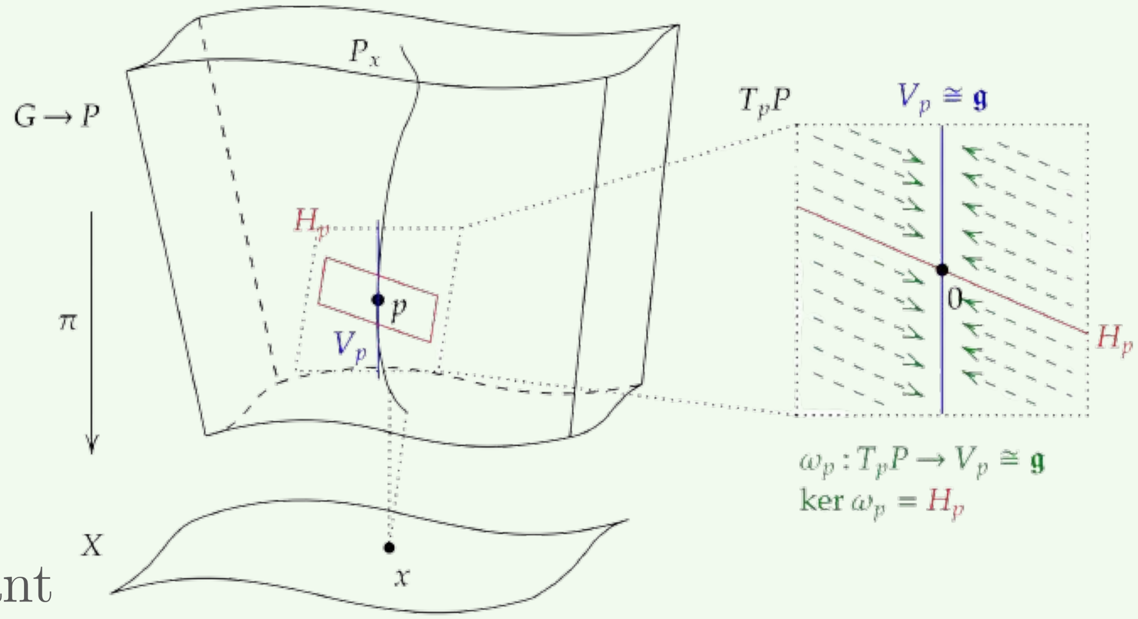
Geometric description of connections

A connection is a horizontal distribution on the total space of a principal G bundle

$$\omega \in \Omega(P, \mathfrak{g})$$

$$\omega(X_v) = v$$

$\omega : T^*P \to \mathfrak{g}$ is equivariant



Flat connections and holonomies

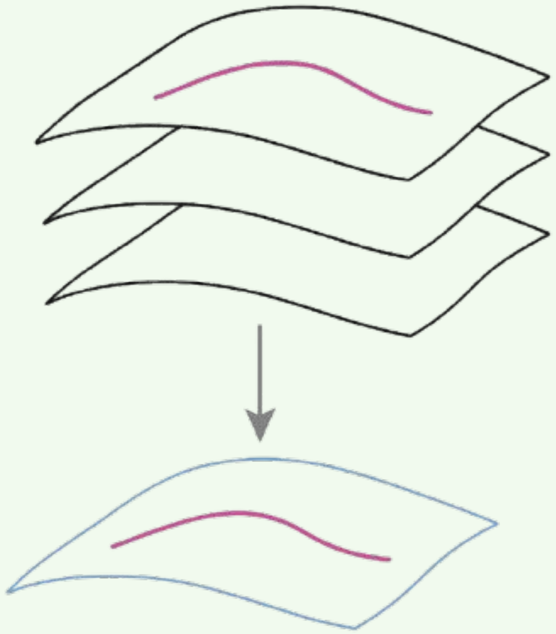
The Maurer Cartan Equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

The topology of the principal bundle can be changed so that the projection becomes a local diffeomorphism...

Holonomy becomes the lifting property for paths in covering spaces.

