# Courant algebroid lifts and curved Courant algebroids

Filip Moučka
(joint with Roberto Rubio)

Universitat Autònoma de Barcelona

Czech Technical University in Prague

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Srní

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## Recalling Courant algebroids

The generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$ is naturally equipped with

- the canonical symmetric form $\langle X+\alpha, Y+\beta\rangle_+ := \alpha(Y)+\beta(X)$,
- the projection $\mathrm{pr}_{TM}\colon \mathbb{T}M \to TM$,
- the Dorfman brakcet $[X+\alpha, Y+\beta] := [X,Y]_{\mathrm{Lie}} + L_X\beta - \iota_Y \mathrm{d}\alpha$.

Properties of this structure motivate the definition [Liu, Weinstein, Xu 1997]:

A **Courant algebroid** over $M$ is a tuple $(\mathbb{E}, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ consisting of

- a vector bundle $\mathbb{E} \to M$,
- a non-degenerate symmetric form $\langle\ ,\ \rangle \in \Gamma(\mathrm{Sym}^2\,\mathbb{E}^*)$, (the **pairing**)
- a vector bundle morphism $\rho\colon \mathbb{E} \to TM$ over $\mathrm{id}_M$, (the **anchor**)
- an $\mathbb{R}$-bilinear map $[\ ,\ ]\colon \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \to \Gamma(\mathbb{E})$. (the **bracket**)

satisfying, for $a, b, c \in \Gamma(\mathbb{E})$ and $\rho^* := \langle\ ,\ \rangle^{-1} \circ \rho^t\colon T^*M \to \mathbb{E}$, the following:

(Ca1) $[a,a] = \rho^*\mathrm{d}\langle a,a\rangle$,

(Ca2) $\rho(a)\langle b,c\rangle = \langle [a,b],c\rangle + \langle b,[a,c]\rangle$,

(Ca3) $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$.

# Transitive Courant algebroids

A Courant algebroid is called **transitive**
if $\rho$ is surjective.

Every transitive Courant algebroid fits into the exact sequence:

$$0 \longrightarrow \ker\rho \hookrightarrow \mathbb{E} \xrightarrow{\rho} TM \longrightarrow 0 \qquad \Rightarrow \qquad \mathbb{E} \cong TM \oplus \ker\rho.$$

As every Courant algebroid satisfies $\rho \circ \rho^* = 0$, we get another exact sequence:

$$0 \longrightarrow \operatorname{im}\rho^* \hookrightarrow \ker\rho \longrightarrow \frac{\ker\rho}{\operatorname{im}\rho^*} \longrightarrow 0$$

## Transitive Courant algebroids

A Courant algebroid is called **transitive** if $\rho$ is surjective. $\Rightarrow$ $\rho^*: T^*M \to \operatorname{im}\rho^*$ is an isomorphism.

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As every Courant algebroid satisfies $\rho \circ \rho^* = 0$, we get another exact sequence:

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} \ker\rho \longrightarrow \underbrace{\frac{\ker\rho}{\operatorname{im}\rho^*}}_{=:\mathcal{G}} \longrightarrow 0 \qquad \Rightarrow \qquad \mathbb{E} \cong \mathbb{T}M \oplus \mathcal{G}.$$

[Chen, Stiénon, Xu 2013]: There is a choice of such vector bundle isomorphism that transports the Courant algebroid structure from $\mathbb{E}$ to $\mathbb{T}M \oplus \mathcal{G}$ such that

- the anchor $\rho\colon \mathbb{T}M \oplus \mathcal{G} \to TM$ is the projection.

$\Rightarrow$ Every transitive Courant algebroid over $M$ is isomorphic to a Courant algebroid on a vector bundle of the form $\mathbb{E} = TM \oplus E$, for some bundle $\mathrm{pr}\colon E \to M$.

