are more diverse than Kähler) came this way:

Take certain hyperbolic ORBIFOLD $M = H^4/\Gamma$

Then Z/Γ is symplectic CY orbifold

Can resolve singularities to get smooth CY 6-manifold.

IV. Minimal surfaces

We are looking at $Z \subseteq \Lambda^+ \to M$, called "twistor space" (Penrose, Atiyah-Hitchin-Singer)

Theorem Eells-Salamon There is an almost complex structure on Z whose J-hol. curves are in 1-1 correspondence with minimal surfaces in M.

- $T_z Z = V_z \oplus H_z$ by LC connection
- $Z \ni z$ is unit-length S.D. 2-form
- "raise an index" get a.c. of $j_z : T_{\pi(z)} M \to T_{\pi(z)} M$
- $H_z \cong T_{\pi(z)} M$
- $J_z := J_V \oplus -j_z$

(Could also use $+j_z$ but that's a completely different story...)

Now let $\Sigma \subseteq M$ be embedded oriented surface. It has a "twistor lift":

Let $e_1, e_2 \in T_\sigma \Sigma$ be oriented o.n. frame and $e_3, e_4 \in (T_\sigma \Sigma)^\perp$ be oriented o.n. frame

Then lift $\sigma \in \Sigma$ to $z = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4) \in Z$

4D analogue of Gauss lift.

Eells – Salamon: $\Sigma \subseteq M$ is minimal iff its twistor lift $\hat{\Sigma} \subseteq Z$ is J-hol.

Conversely if $\hat{\Sigma} \subseteq Z$ is J-hol. then it is the twistor lift of $\pi(\hat{\Sigma}) \subseteq M$ (which is therefore minimal).

Compare Weierstrass's Theorem:

$\Sigma \subseteq \mathbb{R}^3$ has Gauss map $u : \Sigma \to S^2$

Σ is minimal iff u is anti-holomorphic.

(This is why Eells – Salamon use $-j_z$ in their J.)

CONSEQUENCE If $(M^4, g, \text{orientation})$ has LC conn on Λ^+ for which ω_A tames J, then we can use "Gromov-Witten theory" to study minimal surfaces in M!

Asking $\omega_A(u, Ju) > 0$ $\forall u \neq 0$ is a curvature inequality for g:

$\forall v \in \Lambda^+$ of unit length we want

$$|\langle (W^+ + \tfrac{s}{12}) v, v \rangle| > |Ric_0(v)|$$

and $\det (W^+ + \frac{s}{12}) < 0$.

In particular, g hyperbolic has ω_A taming J.

(Remember that for g hyperbolic, $W^+ = 0 = Ric_0$ and $s = -12$.)

V Knots

H^4 has no compact minimal surfaces

(Eg maximum principle applied to distance from a point)

Instead they run out to infinity where they meet $\partial_\infty H^4 = S^3$ in a knot or link, as their "ideal boundary"

Idea: fix an oriented knot or link $L \subseteq S^3$ and count minimal surfaces in H^4 with ideal boundary L. Should get a topological link invariant

This is "Gromov–Witten theory of $(Z, \omega) \to H^4$."

Let $\mathcal{M}_{g,k}$ be set of all oriented minimal surfaces in H^4, of genus g and k boundary components.

Let $\mathcal{L}_k$ be set of all oriented links with k components. (Infinite dim. Banach manifold)

Let $\beta : \mathcal{M}_{g,k} \to \mathcal{L}_k$ send a minimal surface to its ideal boundary.

Theorem (F.)

- $\mathcal{M}_{g,k}$ is infinite dimensional Banach manifold
- $\beta : \mathcal{M}_{g,k} \to \mathcal{L}_k$ is Fredholm index 0. (BECAUSE $c_1(Z) = 0$!)