Flat connections correspond to representations of the fundamental groupoid



Higher holonomies

There is a higher dimensional version of this correspondence

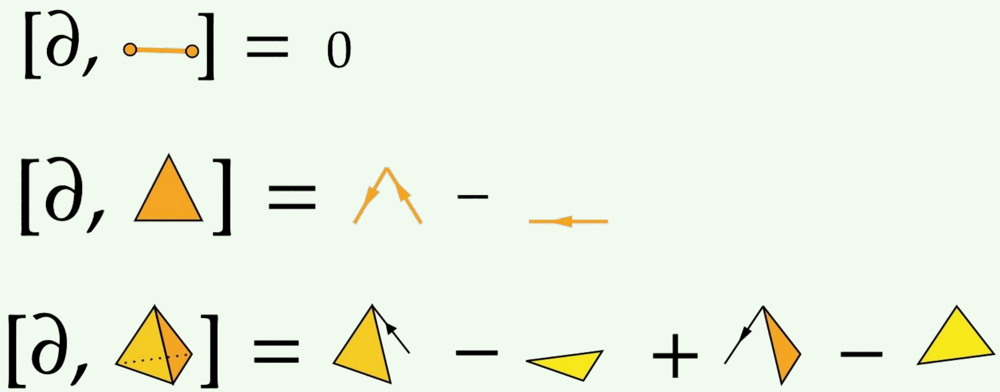
Connections are replaced by superconnections

The fundamental groupoid is replaced by the infinity groupoid of a space...

An instance of Lie theory!

Idea

Assign holonomies to simplices of all dimensions...



Chen's iterated integrals

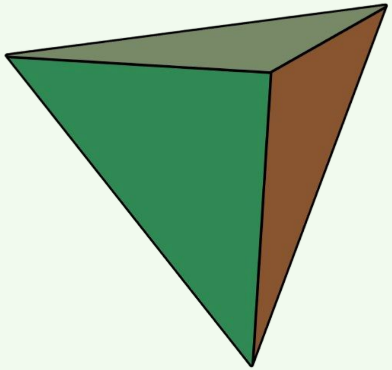
For any manifold M there is a tautological map

$$\text{eval} : \Delta_k \times PM \mapsto M^k$$

$$((t_1, \cdots t_k), \gamma) \mapsto (\gamma(t_1), \cdots, \gamma(t_k))$$

where:

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbb{R}^k : 0 \leq t_1 \leq t_2 \cdots \leq t_k \leq 1\}$$



Chen's iterated integrals

Chen’s map is given by the following composition

$$\begin{CD} \Omega(M)^{\otimes k} @>{\text{Chen}}>> \Omega(PM) \\ @V{\iota}VV @AA{\int_{\Delta_k}}A \\ \Omega(M^k) @>{\text{eval}^*}>> \Omega(\Delta_k \times PM) \end{CD}$$

Where

$$\iota(\omega_1 \otimes \cdots \wedge \omega_k) = \pi_1^*(\omega_1) \wedge \cdots \wedge \pi_k^*(\omega_k)$$

The based loop space of M

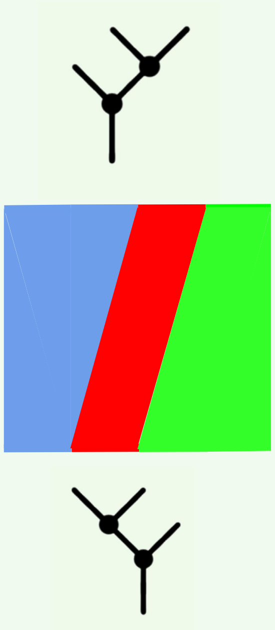
Given a point x in M we consider:

$$P_x(M) \;=\; \{\gamma \in PM \mid \gamma(0) = \gamma(1) = x\}$$

This space wants to be a group. It has a natural binary operation given by composition of loops:

$$(\gamma * \theta)(t) = \begin{cases} \theta(2t) & 0 \le t \le \frac{1}{2}, \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Composition of loops is not quite associative...



