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Moduli spaces of spacefilling branes in symplectic 4-manifolds

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Symplectic manifolds

Let M be a smooth manifold.

Definition

A symplectic form is $\omega \in \Omega^2(M)$ such that

- ω is closed: $d\omega = 0$
- ω is non-degenerate: the following bundle map is injective

$$TM \to T^*M, \; v \mapsto \iota_v \omega$$

Examples

- Any orientable surface
- $(\mathbb{R}^{2n}, \omega)$ where $\omega = \sum_{i=1}^n dx_i \wedge dy_i$
- $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$
- $\mathbb{CP}^n$

Notation

Given any 2-form σ, we denote by the same symbol the bundle map $TM \to T^*M$.

Spacefilling branes

Definition

A spacefilling brane structure on (M, ω) is $F \in \Omega^2_{closed}(M)$ such that $I := \omega^{-1} \circ F$ satisfies

$$I^2 = -1.$$

Lemma

There is a bijection between

- *Spacefilling brane structures* F
- *Complex structures* I *s.t.* $\omega(I\cdot, \cdot)$ *is skew*
- *Complex structures* I *and* $F \in \Omega^2(M)$ *such that* $F + i\omega$ *is holomorphic symplectic w.r.t.* I

$$F \quad \leftrightarrow \quad I := \omega^{-1} \circ F \quad \leftrightarrow \quad F + i\omega$$

Definition

On a complex manifold, a holomorphic symplectic form is $F' + i\omega' \in \Omega^{2,0}_{closed}(M, \mathbb{C})$, so that F' (or ω') is non-degenerate.

Examples of spacefilling branes

Example

Let $M = \mathbb{C}^2$ (or $M = \mathbb{T}^4$), with complex coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Take

$$\omega := Im(dz_1 \wedge dz_2)$$
$$F := Re(dz_1 \wedge dz_2)$$

Example

The $K3$ manifold is

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3.$$

It is a compact, simply connected, 4-dimensional smooth manifold.
Every complex structure on M admits a holomorphic symplectic form $F + i\omega$.

Remarks on branes

Remark

Branes arose in the study of the A-model [Kapustin-Orlov 2001]

Remark

Branes are natural in terms of the generalized geometry [Gualtieri 2003].

A generalized submanifold of M is (Y, F) where

- Y is a submanifold
- $F \in \Omega^2_{closed}(Y)$.

Its generalized tangent bundle sits in $TM \oplus T^*M$:

$$\tau_F Y = \{(X, \xi) : X \in TY, \xi|_{TY} = \iota_X F\}.$$

Let $J\colon TM \oplus T^*M \to TM \oplus T^*M$ be a generalized complex structure.
A brane is a generalized submanifold (Y, F) such that

$$J(\tau_F Y) = \tau_F Y.$$

Notice: ω symplectic form $\rightsquigarrow$ generalized complex structure

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Main result

Let $M = K3$ manifold or $M = \mathbb{T}^4$, with any symplectic form ω admitting a spacefilling brane. Let

- $$\mathcal{M}_\omega := \{\text{Spacefilling branes}\}/\text{Sympl}(M,\omega)_*$$

  where Sympl$(M,\omega)_*$ is the symplectomorphisms inducing $Id_{H^\bullet(M,\mathbb{R})}$.
- $$Q_{[\omega]} := \{[F'] \in H^2(M,\mathbb{R}) : [F'] \wedge [\omega] = 0 \text{ and } [F'] \wedge [F'] = [\omega] \wedge [\omega]\},$$

  a codimension two submanifold of $H^2(M,\mathbb{R})$.

Theorem (Kirchhoff-Lukat, Z.)

- *$\mathcal{M}_\omega$ is a smooth manifold (possibly non-Hausdorff)*
- *This map is a local diffeomorphism:*

$$\mathcal{M}_\omega \to Q_{[\omega]}$$
$$F' \textit{ mod } ... \mapsto [F'].$$

Main result (cont.)