## Courant algebroid lifts

Given a Courant algebroid on $\mathbb{E} = TM \oplus E$ and a vector bundle connection $\nabla$ on $E$,

$$TE = \mathcal{H}_\nabla \oplus \mathcal{V} \cong \mathrm{pr}^* TM \oplus \mathrm{pr}^* E = \mathrm{pr}^* \mathbb{E}.$$

Using the pull-back section map, we get the injective $\mathcal{C}^\infty(M)$-module morphism

$$\phi\colon \Gamma(\mathbb{E}) \longrightarrow \mathfrak{X}(E)$$
$$X + \varphi \longmapsto X^{\mathrm{h}} + \varphi^{\mathrm{v}}$$

such that $\operatorname{im}\phi$ locally generates $\mathfrak{X}(E)$.

Courant algebroid structure $(\mathbb{E}, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ induces

- the (pseudo-)Riemannian metric $g_\nabla$ on $E$ $\qquad g_\nabla(\phi a, \phi b) := \mathrm{pr}^* \langle a, b \rangle$,
- the map $\rho_\nabla\colon\ \operatorname{im}\phi \to \mathfrak{X}(E)$ $\qquad \rho_\nabla(\phi a) := \phi\rho(a) = \rho(a)^{\mathrm{h}}$,
- the map $[\ ,\ ]_\nabla\colon\ \operatorname{im}\phi \times \operatorname{im}\phi \to \operatorname{im}\phi$ $\qquad [\phi a, \phi b]_\nabla := \phi[a, b]$.

The maps $\rho_\nabla$ and $[\ ,\ ]_\nabla$ admit unique extensions to an anchor and a bracket on $TE$ by

- requiring $\rho_\nabla\colon \mathfrak{X}(E) \to \mathfrak{X}(E)$ to be $\mathcal{C}^\infty(E)$-linear,
- requiring $[\ ,\ ]_\nabla\colon \mathfrak{X}(E) \times \mathfrak{X}(E) \to \mathfrak{X}(E)$ to satisfy

$$[V_1, fV_2]_\nabla = \ldots, \qquad [fV_1, V_2]_\nabla = \ldots$$

*Is $(TE, g_\nabla, \rho_\nabla, [\ ,\ ]_\nabla)$ a Courant algebroid?*

## Is $(TE, g_\triangledown, \rho_\triangledown, [\ ,\ ]_\triangledown)$ a Courant algebroid?

**Theorem.** Let $(\mathbb{E} = TM \oplus E, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ be a Courant algebroid over $M$ and $\nabla$ a vector bundle connection on $\mathrm{pr}\colon E \to M$. The tuple $(TE, g_\triangledown, \rho_\triangledown, [\ ,\ ]_\triangledown)$ over $E$ satisfies, for $V_1, V_2, V_3 \in \mathfrak{X}(E)$, the following:

(Ca1) $[V_1, V_1]_\triangledown = \rho_\triangledown^* \mathrm{d} g_\triangledown(V_1, V_1)$,

(Ca2) $\rho_\triangledown(V_1) g_\triangledown(V_2, V_3) = g_\triangledown([V_1, V_2]_\triangledown, V_3) + g_\triangledown(V_2, [V_1, V_3]_\triangledown)$,

(Ca3) the Jacobi identity is **fails** in general.

- However, for $a, b \in \Gamma(\mathbb{E})$, we have that

$$\rho_\triangledown([\phi a, \phi b]_\triangledown) - [\rho_\triangledown(\phi a), \rho_\triangledown(\phi b)]_{\mathrm{Lie}} = R_\triangledown(\rho(a), \rho(b))^\upsilon,$$

where $R_\triangledown \in \Omega^2(M, \mathrm{End}E)$ is the curvature of $\nabla$.

- Moreover, $(TE, g_\triangledown, \rho_\triangledown, [\ ,\ ]_\triangledown)$ becomes a Courant algebroid if and only if

$$R_\triangledown(\rho(a), \rho(b)) = 0.$$

If $E \neq 0$, the tuple $(TE, g_\triangledown, \rho_\triangledown, [\ ,\ ]_\triangledown)$ is always non-transitive.

The vertical lift of the field of endomorphisms $A \in \Gamma(\mathrm{End}E) \rightsquigarrow A^\upsilon \in \mathfrak{X}(E)$.