But different results are homotopic

A different version of based loops

This issue is resolved by Moore's version of the loop space

$$\tilde{P}_x(M) = \{(\gamma, t) \mid \gamma : [0, t] \to M, \gamma(0) = \gamma(t) = x, \ t \geq 0\}$$

The Moore loop space of M is a topological monoid.

This monoid structure gives the space of singular chains special algebraic structure

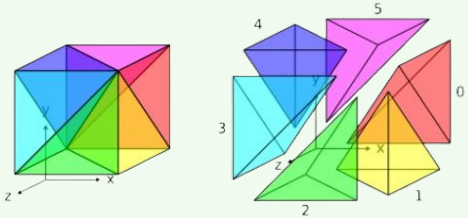
Singular chains on groups

The group structure on G gives the space of singular chains the structure of an algebra with product given by the composition:

$$\mathrm{C}_\bullet(G) \otimes \mathrm{C}_\bullet(G) \xrightarrow{\quad \mathsf{EZ} \quad} \mathrm{C}_\bullet(G \times G) \xrightarrow{\quad \mu_* \quad} \mathrm{C}_\bullet(G)$$

where the Eilenberg-Zilber map is defined by:

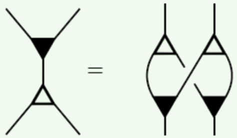
$$\mathsf{EZ}(\sigma \otimes \nu) \;=\; \sum_{\chi \in \mathfrak{S}_{r,s}} (-1)^{|\chi|} (\sigma \times \nu) \circ \chi_*,$$



and

$$\chi_* \colon \Delta_{r+s} \to \Delta_r \times \Delta_s, \quad \chi_*(t_1, \ldots, t_{r+s}) = ((t_{\chi(1)}, \ldots, t_{\chi(r)}), (t_{\chi(r+1)}, \ldots, t_{\chi(r+s)})).$$

These operations give C(G) the structure of a dg-Hopf algebra



The Bar construction

Let A be a commutative differential graded algebra.

The Bar construction of A, is the differential graded Hopf algebra:

$$B(A) = \bigoplus_k A^{\otimes k}$$

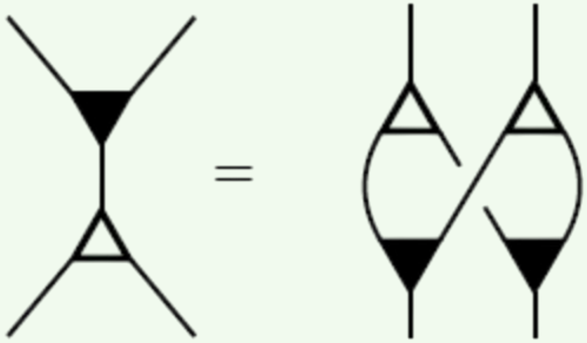
With product and coproduct given by:

$$[a_1 \,|\, \dots \,|\, a_p] \diamond [b_1 \,|\, \dots \,|\, b_q] \;=\; \sum_{\sigma \in S(p,q)} (-1)^{\varepsilon(\sigma;a,b)} \big[\, c_{\sigma^{-1}(1)} \,\big|\, c_{\sigma^{-1}(2)} \,\big|\, \dots \,\big|\, c_{\sigma^{-1}(p+q)} \big]$$

$$\Delta\big[\, a_1 \,|\, a_2 \,|\, \dots \,|\, a_n \big] = \sum_{i=0}^{n} \big[\, a_1 \,|\, \dots \,|\, a_i \big] \otimes \big[\, a_{i+1} \,|\, \dots \,|\, a_n \big]$$

The differentials on the Bar construction

$$d_B = d_{\text{int}} + d_{\text{bar}}, \qquad d_B^2 = 0.$$



$$d_{\text{bar}}[a_1 | \dots | a_n] = \sum_{i=1}^{n-1} \pm [a_1 | \dots | a_i a_{i+1} | \dots | a_n].$$

$$d_{\text{int}}[a_1 | \ . \ \ . | a_n] = \sum_{i=1}^{n} \pm [a_1 | \dots | d_A(a_i) | \dots | a_n].$$