Corollary

$\mathcal{M}_\omega$ is smooth, non-compact,

- *of dimension 20 if $M = K3$ manifold,*
- *of dimension 4 if $M = \mathbb{T}^4$.*

Complex 4-manifolds

Lemma

Let M *be an oriented 4-manifold and* $\Omega \in \Omega^2(M, \mathbb{C})$*.*
Ω *is holomorphic symplectic w.r.t. some complex structure iff*

- HS1 $\Omega \wedge \bar{\Omega} > 0$ *everywhere on* M*,*
- HS2 $\Omega \wedge \Omega = 0$*,*
- HS3 $d\Omega = 0$*.*

The complex structure is recovered by

$$T^{0,1}M := \{X \in T_{\mathbb{C}}M | i_X \Omega = 0\}.$$

Proposition

Let M^4 *be compact, admitting a complex structure and a holomorphic symplectic form. There is a bijection:*

$$\{\text{Complex structures}\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Lines } \mathbb{C}\Omega \subset \Omega^2(M, \mathbb{C}) \\ \text{satisfying HS1-HS3} \end{array} \right\}$$

The local Torelli theorem

Let M^4 compact, admitting a complex structure and a holomorphic symplectic form. Let

$$\mathcal{Q} := \mathbb{P}\{A \in H^2(M, \mathbb{C}) : A \wedge \bar{A} > 0, A \wedge A = 0\},$$

a complex submanifold of $\mathbb{P}(H^2(M, \mathbb{C}))$.

The period map is

$$\begin{aligned} \mathcal{P} \colon \{\text{Complex structures}\} / \text{Diff}_*(M) &\to \mathcal{Q}, \\ I' \text{ mod } \ldots &\mapsto \mathbb{C}[\Omega'] \end{aligned}$$

where Ω' is a holomorphic symplectic form w.r.t. I'.

The local Torelli theorem (cont)

Again: the period map is

$$\mathcal{P}\colon \left\{\text{Complex structures}\right\}/\text{Diff}_*(M) \to \mathcal{Q},$$
$$I' \text{ mod } \ldots \mapsto \mathbb{C}[\Omega']$$

Theorem (Local Torelli)

Let M^4 be compact, admitting a Kähler complex structure I and a holomorphic symplectic structure Ω.
$\mathcal{P}$ yields a bijection between

- *small deformations of the complex structure I, up to Diff$_*(M)$*
- *small deformations of the complex line $\mathbb{C}[\Omega]$ in $\mathcal{Q}$.*

Consequence:
The domain of $\mathcal{P}$ is a smooth manifold (non-Hausdorff), so that $\mathcal{P}$ is a local diffeomorphism.

Remark:
Necessarily $M = K3$ manifold or $M = \mathbb{T}^4$.

Smoothness of the restricted period map

Let (M^4, ω) be compact symplectic admitting a spacefilling brane whose complex structure I is Kähler.
Get a well-defined restricted period map:

$$\mathcal{P}_\omega : \begin{Bmatrix} \text{Complex structures } I' \\ \text{s.t. } \omega \circ I' \text{ is skew} \end{Bmatrix} / \mathrm{Diff}_*(M) \to \{\mathbb{C}[\Omega'] \in \mathcal{Q} \text{ s.t. } Im[\Omega'] = [\omega]\},$$

$$I' \text{ mod } \ldots \mapsto \mathbb{C}[\omega \circ I' + i\omega].$$

Proposition (Kirchhoff-Lukat, Z.)

$\mathcal{P}_\omega$ yields a bijection between:

a) *small deformations I' of the complex structure I s.t. $\omega \circ I'$ is skew, up to Diff$_*(M)$*

b) *small deformations $\mathbb{C}[\Omega']$ of the complex line $\mathbb{C}[\Omega]$ in $\mathcal{Q}$ s.t. $Im[\Omega'] = [\omega]$.*

The codomain of $\mathcal{P}_\omega$ is submanifold of $\mathcal{Q}$.

Consequence:
The domain of $\mathcal{P}_\omega$ is a submanifold of the domain of $\mathcal{P}$, thus smooth.
$\mathcal{P}_\omega$ is a local diffeomorphism.

Back to the moduli space of spacefilling branes

The restricted period map $\mathcal{P}_\omega$ corresponds to the map of the main result:

$$
\begin{array}{ccc}
\dfrac{\left\{\begin{array}{c}\text{Complex structures } I' \\ \text{s.t. } \omega \circ I' \text{ is skew}\end{array}\right\}}{\text{Diff}_*(M)} & \xrightarrow{\mathcal{P}_\omega} & \{\mathbb{C}[\Omega'] \in \mathcal{Q} \text{ s.t. } Im[\Omega'] = [\omega]\} \\
\wr\! \| & & \wr\! \| \\
\mathcal{M}_\omega = \dfrac{\{\text{Spacefilling branes}\}}{\text{Sympl}(M,\omega)_*} & \longrightarrow & Q_{[\omega]}.
\end{array}
$$

- The left map is the bijection $[I'] \mapsto [\omega \circ I']$
- The right map is the diffeomorphism $\mathbb{C}[\Omega'] \mapsto Re[\Omega']$
  (recall: $[\Omega'] \in H^2(M, \mathbb{C})$)

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Thank you for your attention