## Examples of Courant algebroid lifts

The lift of a transitive Courant algebroid (regarded as $\mathbb{T}M \oplus \mathcal{G}$) by a connection $\nabla$ on $E = T^*M \oplus \mathcal{G}$ is a Courant algebroid if and only if $\nabla$ is flat.

**Example 1.** The Courant algebroid $(\mathbb{T}M, \langle \ , \ \rangle_+, \mathrm{pr}_{TM}, [\ ,\ ])$ lifts to a Courant algebroid on $T(T^*M)$ by flat affine connections on $M$. In this case,

$g_\nabla$ is the Patterson-Walker metric [P., W. 1952]

$$g_\nabla|_{T^*U} = \mathrm{d}p_i \odot \mathrm{d}x^i - p_k(\mathrm{pr}^*\Gamma^k_{ij})\,\mathrm{d}x^i \odot \mathrm{d}x^j.$$

The $B_n$-Courant algebroid is $(\mathbb{T}M \oplus (M \times \mathbb{R}), \langle \ , \ \rangle, \mathrm{pr}_{TM}, [\ ,\ ])$, where

- $\langle X + \alpha + f, Y + \beta + \mathrm{g} \rangle_+ := \alpha(Y) + \beta(X) + 2f\mathrm{g}$
- $[X + \alpha + f, Y + \beta + \mathrm{g}] := [X, Y]_{\mathrm{Lie}} + (L_X\beta - \iota_Y \mathrm{d}\alpha + 2f\mathrm{d}\mathrm{g}) + (X\mathrm{g} - Yf)$

$\rightsquigarrow$ The **odd Patterson-Walker metric** on $T^*M \oplus (M \times \mathbb{R})$.

**Example 2.** For an almost complex structure $J$ and a torsion-free connection $\nabla$,

$$\nabla^J_X(\alpha + f) := \nabla_X\alpha + Xf + \alpha(JX)$$

is a connection on $T^*M \oplus (M \times \mathbb{R})$, which is flat if and only if $(J, \nabla)$ is a special complex complex structure, that is, $\nabla$ is flat and $\mathrm{d}_\nabla J = 0$.

# Courant algebroid lift of a non-transitive Courant algebroid

[Liu, Weinstein, Xu 1997]: A pair of Lie algebroids $(A, \rho_A, [\ ,\ ]_A)$ and $(A^*, \rho_{A^*}, [\ ,\ ]_{A^*})$ is said to be a **Lie bialgebroid** if

$$\mathrm{d}_{A^*}[\varphi, \psi]_A = [\mathrm{d}_{A^*}\varphi, \psi]_A + [\varphi, \mathrm{d}_{A^*}\psi]_A.$$

- There is a natural Courant algebroid on the double of a Lie bialgebroud $A \oplus A^*$.

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In particular, by choosing $\rho_{A^*} = 0$ and $[\ ,\ ]_{A^*} = 0$, every Lie algebroid $(A, \rho_A, [\ ,\ ]_A)$ induces the Courant algebroid $(A \oplus A^*, \langle\ ,\ \rangle_+, \rho_A \circ \mathrm{pr}_A, [\ ,\ ])$.

Poisson structure $\pi$ on $M$ $\rightsquigarrow$ Lie algebroid on $T^*M$, anchor: $\pi \colon T^*M \to TM$ $\rightsquigarrow$ Courant algebroid on $TM \oplus T^*M$, anchor: $\pi \circ \mathrm{pr}_{T^*M}$

**Example 3.** The Courant algebroid on $\mathbb{T}M$ induced by a Poisson structure $\pi$ lifts to a Courant algebroid by an affine connection $\nabla$ if and only if $\nabla$ is flat on Hamiltonian vector fields, that is,

$$R_\nabla(\pi(\alpha), \pi(\beta)) = 0$$

$\Rightarrow$ Courant algebroid lifts are not restricted to the transitive case and flat connections.