Chen's de Rham theorem

Let M be a simply connected manifold. The map

$$\begin{array}{ccc} \Omega(M)^{\otimes k} & \xrightarrow{\text{Chen}} & \Omega(P_x M) \\ \downarrow \iota & & \uparrow \int_{\Delta_k} \\ \Omega(M^k) & \xrightarrow{\text{eval}^*} & \Omega(\Delta_k \times P_x M) \end{array}$$

gives a homomorphism of differential graded algebras which induces an isomorphism of Hopf algebras in cohomology.

Adams' Cobar construction

Let C be a differential graded coalgebra. The Cobar construction of C, is the free graded associative algebra on C:

$$F(C) = \bigoplus_k C^{\otimes k}$$

with differential given by $d = d_{\text{int}} + d_{\text{ext}}$

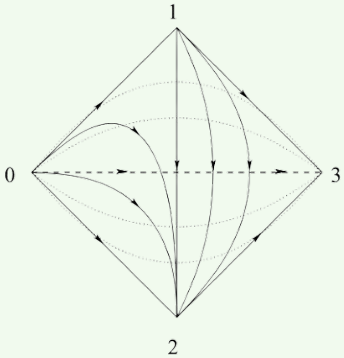
$$d_{\text{ext}}(c_1 \otimes \cdots \otimes c_n) = \sum_{i=1}^{n} \pm \sum_{(c_i)} c_1 \otimes \cdots \otimes c_i' \otimes c_i'' \otimes \cdots \otimes c_n$$

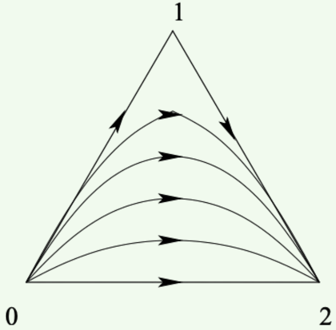
$$d_{\text{int}}(c_1 \otimes \cdots \otimes c_n) = \sum_{i=1}^{n} \pm\, c_1 \otimes \cdots \otimes (dc_i) \otimes \cdots \otimes c_n$$

Adams' Construction

Adams constructed a sequence of maps

$$\theta_n : I^{n-1} \to P\Delta_n$$





which naturally defines a linear map:

$$\text{Adams} : C(M) \mapsto C_{\square}(PM)$$
$$\sigma \mapsto P\sigma \circ \theta_n$$

Adams' theorem

The linear map

$$\text{Adams} : C(M) \mapsto C_{\square}(P_x M)$$

Extends to an algebra map

$$\text{Adams} : F(C(M)) \mapsto C_{\square}(P_x M)$$

Which induces an isomorphism of Hopf algebras in cohomology

De Rham's Theorem

Let M be a smooth manifold. The map

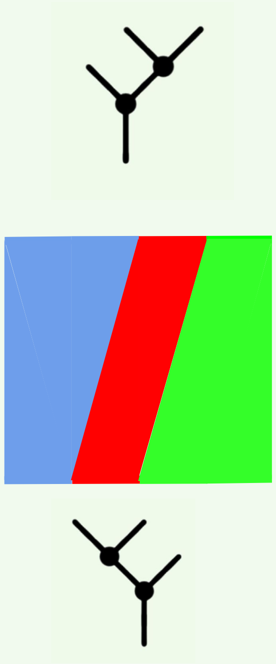
$$\varphi : \Omega(M) \to C^\bullet(M)$$

$$\eta \mapsto \varphi(\eta)$$

where $$\varphi(\eta)(\sigma) := \int_\Delta \sigma^*(\eta)$$

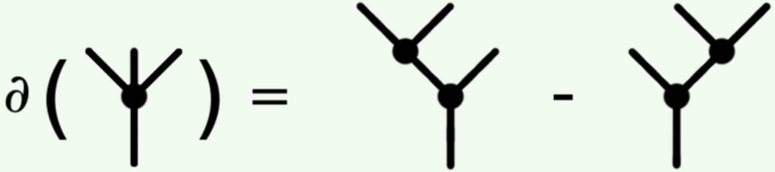
Induces an isomorphism of algebras in cohomology

Homotopy associativity



Homotopy associativity

Associativity



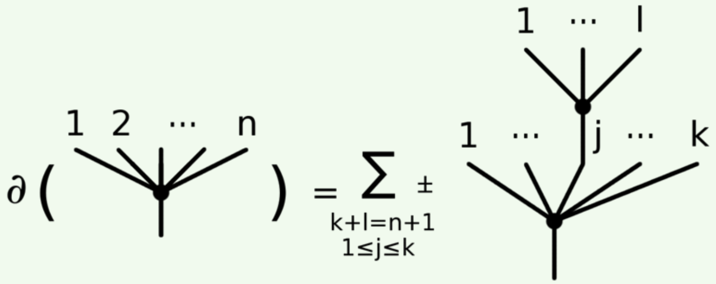
Homotopy associativity

Homotopy associative algebras

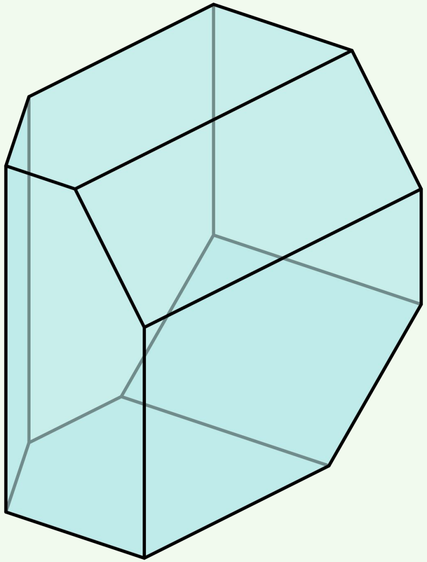
A homotopy associative algebra is a vector space A together with operations:

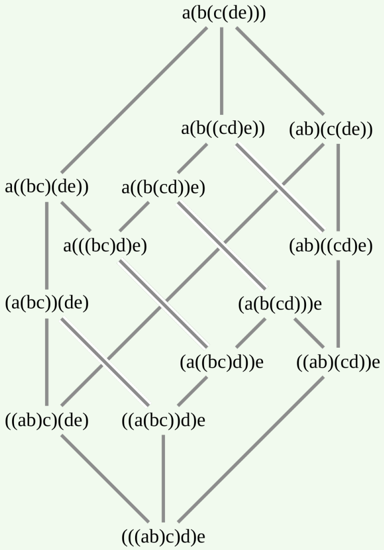
$$m_n : A^{\otimes m} \to A$$

That satisfies



Stasheff Associahedron





Gugenheim's version of de Rham's theorem

The map $G : B(\Omega(M)) \to C^\bullet(M)$

Given by the composition

$$G = Adams^* \circ \varphi \circ \text{Chen}$$

Is an equivalence of homotopy associative algebras.

$$\begin{array}{ccc} \Omega(M)^{\otimes k} & \xrightarrow{\text{Chen}} & \Omega(PM) \\ \downarrow \iota & & \uparrow \int_{\Delta_k} \\ \Omega(M^k) & \xrightarrow{\text{eval}^*} & \Omega(\Delta_k \times PM) \end{array}$$

$$\varphi : \Omega(P_x M) \mapsto C^\bullet_\square(P_x M)$$

$$\text{Adams} : C(M) \mapsto C_\square(PM)$$

A machine that produces holonomies....