## Relation to Courant algebroid actions

[Li-Bland, Meinrenken 2009]: Let $(\mathbb{E}, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ be a Courant algebroid over $M$. A **Courant algebroid action on a manifold** $M'$ is a pair $(\Psi, \varrho)$ consisting of $\Psi\colon M' \to M$ and an $\mathbb{R}$-linear map $\varrho\colon \Gamma(\mathbb{E}) \to \mathfrak{X}(M')$ such that

$$\Psi_{*q}\varrho(a) = \rho(a)_{\Psi(q)}, \qquad \varrho(fa) = (\Psi^* f)\varrho(a), \qquad [\varrho(a), \varrho(b)]_{\mathrm{Lie}} = \varrho([a, b]).$$

- The **stabilizer** of the action at a point $q \in M'$ is the kernel of the map

$$\begin{aligned} \varrho_q\colon \mathbb{E}_{\Psi(q)} &\to T_q M' \\ \mathsf{a} &\mapsto \varrho(a)_q, \end{aligned}$$

where $a$ is an arbitrary section of $\mathbb{E}$ such that $a_{\Psi(q)} = \mathsf{a}$.

- A Courant algebroid action with coisotropic stabilizers naturally induces a Courant algebroid on the pull-back bundle $\Psi^*\mathbb{E} \to M'$.

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Courant algebroid lifts provide large class of exampels of Courant algebroid actions:

**Theorem.** Let $(\mathbb{E} = TM \oplus E, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ be a Courant algebroid over $M$ and $\nabla$ a vector bundle connection on $\mathrm{pr}\colon E \to M$ such that $R_\nabla(\rho(a), \rho(b)) = 0$. Then $(\mathrm{pr}, \rho_\nabla \circ \phi)$ is a Courant algebroid action on $E$ with Lagrangian stabilizers.

## What if $R_\triangledown(\rho(a), \rho(b)) \neq 0$?

Given a tuple $(\mathbb{E}, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ over $M$, we define the map $F\colon \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \to \mathfrak{X}(M)$ by

$$F(a, b) := \rho([a, b]) - [\rho(a), \rho(b)].$$

A **curved Courant algebroid** over $M$ is a tuple $(\mathbb{E}, \langle\ ,\ \rangle, \rho, [\ ,\ ])$ satisfying

(Ca1) $[a, a] = \rho^* \mathrm{d}\langle a, a\rangle$,

(Ca2) $\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$,

and moreover,

($\star$) $F \in \Gamma(\wedge^2 \mathbb{E}^* \otimes TM)$ such that $F(a, b) = 0$ if $a \in \ker \rho$.

- We call $F$ the **curvature** of the curved Courant algebroid.

$\Rightarrow$ The Courant algebroid lift by $\nabla$ is a curved Courant algebroid, whose curvature is

$$F(\phi a, \phi b) = R_\triangledown(\rho(a), \rho(b))^\upsilon.$$

Every curved Courant algebroid still satisfies

- $\rho \circ \rho^* = 0$,
- $[a, fb] = (\rho(a)f)b + f[a, b]$,
- $[fa, b] = -(\rho(b)f)a + f[a, b] + 2\langle a, b\rangle \rho^* \mathrm{d}f$.

## Exact curved Courant algebroids

[Ševera 1998]: Every exact (i.e. transitive such that $\ker\rho = \operatorname{im}\rho^*$) Courant algebroid over $M$ is isomorphic to $(\mathbb{T}M, \langle\ ,\ \rangle_+, \mathrm{pr}_{TM}, [\ ,\ ]_H)$ for some $H \in \Omega^3_{\mathrm{cl}}(M)$, where

$$[X+\alpha, Y+\beta]_H = [X,Y]_{\mathrm{Lie}} + L_X\beta - \iota_X \mathrm{d}\alpha + \iota_Y\iota_X H.$$

- Moreover, $H^3_{\mathrm{dR}}(M) \xleftrightarrow{\sim} \left\{ \text{ exact Courant algebroids over } M \ \right\} \Big/ \sim.$

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### Step 1. Curved exterior derivative

The exterior derivative si fully characterized by the formula

$$\mathrm{d}\varphi = (k+1)\,\mathrm{skew}(\nabla\varphi),$$

where $\varphi \in \Omega^k(M)$ and $\nabla$ is an arbitrary torsion-free connection on $M$.