Consider a connection

$$\omega \in \Omega(M, \mathfrak{g}) \subseteq \Omega(M, U(\mathfrak{g}))$$

By tensoring Gugenheim's map with the enveloping algebra one obtains a map:

$$G \otimes \mathrm{id}_{\mathrm{U}(\mathfrak{g})} : B(\Omega(M)) \otimes U(\mathfrak{g}) \to C^{\bullet}(M) \otimes U(\mathfrak{g})$$

Push forward Maurer-Cartan elements....

$$G \otimes \mathrm{id}_{\mathrm{U}(\mathfrak{g})} : B(\Omega(M)) \otimes U(\mathfrak{g}) \to C^\bullet(M) \otimes U(\mathfrak{g})$$

This is an A-infinity map.

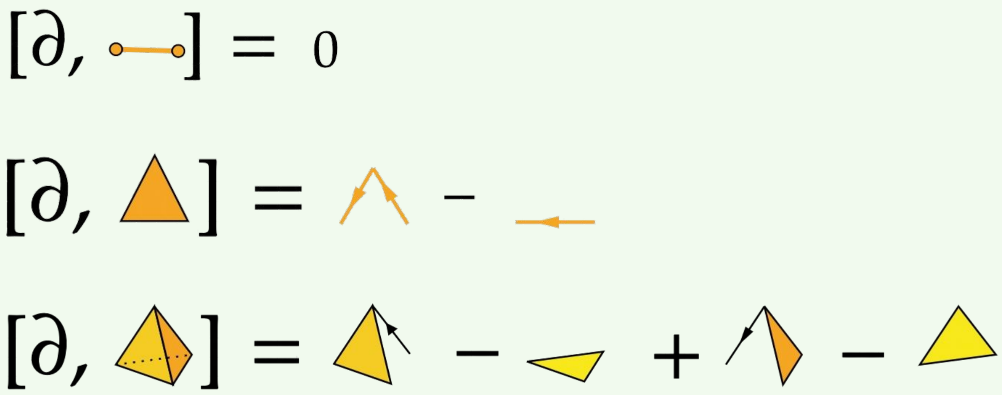
By pushing forward solutions to the Maurer-Cartan equation, one recovers the holonomy.

This applies to higher dimensional situations as well....

For instance one can take the dgla of endomorphisms of a chain complex...

Higher local systems...

Assign holonomies to simplices of all dimensions...



These are the equations for a representation (up to homotopy) of the infinity groupoid.

Lie theory

Consider a Lie algebroid $\pi : A \to M$

The infinity groupoid of A is the simplicial set whose space of k-simplices is

$$(\Pi_\infty(A))_k := \mathrm{Hom}_{\mathsf{DGCA}}(\Omega(A), \Omega(T\Delta_k))$$

Using the same explicit formulas as before, it becomes possible to integrate representations up to homotopy of Lie algebroids.

Higher Riemann-Hilbert Correspondence

The following categories are A-infinity equivalent (Block-Smith, Arias-Schaetz, Holstein, Igusa...)

Point of view	Local systems	Higher local systems
Differential geometry	Flat connections	Flat superconnections
Topology	Reps of fundamental group	Reps of infinity groupoid
Homotopy theory	Modules over the zero-th homology of the based loop space	Modules over chains on loop space

Integrating representations up to homotopy...

Arias Abad - Schaetz

Theorem. *Let $\pi : A \to M$ be a Lie algebroid. There exists an A_∞-functor:*

$$\mathcal{I} : \mathrm{Rep}_\infty(A) \to \mathrm{Rep}_\infty(\pi_\infty(A))$$

which is constructed by computing higher holonomies.

The story

1. **Adams** Algebraic model for the homology of the loop space

2. **Chen** Algebraic model for the cohomology of the loop space

3=1+2 **Gugenheim** Homotopy version of de Rham theorem

4. **Stasheff** Homotopy associativity

5. **Igusa, Block-Smith, Arias-Schaetz** Gugenheim's map is a machine to produce (higher dimensional) holonomies

6. **Arias, Arias-Quintero-Pineda:** Higher local systems on classifying spaces, Chern-Weil theory and the Cartan Relations...

7. ....

A very incomplete list of references…

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Thank you for your attention!