dropping the torsion-freeness $\rightsquigarrow$ operator on $\Omega^\bullet(M)$ depending only on the torsion $T$ of $\nabla$ $\rightsquigarrow$ the **curved exterior derivative** $=: \mathrm{d}^T$

For an arbitrary $T \in \Omega^2(M, TM)$, we still have $\mathrm{d}^T \in \mathrm{gDer}^1(\Omega^\bullet(M))$ and $(\mathrm{d}^T f)(X) = Xf$, but $\mathrm{d}^T \circ \mathrm{d}^T = 0 \quad \Leftrightarrow \quad T = 0$. In fact,

$$\Omega^2(M, TM) \xleftrightarrow{\sim} \left\{ \ D \in \mathrm{gDer}^1(\Omega^\bullet(M)) \text{ such that } (Df)(X) = Xf \ \right\}.$$

## Step 2. Curved Cartan calculus

**curved Lie derivative**

$$L^T_X := [\iota_X, \mathrm{d}^T]_{\mathrm{g}}$$

**curved bracket**

$$[X,Y]^T := [X,Y]_{\mathrm{Lie}} + T(X,Y) = \nabla_X Y - \nabla_Y X$$

Every $T \in \Omega^2(M,TM)$ makes $(\mathrm{End}(\Omega^\bullet(M)), [\ ,\ ]_{\mathrm{g}})$ a curved differential graded Lie algebra with the differential $[\mathrm{d}^T, \ ]_{\mathrm{g}}$ and the curvature $\mathcal{R} := \frac{1}{2}[\mathrm{d}^T, \mathrm{d}^T]_{\mathrm{g}} = \mathrm{d}^T \circ \mathrm{d}^T$.

$$[\iota_X, \iota_Y]_{\mathrm{g}} = 0, \qquad [\iota_X, \mathrm{d}^T]_{\mathrm{g}} = L^T_X, \qquad [\mathrm{d}^T, L^T_X]_{\mathrm{g}} = [\mathcal{R}, \iota_X]_{\mathrm{g}},$$
$$[\mathrm{d}^T, \mathrm{d}^T]_{\mathrm{g}} = 2\mathcal{R}, \quad [L^T_X, \iota_Y]_{\mathrm{g}} = \iota_{[X,Y]^T}, \quad [L^T_X, L^T_Y]_{\mathrm{g}} = L_{[X,Y]^T} + [[\iota_X, \mathcal{R}]_{\mathrm{g}}, \iota_Y]_{\mathrm{g}}.$$

**Theorem.** Every exact curved Courant algebroid over $M$ is isomorphic to $(\mathbb{T}M, \langle\ ,\ \rangle_+, \mathrm{pr}_{TM}, [\ ,\ ]^T_H)$ for some $T \in \Omega^2(M,TM)$ and $H \in \Omega^3(M)$, where

$$[X+\alpha, Y+\beta]^T_H = [X,Y]^T + L^T_X \beta - \iota_X \mathrm{d}^T \alpha + \iota_Y \iota_X H.$$

- Moreover,

$$\bigsqcup_{T \in \Omega^2(M,TM)} \frac{\Omega^3(M)}{\mathrm{im}\,\mathrm{d}^T} \xleftrightarrow{\sim} \left\{ \text{ exact curved Courant algebroids over } M \right\} \Big/ \sim .$$

M., Rubio
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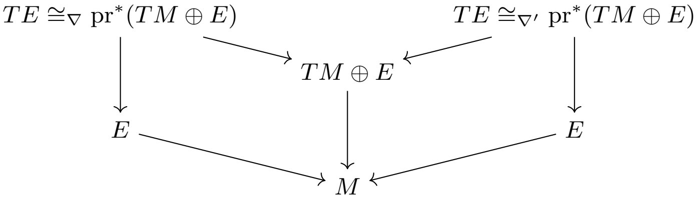
arXiv:2511.04743, 28 pages, November 2025.




# Thank you for your attention!